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Exact solutions of the 2D Ginzburg-Landau equation by the first integral method

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Abstract The first integral method is an efficient method for obtaining exact solutions of some nonlinear partial differential equations. This method can be applied to non integrable equations as well as to integrable ones. In this paper, the first integral method is used to construct exact solutions of the 2D Ginzburg-Landau equation.

Keywords. First integral method, 2D Ginzburg-Landau equation.

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1. Introduction

In the present letter we consider a class of nonlinear partial differential equation with constant coefficients which is called Ginzburg-Landau equation

$$u_t - iu_t + \frac{1}{2}u_{xx} + \frac{1}{2}(\beta - if)u_{yy} + (1 - i\delta)|u|^2 u = i\gamma u. \tag{1.1}$$

where $\beta,\ f,\ \delta,\ \gamma$ are real constants. As we all know Ginzburg-Landau equation is a class of a Schrödinger equation with a nonlinear term [20]. This equation governs the finite amplitude evolution of instability waves in a large variety of dissipative systems which are close to criticality. Various forms of Ginzburg-Landau equation arise in hydrodynamic instability theory: the development of Tollmien- Schlichting waves in plane Poiseuille flows, the nonlinear growth of convection rolls in the Rayleigh-Bénard problem, and appearance of Taylor vortices in the flow between counter rotating circular cylinders [4, 15].

During the past decades, many powerful methods to construct exact solutions of nonlinear evolution equations have been established and developed such as the inverse scattering transform [1], the Hirota's bilinear operators [11], the truncated Painleve expansion, the tanh-function expansion and the Jacobi elliptic function expansion [8, 13], the homogeneous balance method [23], the exp-function expansion method [2, 6, 10], the F-expansion method [12], the Bäcklund transformation method [14, 23], the sine-cosine method [18, 22] and so on. In this paper, using a

new method that is called the first integral method, we obtain some new exact solutions of Eq (1.1). The first-integral method was first proposed by Feng [9] in solving the Burgers-KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method has been widely used by many researchers, such as in [16, 17, 19] and the references therein.
The remainder of this paper is organized as follows. In Section 2, using the first-integral method which is based on the ring theory of commutative algebra, we establish the exact travelling wave solution for Eq. (1.1), which is full agreement with the previously known result in the literature. However, our results provide a good supplement to the existing literatures. Finally, some conclusions are given in Section 3.

2. Exact solutions to the Ginzburg-Landau equation

In this section, we discuss the exact solutions of Ginzburg-Landau equation as following:

Case 1

Assume that Eq. (1.1) has an exact solution in the form

$$u = \exp i(\eta) v(\xi),\ \eta = (px + qy + st). \tag{2.1}$$

where $v(x, y, t)$ is a real function and $p,\ q,\ s$ are constants to be determined. Substituting (2.1) into Eq. (1.1) and canceling $\exp i(\eta)$, gives the partial different equation for v

$$\left\{ \begin{array}{l} iv_t + \frac{1}{2}(v_{xx} + \beta v_{yy}) - \frac{1}{2} i f v_{yy} + i(pv_x + \beta v_y), \\ + fqv_y + i(\frac{1}{2}fq^2 - \gamma)v - [s + \frac{1}{2}(p^2 + \beta q^2)]v + v^3 - i\delta v^3 = 0. \end{array} \right. \tag{2.2}$$

Diving the Eq. (2.2) into real parts and imaginary parts, we have

$$\left\{ \begin{array}{l} \frac{1}{2}(v_{xx} + \beta v_{yy}) + fqv_y + v^3 - [s + \frac{1}{2}(p^2 + \beta q^2)]v = 0, \\ v_t - \frac{1}{2} f v_{yy} + (pv_x + \beta v_y) - \delta v^3 + (\frac{1}{2}fq^2 - \gamma)v = 0. \end{array} \right. \tag{2.3}$$

We seek firstly the traveling wave solutions in the form

$$v(x, y, t) = U(\xi),\ \xi = kx + ly + \nu t, \tag{2.4}$$

where $k,\ l,\ \nu,\ \xi_0$, are constants. Substituting (2.4) into Eqs.(2.3), we have the ordinary differential equations for $U(\xi)$

$$\left\{ \begin{array}{l} \frac{1}{2}(k^2 + \beta l^2)U'' + fqlU' + U^3 - [s + \frac{1}{2}(p^2 + \beta q^2)]U = 0, \\ -\frac{1}{2} f l^2 U'' + (pk + \beta ql + \nu)U' - \delta U^3 + (\frac{1}{2}fq^2 - \gamma)U = 0. \end{array} \right. \tag{2.5}$$

Under the constraint conditions:
$r = -\frac{2fql}{k^2+\beta l^2} = \frac{2(pk+\beta ql+\nu)}{fl^2}$, $b = -\frac{2}{k^2+\beta l^2} = -\frac{2\delta}{fl^2}$, $c = \frac{s+\frac{1}{2}(p^2+\beta q^2)}{k^2+\beta l^2} = \frac{fq^2-2\gamma}{fl^2}$.
we can get

$$U'' = rU' + bU^3 + cU. \tag{2.6}$$

let $z = U(\xi)$, $\omega = U'$, then Eq. (2.6) can be reformulated as a planar dynamic system

$$\left\{ \begin{array}{lcl} \frac{dz}{d\xi} & = & \omega, \\ \frac{d\omega}{d\xi} & = & r\omega + bz^3 + cz. \end{array} \right. \tag{2.7}$$

In order to find the travelling wave solutions of Eq. (1.1), we are now applying the first-integral method, the key idea of which is to utilize the so-called Divison Theorem which is based on the ring theory of commutative algebra and to obtain first integrals to system (2.7) under various parameter conditions. Then using these first integrals, the above two-dimensional autonomous system (2.7) can be reduced to some different first-order integrable differential equations. Finally, through solving these first-order differential equations directly, travelling wave solutions for Eq. (1.1) can be established easily.
Next, let us recall the Divisor Theorem for two variables in the complex domain $\mathbb{C}$:

Theorem 2.1. ***(Divison Theorem)*** *Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$, and that $P(\omega, z)$ is irreductible $\mathbb{C}[\omega, z]$. If $Q(\omega, z)$ vanishes at any zero point of $P(\omega, z)$, then there exists a polynomial $Q(\omega, z)$ in $\mathbb{C}[\omega, z]$ such that*

$$Q(\omega, z) = P(\omega, z).G(\omega, z).$$

It follows immediately from the following theorem in commutative algebra [5]:

Theorem 2.2. ***(Hilbert-Nullstellensatz Theorem)*** *Let k be a field and L an algebraic closure of k. Then*
i) Every ideal γ of $k[X_1, ...X_n]$ not containing 1 admits at least one zero in L^n
ii) Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be two elements of L^n . For the set of polynomials of $k[X_1, ...X_n]$ zero at x to be identical with the set of polynomials of $k[X_1, ...X_n]$ zero at y, it is necessary and sufficient that there exists a k-automorphisms s of L such that $y_i = s_i$ for $1 \leq i \leq n$. iii) For an ideal α of $k[X_1, ...X_n]$ to be maximal, it is necessary and sufficient that there exists x in L such that α is the set of polynomials of $k[X_1, ...X_n]$ zero at x.
iv) For a polynomial Q of $k[X_1, ...X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $k[X_1, ...X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.

Now, we apply the Division Theorem 1 to seek the first integral to (2.7). Suppose that $z = z(\xi)$ and $\omega = \omega(\xi)$ are the nontrivial solutions to (2.7), and $p(\omega, z) = \sum_{i=0}^{m} a_i(z)\omega^i$, is irreducible polynomial in $\mathbb{C}[\omega, z]$ such that

$$p(\omega(\xi), z(\xi)) = \sum_{i=0}^{m} a_i(z(\xi))\omega^i(\xi) = 0, \tag{2.8}$$

where $a_i(z)$ $(i = 0, 1, ..., m)$ are polynomials of z and all relatively prime in $\mathbb{C}[\omega, z]$, $a_m(z) \neq 0$. Equation (2.8) is also called the first integral to (2.7).We start our study by assuming $m = 2$ in (2.8). Note that $\frac{dp}{d\xi}$ is polynomial in z and ω, and $p(\omega(\xi), z(\xi)) = 0$ implies $\frac{dp}{d\xi}\ |_{(2.7)} = 0$. By the Division Theorem, the exists a polynomial $H(z, \omega) =$

$h(z)+g(z)\omega$ in $\mathbb{C}[\omega,z]$ such that

$$\begin{aligned} &\tfrac{dp}{d\xi}\mid_{(2.7)}=(\tfrac{\partial p}{\partial z}\tfrac{\partial z}{\partial \xi}+\tfrac{\partial p}{\partial \omega}\tfrac{\partial \omega}{\partial \xi}\mid_{(2.7)} \\ &=\textstyle\sum_{i=0}^{2}a_i^{'}(z)\omega^{i+1}+\sum_{i=0}^{2}ia_i(z)\omega^{i-1}(r\omega+bz^3+cz^3) \\ &=(h(z)+g(z)\omega)(\textstyle\sum_{i=0}^{2}a_i(z)\omega^i). \end{aligned} \tag{2.9}$$

On equating the coefficients of ω^i $(i=0,\ 1,\ 2,\ 3)$ on both sides of (2.9), we have

$$a_2^{'}(z)=g(z)a_2(z), \tag{2.10}$$

$$a_1^{'}(z)=g(z)a_1(z)+(h(z)-2r)a_2(z), \tag{2.11}$$

$$a_0^{'}(z)=-2a_2(z)[bz^3+cz]+a_1(z)(h(z)-r)+g(z)a_0(z), \tag{2.12}$$

$$a_0(z)h(z)=a_1(z)[bz^3+cz^2]. \tag{2.13}$$

From Eq. (2.10) we obtain that $a_2(z)$ is a constant and $g(z)=0$, and take $a_2(z)=1$, for simplicity. Eqs. (2.11) and (2.12) become

$$a_1^{'}(z)=(h(z)-2r)a_2(z), \tag{2.14}$$

$$a_0^{'}(z)=-2[bz^2+cz+d]+a_1(z)(h(z)-r). \tag{2.15}$$

Balancing the degrees of $a_0(z)$, $a_1(z)$, and $h(z)=0$, we conclude that deg $h(z)=0$, $h(z)=1$ i.e., deg $a_1(z)=1$ or $a_1(z)=2$. Otherwise, if deg $h(z)=k>1$, then we deduce deg $a_1(z)=k+1$ and deg $a_0(z)=2k+2$ from (2.14) and (2.15). This yields a condradiction with Eq. (2.13), for the degree of the polynomial $a_1(z)[bz^3+cz]$ is $k+4$, but the degree of polynomial $a_0(z)h(z)$ is $3k+2$.

In case deg $h(x)=0$, assume that $a_1(z)=A_1z+A_0$, $A_1,\ A_0\in\mathbb{C}$ with $A_1\neq 0$. By (2.14) and (2.15), we deduce that $h(z)=A_1-2r$ and

$$a_0(z)=-\frac{b}{3}z^4+[\frac{A_1}{2}(A_1+r)-c]z^2+[A_0(A_1+r)]z+D,$$

where D is an integration constant.

Substituting $a_0(z)$, $a_1(z)$ and $h(z)$ in (2.13) and setting all the coefficients of powers z^i $(i=0,\ 1,\ 2,\ 3)$ to be zero, we get

$$\begin{cases} A_1b=(A_1+r)(-\frac{b}{2}), \\ A_0b=0, \\ A_1c=(-c+A_1(A_1+r))\frac{4r}{3}, \\ A_0c=A_0(A_1+r)\frac{4r}{3}, \\ \frac{4r}{3}D=0. \end{cases} \tag{2.16}$$

By solving Eq. (2.16), we have

$$A_1=-\tfrac{2r}{3},\ A_0=0,\ D=0,\ c=-\tfrac{2r^2}{9}. \tag{2.17}$$

Now, taking the solution set Eq. (2.17) into account, Eq. (2.8) becomes

$$\omega^2 + (-\tfrac{2r}{3})z\omega - \tfrac{b}{2}z^4 + (\tfrac{r^2}{9})z^2. \tag{2.18}$$

From (2.18), ω can be expressed in terms of z, i.e.,

$$\omega = \frac{r}{3}z \pm \sqrt{\frac{b}{2}}z^2. \tag{2.19}$$

Finally, combining Eq. (2.7) with Eq. (2.19) and changing to the original variables, we obtain traveling wave solutions to Eq. (1.1) as

$$u(x,y,t) = (\tfrac{r}{3}C_0\frac{\exp(\frac{r}{3}\xi)}{1\mp C_0\sqrt{\frac{b}{2}}\exp(\frac{r}{3}\xi)})\exp i(\eta), \tag{2.20}$$

where $r = -\frac{2fql}{k^2+\beta l^2} = \frac{2(pk+\beta ql+\nu)}{fl^2}$, $b = -\frac{2}{k^2+\beta l^2} = -\frac{2\delta}{fl^2}$,
$c = \frac{s+\frac{1}{2}(p^2+\beta q^2)}{k^2+\beta l^2} = \frac{fq^2-2\gamma}{fl^2}$, $\eta = (px + qy + [q^2(f-\frac{1}{2}\beta) - (2\gamma+\frac{1}{2}p^2)]t)$,
$\xi = kx + ly + [(2\delta f - \beta)ql - \beta k]t$ and C_0 remain arbitrary.
In case deg $g(x) = 1$, the argument is identical, so we omit it.

Case 2

Assume that Eq. (1.1) has an exact solution in the form

$$u = \exp i(\eta)v(\xi),\ \eta = (px + st), \tag{2.21}$$

where $v(x,y,t)$ is a real function and $p,\ s$ are constants to be determined. Substituting (2.21) into (1.1) yields

$$\left\{\begin{array}{l} iv_t + \frac{1}{2}(v_{xx} + \beta v_{yy}) - \frac{1}{2}ifv_{yy} + i(pv_x + \beta v_y), \\ -i\gamma v - [s + \frac{1}{2}(p^2]v + v^3 - i\delta v^3 = 0. \end{array}\right. \tag{2.22}$$

Separating the real part and imaginary part of (2.22), we have

$$\left\{\begin{array}{l} \frac{1}{2}(v_{xx} + \beta v_{yy}) + v^3 - [s + \frac{1}{2}p^2]v = 0, \\ v_t - \frac{1}{2}fv_{yy} + (pv_x + \beta v_y) - \delta v^3 - \gamma v = 0. \end{array}\right. \tag{2.23}$$

Suppose

$$v(x,y,t) = U(\xi),\ \xi = kx + ly + \nu t, \tag{2.24}$$

where $k,\ l,\ \nu$, are constants. Substituting (2.24) into Eqs.(2.23), we have the ordinary differential equations for $U(\xi)$

$$\left\{\begin{array}{l} \frac{1}{2}(k^2 + \beta l^2)U'' + U^3 - [s + \frac{1}{2}p^2]U = 0, \\ -\frac{1}{2}fl^2U'' + (pk+\nu)U' - \delta U^3 + (-\gamma)U = 0. \end{array}\right. \tag{2.25}$$

Under the constrain conditions:
$\nu = -pk$, $b = -\frac{2}{k^2+\beta l^2} = -\frac{2\delta}{fl^2}$, $c = 2\frac{s+\frac{1}{2}p^2}{k^2+\beta l^2} = \frac{-2\gamma}{fl^2}$.
Let $z = U(\xi)$, $\omega = U'$, then Eq. (2.25) can be reformulated as a planar dynamic system

$$\left\{\begin{array}{rl} \frac{dz}{d\xi} = & \omega, \\ \frac{d\omega}{d\xi} = & bz^3 + cz. \end{array}\right. \tag{2.26}$$

Now, we apply the Division Theorem to seek the first integral to (2.26). Suppose that $z = z(\xi)$ and $\omega = \omega(\xi)$ are the nontrivial solutions to (2.26), and $p(\omega, z) = \sum_{i=0}^{m} a_i(z)\omega^i$, is irreducible polynomial in $\mathbf{C}[\omega, z]$ such that

$$p(\omega(\xi), z(\xi)) = \sum_{i=0}^{m} a_i(z(\xi))\omega^i(\xi) = 0, \tag{2.27}$$

where $a_i(z)$ $(i = 0, 1, ..., m)$ are polynomials of z and all relatively prime in $\mathbf{C}[\omega, z]$, $a_m(z) \neq 0$. Equation (2.27) is also called the first integral to (2.26). We start our study by assuming $m = 1$ in (2.27). Note that $\frac{dp}{d\xi}$ is polynomial in z and ω, and $p(\omega(\xi), z(\xi)) = 0$ implies $\frac{dp}{d\xi} |_{(2.26)}= 0$. By the Division Theorem, the exists a polynomial $H(z, \omega) = h(z) + g(z)\omega$ in $\mathbf{C}[\omega, z]$ such that

$$\begin{aligned} &\frac{dp}{d\xi} |_{(2.26)}= (\frac{\partial p}{\partial z}\frac{\partial z}{\partial \xi} + \frac{\partial p}{\partial \omega}\frac{\partial \omega}{\partial \xi} |_{(2.26)}, \\ &= \textstyle\sum_{i=0}^{1} a_i^{'}(z)\omega^{i+1} + \sum_{i=0}^{1} i a_i(z)\omega^{i-1}(cz + bz^3), \\ &= (h(z) + g(z)\omega)(\textstyle\sum_{i=0}^{1} a_i(z)\omega^i). \end{aligned} \tag{2.28}$$

where prime denotes differentiating with respect to the variable z. On equating the coefficients of ω^i $(i = 0,\ 1,\ 2)$ on both sides of (2.28), we have

$$a_1^{'}(z) = g(z)a_1(z), \tag{2.29}$$

$$a_0^{'}(z) = g(z)a_0(z) + h(z)a_1(z), \tag{2.30}$$

$$h(z)a_0(z) = a_1(z)[cz + bz^3]. \tag{2.31}$$

Since, $a_1(z)$ and $g(z)$ are polynomials, from (2.29) we conclude that $a_1(z)$ is a constant and $g(z) = 0$. for simplicity, we take $a_1(z) = 1$, and balancing the degrees of $a_0(z)$, and $h(z)$, we conclude that $deg\ h(z) = 1$ only. Suppose that $h(z) = Az + B$, then from (2.30) we find

$$a_0(z) = \frac{1}{2}Az^2 + Bz + C\ (A \neq 0), \tag{2.32}$$

where C is an constants to be determined. Substituting $a_0(z)$, $a_1(z)$ and $h(z)$ in (2.31) and setting all the coefficients of powers z to be zero, we obtain a system of nonlinear algebraic equations, and by solving it, we obtain the following solutions:

$$B = 0,\ A = \sqrt{2b},\ C = \tfrac{c}{\sqrt{2b}}. \tag{2.33}$$

$$B = 0,\ A = -\sqrt{2b},\ C = -\tfrac{c}{\sqrt{2b}}. \tag{2.34}$$

Using the conditions (2.33) and (2.34) in (2.26), we obtain

$$\omega = \frac{c}{\sqrt{2b}} + \frac{\sqrt{2b}}{2}z^2, \tag{2.35}$$

$$\omega = -\frac{c}{\sqrt{2b}} - \frac{\sqrt{2b}}{2} z^2, \tag{2.36}$$

respectively. Combining (2.35) with (2.26), we obtain the exact solution to Eq. (1.1) and the exact solution to (1.1) can be written as :

$$u(x,t) = \exp(i\eta)\sqrt{\frac{\gamma}{\delta}}\tan(\sqrt{[s+\frac{1}{2}p^2]}[k(x-pt)]) + C_0, \tag{2.37}$$

where $\eta = px - (\frac{\gamma}{fl^2}(k^2+\beta l^2)+p^2)t$ and C_0 remain arbitrary.
Similarly, for the cases of (2.36), we have anther exact solution to (1.1) can be written as

$$u(x,t) = \exp(i\eta)\sqrt{\tfrac{\gamma}{\delta}}\tan -(\sqrt{[s+\tfrac{1}{2}p^2]}[k(x-pt)]) + C_1, \tag{2.38}$$

where $\eta = px - (\frac{\gamma}{fl^2}(k^2+\beta l^2)+p^2)t$ and C_1 remain arbitrary.
Notice that the results in this paper are based on the assumption of $m = 1,\ 2$ in Eq. (2.27), respectively. The discussion becomes more complicated for the cases $m = 3,\ 4$ since the hyper-elliptic integrals, the irregular singular point theory, and the elliptic integrals of the second kind are involved. We do not need to consider the case $m \geq 5$ because of the fact that an algebraic equation with the degree greater than or equal to 5 is generally not solvable.

3. Conclusions

In this paper, the first integral method was applied successfully for solving the 2D Ginzburg-Landau equation. Thus, we conclude that the proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas.

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