Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-9
DOI:10.22034/cmde.2024.60228.2566

Computing high-index eigenvalues for the Sturm-Liouville equation with Robin boundary conditions

Yasser Khalili[1,*] **and Mohammad Dehghan**[2]
[1]Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, 578 Sari, Iran.
[2]Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran.

Abstract

The calculation of the high-indexed eigenvalues of Sturm-Liouville problems tends to be a complex job. The larger eigenvalues are estimated, the greater scaled errors we gain. In this paper, we study the Sturm-Liouville problems subject to Rubin boundary conditions in which the high-indexed eigenvalues are computed by means of an efficient method. On the contrary of previous methods, estimated errors of further eigenvalues are less than primary ones. A good illustration of the accuracy of our method can be delineated by some numerical examples.

Keywords. Sturm-Liouville operator, Robin boundary condition, High-index eigenvalue, Quadrature method.
2010 Mathematics Subject Classification. 34L15, 34B24.

1. Introduction

In this paper, the Sturm-Liouvile equation

$$-y''(x) + p(x)y(x) = \lambda y(x), \quad 0 \leq x \leq T, \tag{1.1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \tag{1.2}$$

$$V(y) := y'(T) + Hy(T) = 0, \tag{1.3}$$

is considered. Here, the parameters h and H are real, and the smooth function $p(x) \in L^2[0, T]$. Also, $\lambda = \mu^2$ is a spectral parameter.

Sturm-Liouville problems (SLPs) appear in geophysical models, acoustics, etc. For an example, Sturm-Liouville equations are taken to model the wave problem. Regarding the wave propagation problems in waveguides, knowing whole eigenvalues that are accord with the radiation and subsidence is required. Furthermore, for obtaining a reasonably exact solution of a partial differential equations by Fourier method of separation of variables, one has to consider a large number of eigenvalues with a reliable estimation.

It is worth noting that finding the eigenvalues in most boundary value problem tends to be a difficult task. Having said that, some scientists have given it their best shot to deal with this challenging problem. The existing numerical methods such as Variational Iteration Method (VIM) [12], Homotopy Analysis Method (HAM) [1], Boundary Value Method (BVM) [2, 5], Sinc-Galerkin Method [8], Differential Transform Method [6] and Shooting-type algorithm [10] help us get a suitable accuracy in calculating the eigenvalues.

However, the aforementioned numerical methods, apart from their competence, have a drawback in which the errors of the calculating of the eigenvalues increase when large-index eigenvalues are desired. Unlike the existing numerical methods, the method presented in this paper resolves this problem. That is to say, the more accurate approximations are obtained for higher eigenvalues. We applied this technique for Sturm-Liouville equations with respect to Dirichlet

Received: 23 January 2024 ; Accepted: 12 November 2024.
* Corresponding author. Email: y.khalili@sanru.ac.ir.

boundary conditions in [3]. We note that this asymptotic method is a complementary to other numerical methods, and their combination provides us with a uniform accurate estimates of all eigenvalues.

The existence and stability of the solution of the boundary value problem (1.1)-(1.3) have been discussed in [4]. We present the following theorem, which is important for us to show the main results of this paper (see [4] for more details).

Theorem 1.1. *Let $\varphi(x;\mu)$ be the solution of the Sturm-Liouville Equation (1.1) with initial conditions*

$$\varphi(0;\mu)=1, \quad \varphi'(0;\mu)=h, \tag{1.4}$$

then $\varphi(x;\mu)$ has the representation:

$$\varphi(x;\mu)=\cos\mu x+\frac{h\sin\mu x}{\mu}+\int_0^x \frac{\sin\mu(x-t)}{\mu}p(t)\varphi(t;\mu)dt.$$

The eigenvalues of the boundary value problem (1.1)-(1.3) have been computed by finding the roots of the characteristic function $\Delta(\mu)=\varphi'(T;\mu)+H\varphi(T;\mu)$ with respect to μ.

2. Analysis of the method

Taking the successive approximation's method, we have

$$\varphi(x;\mu)=\sum_{n=0}^{\infty}\varphi_n(x;\mu), \tag{2.1}$$

as the solution of (1.1) satisfying initial conditions (1.4), in which

$$\varphi_0(x;\mu) = \cos\mu x+\frac{h\sin\mu x}{\mu}, \tag{2.2}$$

$$\varphi_{n+1}(x;\mu) = \int_0^x \frac{\sin\mu(x-t)}{\mu}p(t)\varphi_n(t;\mu)dt. \tag{2.3}$$

The kth partial sum of series (2.1),

$$\varphi^{(k)}(x;\mu)=\sum_{n=0}^{k}\varphi_n(x;\mu), \tag{2.4}$$

satisfying

$$\varphi^{(k)}(0;\mu)=1, \quad \varphi^{(k)'}(0;\mu)=h, \quad k\geq 0,$$

can be regarded as the asymptotic solution of the problem (1.1) and (1.4).

Owing to the fact that μ has a chance to be large enough, the integrand of (2.3) ends up the high oscillation. We come up with a quadrature method established by Iserles in [7] in order to remove this predicament. The following theorems support this idea.

Theorem 2.1. *Consider the continuously differentiable functions f and g. Let*

$$\sigma_0[f](t) = f(t),$$

$$\sigma_{k+1}[f](t) = \frac{d}{dt}\frac{\sigma_k[f](t)}{g'(t)}, \qquad k=0,1,\cdots,$$

$$I[f](x) := \int_0^x f(t)e^{i\mu g(t)}dt, \quad \text{as} \quad \mu\longrightarrow\infty.$$

Thus, a suitable estimation for above integral as sufficiently large μ can be established via

$$I[f](x)\sim P_l^A[f]=-\sum_{m=1}^{l}\frac{1}{(-i\mu)^m}\left\{\frac{e^{i\mu g(x)}}{g'(x)}\sigma_{m-1}[f](x)-\frac{e^{i\mu g(0)}}{g'(0)}\sigma_{m-1}[f](0)\right\}, \tag{2.5}$$

for any fixed x and $l\geq 1$.

Proof. The reader is referred to [7]. □

Theorem 2.2. *For any continuously differentiable functions f and g, we have*

$$P_l^A[f] - I[f] = O\left(\frac{1}{\mu^{l+1}}\right). \tag{2.6}$$

Proof. The reader is referred to [7]. □

3. Application of the method to Sturm-Liouville equations

In our approach, the Sturm-Liouville equation subject to the initial condition at zero has been reduced to an integral equation of the Volterra type in which the solution is asymptotically given by the method of the so called successive approximation. Then, substituting this solution into (1.3) followed by calculating the characteristic function, eigenvalues of the original problem can be computed, approximately.

3.1. **Quadrature term with $l = 1$.** Supposing that the first phrase of (2.5) is considered, i.e., $l = 1$, afterwards taking (2.6), we have

$$I[f](x) = -\frac{i}{\mu}\left\{\frac{e^{i\mu g(x)}}{g'(x)}f(x) - \frac{e^{i\mu g(0)}}{g'(0)}f(0)\right\} + O\left(\frac{1}{\mu^2}\right). \tag{3.1}$$

Using Euler's formula, we can write

$$\int_0^x f(t)e^{i\mu g(t)}dt = \int_0^x f(t)\cos\mu g(t)dt + i\int_0^x f(t)\sin\mu g(t)dt,$$

and then (3.1), regarding $g(t) = x - 2t$, turns to

$$\int_0^x f(t)\cos\mu(x-2t)dt = \frac{\sin\mu x}{2\mu}f_{+0}(x) + O\left(\frac{1}{\mu^2}\right), \tag{3.2}$$

$$\int_0^x f(t)\sin\mu(x-2t)dt = \frac{\cos\mu x}{2\mu}f_{-0}(x) + O\left(\frac{1}{\mu^2}\right), \tag{3.3}$$

for $f_{\pm 0}(x) = f(x) \pm f(0)$ and sufficiently large μ.

Now applying (2.2), (2.3), and the triangle relations, one gives

$$\begin{aligned}\varphi_1(x;\mu) &= \frac{\sin\mu x}{2\mu}P(x) - \frac{h\cos\mu x}{2\mu^2}P(x)\\ &+ \frac{1}{2\mu}\int_0^x \sin\mu(x-2t)p(t)dt + \frac{h}{2\mu^2}\int_0^x \cos\mu(x-2t)p(t)dt + o\left(\frac{1}{\mu^3}\right),\end{aligned} \tag{3.4}$$

where $P(x) = \int_0^x p(t)dt$. Taking (3.2) and (3.3) in the right hand side of (3.4), we will have

$$\varphi_1(x;\mu) = \frac{\sin\mu x}{2\mu}P(x) + \frac{\cos\mu x}{4\mu^2}\left(-2hP(x) + p_{-0}(x)\right) + h\frac{\sin\mu x}{4\mu^3}p_{+0}(x) + O\left(\frac{1}{\mu^4}\right), \tag{3.5}$$

as $\mu \to \infty$.

Regarding third terms of (3.5) vanishes asymptotically after substituting it in (2.3), it gives

$$\varphi_2(x;\mu) = \frac{1}{\mu}\int_0^x \sin\mu(x-t)p(t)\left(\frac{\sin\mu t}{2\mu}P(t) + \frac{\cos\mu t}{4\mu^2}\left(-2hP(t) + p_{-0}(t)\right)\right)dt + O\left(\frac{1}{\mu^4}\right).$$

Using (3.3), we get

$$\varphi_2(x;\mu) = \frac{-\cos\mu x}{4\mu^2}\int_0^x p(t)P(t)dt + \frac{\sin\mu x}{8\mu^3}\left(\int_0^x p(t)\left(p_{-0}(t) - 4hP(t)\right)dt + p(x)P(x)\right) + O\left(\frac{1}{\mu^4}\right), \tag{3.6}$$

as $\mu \to \infty$.

Continuing this iterative process, we get

$$\varphi_3(x;\mu) = \frac{-\sin\mu x}{8\mu^3}\int_0^x p(t)\left(\int_0^t p(r)P(r)dr\right)dt + O\left(\frac{1}{\mu^4}\right), \tag{3.7}$$

and $\varphi_4(x;\mu)$ is included in $O\left(\frac{1}{\mu^4}\right)$, as $\mu \to \infty$.
Therefore, considering $l = 1$ in quadrature method, we can just use four phrases of (2.4), i.e.,

$$\varphi^{(3)}(x;\mu) = \sum_{n=0}^{3}\varphi_n(x,\mu).$$

So, from (2.2), (3.5), (3.6), and (3.7), one gives

$$\begin{aligned}
\varphi^{(3)}(x;\mu) &= \cos\mu x + \frac{\sin\mu x}{2\mu}(2h + P(x)) \\
&+ \frac{\cos\mu x}{4\mu^2}\left(-2hP(x) + p_{-0}(x) - \int_0^x p(t)P(t)dt\right) \\
&+ \frac{\sin\mu x}{8\mu^3}\left(2hp_{+0}(x) + \int_0^x p(t)\left(p_{-0}(t) - 4hP(t)\right)dt\right. \\
&+ \left. p(x)P(x) - \int_0^x p(t)\left(\int_0^t p(r)P(r)dr\right)dt\right) + O\left(\frac{1}{\mu^4}\right) \\
&\equiv \hat{\varphi}^{(3)}(x;\mu) + O\left(\frac{1}{\mu^4}\right),
\end{aligned} \tag{3.8}$$

as $\mu \to \infty$.

3.2. **Quadrature term with $l = 2$.** Considering the first and second phrases of (2.5), i.e., $l = 2$,

$$\begin{aligned}
P_2^A[f] = &-\frac{i}{\mu}\left\{\frac{e^{i\mu g(x)}}{g'(x)}f(x) - \frac{e^{i\mu g(0)}}{g'(0)}f(0)\right\} \\
&+ \frac{1}{\mu^2}\left\{\frac{e^{i\mu g(x)}}{g'(x)}\left(-\frac{g''(x)}{g'^2(x)}f(x) + \frac{1}{g'(x)}f'(x)\right) - \frac{e^{i\mu g(0)}}{g'(0)}\left(-\frac{g''(0)}{g'^2(0)}f(0) + \frac{1}{g'(0)}f'(0)\right)\right\},
\end{aligned} \tag{3.9}$$

and taking the real and imaginary parts of $I[f](x)$ and Equation (3.9), we get

$$\int_0^x f(t)\cos\mu(x-2t)dt = \frac{\sin\mu x}{2\mu}f_{+0}(x) + \frac{\cos\mu x}{4\mu^2}f'_{-0}(x) + O\left(\frac{1}{\mu^3}\right), \tag{3.10}$$

$$\int_0^x f(t)\sin\mu(x-2t)dt = \frac{\cos\mu x}{2\mu}f_{-0}(x) - \frac{\sin\mu x}{4\mu^2}f'_{+0}(x) + O\left(\frac{1}{\mu^3}\right), \tag{3.11}$$

as $\mu \to \infty$.

On the other hand, using the quadrature method (3.10) and (3.11) on the third and fourth integrals of (3.4), we get

$$\begin{aligned}
\varphi_1(x;\mu) &= \frac{\sin\mu x}{2\mu}P(x) + \frac{\cos\mu x}{4\mu^2}(-2hP(x) + p_{-0}(x)) \\
&+ \frac{\sin\mu x}{8\mu^3}\left(2hp_{+0}(x) - p'_{+0}(x)\right) + h\frac{\cos\mu x}{8\mu^4}p'_{-0}(x) + O\left(\frac{1}{\mu^5}\right).
\end{aligned} \tag{3.12}$$

Analogously, substituting first three phrases of (3.12) into (2.3), one can write

$$\begin{aligned}\varphi_2(x;\mu) &= -\frac{1}{4\mu^2}\left(\cos\mu x\int_0^x p(t)P(t)dt+\int_0^x\cos\mu(x-2t)p(t)P(t)dt\right)\\ &+ \frac{1}{8\mu^3}\left(\sin\mu x\left(-2h\int_0^x p(t)P(t)dt+\int_0^x p(t)p_{-0}(t)dt\right)\right.\\ &- \left.2h\int_0^x\sin\mu(x-2t)p(t)P(t)dt+\int_0^x\sin\mu(x-2t)p(t)p_{-0}(t)dt\right)\\ &+ \frac{1}{16\mu^4}\left(\cos\mu x\left(\int_0^x(p(t)p'_{+0}(t)-2hp(t)p_{+0}(t))dt\right)\right.\\ &+ \left.\int_0^x\cos\mu(x-2t)(2hp(t)p_{+0}(t)-p(t)p'_{+0}(t))dt\right)+O\left(\frac{1}{\mu^5}\right).\end{aligned}$$

The above equation can be rewritten by means of (3.10) and (3.11) as follows

$$\begin{aligned}\varphi_2(x;\mu) &= -\frac{\cos\mu x}{4\mu^2}\int_0^x p(t)P(t)dt\\ &+ \frac{\sin\mu x}{8\mu^3}\left(p(x)P(x)-2h\int_0^x p(t)P(t)dt+\int_0^x p(t)p_{-0}(t)dt\right)\\ &+ \frac{\cos\mu x}{16\mu^4}\left((p(x)P(x))'-p^2(0)-2hp(x)P(x)+p(x)p_{-0}(x)\right.\\ &+ \left.\int_0^x(p(t)p'_{+0}(t)-2hp(t)p_{+0}(t))dt\right)+O\left(\frac{1}{\mu^5}\right).\end{aligned} \tag{3.13}$$

Substituting two first terms of φ_2 in the iterative relation (2.3), we can get

$$\begin{aligned}\varphi_3(x;\mu) &= \frac{-\sin\mu x}{8\mu^3}\int_0^x p(t)\left(\int_0^t p(r)P(r)dr\right)dt\\ &- \frac{\cos\mu x}{16\mu^4}\left((1-2h)\int_0^x p(t)\left(\int_0^t p(r)P(r)dr\right)dt\right.\\ &+ \left.\int_0^x\left(p^2(t)P(t)+p(t)\int_0^t p(r)p_{-0}(r)dr\right)dt\right)+O\left(\frac{1}{\mu^5}\right).\end{aligned} \tag{3.14}$$

Analogously, taking the first term of the previous equation into (2.3), we have

$$\varphi_4(x;\mu)=\frac{\cos\mu x}{16\mu^4}\int_0^x p(t)\left(\int_0^t p(r)\left(\int_0^r p(s)P(s)ds\right)dr\right)dt+O\left(\frac{1}{\mu^5}\right). \tag{3.15}$$

Doing the same argument as for $l=1$, five phrases of (2.4) must be written in the following form

$$\varphi^{(4)}(x;\mu)=\sum_{n=0}^{4}\varphi_n(x,\mu).$$

So, from (3.12), (3.13), (3.14), and (3.15) we give

$$\begin{aligned}
\varphi^{(4)}(x;\mu) &= \cos\mu x + \frac{\sin\mu x}{2\mu}\left(2h + P(x)\right) \\
&+ \frac{\cos\mu x}{4\mu^2}\left(-2hP(x) + p_{-0}(x) - \int_0^x p(t)P(t)dt\right) \\
&+ \frac{\sin\mu x}{8\mu^3}\Bigg(2hp_{+0}(x) - p'_{+0}(x) + p(x)P(x) - 2h\int_0^x p(t)P(t)dt \\
&+ \int_0^x p(t)p_{-0}(t)dt - \int_0^x p(t)\left(\int_0^t p(r)P(r)dr\right)dt\Bigg) \\
&+ \frac{\cos\mu x}{16\mu^4}\Bigg(2hp'_{-0}(x) + (p(x)P(x))' - p^2(0) - 2hp(x)P(x) \\
&+ p(x)p_{-0}(x) + \int_0^x (p(t)p'_{+0}(t) - 2hp(t)p_{+0}(t))dt \\
&- (1-2h)\int_0^x p(t)\left(\int_0^t p(r)P(r)dr\right)dt \\
&+ \int_0^x \left(p^2(t)P(t) + q(t)\int_0^t p(r)p_{-0}(r)dr\right)dt \\
&+ \int_0^x p(t)\left(\int_0^t p(r)\left(\int_0^r p(s)P(s)ds\right)dr\right)dt\Bigg) + O\left(\frac{1}{\mu^5}\right) \\
&\equiv \hat{\varphi}^{(4)}(x;\mu) + O\left(\frac{1}{\mu^5}\right).
\end{aligned} \tag{3.16}$$

Remark 3.1. Imposing the second boundary condition (1.3) on the approximating solution (3.8) or (3.16) we have

$$\hat{\varphi}^{(k)'}(T;\mu) + H\hat{\varphi}^{(k)}(T;\mu) = 0, \tag{3.17}$$

for $k = 3, 4$. Now, we can employ root-finder packages in some mathematical software such as Mathematica and Maple on (3.17) in order to get the eigenvalues λ of (1.1)-(1.3). The symbol $\lambda_n^{(k)}$ stands for the eigenvalues of (3.17).

Remark 3.2. The solutions $\hat{\varphi}^{(k)}(x;\mu)$, $k = 3, 4$, obtained from (3.8) and (3.16) asymptotically satisfy the boundary condition (1.2) and the Sturm-Liouville Equation (1.1) in the following form as sufficiently large μ

$$\hat{\varphi}^{(k)''}(x;\mu) + (\mu^2 - p(x))\hat{\varphi}^{(k)}(x;\mu) = O\left(\frac{1}{\mu^{k-1}}\right).$$

Proof. It is similar to Theorem 3.1 in [3]. □

4. Numerical Results

In this section, eigenvalues of some Sturm-Liouville problems have been calculated through the given method in the previous section. As a result of comparing our results with those obtained from a Matlab package so-called *Matslise*, the efficiency of our method is observed. We note that Matslise has been extracted from the Constant Perturbation Method (CPM) [9].

Example 4.1. Let the boundary value problem be given

$$\begin{cases} -y'' + e^x y = \lambda y, \\ y'(0) - y(0) = 0, \\ y(\pi) = 0. \end{cases}$$

By means of (3.8) and (3.16), one can give

$$\begin{aligned}\hat{\varphi}^{(3)}(x;\mu) &= \cos(\mu x) + \frac{\sin(\mu x)}{2\mu}(1+e^{x}) + \frac{\cos(\mu x)}{8\mu^2}(1-e^{2x}) \\ &+ \frac{\sin(\mu x)}{4\mu^3}\left(1/3 + 7/4\,e^{x} - 1/12\,e^{3x}\right), \\ \hat{\varphi}^{(4)}(x;\mu) &= \cos(\mu x) + \frac{\sin(\mu x)}{2\mu}(1+e^{x}) + \frac{\cos(\mu x)}{8\mu^2}(1-e^{2x}) \\ &+ \frac{\sin(\mu x)}{4\mu^3}\left(1/3 + 1/4\,e^{x} - 1/12\,e^{3x}\right) \\ &+ \frac{\cos(\mu x)}{8\mu^4}\left(-39/48 + 5/12\,e^{x} + 1/8\,e^{2x} - 1/4\,e^{3x} + 1/48\,e^{4x}\right),\end{aligned}$$

respectively. Table 1 shows the eigenvalues $\lambda_n^{(k)}$ for $k = 3, 4$ and their accuracy in comparison with ones calculated by Matslise software [9] ($\lambda_n^{(CPM)}$).

Example 4.2. Let the following problem be given

$$\begin{cases} -y'' + x^2 y = \lambda y, \\ y'(0) + 2y(0) = 0, \\ y'(1) - y(1) = 0 \end{cases}$$

(see [11]).

The solution of this problem can be written as follows

$$\begin{aligned}\hat{\varphi}^{(3)}(x;\mu) &= \cos(\mu x) + \frac{\sin(\mu x)}{\mu}(-2 + 1/6\,x^3) + \frac{\cos(\mu x)}{2\mu^2}(1/2\,x^2 + 2/3\,x^3 - 1/36\,x^6) \\ &- \frac{\sin(\mu x)}{2\mu^3}(x^2 - 2/15\,x^5 - 1/9\,x^6 + 1/648\,x^9), \\ \hat{\varphi}^{(4)}(x;\mu) &= \cos(\mu x) + \frac{\sin(\mu x)}{\mu}(-2 + 1/6\,x^3) + \frac{\cos(\mu x)}{2\mu^2}(1/2\,x^2 + 2/3\,x^3 - 1/36\,x^6) \\ &- \frac{\sin(\mu x)}{2\mu^3}(1/2\,x + x^2 - 2/15\,x^5 - 1/18\,x^6 + 1/648\,x^9) \\ &- \frac{\cos(\mu x)}{2\mu^4}(x - 19/48\,x^4 - 4/15\,x^5 + 1/120\,x^8 + 5/1296\,x^9 - 1/15552\,x^{12}),\end{aligned}$$

by (3.8) and (3.16). The result of the eigenvalues $\lambda_n^{(k)}, k = 3, 4$ for this problem and their accuracy in comparison with those computed via Matslise [9] ($\lambda_n^{(CPM)}$) has been shown in Table 2.

Example 4.3. Let the following Paine problem be given

$$\begin{cases} -y'' + (x+0.1)^{-2}\,y = \lambda y, \\ y'(0) + y(0) = 0, \\ y(\pi) = 0. \end{cases}$$

The eigenvalues of this problem have been shown in Table 3.

Remark 4.4. We know that for $q \in C^2[0, T]$,

$$\lambda_n = n^2 + n + \frac{1}{4} + \frac{2}{\pi}\left(1 + \frac{1}{2}P(T)\right) + O(n^{-1}) \equiv \lambda_n^* + O(n^{-1}), \quad n \gg 1$$

TABLE 1. Computed eigenvalues for $p(x) = e^x$.

n	$\lambda_n^{(3)}$	$\lambda_n^{(4)}$	$\lvert\lambda_n^{(3)} - \lambda_n^{(CPM)}\rvert$	$\lvert\lambda_n^{(4)} - \lambda_n^{(CPM)}\rvert$
1	3.374345928854	1.6781185449602	0.039	1.656
5	28.69748406719	27.416790082342	0.628	0.652
10	97.980629093769	97.863487980023	0.0041	0.121
20	387.87945174495	387.936894173193	0.067	0.009
50	2457.9217994528	2457.93588074950	0.014	0.00025
100	9907.9309983176	9907.934679068410	0.003	0.000015
500	2.495079340913e5	2.495079342398e5	0.00014	1.218e-7
1000	9.9900793418e5	9.99007934225817e5	0.000045	2.19e-9

TABLE 2. Computed eigenvalues for $p(x) = x^2$.

n	$\lambda_n^{(3)}$	$\lambda_n^{(4)}$	$\lvert\lambda_n^{(3)} - \lambda_n^{(CPM)}\rvert$	$\lvert\lambda_n^{(4)} - \lambda_n^{(CPM)}\rvert$
1	3.71109504383	3.70269935002848	0.039	1.656
5	241.06548271016715	241.063243842134	0.628	0.652
10	981.29181538211844	981.2912532392090	0.0041	0.121
20	3942.174606214847	3942.174465537377	0.067	0.009
50	24668.344258135387	24668.34423562075	0.014	0.00025
100	98690.377324749698	98690.37731912081	0.003	0.000015
500	2.467395433604893e6	2.46739543360466e6	0.00014	1.218e-7
1000	9.869598734422497e6	9.86959873442244e6	0.000045	2.19e-9

TABLE 3. Computed eigenvalues for $p(x) = (x + 0.1)^{-2}$.

n	$\lambda_n^{(3)}$	$\lambda_n^{(4)}$	$\lvert\lambda_n^{(3)} - \lambda_n^{(CPM)}\rvert$	$\lvert\lambda_n^{(4)} - \lambda_n^{(CPM)}\rvert$
1	3.3420527305913194870	2.158316160814778	1.843	0.659
5	21.276094825863842783	19.20226841134420	3.447	5.521
10	92.763998945053579297	95.750585285834677	1.055	1.930
20	382.70314462001406755	383.158785314918861	0.331	0.123
50	2452.6987021184157335	2452.76184602488616	0.059	0.0039
100	9902.698376229433721	9902.71377238766073	0.015	0.00026
500	2.495026982871074e5	2.49502698895181e5	0.0006	4.42e-7
1000	9.99002698284449719e5	9.99002698436294e5	0.00015	2.75e-8

(see [4]). Having said that, although the estimation of eigenvalues λ_n^* in the classical form is more exact than $\lambda_n^{(3)}$, by calculating one additional estimate of order $O\left(\frac{1}{\mu^4}\right)$ in $\hat{\varphi}^{(4)}(x;\mu)$, we are able to calculate $\lambda_n^{(4)}$ which discloses the much better estimations.

5. CONCLUSION

In this article, the eigenvalue problem for the Sturm-Liouville operator under Robin boundary conditions on the segment $[0, T]$ is studied. As the main core of this article, we present the asymptotic form of the solution through the quadrature method which is efficient for solving high-oscillating integrals. This solution which satisfies the first

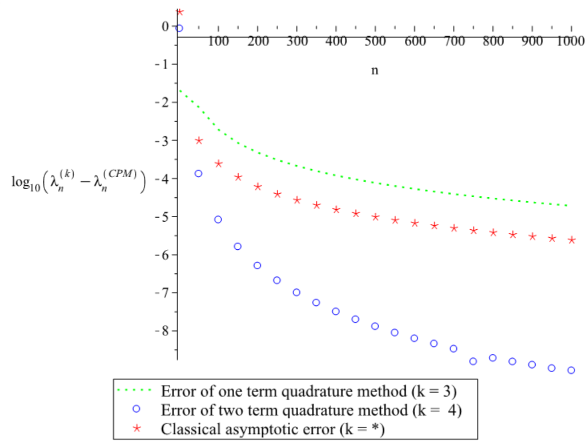

FIGURE 1. Errors approximated in Examples 4.1.

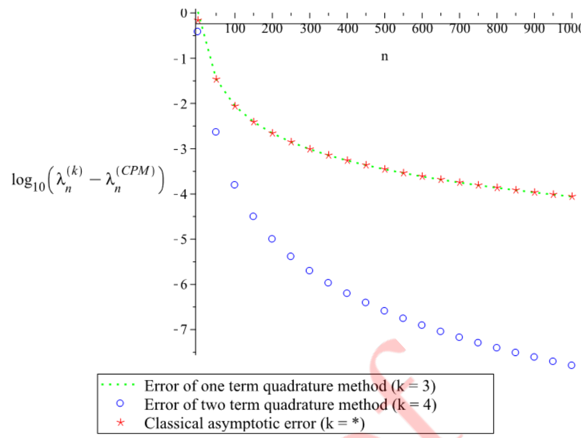

FIGURE 2. Errors approximated in Examples 4.3.

boundary condition enables us to obtain the eigenvalues taking the second boundary condition at the point T. On the contrary of other existing methods, the advantage of the method applied in this paper is that with increasing the eigenvalue index, the error magnitude becomes less. We recommend this efficient method for other forms of the Sturm-Liouville problems such as Sturm-Liouville equations with an impulse.

References

[1] S. Abbasbandy and A. Shirzadi, *A new application of the homotopy analysis method: Solving the Sturm-Liouville problems,* Commun Nonlinear Sci Numer Simulat, *16* (2011), 112–126.

[2] L. Aceto, P. Ghelardoni, and C. Magherini, *Boundary Value Methods asan extension of Numerovs method for Sturm-Liouville eigenvalue estimates,* Applied Numerical Mathematics, *59* (2009), 1644–1656.

[3] M. Dehghan, *An efficient method to approximate eigenfunctions and high-index eigenvalues of regular Sturm-Liouville problems,* Applied Mathematics and Computation, *279* (2016), 249–257.

[4] G. Freiling and V. A. Yurko, *Inverse Sturm-Liouville Problems and their Applications,* NOVA Science Publ., New York, 2001.

[5] P. Ghelardoni, *Approximations of Sturm-Liouville eigenvalues using Boundary Value Methods,* Applied Numerical Mathematics, *23* (1997), 311–325.

[6] I. A. H. Hassan, *On solving some eigenvalue problems by using a differential transformation,* Applied Mathematics and Computation, *127*(1) (2002), 1–22.

[7] A. Iserles and S. P. Norsett, *Efficient quadrature of highly oscillatory integrals using derivatives,* Proc. R. Soc. A, *461* (2005), 1383–1399.

[8] M. Jarratt, J. Lund, and K.L. Bowers, *Galerkin schemes and the Sinc-Galerkin method for singular Sturm-Liouville problems,* J. Computational Physics, *89*(1) (1990), 41–62.

[9] V. Ledoux, M. Van Daele, and G. Vanden Berghe, *Matslise: A Matlab package for the numerical solution of Sturm-Liouville and Schrodinger equations,* ACM Transactions on Mathematical Software, *31* (2005), 532–554.

[10] J. D. Pryce, *Classical and vector Sturm-Liouville problems: recent advances in singular-point analysis and shooting-type algorithms,* J. Computational and Applied Mathematics, *50*(1-3) (1994), 455–470.

[11] J. D. Pryce, *Numerical Solution of Sturm-Liouville Problems,* Oxford University Press, 1993.

[12] A. H. S. Taher and A. Malek, *An efficient algorithm for solving high-order Sturm-Liouville problems using variational iteration method,* Fixed Point Theory, *14* (2013), 193–210.