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Conservation law, exact solutions and numerical approximate of the Barles-Soner non-linear equation

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Abstract

Solving option pricing equations is one of the most important challenges facing financial mathematics. In this article, a non-linear model is assumed for the market and the European option is priced under this model. To solve the pricing problem accurately, the Lie algebra method has been used. The conservation laws of the model have been calculated using Lie direct method, as well. Numerical simulation of the model has been done using the finite difference method.

Keywords. Non-linear market, Barles-Soner equation, Lie groups, Finite differences method, Conservation laws.
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1. Introduction

Choosing the right model for option pricing is important in creating and controlling volatility, creating an arbitrage environment, and risk hedging. The Black-Scholes model is one of the main pricing models that presents a mathematical model for a financial market using a stochastic differential equation. This modeling is done with riskless and risky assets as defined by the following dynamics.

$$dN_t = rN_t dt, \tag{1.1}$$

$$dX_t = \mu(t, X_t)X_t dt + \sigma(t, X_t)X_t dW_t, \tag{1.2}$$

where N_t is riskless asset at time t, X_t risky asset, r interest rate, μ expected rate of return, σ volatility and $\{W_t\}_t$ a Winner process. As seen in this market, volatility and interest rate are assumed to be risk-free and constant. The linear Black-Scholes equation related to European option pricing is

$$\begin{aligned} &Z_t(t,x) + rxZ_x(t,x) + \frac{1}{2}x^2\sigma^2(t,x)Z_{xx}(t,x) - rZ(t,x) = 0,\\ &Z(T,x) = \Phi(x), \end{aligned} \tag{1.3}$$

where x is underlying asset price and $Z_x(t,x) = \frac{\partial Z}{\partial x}(t,x)$ [5].

Black-Scholes model was used by traders for a long time but the assumption of stability of volatility and interest rate caused it to lose its efficiency gradually [11]. So the researchers tried to get a mathematical model much closer to reality by eliminating these assumptions. Heston's model, for instance, assumes that volatility is a stochastic process. By including a stochastic process for the interest rate, it is also possible to remove the interest rate from its fixed state [24]. Using the fractional Brownian process in place of the Brownian process is another method to enhance the Black-Scholes model [10]. Assuming that transaction costs are zero is another way to violate the Black-Scholes model. Transaction costs are calculated in different ways, which is used in this article from Barles and Soner model. Barles and Soner extracted a more complicated model using the approach of the utility function of Hodges and Neuberger

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[16] which was developed in [12]. Applying an exponential utility function and the theory of stochastic optimal control they proved Z satisfies the following non-linear PDE, when ϵ and κ tend to zero [14].

$$\begin{aligned} &Z_t + \frac{1}{2}x^2\tilde{\sigma}^2 Z_{xx} + rxZ_x - rZ = 0, \\ &\tilde{\sigma}^2 = \sigma^2(1 + \Psi(e^{r(T-t)}a^2x^2Z_{xx})), \end{aligned} \tag{1.4}$$

where $a = \frac{\kappa}{\sqrt{\epsilon}}$ and κ denotes transaction cost for per unit dollar traded. $\Psi(x)$ is the solution of non-linear ODE,

$$\Psi'(x) = \frac{\Psi(x)+1}{2\sqrt{x\Psi(x)} - x}, \quad x \neq 0, \tag{1.5}$$
$$\Psi(0) = 0.$$

We have

$$\lim_{x\to\infty} \frac{\Psi(x)}{x} = 1, \tag{1.6}$$
$$\lim_{x\to-\infty} \Psi(x) = -1.$$

This property allows the treatment of the function $\Psi(.)$ as the identity for large arguments. Following [3], let

$$\tilde{\sigma}^2 = \sigma^2(1 + e^{r(T-t)}a^2x^2Z_{xx}). \tag{1.7}$$

Substituting (1.7) into (1.4) yields

$$Z_t + \frac{1}{2}e^{r(T-t)}a^2x^4\sigma^2Z_{xx}^2 + \frac{1}{2}\sigma^2x^2Z_{xx} + rxZ_x - rZ = 0. \tag{1.8}$$

Equation (1.8) is called non-linear Black-Scholes PDE. Initial and boundary conditions of considered European call option with strike price K respectively are

$$Z(x,0) = max(x-K,0), \quad 0 \leqslant x \leqslant X_{max}, \tag{1.9}$$

$$\begin{cases} Z(0,t) = 0, \\ \lim_{x\to\infty} Z(x,t) = x, \end{cases} \tag{1.10}$$

and for European put option,

$$Z(x,0) = max(K-x,0), \quad 0 \leqslant x \leqslant X_{max}, \tag{1.11}$$

$$\begin{cases} Z(0,t) = Ke^{-r(T-t)}, \\ \lim_{x\to\infty} Z(x,t) = 0. \end{cases} \tag{1.12}$$

X_{max} is upper bound of price x which is assumed three or four times K [13].

There are two exact and numerical methods to solve Equation (1.8). One of the precise solution methods is to use Lie symmetries [8, 9, 11, 25]. In this paper, the exact and approximate solutions as well as the conservation laws of Eq. (1.8) are found. The exact solutions are found using Lie algebra, especially by points symmetries and one-parameter groups. Using Lie symmetries, the parameters of considered PDE are reduced. Usually reducing the parameters, PDEs are transformed into ODEs which are solved easily. Conservation Laws for PDEs in reality are mathematical expressions for physical principles. These laws are important to prove the uniqueness and existence of solutions [7, 21].In this paper, we calculate the conservation laws with a direct method.

There are also different methods to solve a PDE numerically. Numerical solutions for PDEs can also be found in various ways. In [2], the full discretization in the second level for the pricing of catastrophic bonds is generated using the spectral collocation method approach, which is based on the Chebyshev basis of the second kind. The approximate solution of the temporal fractional Black-Scholes model with beginning and boundary conditions, incorporating the time derivative in the Caputo sense, has been examined in [20]. The orthogonal polynomials utilized for spatial discretization serve as the foundation for the Chebyshev collocation, while time discretization has been accomplished by linear interpolation with temporally order accuracy. This article considers the time-fractional Black-Scholes model regulating European options, as described in [19], where the temporal derivative is concentrated on the Caputo fractional derivative. First, the semi-discrete was produced in the temporal sense using a quadratic interpolation with accuracy order, and then the unconditional stability and convergence order were examined. This is how the approximate numerical scheme was derived. Using the composition of the orthogonal Gegenbauer polynomials (GB polynomials) and the approximation of the fractional derivative dependent on the Caputo derivative, [1] proposes an efficient procedure to estimate the fractional Black-Scholes model in time dependent on the market prices of European options. The numerical method's speed and computation time reduction are attributed to the orthogonality of GB polynomials and operational matrices.

In the sequel, an approximate solution is obtained using the finite differences method.

Some notations and definitions were presented in section 2. The exact solutions of the non-linear Black-Scholes model are found in the second section, as well. The conservation law of the considered equation is given in the third section. In the next section, the approximate solution of European options pricing is g using the finite differences method. The paper concluded in section 5.

2. Lie symmetries

Assume the differential equation system of order β with independent variable $x^i, 1 \leqslant i \leqslant n$ and dependent variable $u^\beta, 1 \leqslant \beta \leqslant m$,

$$\Delta^f(x^i, u^\beta, u_i^\beta, u_{ij}^\beta, \dots) = 0, \quad 1 \leqslant f \leqslant l, \tag{2.1}$$

where $u_{ij}^\beta = \partial^2 u^\beta / \partial x^i \partial x^j$ [22]. The infinitesimal Lie transformations for (2.1), are

$$\begin{aligned} \tilde{x}^i &= x^i + \epsilon \xi^i + \mathcal{O}(\varepsilon^2), \\ \tilde{u}^\beta &= u^\beta + \epsilon \lambda^\beta + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{2.2}$$

which do not change the system of equations to $\mathcal{O}(\varepsilon^2)$. In Lie point symmetry, the infinitesimal generators of $\xi^i = \xi^i(x^i, u^\beta)$ and $\lambda^\beta = \lambda^\beta(x^i, u^\beta)$ depend only on x^i and λ^β and not on the derivatives or integrals of λ^β.
When the transformations (2.2) are dependent on derivatives or integrals of λ^β we use generalized Lie symmetries. The infinitesimal transformations for the first and second order derivatives to $\mathcal{O}(\varepsilon^2)$ are

$$\tilde{u}_i^\beta = u_i^\beta + \epsilon \eta_i^\beta, \quad \tilde{u}_{ij}^\beta = u_{ij}^\beta + \epsilon \eta_{ij}^\beta, \tag{2.3}$$

where

$$\eta_i^\beta = D_i \hat{\lambda}^\beta + \xi^f u_{fi}^\beta, \quad \eta_{ij}^\beta = D_i D_j \hat{\lambda}^\beta + \xi^f u_{fij}^\beta, \tag{2.4}$$

and

$$\hat{\lambda}^\beta = \lambda^\beta - \xi^f u_f^\beta, \tag{2.5}$$

are related to Lie transformations in which $\tilde{x}^i = x^i$ and $\tilde{u}^\beta = u^\beta + \epsilon \hat{\lambda}^\beta$ [9]. The operator D_i denotes the total derivative w.r.t. x^i and

$$D_i = \frac{\partial}{\partial x_i} + u_i^\beta \frac{\partial}{\partial u^\beta} + u_{ij}^\beta \frac{\partial}{\partial u_j^\beta} + \dots [6] \tag{2.6}$$

The following relation must hold for the transformations of (2.2) not to change the hypothetical system of differential equations to $\mathcal{O}(\varepsilon^2)$,

$$\mathcal{L}_{\tilde{\mathbf{v}}} \Delta^f \equiv \tilde{\mathbf{v}}(\Delta^f) = 0 \quad \text{whenever} \quad \Delta^f = 0, \quad 1 \leqslant f \leqslant l, \tag{2.7}$$

where

$$\tilde{\mathbf{v}} = \mathbf{v} + \eta_i^\beta \frac{\partial}{\partial u_i^\beta} + \eta_{ij}^\beta \frac{\partial}{\partial u_{ij}^\beta} + \dots \ , \tag{2.8}$$

the prolongation of the vector field

$$\mathbf{v} = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} \lambda^\beta \frac{\partial}{\partial u^\beta}, \tag{2.9}$$

for infinitesimal transformations (2.2) [11]. $\mathcal{L}_{\mathbf{v}}\Delta^f$ represents the derivative of Δ^f w.r.t. the vector field $\mathbf{v}$. The partial differential equation (1.8) which can be represented as

$$\Delta(x, t, Z, Z_x, Z_t, Z_{xx}, Z_{tt}, Z_{xt}) = 0, \tag{2.10}$$

has the following infinitesimal transformations

$$\begin{aligned} \tilde{x} &= x + \epsilon\xi^x + \mathcal{O}(\varepsilon^2),\\ \tilde{t} &= t + \epsilon\xi^t + \mathcal{O}(\varepsilon^2),\\ \tilde{Z} &= Z + \epsilon\lambda + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{2.11}$$

The prolongation of vector field of transformations (2.11) is as follows.

$$\mathbf{v}^{(2)} = \mathbf{v} + \lambda^x \frac{\partial}{\partial Z_x} + \lambda^t \frac{\partial}{\partial Z_t} + \lambda^{xx} \frac{\partial}{\partial Z_{xx}} + \lambda^{tt} \frac{\partial}{\partial Z_{tt}} + \lambda^{xt} \frac{\partial}{\partial Z_{xt}}, \tag{2.12}$$

where

$$\mathbf{v} = \xi^x(x, t, Z)\frac{\partial}{\partial x} + \xi^t(x, t, Z)\frac{\partial}{\partial t} + \lambda(x, t, Z)\frac{\partial}{\partial Z}, \tag{2.13}$$

and

$$\begin{aligned} \lambda^x &= D_x(\lambda - \xi^x Z_x - \xi^t Z_t) + \xi^x Z_{xx} + \xi^t Z_{xt},\\ \lambda^t &= D_t(\lambda - \xi^x Z_x - \xi^t Z_t) + \xi^x Z_{xt} + \xi^t Z_{tt},\\ \lambda^{xx} &= D_x\left[D_x(\lambda - \xi^x Z_x - \xi^t Z_t)\right] + \xi^x Z_{xxx} + \xi^t Z_{xxt},\\ \lambda^{tt} &= D_t\left[D_t(\lambda - \xi^x Z_x - \xi^t Z_t)\right] + \xi^x Z_{xtt} + \xi^t Z_{ttt},\\ \lambda^{xt} &= D_x\left[D_t(\lambda - \xi^x Z_x - \xi^t Z_t)\right] + \xi^x Z_{xxt} + \xi^t Z_{xtt}. \end{aligned} \tag{2.14}$$

In the sequel the parameters of the considered equation is reduced. For this, invariant of the vector field (2.9) is calculated. So the following characteristic device should be solved [8, 15].

$$\frac{dx^1}{\xi^1} = \ldots = \frac{dx^n}{\xi^n} = \frac{dZ^1}{\lambda^1} = \ldots = \frac{dZ^m}{\lambda^m}. \tag{2.15}$$

Some vector fields do not find a suitable solution for the differential equation. So calculating flow and substituting it into one of the found solutions yield a new solution. The flows of the vector field $\mathbf{v}$ is

$$(\bar{x}, \bar{t}, \bar{Z}) = \exp(\epsilon\mathbf{v})(x, t, Z), \tag{2.16}$$

where

$$\exp(\epsilon\mathbf{v}) = 1 + \frac{\epsilon}{1!}\mathbf{v} + \frac{\epsilon^2}{2!}\mathbf{v}^2 + \ldots, \tag{2.17}$$

and

$$\mathbf{v}^2(x, t, Z) = \mathbf{v}(\mathbf{v}(x, t, Z)), \text{ [21]}. \tag{2.18}$$

The prolongation of vector field of the second order Eq. (1.8) is

$$\begin{aligned} &\xi^x(2a^2x^3\sigma^2 Z_{xx}^2 + \sigma^2 x Z_{xx} r Z_x) - \frac{1}{2}\xi^t e^{r(T-t)} r a^2 x^4 \sigma^2 Z_{xx}^2 - \lambda r\\ &\quad + \lambda^x r x + \lambda^t + \lambda^{xx}(e^{r(T-t)} a^2 x^4 \sigma^2 Z_{xx} + \frac{1}{2}\sigma^2 x^2) = 0. \end{aligned} \tag{2.19}$$

So

$$
\begin{aligned}
&\xi^x_Z = \xi^t_x = \xi^t_Z = \xi^t_{tt} = 0,\\
&\lambda_{xx} = -\frac{\xi^t_t}{2a^2x^2e^{r(T-t)}}, \quad \lambda_Z = r\xi^t - \xi^t_t,\\
&\lambda_t = \frac{8\big(-\frac{1}{2}rZ\xi^t + Z\xi^t_t - \frac{1}{2}x\lambda_x + \frac{1}{2}\lambda\big)a^2re^{r(T-t)} + \sigma^2\xi^t_t}{4a^2e^{r(T-t)}},\\
&\xi^x_t = rx\xi^t_t, \quad \xi^x_x = \frac{1}{x}\xi^x.
\end{aligned} \tag{2.20}
$$

From (2.20) the vector field coefficients $\mathbf{v}$ are

$$
\begin{aligned}
&\xi^x = x(c_1rt + c_3), \quad \xi^t = c_1t + c_2,\\
&\lambda = \frac{1}{4ra^2}\Big[c_1(\sigma^2 - 2r)(rt-1)e^{-r(T-t)} + 2c_1r\ln(x)e^{-r(T-t)}\\
&\quad + 4a^2\Big(c_4e^{rt} + \big(rZ(c_1t+c_2) + c_5x - c_1Z\big)r\Big)\Big].
\end{aligned} \tag{2.21}
$$

So the infinitesimal generators of the Barles and Soner equation are

$$
\begin{aligned}
&\mathbf{v}_1 = x\frac{\partial}{\partial x}, \quad \mathbf{v}_2 = e^{rt}\frac{\partial}{\partial Z}, \quad \mathbf{v}_3 = x\frac{\partial}{\partial Z}, \quad \mathbf{v}_4 = \frac{\partial}{\partial t} + rZ\frac{\partial}{\partial Z},\\
&\mathbf{v}_5 = xrt\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{2e^{-r(T-t)}\ln(x) - 2te^{r(t-T)}(-\frac{\sigma^2}{2} + r) + 4Z(rt-1)a^2}{4a^2}\frac{\partial}{\partial Z}.
\end{aligned} \tag{2.22}
$$

In addition to the above vector fields, their linear combinations can be used. We can not use vector fields $\mathbf{v}_2$ and $\mathbf{v}_3$ in which the infinitesimal generators ξ^x and ξ^t are zero. So we combine them with $\mathbf{v}_1$ or $\mathbf{v}_4$ vectors. So

$$
\begin{aligned}
&\mathbf{v}_6 = \mathbf{v}_1 + \mathbf{v}_2 = x\frac{\partial}{\partial x} + e^{rt}\frac{\partial}{\partial Z},\\
&\mathbf{v}_7 = \mathbf{v}_1 + \mathbf{v}_3 = x\frac{\partial}{\partial x} + x\frac{\partial}{\partial Z}.
\end{aligned} \tag{2.23}
$$

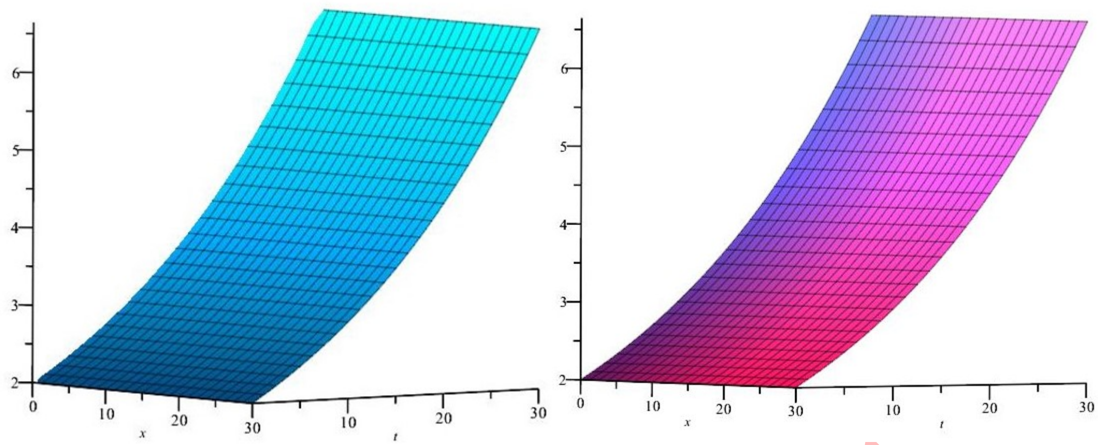
2.1. **The exact solution for the Barles-Soner equation.** To calculate the exact solutions of the Barles and Soner equation using Lie groups, the invariant of vector fields should be obtained. Substituting them into the considered equation, the number of parameters of the equation is reduced. We apply the parameter reduction process until the reduced equation is solvable.

- For the generator $\mathbf{v}_1$ the invariant transformations is $Z = u(y)$, $t = y$. So substituting obtained invariant into (1.8) and reducing its parameters, it becomes a linear first-order differential equation as follows.

  $$u' - ru = 0, \tag{2.24}$$

  so we have $Z = c.e^{rt}$ where c is the integrating coefficient. In the sequel, new solutions for remained generators are presented using one-parameter groups (flow).
- The flow of generator $\mathbf{v}_5$ is $(\bar{x}, \bar{t}, \bar{Z}) = \exp(\epsilon\mathbf{v}_5)(x, t, Z)$ and so

  $$
  \begin{aligned}
  &\bar{x} = xe^{rt\epsilon}, \quad \bar{t} = te^{\epsilon},\\
  &\bar{Z} = -\frac{1}{8a^2}\Big[(\sigma^2te^{-rT} - 6rte^{-rT} - \sigma^2te^{-Tr+2\epsilon} - 8e^{-rt}a^2Z\\
  &\quad + 4\ln(x)e^{-rT} + 4e^{-Tr+\epsilon}rt + 2rte^{-Tr+2\epsilon} - 4\ln(xe^{rt\epsilon})e^{-Tr+\epsilon})e^{-\epsilon+rte^{\epsilon}}\Big].
  \end{aligned} \tag{2.25}
  $$



(A) (2.26).

(B) (2.29).

FIGURE 1. The solutions of Eqs. (2.26) and (2.29) for $a = 0.6$, $r = 0.04$ and $\sigma = 0.45$.

Substituting the obtained flow into $Z = c.e^{rt}$, yields

$$Z(x,t) = -\frac{e^{rt-\epsilon}}{2a^2}\Bigg[-\ln(xe^{rt\epsilon(-1+\exp(-\epsilon))})e^{-Tr+\epsilon} - \frac{3(-\frac{\sigma^2}{6}+r)te^{-Tr-\epsilon}}{2} + e^{-rT}\ln(xe^{-rt\epsilon}) + \frac{t(-\frac{\sigma^2}{2}+r)e^{-Tr+\epsilon}}{2} + rte^{-rT} - 2a^2c\Bigg]. \tag{2.26}$$

- Similarly the flow of generator $\mathbf{v}_6$ is $(\bar{x}, \bar{t}, \bar{Z}) = \exp(\epsilon\mathbf{v}_6)(x,t,Z)$ and so

$$\bar{x} = xe^{\epsilon}, \quad \bar{t} = t, \quad \bar{Z} = Z + \epsilon e^{rt}. \tag{2.27}$$

yields

$$Z(x,t) = (c-\epsilon)e^{rt}. \tag{2.28}$$

- Finally the last generator flow gives the following solution,

$$Z(x,t) = ce^{rt} - xe^{-\epsilon} + x. \tag{2.29}$$

Figure 1 shows the solutions of Eqs. (2.26) and (2.29) when $a = 0.6$, $r = 0.04$ and $\sigma = 0.45$.

3. Conservation laws

The conservation laws of a system of differential equations with partial derivatives is a divergence expression that is constant on all solutions of the system. Each non-trivial divergence expression that results in a local law of the system of differential equations is obtained from the local multipliers of the independent and dependent variables and the derivatives of the dependent variables. In the direct method to find divergence expressions, we replace the dependent variables and their derivatives in the system of differential equations and functional multipliers with arbitrary functions. This makes zero the divergence expressions on all system solutions [6, 18, 21].

A conservation law for a PDE (2.1) is a divergence expression

$$D_i\Phi^i[u] = \sum_{j=1}^{n} D_j\Phi^j[u] = 0, \tag{3.1}$$

which must be satisfied all solutions system (2.1). In the above expression, $\Phi^i[u]$ is called the fluxes of conservation laws and

$$\Phi^i[u] = \Phi^i(x, u, \partial u, \dots, \partial^r u). \tag{3.2}$$

The highest order of the derivative seen in the fluxes is called the order of the conservation law [6]. Multipliers

$$\{\Lambda_\nu[U]\}_{\nu=1}^N = \{\Lambda_\nu(x, U, \partial U, \dots, \partial^s U)\}_{\nu=1}^N, \tag{3.3}$$

where $U(x)$ are arbitrary functions, produces a divergence expression, provided

$$\Lambda_\nu[U]R^\nu[U] \equiv D_i\Phi^i[U], \tag{3.4}$$

multipliers (3.3) with arbitrary order s yields a conservation law for

$$R(x^i, u) = \Delta^f(x^i, u^\beta, u_i^\beta, u_{ij}^\beta, \dots) = 0, \quad 1 \leqslant f \leqslant l, \tag{3.5}$$

if for each arbitrary function $U(x)$, we have

$$E_{U^\beta}[\Lambda_\nu(x, U, \partial U, \dots, \partial^s U)R^\nu(x, U, \partial U, \dots, \partial^s U)] \equiv 0, \quad 1 \leqslant \beta \leqslant m, \tag{3.6}$$

where $E_{U^\beta}[.]$ is the Euler-Lagrange operator w.r.t. U^β, that for $1 \leqslant \beta \leqslant m$ is defined as follows.

$$E_{U^\beta} = \frac{\partial}{\partial U^\beta} - D_i\frac{\partial}{\partial U_i^\beta} + \dots + (-1)^s D_{i_1}\dots D_{i_s}\frac{\partial}{\partial U_{i_1\dots i_s}^\beta} + \dots . \tag{3.7}$$

This operator makes zero the divergence expression $D_i\Phi^i[U]$ [6, 27].

3.1. **The conservation law for Barles-Soner equation.** One of the independent variables of the equation

$$R[u] = u_t + \frac{1}{2}e^{r(T-t)}a^2x^4\sigma^2u_{xx}^2 + \frac{1}{2}\sigma^2x^2u_{xx} + rxu_x - ru, \tag{3.8}$$

is the time variable t. So its conservation law is

$$D_t\Psi[u] + \sum_{i=1}^{n-1} D_i\Phi^i[u] = 0. \tag{3.9}$$

$\Psi[u]$ is the density and $\Phi^i[u]$ spatial fluxes of conservation law of Eq. (3.8) [4].

To find conservation law, we find the multipliers Λ which are obtained from Eq. (3.6) for an arbitrary order s. Then from relation

$$D_t\Psi[u] + D_x\Phi^x[u] \equiv \Lambda(x, U, \partial U, \dots, \partial^s U)R[u], \tag{3.10}$$

the conservation law of the order s is obtained, where $U(x,t)$ are arbitrary functions. For the zero-order conservation law, the local multipliers $\Lambda(x,t,u)$ is obtained from

$$E_U[\Lambda(x,t,U)(U_t + \frac{1}{2}e^{r(T-t)}a^2x^4\sigma^2U_{xx}^2 + \frac{1}{2}\sigma^2x^2U_{xx} + rxU_x - rU)] \equiv 0, \tag{3.11}$$

for arbitrary functions $U(x,t)$. Applying the Euler-Lagrange operator, the above relation is written as follows.

$$\begin{aligned}
&8x^2a^2\sigma^2e^{r(T-t)}\Big(\frac{1}{8}\Lambda_{UU}x^2U_x^2U_{xx} + \frac{1}{4}\Lambda_{xU}x^2U_xU_{xx} + \frac{1}{8}\Lambda_{xx}x^2U_{xx}\\
&+ \Lambda_U\Big(\frac{1}{4}x^2(U_{xxx}U_x + \frac{3}{4}U_{xx}^2) + xU_xU_{xx}\Big) + \Lambda_x(\frac{1}{4}x^2U_{xxx} + xU_{xx})\\
&+ \Lambda(\frac{1}{8}x^2U_{xxxx} + xU_{xxx} + \frac{3}{2}U_{xx})\Big) + \frac{1}{2}\Lambda_{UU}x^2U_x^2\sigma^2 + \Lambda_{xU}x^2U_x\sigma^2\\
&+ \frac{1}{2}\Lambda_{xx}x^2\sigma^2 + \Lambda_U\Big((x^2U_{xx} + 2xU_x)\sigma^2 - rU\Big) - \Lambda_x x(-2\sigma^2 + r)\\
&+ \Lambda(\sigma^2 - 2r) - \Lambda_t \equiv 0.
\end{aligned} \tag{3.12}$$

The above equation is a polynomial in terms of U_x, U_{xx}, U_{xxx} and U_{xxxx} variables. From (3.12) we get the local multiplier $\Lambda = 0$. From relationship (3.10) for the local multiplier $\Lambda = 0$, the conservation law from the zeroth order of Eq. (3.8) is

$$D_t u - D_x x u_t = 0. \tag{3.13}$$

4. Numerical approximate

In this paper, using the finite differences method, a numerical solution is found. For this, the explicit method is applied to approximate Equation (1.8). In this method, the forward difference is used for the first derivative w.r.t. time, the central difference for the first derivative w.r.t. underlying price, and the central symmetric difference for the second derivative w.r.t. x, These differences are computed having three points [13, 23].

For the European option pricing that follows a non-linear Black-Scholes market, the remaining time until the maturity which is denoted by t, at the moment of concluding the contract, is T. With approaching the maturity time, t decreases from T to 0. At the moment of contact performance, it is 0. Decreasing t from T to 0 transforms the Equation (1.8) into a backward equation. Changing of variable $\rho = T - t$, yields a forward equation in which ρ increase to T [11].

$$\frac{\partial Z(x,t)}{\partial t} = \frac{\partial Z(x,\rho(t))}{\partial \rho(t)}\rho'(t) = -\frac{\partial Z(x,\rho)}{\partial \rho}. \tag{4.1}$$

Replacing ρ with t,

$$Z_t - \frac{1}{2}e^{rt}a^2x^4\sigma^2 Z_{xx}^2 - \frac{1}{2}\sigma^2x^2Z_{xx} - rxZ_x + rZ = 0. \tag{4.2}$$

To approximate the derivatives, time and price intervals should be discretized. Divide $[0,T]$ into Q sub-intervals of length Δt. Limit interval $[0,\infty)$ which its maximum amount (X_{max}) has been assumed to be three or four times the strike price. Divide $[0, X_{max}]$ into P sub-intervals of length Δx. Each point of obtained grid of space $[0, X_{max}] \times [0,T]$ for $0 \leqslant p \leqslant P$ and $0 \leqslant q \leqslant Q$ is $(p\Delta x, q\Delta t)$. For discretization and approximation of derivatives in finite differences, the following approximations in an explicit way are needed [13, 26].

$$\begin{aligned}
\frac{\partial Z}{\partial t}(p\Delta x, q\Delta t) &= \frac{z_p^{q+1} - z_p^q}{\Delta t} + O(\Delta t), \\
\frac{\partial Z}{\partial x}(p\Delta x, q\Delta t) &= \frac{z_{p+1}^q - z_{p-1}^q}{2\Delta x} + O((\Delta x)^2), \\
\frac{\partial^2 Z}{\partial x^2}(p\Delta x, q\Delta t) &= \frac{z_{p+1}^q - 2z_p^q + z_{p-1}^q}{(\Delta x)^2} + O((\Delta x)^2).
\end{aligned} \tag{4.3}$$

For Eq. (1.8),

$$\begin{aligned}
\frac{z_p^{q+1} - z_p^q}{\Delta t} \quad &- \quad e^{rq\Delta t}a^2\sigma^2p^4\frac{(z_{p+1}^q - 2z_p^q + z_{p-1}^q)^2}{2} - \sigma^2p^2\frac{(z_{p+1}^q - 2z_p^q + z_{p-1}^q)}{2} \\
&- \quad rp\frac{(z_{p+1}^q - z_{p-1}^q)}{2} + rz_p^q = 0, \quad 1 \leqslant p \leqslant P-1, \quad 0 \leqslant q \leqslant Q-1.
\end{aligned}$$

From (4.4),

$$\begin{aligned}
z_p^{q+1} = z_p^q &+ \frac{\Delta t e^{rq\Delta t}a^2\sigma^2p^4}{2}\Big[4(z_p^q)^2 - 4z_{p-1}^qz_p^q - 4z_p^qz_{p+1}^q + (z_{p-1}^q)^2 + 2z_{p-1}^qz_{p+1}^q \\
&+ (z_{p+1}^q)^2\Big] + \frac{\Delta t\sigma^2p^2}{2}\Big[z_{p+1}^q - 2z_p^q + z_{p-1}^q\Big] + \frac{\Delta trp}{2}\Big[z_{p+1}^q - z_{p-1}^q\Big] \\
&- r\Delta t z_p^q, \quad 1 \leqslant p \leqslant P-1, \quad 0 \leqslant q \leqslant Q-1.
\end{aligned} \tag{4.4}$$

Take the change of variables

$$\alpha_1 = \frac{\Delta t a^2\sigma^2}{2}, \qquad \alpha_2 = \frac{\Delta t\sigma^2}{2}, \qquad \alpha_3 = \frac{\Delta t r}{2}, \qquad \alpha_4 = r\Delta t, \tag{4.5}$$

for $1 \leqslant p \leqslant P-1$ and $0 \leqslant q \leqslant Q-1$. So Eq. (4.4) can be rewritten as follows.

$$\begin{aligned} z_p^{q+1} &= z_p^q + \alpha_1 e^{rq\Delta t} p^4\Big[4(z_p^q)^2 - 4z_{p-1}^q z_p^q - 4z_p^q z_{p+1}^q + (z_{p-1}^q)^2 + 2z_{p-1}^q z_{p+1}^q \\ &+ (z_{p+1}^q)^2\Big] + \alpha_2 p^2\Big[z_{p+1}^q - 2z_p^q + z_{p-1}^q\Big] + \alpha_3 p\Big[z_{p+1}^q - z_{p-1}^q\Big] \\ &- \alpha_4 z_p^q = \big(1 - 2\alpha_2 p^2 - \alpha_4\big) z_p^q + \big(\alpha_2 p^2 + \alpha_3 p\big) z_{p+1}^q + \big(\alpha_2 p^2 - \alpha_3 p\big) z_{p-1}^q \\ &+ \alpha_1 e^{rq\Delta t} p^4 \big(z_{p+1}^q - 2z_p^q + z_{p-1}^q\big)^2. \end{aligned} \tag{4.6}$$

The equation (4.6) can be rewritten as the following matrix.

$$z^{q+1} = Az^q + g_2^q (Bz^q + g_3^q)^2 + g_1^q, \quad 0 \leqslant q \leqslant Q-1, \tag{4.7}$$

where the power 2, refers to the power of each element of the matrix $Bz^q + g_3^q$ and we have

$$A = \begin{pmatrix} d_1 & u_2 & 0 & \cdots & 0 \\ l_1 & d_2 & u_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & u_{P-1} \\ 0 & \cdots & \cdots & l_{P-2} & d_{P-1} \end{pmatrix}, \quad B = \begin{pmatrix} d'_1 & u'_2 & 0 & \cdots & 0 \\ l'_1 & d'_2 & u'_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & u'_{P-1} \\ 0 & \cdots & \cdots & l'_{P-2} & d'_{P-1} \end{pmatrix}, \tag{4.8}$$

where $A, B \in \mathbb{R}^{(P-1)\times(P-1)}$ and

$$\begin{aligned} d_p &= 1 - 2\alpha_2 p^2 - \alpha_4, & d'_p &= -2p^2, & 1 \leqslant p \leqslant P-1, \\ u_p &= \alpha_2 (p-1)^2 + \alpha_3 (p-1), & u'_p &= (p-1)^2, & 2 \leqslant p \leqslant P, \\ l_p &= \alpha_2 (p+1)^2 - \alpha_3 (p+1), & l'_p &= (p+1)^2, & 0 \leqslant p \leqslant P-2, \end{aligned} \tag{4.9}$$

and

$$z^{q+1} = \begin{pmatrix} z_1^{q+1} \\ z_2^{q+1} \\ \vdots \\ z_{P-1}^{q+1} \end{pmatrix}, \quad z^q = \begin{pmatrix} z_1^q \\ z_2^q \\ \vdots \\ z_{P-1}^q \end{pmatrix}, \quad g_1^q = \begin{pmatrix} l_0 z_0^q \\ 0 \\ \vdots \\ 0 \\ u_P z_P^q \end{pmatrix}, \tag{4.10}$$

$$g_3^q = \begin{pmatrix} l'_0 z_0^q \\ 0 \\ \vdots \\ 0 \\ u'_P z_P^q \end{pmatrix}, \quad g_2^q = \alpha_1 e^{rq\Delta t}.$$

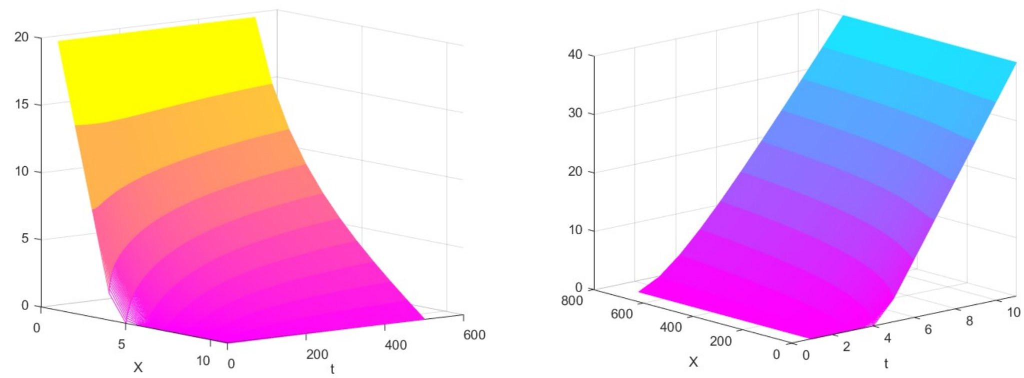
So the amount of z_p^{q+1} at each moment can be calculated with three points $z_{p-1}^q, z_p^q, z_{p+1}^q$ at a moment before $q+1$. The explicit method is easy to implement with any programming language capable of storing arrays of data [13]. The following condition of stability

$$0 < \Delta t < \frac{1}{\sigma^2 (N-1) + \frac{1}{2} r}, \tag{4.11}$$

has been proved in [17].

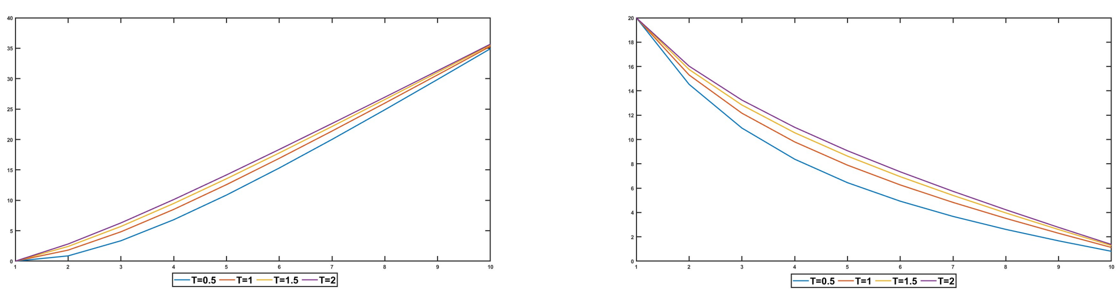
TABLE 1. Numerical solution for Barles and Soner equation using finite differences method.

X	Call option	Put option
12	1.812	12.162
18	8.503	9.801
30	16.902	6.265
42	25.973	3.516
54	35.309	1.127
CPU Time (sec)	0.02619	0.02256



(A) Put option. (B) Call option.

FIGURE 2. Explicit solution of Barles and Soner model.



(A) Call option. (B) Put option.

FIGURE 3. Comparison between obtained prices at different maturity times.

4.1. **European option pricing.** In this subsection, we investigate the approximate solution of the European call and put option (4.7) with initial and boundary conditions (1.9)–(1.12). Take $K = 20$, $r = 0.04$, $\sigma = 0.45$, $a = 0.6$ and $T = 1$. The European call and put option pricing results for $Q = 700$ and $P = 10$ have been shown in Table 1.

Figure 2 shows the diagrams of European options pricing of the non-linear Black-Scholes market (Barles and Soner model). In Fig. 3 options pricing with different maturities; 6 months, one year, one and a half years and two years have been compared.

5. Conclusion

In this paper, using Lie symmetries, four exact solutions for European options pricing under the non-linear Black-Scholes market (Barles and Soner model) have been found. One of them is obtained by applying invariant and the other by flow vector fields. In the sequel, a conservation law with order zero for the considered equation has been presented. The numerical solution of the assumed solution is computed, as well. Finally, the comparison between European option prices in different maturity times has been shown via diagram.

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