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A robust numerical method to find the solutions of time-fractional Klein-Gordon equation

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Abstract

In mathematical physics and physical applications, solving approximate solutions to fractional partial differential equations (FPDEs), especially for constant-coefficient FPDE (cc-FPDE) system is an interesting and challenging work. A robust numerical method is proposed to approximate the solutions of time-fractional Klein-Gordon equation (TFKG) which is based on the Lorentz group. The presented technique, namely the fictitious time integration technique (FTIT) converts the undetermined dependent variable $u(x,t)$ into a new variable with one dimension more. Then the group preserving technique (GPT) is implemented to integrate the new FPDEs. By appropriately assigning values to free parameters, the dynamic wave structures of certain analytical solutions are illustrated through three-dimensional and contour graphics. The consequences of numerical experiments are displayed to affirm the accuracy and efficiency of the offered scheme. The findings look at how the equation responds to slight changes in initial conditions. These outcomes can enhance the understanding of nonlinear dynamics in various fields like mathematical physics and fluid dynamics. Additionally, they validate the effectiveness of the utilized approaches, warranting further validation. The findings of this study could be more useful in a range of scientific disciplines and higher-level research.

Keywords. Fictitious time integration technique, Group preserving technique, Time fractional Klein-Gordon equation, Caputo derivative.
2010 Mathematics Subject Classification. 02.60.Lj, 02.70.Wz, 02.90.+p.

1. Introduction

Fractional calculus has a long history that dates back to the days of differential calculus, which was originated by Newton and Leibniz. Differential equations of fractional order are extensions of classical differential equations of integer order. It has been established that fractional differential equations provide a more effective explanation for physical phenomena than integer-order models, and that fractional order models better characterize dynamical systems [2, 3]. In mathematical physics, water wave concepts, dispersion processes, etc., nonlinear fractional-order differential equations are encountered [19, 20]. It is difficult to find solutions to fractional issues, especially fractional-order problems with higher dimensions involving novel derivatives, because fractional derivative operators are non-local and have complex features. To the best of our knowledge, there is no conventional approach for handling fractional nonlinear issues with larger dimensions. Using concepts from integer order calculus, a number of novel numerical [21]

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and analytical techniques are presented for fractional order problems; nonetheless, different approaches are being used by various researchers to investigate fractional order problems [37, 38].

There is a deep past of fractional differential equations. Its past is as long as classical calculus and up to date since 1695. Due to its broad application areas such as biology, physics, fluid dynamics, engineering etc, many mathematicians and physicists have paid great attention to this issue [2, 38]. Also, numerous researchers have worked on numerical and analytical solution of nonlinear fractional partial differential equations in which can be pointed to interesting finding such as [15, 30]. The development and obtaining the numerical and exact solutions of the equations, containing fractional derivative and integral, have gained great and significant importance. So, various methods have been investigated for this purpose. Among others, some of them are [24, 47]. In this study, we are going to gain the numerical solutions of time fractional Klein-Gordon equation according to Caputo sense derivative which is one of the fundamental equations considered in fractional calculus.

$$\begin{cases} \mathcal{D}_t^{\alpha}\Lambda(x,t) + \kappa\frac{\partial\Lambda}{\partial x}^2(x,t) = -\eta\Lambda(x,t) + \zeta\Lambda^{\beta}(x,t)\Lambda(x,t) + \mathcal{H}(x,t), \ (\mathbf{x},t) \in \Omega \subset \mathbb{R}^2, \\ \Lambda(x,0) = f_1(x), \ x \in \Omega_{\mathbf{x}}, \\ \Lambda(x,T) = f_2(x), \ x \in \Omega_{\mathbf{x}}, \\ \Lambda(0,t) = p_1(t), \quad t \in \Omega_t, \\ \Lambda(b,t) = p_2(t), \quad t \in \Omega_t, \end{cases} \tag{1.1}$$

where Ω_t and $\Omega_{\mathbf{x}}$ are boundaries of $\Omega := \{(x,t) : a \leq x \leq b, \ 0 \leq t \leq T\}$ in t and x, respectively. $\mathcal{H}(x,t)$ is a known forced term and in addition to these terms κ, η, ζ and β are real constants and also ζ can presented as a variable coefficients in some examples. Moreover $\mathcal{D}_t^{\alpha}\Lambda(x,t)$ is the Caputo fractional derivative with order of α defined as

$$\mathcal{D}_t^{\alpha}\Lambda(x,t) = \frac{\partial^{\alpha}\Lambda(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\sigma)^{m-\alpha-1}\frac{\partial^m\Lambda(x,\sigma)}{\partial\sigma^m}\mathrm{d}\sigma, & m-1 < \alpha < m, \\ \frac{\partial\Lambda(x,t)}{\partial t^m}, & \alpha = m. \end{cases} \tag{1.2}$$

The time fractional Klein-Gordon equation (TFKG) are inspected by many authors. Many mathematicians and physicists have been interested in finding robust and accurate numerical methods for solving TFKG. Examples of such methods include, for instance, a high-order difference scheme [45], Chebyshev spectral method [27], spectral collocation method [8], Wavelet method [23], implicit RBF meshless approach [17], linearized second-order scheme [36]. The more relevant and recent valuable works in applications of fractional calculus in the real-life applications there are such: fractional-Legendre spectral Galerkin method [5], Fractional-order Legendre-collocation method [6], fractional Walter's B fluid with applications [7], fractional Maxwell fluid in a porous medium [9], fractional logistic models in the frame [1].

Despite the various methods available, researchers continue to develop new methods to investigate NLPDEs [43, 44]. The set of solutions in most nonlinear models is challenging to characterize, prompting researchers to present families of new solutions [39, 41]. Also, researchers have developed several approaches for finding analytical solutions to NLPDEs [31, 40]. A plethora of computational techniques have been devised to delve into the essence of solutions for such equations, each offering its unique advantages. Among these methodologies is the fractional generalized CBS-BK equation [54], the (3+1)-D Burger system [22], the reduced differential transform method [46], the generalized Hirota bilinear scheme [**?**] and the analytical expansion technique [4]. In this work, we will first investigate time-fractional Klein-Gordon equation using the FTIT where is given to model. Furthermore, different examples will be obtained as solutions by applying the above mentioned technique.

The structure of this paper is given as under: This paper is formed because the section 2 contains fictitious time integration method. In section 3, the numerical results is presented. Finally, a concluding summary of the research is presented.

2. Fictitious time integration technique

The fictitious time integration technique is applied for solving time and space fractional Burger equation [34]. The construction of FTIT for TFKG is in the following:

According to the Caputo fractional derivative definition (1.2), Eq. (1.1) can be written as:

$$\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\Lambda_{\sigma\sigma}(x,\sigma)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \kappa\Lambda_{xx} + \eta\Lambda - \zeta\Lambda^{\beta} - \mathcal{H}(x,t) = 0. \tag{2.1}$$

Then, by multiplying the Eq. (2.1) into the amount ρ as a fictitious damping coefficient where can handle the stability of numerical integration, we have:

$$\frac{\rho}{\Gamma(2-\alpha)}\int_0^t \frac{u_{\sigma\sigma}(x,\sigma)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \rho\kappa\Lambda_{xx} + \rho\eta\Lambda - \rho\zeta\Lambda^{\beta} - \rho\mathcal{H}(x,t) = 0. \tag{2.2}$$

Imposing the following transformation:

$$\omega(x,t,\chi) = (1+\chi)^c\Lambda(x,t),\ 0 < c \leq 1, \tag{2.3}$$

Eq. (2.2) converts to a following form:

$$\frac{\rho}{(1+\chi)^c}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\omega_{\sigma\sigma}(x,\sigma,\chi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \kappa\omega_{xx}(x,t,\chi) + \eta\omega(x,t,\chi) - \zeta\omega^{\beta}(x,t,\chi)(1+\chi)^{c(1-\beta)}\right] - \rho\mathcal{H}(x,t) = 0. \tag{2.4}$$

From Eq. (2.3) it is easy to find:

$$\frac{\partial\omega}{\partial\chi} = c(1+\chi)^{c-1}\Lambda(x,t). \tag{2.5}$$

Combination of Eqs. (2.5) and (2.4) leads to:

$$\begin{aligned}\frac{\partial\omega}{\partial\chi} = &\frac{\rho}{(1+\chi)^c}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\omega_{\sigma\sigma}(x,\sigma,\chi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \kappa\omega_{xx}(x,t,\chi)\right.\\ &\left.+\eta\omega(x,t,\chi) - \zeta\omega^{\beta}(x,t,\chi)(1+\chi)^{c(1-\beta)}\right]\\ &-\rho\mathcal{H}(x,t) + c(1+\chi)^{c-1}u.\end{aligned} \tag{2.6}$$

Eq. (2.6) transforms into a new type of PDE for ω, by setting $u = \omega/(1+\chi)^c$:

$$\begin{aligned}\frac{\partial\omega}{\partial\chi} = &\frac{\rho}{(1+\chi)^c}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\omega_{\sigma\sigma}(x,\sigma,\chi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \kappa\omega_{xx}(x,t,\chi)\right.\\ &\left.+\eta\omega(x,t,\chi) - \zeta\omega^{\beta}(x,t,\chi)(1+\chi)^{c(1-\beta)}\right] - \rho\mathcal{H}(x,t) + \frac{c\omega}{1+\chi}.\end{aligned} \tag{2.7}$$

Using

$$\frac{\partial}{\partial\chi}\left(\frac{\omega}{(1+\chi)^c}\right) = \frac{\omega_\chi}{(1+\chi)^c} - \frac{c\omega}{(1+\chi)^{1+c}}, \tag{2.8}$$

and by multiplying the integrating factor $1/(1+\chi)^c$ on both sides of Eq. (2.7), results:

$$\begin{aligned}\frac{\partial}{\partial\chi}\left(\frac{\omega}{(1+\chi)^c}\right) &= \frac{\rho}{(1+\chi)^{2c}}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\omega_{\sigma\sigma}(x,\sigma,\chi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \kappa\omega_{xx}(x,t,\chi)\right.\\ &+ \left.\eta\omega(x,t,\chi) - \zeta\omega^{\beta}(x,t,\chi)(1+\chi)^{c(1-\beta)}\right] - \frac{\rho\mathcal{H}(x,t)}{(1+\chi)^c}.\end{aligned} \tag{2.9}$$

Using again the relation $\Lambda = \frac{\omega}{(1+\chi)^c}$, results:

$$\begin{aligned}\Lambda_\chi = &\frac{\rho}{(1+\chi)^c}\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\Lambda_{\sigma\sigma}(x,\sigma,\chi)}{(t-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \kappa\Lambda_{xx}(x,t,\chi)\right.\\ &\left.+\eta\Lambda(x,t,\chi) - \zeta\Lambda^{\beta}(x,t,\chi) - \mathcal{H}(x,t)\right].\end{aligned} \tag{2.10}$$

Supposing $\Lambda_i^j(\chi) := \Lambda(x_i, t_j, \chi)$ and $\mathcal{H}_i^j := \mathcal{H}(x_i, t_j)$. By using this notations, Eq.(2.10) can be written to the following form:

$$\frac{d}{d\chi}\Lambda_i^j(\chi) = \frac{\rho}{(1+\chi)^c}\Bigg[\frac{1}{\Gamma(2-\alpha)}\int_0^{t_j}\frac{\Lambda_{\sigma\sigma}(x_i,\sigma,\chi)}{(t_j-\sigma)^{\alpha-1}}\mathrm{d}\sigma + \kappa\frac{\Lambda_{i+1}^j(\chi) - 2\Lambda_i^j(\chi) + \Lambda_{i-1}^j(\chi)}{\Delta x^2} - \eta\Lambda_i^j(\chi) - \zeta(\Lambda_i^j(\chi))^\beta - \mathcal{F}_i^j\Bigg], \tag{2.11}$$

where Δx and Δt are the uniform spatial lengths in the x and t directions, $\Lambda_i^j(\xi) := \Lambda(x_i, t_j, \chi)$, $\mathcal{H}_i^j := \mathcal{H}(x_i, t_j)$, and u_χ indicates the differential of u subjected to χ. Fully discretization of above equation needs to calculate the approximation of appeared integral as follows:

$$\int_0^{t_j}\frac{\Lambda_{\sigma\sigma}(x_i,\sigma,\chi)}{(t_j-\sigma)^{\alpha-1}}\mathrm{d}\sigma \approx \frac{\Lambda(x_i,t_3,\chi) - 2\Lambda(x_i,t_2,\chi) + \Lambda(x_i,t_1,\chi)}{\Delta t^2(t_j-t_1)^{\alpha-1}} + \sum_{l=2}^{j-1}\frac{\Lambda(x_i,t_{l+1},\chi) - 2\Lambda(x_i,t_l,\chi) + \Lambda(x_i,t_{l-1},\chi)}{\Delta t^2(t_j-t_l)^{\alpha-1}}, \tag{2.12}$$

where $\Delta x = \frac{b-a}{N_1}$, $\Delta t = \frac{T}{N_2}$, $x_i = a + i\Delta x$ and $t_j = j\Delta t$.
Considering $\Lambda = (\Lambda_1^1, \Lambda_1^2, ..., \Lambda_{N_1}^{N_2})^T$, we can write the Eq. (2.11) in the vector form as:

$$\Lambda' = \mathbf{G}(\Lambda, \chi),\ \Lambda \in \mathbb{R}^M, \chi \in \mathbb{R}, \tag{2.13}$$

where Λ is an M-dimensional state vector, $\mathbf{F} \in \mathbb{R}^M$ is a vector valued function of Λ and χ and $M = N_1 \times N_2$. Now, we implement the following scheme namely the group preserving technique (GPT) which is introduced by Liu in [33] to solve Eq. (2.11) by taking the initial value $\Lambda_i^j(0)$:

$$\Lambda_{k+1} = \Lambda_k + \frac{\left[\cosh\left(\frac{\Delta\chi\|\mathbf{G}_k\|}{\|\Lambda_k\|}\right) - 1\right]\mathbf{G}_k.\Lambda_k + \sinh\left(\frac{\Delta\chi\|\mathbf{G}_k\|}{\|\Lambda_k\|}\right)\|\Lambda_k\|\|\mathbf{G}_k\|}{\|\mathbf{G}_k\|^2}\mathbf{G}_k. \tag{2.14}$$

Supposing $\Lambda_i^j(0)$, to solve Eq. (2.11) from the fictitious initial $\chi = 0$ to a chosen fictitious final χ_f, we can implement the proposed method. Terminating criterion for this scheme is:

$$\sqrt{\sum_{i,j=1}^{N_1,N_2}[\Lambda_i^j(k+1) - \Lambda_i^j(k)]^2} \leq \varepsilon, \tag{2.15}$$

where ε is a chosen convergence measure. The solution of u is gettable from

$$\Lambda_i^j = \frac{\Lambda_i^j(\chi_0)}{(1+\chi_0)^d}, \tag{2.16}$$

where $\chi_0(\leq \chi_f)$ satisfies at the mentioned measure. By choosing the appropriate ρ and c, we can raise the stableness of solution and increasing the convergence rate of numerical integration, respectively.

3. Numerical examples

In this part, we analyze the performance of the proposed method by solving the following four examples as:

Example 3.1. As the first example, we consider the fractional TFKG Equation (1.1), [26] with $\alpha = 1.9, \kappa = -1, \eta = 0, \zeta = 2.5, \beta = 3/2$,

$$\mathcal{H}(x,t) = \frac{\Gamma(3+\alpha)}{2}t^2x^3(1-x)^3 + (30x^4 - 60x^3 + 36x^2 - 6x)t^{2+\alpha} + 2.5e^x x^{4.5}(1-x)^{4.5}t^{3+1.5\alpha}.$$

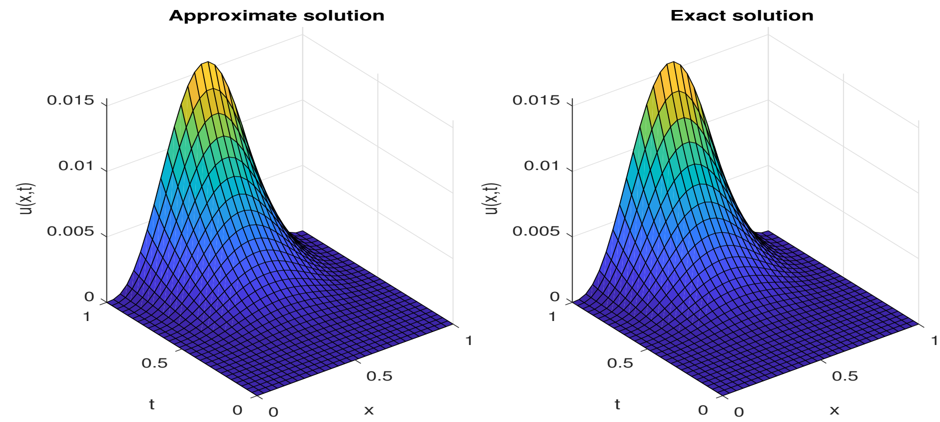

FIGURE 1. Plot of the analytical and approximate solutions of Example 3.1.

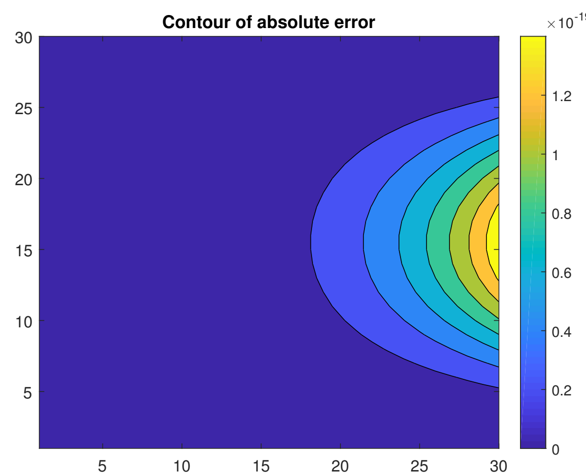

FIGURE 2. Contour plot of absolute error of Example 3.1.

The analytical solution of the Eq. (1.1) for this instance is $\Lambda(x,t) = t^{2+\alpha}x^3(1-x)^3$. When the semi-discretization procedure, we employ the number of knots $N_1 = N_2 = 30$ in each coordinates of space and time respectively. Mentioned domain in this instance is $\Omega = [0,1] \times [0,1]$. The boundary conditions in Ω_x are stated to be homogeneous. The initial assumption and step size for χ are mentioned as $\Lambda_i^j(0) = 1e-4$ and $\Delta\chi = 1e-5$. In order to control the convergency and stability of FTIT for the current instance, we consider $\rho = 47$ and $c = 1$. Figure 1 shows the analytical and numerical solutions obtained by our method. Numerical error of the current scheme can be found in Figure 2. Table 1 shows that the obtained maximum absolute numerical errors which are 1.2×10^{-19} obviously are better than error 8.19×10^{-7} of the the method [26]

Table 1. The comparison of the analytical solutions with the numerical solutions for $\alpha = 1.9, N_1 = N_2 = 30$, $T = 1$ and the maximum absolute error for Example 3.1.

(x, t)	Numerical	Exact	Error
(0.1,0.1)	1.146348117901298e-07	1.146348117901298e-07	1.146348117901298e-24
(0.2,0.2)	9.477350807799033e-06	9.477350807799033e-06	9.477350807799034e-23
(0.3,0.3)	1.022405372471378e-04	1.022405372471378e-04	1.022405372471378e-21
(0.4,0.4)	4.570431272556304e-04	4.570431272556304e-04	4.570431272556304e-21
(0.5,0.5)	0.001190347092027	0.001190347092027	1.190347092026936e-20
(0.6,0.6)	0.002031506243924	0.002031506243924	2.031506243924486e-20
(0.7,0.7)	0.002263825332587	0.002263825332587	2.263825332586649e-20
(0.8,0.8)	0.001388760647375	0.001388760647375	1.388760647375236e-20
(0.9,0.9)	2.003363608618149e-04	2.003363608618149e-04	2.003363608618149e-21

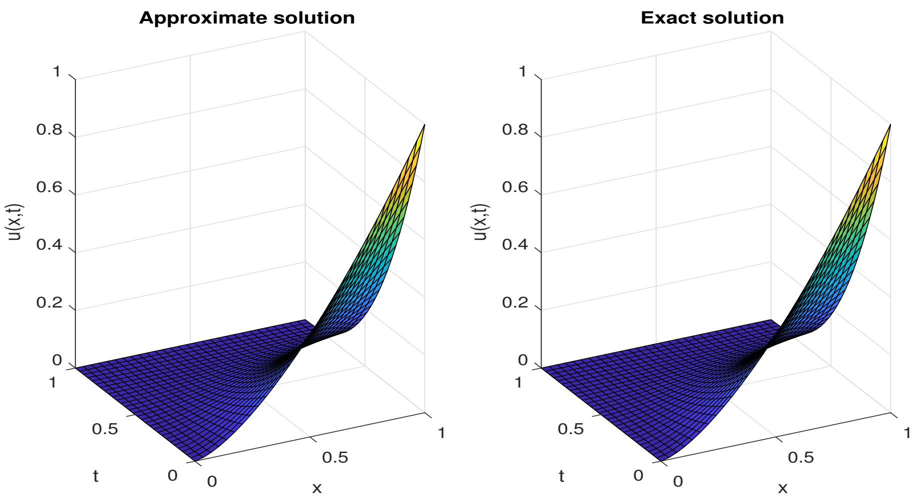

Figure 3. Plot of the analytical and approximate solutions of Example 3.2.

Example 3.2. Considering the nonlinear Klein Gordon Eq. (1.1) where the constants are $\kappa = 1, \eta = 0, \zeta = 1$ and $\beta = 2$, $0 < x < 1$ and $0 < t \leq 1$ with the source term [26]

$$\mathcal{H}(x,t) = \frac{\Gamma(5/2)}{\Gamma(8/2-\alpha)}(1-x)^{5/2}t^{3/2-\alpha} - \frac{15}{4}(1-x)^{1/2}t^{3/2} + (1-x)^5 t^3.$$

Initial assumption is $\Lambda_i^j(0) = 0.01$, and the values of parameters are $\rho = 0.2$ and $c = 0.1$. The analytical solution, $\Lambda(x,t) = (1-x)^{5/2}t^{3/2}$, and approximate solutions are plotted in Figure 3. The power of the proposed method can be seen in Figure 4. Table 2 is dedicated to show the exact and numerical solutions point by point and the gained absolute errors.

Example 3.3. We take the Eq. (1.1) with $\kappa = 1, \eta = -1, \zeta = 3/2$ and $\beta = 3$ and

$$\mathcal{H}(x,t) = \frac{\Gamma(3+\alpha)}{2} sin(\pi x)t^2 + (1+\pi^2) sin(\pi x)t^{2+\alpha} + \frac{3}{2}(sin(\pi x)t^{2+\alpha})^3.$$

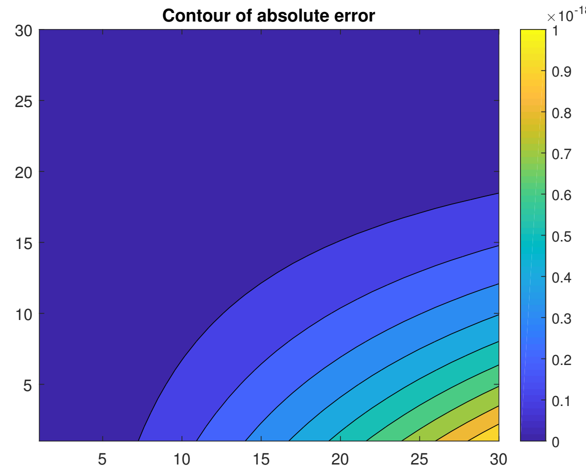

FIGURE 4. Contour plot of the maximum absolute error of Example 3.2.

TABLE 2. The comparison of the analytical solutions with the numerical solutions for $\alpha = 1.9, N_1 = N_2 = 30$, $T = 1$ and the maximum absolute error for Example 3.2.

(x, t)	Numerical	Exact	Error
(0.1,0.1)	0.025323472618256	0.025323472618256	2.532347261825621e-20
(0.2,0.2)	0.052717459616574	0.052717459616574	5.271745961657405e-20
(0.3,0.3)	0.068288372392296	0.068288372392296	6.828837239229595e-20
(0.4,0.4)	0.070032911361076	0.070032911361076	7.003291136107603e-20
(0.5,0.5)	0.060237229098190	0.060237229098190	6.023722909819004e-20
(0.6,0.6)	0.043330976761268	0.043330976761268	4.333097676126785e-20
(0.7,0.7)	0.024629858567274	0.024629858567274	2.462985856727429e-20
(0.8,0.8)	0.009292870429238	0.009292870429238	9.292870429238161e-21
(0.9,0.9)	0.001122092459237	0.001122092459237	1.122092459237206e-21

The approximate and the analytical solutions $\Lambda(x,t) = sin(\pi x)t^{2+\alpha}$ of this example are displayed in Figure 5 for applied FTIT subjected to the parameters: $\rho = 0.002, c = 0.1, \Delta\chi = 1e-9$. We choose the initial assumption $\Lambda_i^j(0) = 0.01$. High accuracy of presented method for this example can be detected from Figure 6 which shows relative error for Example 3.3. Accuracy of our method can be found from Table 3.

Example 3.4. ([51]). As the third example, we consider the Eq. (1.1) with $\alpha = 1.7$, $\kappa = -1, \eta = 0, \zeta = 1$ and $\beta = 0$ as follows

$$\mathcal{H}(x,t) = \frac{2t^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)}(e - e^x)sin(x) + t^2(2e - e^x)sin(x) + 2t^2e^x cos(x).$$

Let $\Lambda_i^j(0) = 0.01$. Domain of the problem and Dirichlet boundary conditions are taken as the Example 3.2. By taking $\rho = 102$, $c = 0.6$, and $\Delta\chi = 1e-2$, the low obtained numerical error is shown in Figure 8 which shows FTIT is suitable to solve TFKG equation. The analytical solution of TFKG equation for this example is $\Lambda(x,t) = t^2(e - e^x)sin(x)$. The exact solution and obtained approximate solutions are plotted in Figure 7. The low absolute errors, the exact and

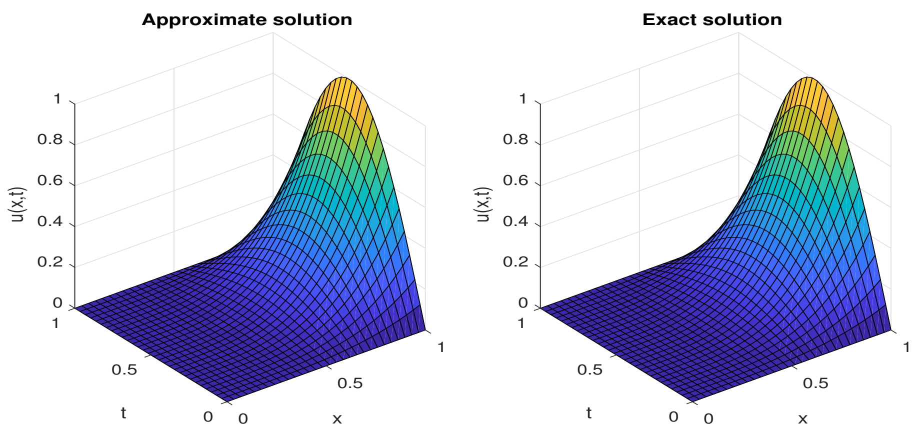

FIGURE 5. Plot of the analytical and approximate solutions of instance 3.

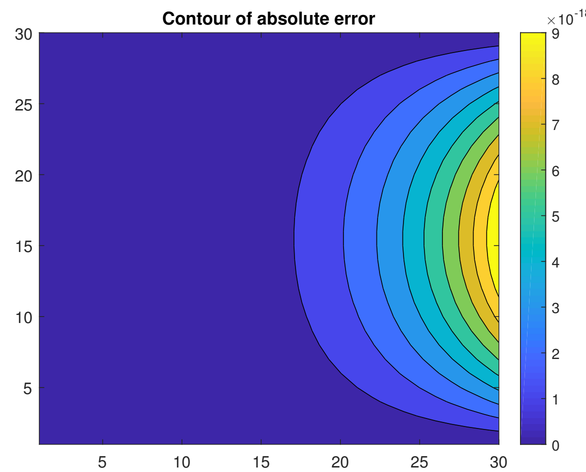

FIGURE 6. Contour plot of the maximum absolute error of instance 3.

numerical values at points are presented in Table 4 which the errors obtained via GPT are more better than errors gained by the method of [51].

4. Conclusion

In this work, we have converted the TFKG equation into a new type of functional PDE in a new dimension with one extra by proposing a fictitious coordinate. After applying the semi-discretization for the original equation, the GPT as a geometric technique is utilized to integrate the equations of first-order ODEs. Some numerical instances were performed, which illustrate that the offered technique is reliable and applicable to gain the numerical solutions of TFKG equation. By employing different parametric values, new approximate solutions were produced at certain fractional order levels α. These findings and technique can be extended to investigate other higher-dimensional fractional-order models in nonlinear wave theory, encompassing fields such as optics, quantum gases, hydro-dynamics,

TABLE 3. The comparison of the analytical solutions with the numerical solutions for $\alpha = 1.9, N_1 = N_2 = 30$, $T = 1$ and the maximum absolute error for instance 5.

(x, t)	Numerical	Exact	Error
(0.1,0.1)	4.587981957479481e-05	4.587981957479481e-05	4.587981957479481e-23
(0.2,0.2)	0.001298129742021	0.001298129742021	1.298129742020981e-21
(0.3,0.3)	0.008630993260602	0.008630993260602	8.630993260601923e-21
(0.4,0.4)	0.030855239384296	0.030855239384296	3.085523938429597e-20
(0.5,0.5)	0.076342489974861	0.076342489974861	7.634248997486077e-20
(0.6,0.6)	0.144615688247927	0.144615688247927	1.446156882479273e-19
(0.7,0.7)	0.216446406988547	0.216446406988547	2.164464069885473e-19
(0.8,0.8)	0.246459572764354	0.246459572764354	2.464595727643535e-19
(0.9,0.9)	0.162684146882440	0.162684146882440	1.626841468824396e-19

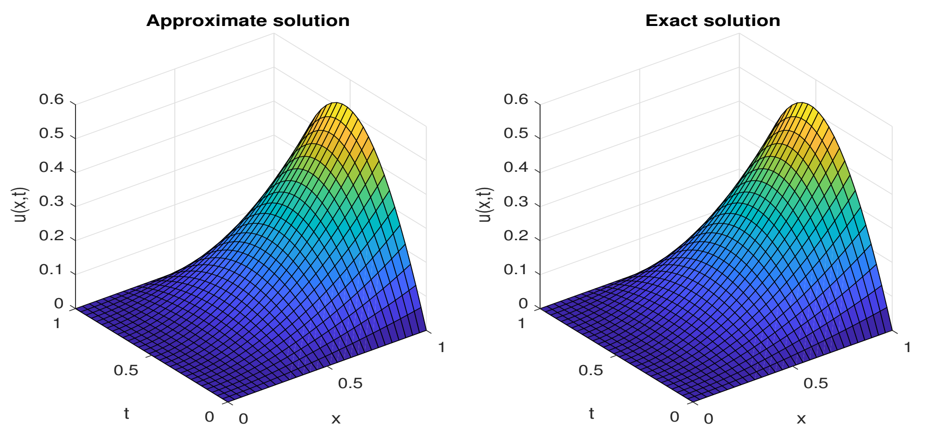

FIGURE 7. Plot of the analytical and approximate solutions of instance 4.

TABLE 4. The comparison of the analytical solutions with the numerical solutions for $\alpha = 1.7, N_1 = N_2 = 30$, $T = 1$ and the maximum absolute error for instance 4.

(x, t)	Numerical	Exact	Error
(0.1,0.1)	0.001778402744399	0.001778402744399	1.778402744398820e-20
(0.2,0.2)	0.013088331917926	0.013088331917926	1.308833191792626e-19
(0.3,0.3)	0.039836593476932	0.039836593476932	3.983659347693239e-19
(0.4,0.4)	0.083011335376355	0.083011335376355	8.301133537635463e-19
(0.5,0.5)	0.137702854681641	0.137702854681641	1.377028546816410e-18
(0.6,0.6)	0.144615688247927	0.144615688247927	1.446156882479273e-18
(0.7,0.7)	0.227657449061668	0.227657449061668	2.276574490616675e-18
(0.8,0.8)	0.217093344003187	0.217093344003187	2.170933440031868e-18
(0.9,0.9)	0.125971300346882	0.125971300346882	1.259713003468815e-18

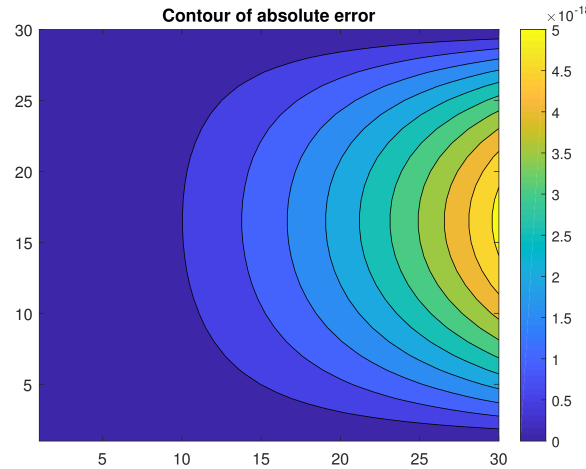

Figure 8. Contour plot of the maximum absolute error of instance 4.

photonics, plasmas, and solid-state physics. This research thus opens new avenues for understanding and applying high-dimensional fractional equations across various scientific disciplines.

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