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High-order numerical solution for a class of nonlinear Fredholm integro-differential equations

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Abstract

The main objective of this work is to present a high-order numerical method to solve a class of nonlinear Fredholm integro-differential equations. By multiplying appropriate efficient factors and constructing an appropriate approximate function, as well as employing a numerical integration method of order γ, the above-mentioned problem can be simplified to a nonlinear system of algebraic equations. Furthermore, we discuss the convergence analysis of the presented method in detail and demonstrate that it converges with an order $\mathcal{O}(h^{3.5})$ in the L^2-norm. Some test examples are provided to demonstrate that the claimed order of convergence is obtained.

Keywords. Efficient factors, Approximate function, Nonlinear algebraic system, Convergence analysis, Order of convergence.
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1. Introduction

There are numerous applications of integro-differential equations in the fields of natural sciences and engineering. Some of these equations can be seen in the mathematical modeling of spatiotemporal developments, epidemic modeling [25], heat and mass transfer [14], scattering in quantum physics and population models [23].

Since the extraction of exact solutions of integro-differential equations is very difficult, researchers have been compelled to solve them numerically. Therefore, various numerical procedures have been developed to widely approximate their solutions. Recently, several approaches have been proposed for solving integral and, in particular, integro-differential equations either numerically or approximately. Some of the well-known methods that have been recently used for these equations are Taylor collocation method [27, 28], iterative method [31], rational Chebyshev functions approach [20], sinc-collocation method [30], wavelet method [3, 12], Legendre wavelets method [21], Walsh function method [18], improved reproducing kernel method [26], differential transform method [4], Bell polynomials method [16], homotopy analysis method [24], Haar wavelet method [1], Chebshyev series and power series technique [11], Chebyshev finite difference method [9], direct computation method [17], non-standard difference method [19], exponential spline method [13], Legendre polynomial method [22], parametrization method [10], and Bernoulli matrix method [5]. It should be pointed out that the convergence rate of the above-mentioned methods is less than or equal two.

The multiscale Galerkin method is one of the practical tools that researchers have used in many mathematical applications. For example, the authors of [6–8, 15, 29] have used the multiscale Galerkin method for various kinds of integral or integro-differential equations. It should be noted that, except for the method used in [7], the convergence order of the other applied methods is at most 2. In fact, the convergence order of the method utilized in [7] is 3. Due to the lack of numerical methods with a high convergence order for two-point boundary value integro-differential equations, in this research, a suitable high-order numerical method is introduced to solve a class of nonlinear second-order Fredholm integro-differential equations (FIDEs). The proposed method is based on multiplying some appropriate exponential functions on both sides of the problem under consideration, applying numerical integration methods, and employing an approximate function for solving the second-order boundary value problem governed by the nonlinear FIDEs. The established method is straightforward to implement in fully discretizing the considered FIDEs. By

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implementing this method, the problem under consideration is transformed into a system of nonlinear algebraic equations. The convergence analysis of the proposed method is established. It will be shown that the convergence order of the developed method is $\mathcal{O}(h^{\min\{\frac{7}{2},\gamma-\frac{1}{2}\}})$ in L^2-norm.

This paper is structured as follows. In section 2, a class of FIDEs is considered and then an effective numerical method is presented. Error analysis of the new method is discussed in section 3 in L^2-norm. In addition, we demonstrate in this section that the order of convergence of the proposed method can reach up to $\mathcal{O}(h^{\frac{7}{2}})$ in L^2-norm. The efficiency of the method presented in section 2 is verified through comparative numerical examples in section 4.

2. Description of the proposed method

In light of the significance of integro-differential equations mentioned in the preceding section, the following class of the second-order two-point boundary value problems of FIDEs is considered:

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t) + \int_0^1 v(t,s)u(y(s))ds, \quad t \in (0,1], \quad y(0) = a, \quad y(1) = b, \tag{2.1}$$

where p, q, f, v, and u are given functions, and y is an unknown function. All of them belong to $C^4([0,1])$.

Consider the partition $\{t_k = kh : \; k = 0, 1, \ldots, N\}$ for the closed interval $[0,1]$, with $h = \frac{1}{N}$ and $t_0 = 0$, $t_N = 1$. Assume that Y_k and $V_{k,n}$ are the approximations of $y_k := y(t_k)$ and $v_{k,n} := v(t_k, t_n)$ for $k = 0, 1, \ldots, N$, respectively. Also, let $\mathcal{I}_k^{\pm} = [\min\{t_k, t_{k\pm1}\}, \max\{t_k, t_{k\pm1}\}]$. In the next section, a new numerical approach is presented to find the solution of the above-mentioned problem. Before stating the main result, we provide some preliminary requirements. First, suppose that $p(t) \geq 0$ is integrable and that there exists a function $\phi(t)$ such that:

$$\begin{cases} \phi'(t) = p(t), & t \in (0,1], \\ \phi(0) = 0. \end{cases}$$

Then, with the help of the following lemma, we try to find an appropriate approximation for the integral forms $\varphi_k^+(s) := \int_s^{t_{k+1}} \phi^-(t)dt$ and $\varphi_k^-(s) := -\int_{t_{k-1}}^s \phi^-(t)dt$ for $k = 1, \cdots, N-1$, where $\phi^-(t) = e^{-\phi(t)}$.

Lemma 2.1. *If $g \in L^2([0,1])$ is a differentiable function, and its derivatives up to order κ are continuous, then we have:*

$$\int_s^{t_{k\pm1}} g(t)dt = \sum_{m=0}^{\kappa-1} \frac{(-1)^m (t_k - s)^{m+1} + (\pm h)^{m+1}}{(m+1)!} \, g_k^{(m)} + \frac{(-1)^\kappa (t_k - s)^{\kappa+1} + (\pm h)^{\kappa+1}}{(\kappa+1)!} \, g^{(\kappa)}(\eta_k^{\pm}), \tag{2.2}$$

$$g_k^{(m)} = g^{(m)}(t_k), \eta_k^{\pm} \in [0,1], \qquad m = 0, 1, \cdots, \kappa, \qquad k = 1, \cdots, N-1. \tag{2.3}$$

Proof. According to the expressed assumptions, if the function g is expanded around t_k, then there exists $\eta_k(t) \in [0,1]$, such that:

$$g(t) = \sum_{m=0}^{\kappa-1} \frac{(t-t_k)^m}{m!} \, g_k^{(m)} + \frac{(t-t_k)^\kappa}{\kappa!} \, g^{(\kappa)}(\eta_k(t)).$$

So, by applying the mean value theorem for definite integrals, there exist constants $\eta_k^+, \eta_k^- \in [0,1]$ such that:

$$\begin{aligned}
\int_s^{t_{k+1}} g(t)dt &= \sum_{m=0}^{\kappa-1} \int_s^{t_{k+1}} \frac{(t-t_k)^m}{m!} \, g_k^{(m)} dt + \int_s^{t_{k+1}} \frac{(t-t_k)^\kappa}{\kappa!} g^{(\kappa)}(\eta_k(t))dt \\
&= \sum_{m=0}^{\kappa-1} \frac{h^{m+1} - (s-t_k)^{m+1}}{(m+1)!} \, g_k^{(m)} + \frac{h^{\kappa+1} - (s-t_k)^{\kappa+1}}{(\kappa+1)!} g^{(\kappa)}(\eta_k^+) \\
&= \sum_{m=0}^{\kappa-1} \frac{(-1)^m (t_k - s)^{m+1} + h^{m+1}}{(m+1)!} \, g_k^{(m)} + \frac{(-1)^\kappa (t_k - s)^{\kappa+1} + h^{\kappa+1}}{(\kappa+1)!} g^{(\kappa)}(\eta_k^+),
\end{aligned}$$

$$\begin{aligned}\int_{t_{k-1}}^{s} g(t)dt &= \sum_{m=0}^{\kappa-1}\int_{t_{k-1}}^{s}\frac{(t-t_k)^m}{m!}\, g_k^{(m)}dt + \int_{t_{k-1}}^{s}\frac{(t-t_k)^\kappa}{\kappa!}g^{(\kappa)}(\eta_k(t))dt\\ &= \sum_{m=0}^{\kappa-1}\frac{(s-t_k)^{m+1}-(-h)^{m+1}}{(m+1)!}\, g_k^{(m)} + \frac{(s-t_k)^{\kappa+1}-(-h)^{\kappa+1}}{(\kappa+1)!}g^{(\kappa)}(\eta_k^-)\\ &= \sum_{m=0}^{\kappa-1}\frac{(-1)^{m+1}(t_k-s)^{m+1}-(-h)^{\kappa+1}}{(m+1)!}\, g_k^{(m)} + \frac{(-1)^{\kappa+1}(t_k-s)^{\kappa+1}-(-h)^{\kappa+1}}{(\kappa+1)!}g^{(\kappa)}(\eta_k^-).\end{aligned}$$

This completes the proof. □

Therefore, if we apply Lemma 2.1 with $\kappa = 4$ for $\varphi_k^{\pm}(s)$, then we conclude that

$$\varphi_k^{\pm}(s) = \sum_{m=0}^{3}\frac{(-1)^m(t_k-s)^{m+1}+(\pm h)^{m+1}}{(m+1)!}\phi_k^{-(m)} + \frac{(t_k-s)^5 \pm h^5}{5!}\phi^{-(4)}(\eta_k^{\pm}), \qquad \eta_k^{\pm}\in[0,1], \tag{2.4}$$

in which $\phi_k^{-(m)} = \frac{d^m}{dt^m}\phi^-(t)\big|_{t=t_k}$.

Here, the main aim of these calculations is to discretize FIDE (2.1) and to formulate the proposed method. Since we will encounter integral expressions of the form $\int_{t_k}^{t_{k\pm1}} g(t)\varphi_k^{\pm}(t)dt$ for some functions g, we first propose a pair of approximate functions for $g(t)$ on $\mathcal{I}k^-$ and $\mathcal{I}k^+$ as follows:

$$\begin{cases}\mathcal{L}_k^-(t) := g_{k-1} + \dfrac{g_k-g_{k-1}}{h}(t-t_{k-1}) + \dfrac{g_{k-1}-2g_k+g_{k+1}}{2h^2}(t-t_k)(t-t_{k-1}), & t\in\mathcal{I}_k^-,\\[2mm] \mathcal{L}_k^+(t) := g_{k+1} + \dfrac{g_{k+1}-g_k}{h}(t-t_{k+1}) + \dfrac{g_{k-1}-2g_k+g_{k+1}}{2h^2}(t-t_k)(t-t_{k+1}), & t\in\mathcal{I}_k^+.\end{cases} \tag{2.5}$$

The error of the approximate functions $\mathcal{L}_k^{\pm}(t)$ for $g(t)$ on $\mathcal{I}_k^-$ and $\mathcal{I}_k^+$ is stated and proved in the following lemma.

Lemma 2.2. *Let $g\in\mathbb{C}^4([0,1])$. Then, the error of $\mathcal{L}_k^{\pm}(t)$ in approximating $g(t)$ is given by:*

$$\begin{cases} g(t)-\mathcal{L}_k^-(t) = (t-t_k)(t-t_{k-1})\left(\frac{1}{3!}(t-t_{k+1})g^{(3)}(\xi_{0,k}^-) - \frac{h^2}{4!}g^{(4)}(\xi_{1,k}^-)\right), & t\in\mathcal{I}_k^-,\\ g(t)-\mathcal{L}_k^+(t) = (t-t_k)(t-t_{k+1})\left(\frac{1}{3!}(t-t_{k-1})g^{(3)}(\xi_{0,k}^+) - \frac{h^2}{4!}g^{(4)}(\xi_{1,k}^+)\right), & t\in\mathcal{I}_k^+,\end{cases} \tag{2.6}$$

where $\xi_{0,k}^{\pm},\ \xi_{1,k}^{\pm}\in(t_{k-1},t_{k+1})$ and $k=1,\dots,N-1$.

Proof. The proof is omitted for brevity. □

In the next step, we attempt to find an appropriate approximation for the integral expressions

$$I_k^{\pm}(g) := \int_{t_k}^{t_{k\pm1}} g(t)\varphi_k^{\pm}(t)dt,$$

where $g\in C^4([0,1])$.

Lemma 2.3. *Let $g\in C^4([0,1])$. If one applies (2.4) for $\varphi_k^{\pm}(t)$ and then utilizes the approximate method (2.5) for the integrand of $I_k^{\pm}(g) = \int_{t_k}^{t_{k\pm1}} g(t)\varphi_k^{\pm}(t)dt$, then there exist constants $C_{\mathfrak{g}_k}^{\pm}\in\mathbb{R}$ such that an appropriate approximation for $I_k^{\pm}(g)$ can be written as*

$$I_k^{\pm}(g) = h^2\mathbf{M}_k^{\pm}\mathbf{g}_\mathbf{k}^{\pm} + C_{\mathfrak{g}_k}^{\pm}h^5, \tag{2.7}$$

where $\mathbf{M}_k^{\pm} = (H.\widehat{\phi}_k)\ A^{\pm\top}$, $\mathbf{g}_{\mathbf{k}}^{\pm} = \left[g_0{}_k^{\pm}, g_1{}_k^{\pm}, g_2{}_k^{\pm}\right]^{\top}$, *and* $C_{\mathfrak{g}_k}^{\pm}$ *depends on the fourth-order derivatives of function* g*, and*

$$\begin{cases} H = \left[1, h, h^2, h^3\right], \widehat{\phi}_k = \left[\phi_k^-, \phi_k^{-(1)}, \phi_k^{-(2)}, \phi_k^{-(3)}\right], \\ A^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & \frac{1}{30} \\ \frac{1}{3} & \frac{5}{24} & \frac{3}{40} & \frac{7}{360} \\ \frac{-1}{24} & \frac{-7}{240} & \frac{-1}{90} & \frac{-1}{336} \end{bmatrix}, \quad A^- = \begin{bmatrix} \frac{1}{2} & \frac{-1}{3} & \frac{1}{8} & \frac{1}{30} \\ \frac{1}{3} & \frac{-5}{24} & \frac{3}{40} & \frac{7}{360} \\ \frac{-1}{24} & \frac{7}{240} & \frac{-1}{90} & \frac{-1}{336} \end{bmatrix}, \\ g_0{}_k^{\pm} = g_{k\pm1}, \quad g_1{}_k^{\pm} = g_k - g_{k\pm1}, \quad g_{2k} = g_{k-1} - 2g_k + g_{k+1}. \end{cases}$$

Proof. By applying (2.4) together with (2.6) and then rearranging the coefficients, the proof follows straightforwardly. □

Now, we are ready to solve FIDE (2.1). First, we multiply both sides of (2.1) by the function $\phi^+(t) = e^{\phi(t)}$. Thus, the following equation is obtained:

$$\left(\phi^+(t)y'(t)\right)' + \phi^q(t)y(t) = \phi^f(t) + \phi^+(t)\int_0^1 v(t,s)u(y(s))ds, \tag{2.8}$$

where $\phi^q(t) = \phi^+(t)q(t), \phi^f(t) = \phi^+(t)f(t)$. Next, by integrating (2.8) over $[t_k, t]$ and applying integration by parts, we obtain

$$\phi^+(t)y'(t) - \phi_k^+ y_k' + \int_{t_k}^t \phi^q(\xi)y(\xi)d\xi = \int_{t_k}^t \phi^f(\xi)d\xi + \int_{t_k}^t\int_0^1 \phi^+(\xi)v(\xi,s)u(y(s))dsd\xi, \quad k = 1,\dots,N-1. \tag{2.9}$$

Now, by dividing both sides of (2.9) by $\phi^+(t)$, and then integrating over the intervals $\mathcal{I}_k^{\pm}$, the following system is derived:

$$\begin{aligned} y_{k+1} - y_k - \phi_k^+\vartheta_k y_k' + \int_{t_k}^{t_{k+1}}\int_{t_k}^t \phi^-(t)\phi^q(\xi)y(\xi)d\xi dt \\ = \int_{t_k}^{t_{k+1}}\int_{t_k}^t \phi^-(t)\phi^f(\xi)d\xi dt + \int_{t_k}^{t_{k+1}}\int_{t_k}^t \phi^-(t)\int_0^1 \phi^+(\xi)v(\xi,s)u(y(s))dsd\xi dt, \end{aligned} \tag{2.10}$$

$$\begin{aligned} y_{k-1} - y_k + \phi_k^+\vartheta_{k-1} y_k' + \int_{t_k}^{t_{k-1}}\int_{t_k}^t \phi^-(t)\phi^q(\xi)y(\xi)d\xi dt \\ = \int_{t_k}^{t_{k-1}}\int_{t_k}^t \phi^-(t)\phi^f(\xi)d\xi dt + \int_{t_k}^{t_{k-1}}\int_{t_k}^t \phi^-(t)\int_0^1 \phi^+(\xi)v(\xi,s)u(y(s))ds\ d\xi\ dt, \end{aligned} \tag{2.11}$$

where $k = 1,\dots,N-1$, and $\vartheta_k := \int_{t_k}^{t_{k+1}} \phi^-(t)dt$.

Next, using Fubini's theorem, the system of Equations (2.10) can be expressed as follows:

$$y_{k+1} - y_k - \phi_k^+\vartheta_k y_k' + I_k^+(\phi^q y) = I_k^+(\phi^f) + \int_0^1 u(y(s))I_k^+(\phi^+(.)v(.,s))ds, \tag{2.12}$$

$$y_{k-1} - y_k - \phi_k^+\vartheta_{k-1} y_k' + I_k^-(\phi^q y) = I_k^-(\phi^f) + \int_0^1 u(y(s))I_k^-(\phi^+(.)v(.,s))ds, \tag{2.13}$$

for $k = 1,\dots,N-1$, where:

$$I_k^{\pm}(\phi^q y) = \int_{t_k}^{t_{k\pm1}} \phi^q(t)y(t)\varphi_k^{\pm}(t)dt, \quad I_k^{\pm}(\phi^f) = \int_{t_k}^{t_{k\pm1}} f(t)\varphi_k^{\pm}(t)dt,$$

$$I_k^{\pm}(\phi^+(.)v(.,s)) = \int_{t_k}^{t_{k\pm1}} \phi^+(t)v(t,s)\varphi_k^{\pm}(t)dt.$$

By adding the equations in (2.12) together, the following system of integral equations is obtained:

$$\tau_k y_{k+1} - (\tau_{k-1} + \tau_k) y_k + \tau_{k-1} y_{k-1} + I_k(\phi^q y) = I_k(\phi^f) + \int_0^1 u(y(s)) I_k(\phi^+(.)v(.,s))ds, \tag{2.14}$$

where $\tau_k = \frac{1}{\vartheta_k}, k = 1, \dots, N-1$, and

$$I_k(\phi^q y) := \tau_{k-1} I_k^-(\phi^q y) + \tau_k I_k^+(\phi^q y), \tag{2.15}$$

$$I_k(\phi^f) := \tau_{k-1} I_k^-(\phi^f) + \tau_k I_k^+(\phi^f), \tag{2.16}$$

$$I_k(\phi^+(.)v(.,s)) := \tau_{k-1} I_k^-(\phi^+(.)v(.,s)) + \tau_k I_k^+(\phi^+(.)v(.,s)), \quad k = 1, \dots, N-1. \tag{2.17}$$

According to the boundary conditions of the original problem (2.1), we have $y_0 = a$ and $y_N = b$. Therefore, to find the unknown values $y_1, \dots, y_{N-1}$, we must solve system (2.14). It should be mentioned that this nonlinear system is difficult to solve analytically and requires the use of approximate or numerical methods. Hence, to obtain an approximate solution to system (2.14), let us use some numerical integration techniques for the integrals in (2.15) and the integral term in the right-hand side of (2.14).

To this end, we employ the numerical integration method (2.7) to approximate the integrals given by (2.15). Furthermore, if a numerical integration method of order γ is used to approximate the integral $\int_0^1 u(y(s)) I_k(\phi^+(.)v(.,s))ds$ in the right-hand side of (2.14), with integration points $\{t_j\}_{j=0}^N$ and weights $\mathbf{w} = (\ell_0, \ell_1, \dots, \ell_N)^\top$, then it can inferred that there exist constants $c_{0_k}^\pm \in \mathbb{R}$ such that

$$\begin{aligned}
\int_0^1 u(y(s)) I_k(\phi^+(.)v(.,s))ds &= \tau_{k-1} \int_0^1 u(y(s)) I_k^-(\phi^+(.)v(.,s))ds + \tau_k \int_0^1 u(y(s)) I_k^+(\phi^+(.)v(.,s))ds \\
&= h\tau_{k-1} \sum_{j=0}^N \ell_j u(y_j) I_k^-(\phi^+(.)v(.,t_j)) + h\tau_k \sum_{j=0}^N \ell_j u(y_j) I_k^+(\phi^+(.)v(.,t_j)) \\
&\quad + (\tau_{k-1} c_{0_k}^- + \tau_k c_{0_k}^+) h^{\gamma+2}.
\end{aligned}$$

So, according to (2.7), we conclude that there exist constants $c_{l,k}^\pm \in \mathbb{R}, k = 1, \dots, N-1, l = 0, \dots, 3$, such that

$$\begin{cases} I_k(\phi^q y) := h^2 \Phi_{0_k} + c_{1_k} h^5, \qquad I_k(\phi^f) := h^2 \Phi_{1_k} + c_{2_k} h^5, \\ \displaystyle\int_0^1 u(y(s)) I_k(\phi^+(.)v(.,s))ds := h^3 \sum_{j=0}^N \ell_j u_j \Phi_{2_{k,j}} + h^6 c_{3_k} \sum_{j=0}^N \ell_j u_j + c_{0_k} h^{\gamma+2}, \quad k = 1, \dots, N-1, \end{cases}$$

where for $k = 1, \dots, N-1, i = 0, 1, l = 0, \dots, 3$,

$$\begin{cases} \Phi_{i_k} = \tau_{k-1} \mathbf{M}_k^- \Phi_{i_k}^- + \tau_k \mathbf{M}_k^+ \Phi_{i_k}^+, \\ \Phi_{2_{k,j}} = \tau_{k-1} \mathbf{M}_k^- \Phi_{2_{k,j}}^- + \tau_k \mathbf{M}_k^+ \Phi_{2_{k,j}}^+, \\ c_{l_k} = \tau_{k-1} c_{l,k}^- + \tau_k c_{l,k}^+, \end{cases}$$

and for $k = 1, \dots, N-1, j = 0, \dots, N$,

$$\begin{cases} \Phi_{0_k}^\pm = \mathcal{C}_0^\pm \phi_{k-1}^q y_{k-1} + \mathcal{C}_1 \phi_k^q y_k + \mathcal{C}_0^\mp \phi_{k+1}^q y_{k+1}, \\ \Phi_{1_k}^\pm = \mathcal{C}_0^\pm f_{k-1} + \mathcal{C}_1 f_k + \mathcal{C}_0^\mp f_{k+1}, \\ \Phi_{2_{k,j}}^\pm = \mathcal{C}_0^\pm \phi_{k-1}^+ v_{k-1,j} + \mathcal{C}_1 \phi_k^+ v_{k,j} + \mathcal{C}_0^\mp \phi_{k+1}^+ v_{k+1,j}, \\ \mathcal{C}_0^+ = [0, 0, 1]^\top, \quad \mathcal{C}_0^- = [1, -1, 1]^\top, \quad \mathcal{C}_1 = [0, 1, -2]^\top. \end{cases}$$

Hence, the following fully discrete method is developed to solve the two-point boundary value FIDE (2.1):

$$a_k y_{k-1} + b_k y_k + c_k y_{k+1} + h^3 \sum_{j=0}^N \ell_j u_j \Phi_{2_{k,j}} = -h^2 \Phi_{1_k} + C_k h^5 + c_{0_k} h^{\gamma+2}, \tag{2.18}$$

where

$$\begin{cases} a_k = -\tau_{k-1} - h^2\left(\tau_{k-1}\mathbf{M}_k^-\mathcal{C}_0^- + \tau_k\mathbf{M}_k^+\mathcal{C}_0^+\right)\phi_{k-1}^q, \\ b_k = (\tau_{k-1}+\tau_k) - h^2\left(\tau_{k-1}\mathbf{M}_k^- + \tau_k\mathbf{M}_k^+\right)\mathcal{C}_1\phi_k^q, \\ c_k = -\tau_k - h^2\left(\tau_{k-1}\mathbf{M}_k^-\mathcal{C}_0^+ + \tau_k\mathbf{M}_k^+\mathcal{C}_0^-\right)\phi_{k+1}^q, \\ C_k = c_{1_k} - c_{2_k} - hc_{3_k}\displaystyle\sum_{j=0}^{N}\ell_j u_j. \end{cases}$$

Thus, if we set

$$\begin{cases} \widehat{\Phi}_2^k = \left[\Phi_{2_{1,k}}, \Phi_{2_{2,k}}, \dots, \Phi_{2_{N-1,k}}\right]^\top, \widehat{\Phi}_2 = \left[\widehat{\Phi}_2^1, \dots, \widehat{\Phi}_2^{N-1}\right], \widehat{\Phi}_1 = \left[\Phi_{1_1}, \dots, \Phi_{1_{N-1}}\right]^\top, \\ \mathbf{W} = \operatorname{diag}(\ell_1, \dots, \ell_{N-1}), \mathbf{L} = \widehat{\Phi}_2\mathbf{W}, \\ \bar{\mathbf{C}} = (C_1, \dots, C_{N-1})^\top, \bar{\mathbf{c}}_0 = (c_{0_1}, \dots, c_{0_{N-1}})^\top, \quad \mathbf{y}^h = (y_1, \dots, y_{N-1})^\top, \end{cases}$$

then a matrix formulation of the proposed method (2.18) can be demonstrated as follows:

$$\Omega^{N-1}\ \mathbf{y}^h + h^3\ \mathbf{L}\ u(\mathbf{y}^h) = -h^2\widehat{\Phi}_1 + \mathbf{b_0} + h^5\bar{\mathbf{C}} + h^{\gamma+2}\bar{\mathbf{c}}_0, \tag{2.19}$$

in which

$$\mathbf{b_0} = -a_1 y_0 \mathbf{I}_1 - c_{N-1}y_N\mathbf{I}_{N-1} - h^3\ell_0 u_0\widehat{\Phi}_2^0 - h^3\ell_N u_N\widehat{\Phi}_2^N,$$

and $\Omega^{N-1} = \operatorname{tridiag}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an $(N-1)\times(N-1)$ tridiagonal matrix, where

$$\mathbf{a} = (a_2, \dots, a_{N-1})^\top, \mathbf{b} = (b_1, \dots, b_{N-1})^\top, \mathbf{c} = (c_1, \dots, c_{N-2})^\top,$$

and $\mathbf{I}_i, i = 1, N-1$, is a column vector composed of $(N-1)$ elements, all of which are zero except for one element at position i that is equal to 1. So, the corresponding proposed numerical method can be expressed as follows

$$\Omega^{N-1}\ \mathbf{Y}^h + h^3\ \mathbf{L}\ u(\mathbf{Y}^h) = \mathbf{b}, \tag{2.20}$$

where $\mathbf{Y}^h = (Y_1, \dots, Y_{N-1})^\top$ and $\mathbf{b} = \mathbf{b_0} - h^2\widehat{\Phi}_1$. Note that the derived nonlinear equations can be solved using Newton's iteration solver or other iterative techniques.

3. Error analysis and convergence rate of the proposed method

This section examines the error analysis of the numerical method for FIDE (2.1), as well as its convergence properties, as outlined in the previous section. To this end, we focus on the matrix formulations (2.19) and (2.20), which correspond to the exact and approximate solutions of FIDE (2.1), respectively. According to (2.19) and (2.20), the error equation can be derived as follows:

$$\Omega^{N-1}(\mathbf{y}^h - \mathbf{Y}^h) + h^3\ \mathbf{L}\big(u(\mathbf{y}^h) - u(\mathbf{Y}^h)\big) = \mathbf{E_0}, \tag{3.1}$$

where $\mathbf{E_0} = h^5\bar{\mathbf{C}} + h^{\gamma+2}\bar{\mathbf{c}}_0$.

Let $\mathbf{J}_U$ denote the Jacobian matrix of the kernel function $u(y)$. A linearized form of the error system can then be written as:

$$(\Omega^{N-1} + h^3\ \mathbf{L}\mathbf{J}_U)\mathbf{E}(h) = \mathbf{E_0}, \tag{3.2}$$

where $\mathbf{E}(h) = \mathbf{y}^h - \mathbf{Y}^h$. Note that the $\mathbf{J}_U$ is a diagonal matrix defined by:

$$\mathbf{J}_U = \operatorname{diag}\Big(\Big[\frac{\partial}{\partial y}u(y)\big|_{y=y_k}\Big]_{k=1}^{N-1}\Big).$$

Using the fixed-point theorem, the iterative solutions of the linear system (3.2) converge to the solution of the nonlinear system (3.1). We can also represent the matrix $\Omega^{N-1} = \mathbf{T} - h^2\mathbf{Q}$, where

$$\mathbf{T} = \operatorname{tridiag}(-\mathbf{T}_{N-2}^1, \mathbf{T}_{N-2}^0 + \mathbf{T}_{N-1}^1, -\mathbf{T}_{N-2}^1), \tag{3.3}$$

$$\mathbf{Q} = \operatorname{tridiag}(\mathbf{Q}_{N-1}^{2^-}, \mathbf{Q}_{N-1}^1, \mathbf{Q}_{N-2}^{1^+}), \tag{3.4}$$

are tridiagonal matrices with

$$\mathbf{T}_n^k = [\tau_k, \tau_{k+1}, \dots, \tau_n]^\top, \mathbf{Q}_n^{k^\pm} = \left[\alpha_k^\pm, \alpha_{k+1}^\pm, \dots, \alpha_n^\pm\right]^\top, \mathbf{Q}_n^k = [\alpha_k, \alpha_{k+1}, \dots, \alpha_n]^\top,$$

for $k = 1, \dots, N-1$, where

$$\alpha_k^\pm = \left(\tau_{k-1}\mathbf{M}_k^- \mathcal{C}_0^\pm + \tau_k \mathbf{M}_k^+ \mathcal{C}_0^\mp\right) \phi_{k\pm 1}^q, \alpha_k = \left(\tau_{k-1}\mathbf{M}_k^- + \tau_k \mathbf{M}_k^+\right) \mathcal{C}_1 \phi_k^q.$$

We now investigate some properties of the matrix $\mathbf{T}$ as defined in (3.3). First, we know that the matrix $\mathbf{T}$ satisfies the propert of being weakly diagonally dominant and symmetric. For this reason, it can be concluded that it is a semidefinite matrix due to the positive values of its diagonal elements. We can observe that, the elements of $\mathbf{T}_n^k$ do not vanish. For this matrix, considering an LU-decomposition in [2] (p:7, Eqs: (16)-(17)) and utilizing Lemma 3 of [2], it follows that $|\mathbf{T}| \neq 0$. This results in positive singular values σ_k and positive eigenvalues λ_k for matrix $\mathbf{T}$. In the next section, we only need to prove that $\lambda_{min}(\mathbf{T}) > h$ where $\lambda_{min}(\mathbf{T})$ is the smallest eigenvalue of $\mathbf{T}$. In order to do so, first, we consider $\mathbf{M} = \mathbf{T} - h\mathbf{I}$, where $\mathbf{I}$ is an identity matrix with dimension $(N-1) \times (N-1)$. Taking λ as an eigenvalue of $\mathbf{T}$ implies that $\lambda - h$ be the eigenvalue of $\mathbf{M}$. Therefore, all we have to do is indicate that $\lambda_{min}(\mathbf{M}) > 0$. If we consider $\bar{a}_k, \bar{c}_k$ as lower- and upper-diagonal elements and $\bar{d}_k$ as diagonal elements of the matrix $\mathbf{M}$, then we have,

$$\begin{cases} \bar{d}_k = \tau_{k-1} + \tau_k - h, & k = 1, 2, \cdots, N-1, \\ \bar{a}_k = \bar{c}_k = -\tau_k, & k = 1, 2, \cdots, N-2. \end{cases}$$

If there exists a matrix decomposition, according to the corresponding LU-factorization [2] (p:7, Eqs: (16)-(17)), for the tridiagonal matrix $\mathbf{M}$, then we get

$$\underbrace{\begin{bmatrix} \bar{d}_1 & \bar{c}_1 & & & \\ \bar{a}_1 & \bar{d}_2 & \bar{c}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \bar{a}_{N-2} & \bar{d}_{N-1} & \bar{c}_{N-1} \\ & & & \bar{a}_{N-1} & \bar{d}_N \end{bmatrix}}_{\mathbf{M}} = \underbrace{\begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & \ddots & \ddots & \\ & & l_{N-1} & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} p_1 & \bar{c}_1 & & \\ & p_2 & \ddots & \\ & & \ddots & \bar{c}_{N-1} \\ & & & p_N \end{bmatrix}}_{U_1}, \tag{3.5}$$

and also we obtain,

$$p_1 = \bar{d}_1, \quad l_k = \frac{\bar{a}_k}{p_k}, \quad p_{k+1} = \bar{d}_{k+1} - l_k \bar{c}_k, \quad k = 1, 2, \cdots, N-1. \tag{3.6}$$

It should be pointed out that, according to (2.2) with $s = t_k$, we can obtain an approximation for τ_k as

$$\tau_k \sim \frac{1}{h \displaystyle\sum_{m=0}^{\kappa} \frac{h^m}{(m+1)!} \phi_k^{-(m)}}. \tag{3.7}$$

Remark 3.1. Since $h = \frac{1}{N}$, from the definition of τ_k and according to (3.7), we conclude that

$$\tau_k \sim \mathcal{O}\left(\frac{1}{h}\right),$$

as $h \to 0$. Therefore, using the property $p(t) \geq 0$, it is easy to show that $\tau_k > 1 + h$.

Lemma 3.2. *Suppose that for the matrix* $\mathbf{M}$*, there exists a matrix decomposition according to LU-factorization* (3.5)*, then we can conclude that* $|\mathbf{M}| > 0$.

Proof. Taking $p_1 = \bar{d}_1, \quad l_k = \frac{\bar{a}_k}{p_k}, \quad p_{k+1} = \bar{d}_{k+1} - l_k \bar{c}_k, \quad k = 1, 2, \cdots, N-1$, yields $p_{k+1} = \tau_{k+1} + \tau_k - h - \tau_k \bar{l}_k$, in which $\bar{l}_k = -l_k$. Therefore, we get

$$\bar{l}_{k+1} = \frac{\gamma_{k+1}}{\gamma_{k+1} + 1 - \bar{l}_k - h/\tau_k},$$

in which $\gamma_{k+1} = \frac{\tau_{k+1}}{\tau_k}, k = 1, \cdots, N-1$. Since $\tau_k \sim \mathcal{O}\left(\frac{1}{h}\right)$, we conclude that $\bar{l}_1 = \frac{\tau_1}{\tau_0+\tau_1-h} < 1$ and $\lim_{k\to\infty} \bar{l}_k = 1$. So, form the results expressed in Remark 3.1, we conclude that $h\left(\frac{1}{\tau_0-h} + \frac{1}{\tau_1}\right) < 1$ which in turn infers $\bar{l}_1 < 1 - \frac{h}{\tau_1} < 1$. Thus, as $h \to 0$, assuming $\bar{l}_k < 1 - \frac{h}{\tau_k} < 1$ leads to $\bar{l}_{k+1} < 1 - \frac{h}{\tau_{k+1}} < 1$. Finally, we get $0 < \bar{l}_k < 1, k = 1, \cdots, N-1$. Hence, we obtain

$$p_{k+1} > \tau_{k+1} - h > 1.$$

Therefore, the proof is complete via the decomposition (3.5). □

It follows from Theorem 7.2 in [32] that matrix $\mathbf{M}$ is positive definite and $\lambda_{min}(\mathbf{M}) > 0$. This implies that eigenvalues of $\mathbf{T}$ satisfy $\lambda > h$. In summary, setting $A = \mathbf{T} - h^2\mathbf{Q} + h^3\ \mathbf{LJ}_U$ and $\mathbf{x} = \mathbf{y}^h - \mathbf{Y}^h$ and applying Lemma 1 in [2] to (3.2), we conclude that

$$\|\mathbf{E}(h)\|_2 \le \frac{\|\mathbf{E_0}\|_2}{\sigma_{min}\left(\mathbf{T} - h^2\mathbf{Q} + h^3\ \mathbf{LJ}_U\right)}. \tag{3.8}$$

Because the matrix $\mathbf{T}$ is tridiagonal, non-singular, and positive semidefinite, then for sufficiently small h, we can easily deduce that

$$\sigma_{min}\left(\mathbf{T} - h^2\mathbf{Q} + h^3\ \mathbf{LJ}_U\right) \sim \sigma_{min}(\mathbf{T}), \quad as \quad h \to 0.$$

Theorem 3.3. *Suppose that the functions p, q, f, v, u have continuous fourth-order derivatives. If $\boldsymbol{E}(h)$ is the solution of the error system (3.2), then we have $\|\mathbf{E}(h)\|_2 = \mathcal{O}(h^{\min\{\frac{7}{2}, \gamma-\frac{1}{2}\}})$.*

Proof. Since the mentioned functions are continuously differentiable up to order 4, then from the error Equations (3.1) and (3.2), one can observe that there are constants $\mathbf{d}, \bar{\mathbf{d}} \in \mathbb{R}$ where

$$\|\mathbf{E_0}\|_2 \le \bar{\mathbf{d}}h^{\frac{9}{2}} + \mathbf{d}h^{\gamma+\frac{1}{2}},$$

and so, from (3.8) we get

$$\|\mathbf{E}(h)\|_2 \le \frac{\|\mathbf{E_0}\|_2}{\lambda_{min}(\mathbf{T})} \le \frac{\bar{\mathbf{d}}h^{\frac{9}{2}} + \mathbf{d}h^{\gamma+\frac{1}{2}}}{h} = \bar{\mathbf{d}}h^{\frac{7}{2}} + \mathbf{d}h^{\gamma-\frac{1}{2}}.$$

□

Theorem 3.3 states that the maximum order of convergence is attained for $\gamma \ge 4$. To make sure the method converges with the highest order, all that is required is to apply a quadrature rule with $\gamma \ge 4$ for integral parts of (2.1).

4. Numerical results

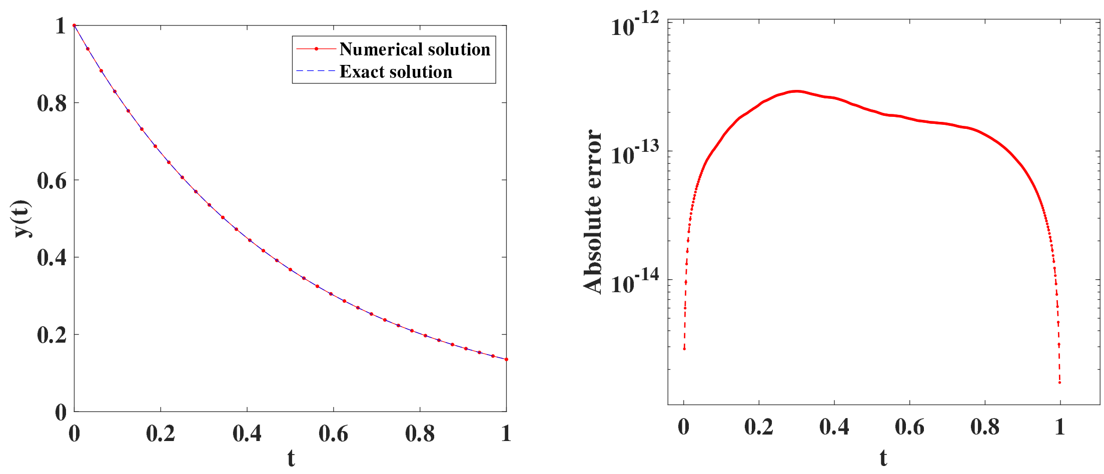
This section includes some numerical examples to illustrate the efficiency of the technique presented in section 2. For the numerical simulations, we consider step sizes $h = 2^{-k}, k = 2, 3, \ldots$, to calculate the errors $\|\mathbf{E}(h)\|_2$ or $\|\mathbf{E}(h)\|_\infty$ (maximum norm). Additionally, the order of convergence $\log_2\left(\frac{\|\mathbf{E}(h)\|}{\|\mathbf{E}(\frac{h}{2})\|}\right)$ is computed where appropriate.

Example 4.1. In the first example, consider the following FIDE:

$$y''(t) + t^2y'(t) + \sin(\pi t)y(t) = f(t) + \int_0^1 s(t+2s)y(s)ds, \quad y(0) = 1, \quad y(1) = e^{-2}, \quad t \in [0, 1],$$

where $f(t) = \frac{1}{4}\left(-8e^{-2t}\left(t^2 - 2\right) - t + \frac{3t+10}{e^2} - 2\right) + e^{-2t}\sin(\pi t)$, and the exact solution is $y(t) = e^{-2t}$. Here, $p(t) = t^2$ and consequently, we have $\bar{\phi}(t) = e^{-\frac{t^3}{3}}$. We solved this problem numerically using the presented in (2.20). In Table 1 shows the obtained L^2- and maximum-norm errors as well as the order of convergence of the proposed technique. The computational results demonstrate that the proposed method achieves high accuracy, with convergence orders of $\frac{7}{2}$ and 4 for the L^2- and maximum-norms, respectively.

This indicates that the expected convergence orders have been attained, which is in good agreement with the theoretical results stated in Theorem 3.3. Additionally, a comparison between the exact and numerical solution for



(a) Exact and approximate solutions with $h = 2^{-9}$. (b) Absolute error.

FIGURE 1. Numerical results obtained of the proposed method for Example 4.1 with $h = 2^{-9}$.

$h = 2^{-9}$ is shown in Figure 1(a). The absolute error of the obtained solution for this example is plotted in Figure 1(b).

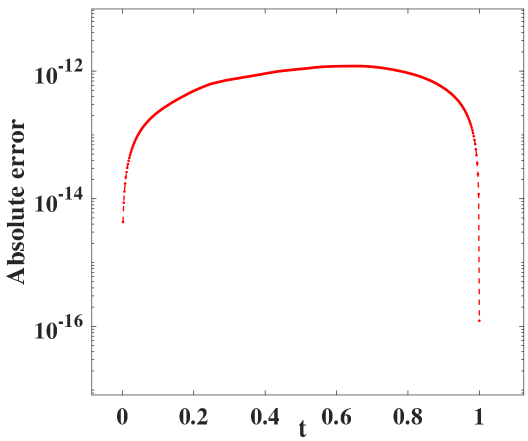
TABLE 1. Errors and convergence order of the results obtained by the method proposed in (2.20) for Example 4.1.

N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
4	1.9400e-04	–	1.1428e-04	–	8.1978e-05	–
8	2.5192e-05	2.9450	1.0125e-05	3.4965	5.1354e-06	3.9967
16	3.1811e-06	2.9853	8.9647e-07	3.4976	3.2176e-07	3.9964
32	3.9867e-07	2.9963	7.9276e-08	3.4993	2.0217e-08	3.9923
64	4.9866e-08	2.9991	7.0080e-09	3.4998	1.2637e-09	3.9998
128	6.2334e-09	2.9999	6.1936e-10	3.5001	7.8975e-11	4.0001
256	7.8274e-10	2.9934	5.4735e-11	3.5002	4.9301e-12	4.0017

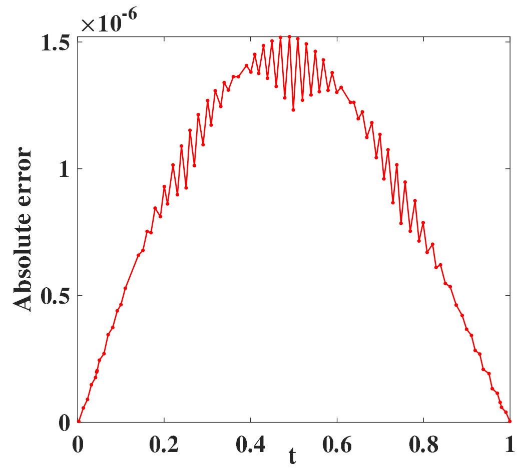
Example 4.2. ([6, 8]) In the second example, consider the test problem

$$y''(t) + ty'(t) + \pi^2 y(t) = \pi t \cos(\pi t) - \frac{2t+1}{\pi} + \int_0^1 (s+t)y(s)ds, \quad t \in [0,1],$$
$$y(0) = y(1) = 0,$$

whose exact solution is given by $y(t) = \sin(\pi t)$. This FIDE is solved using the method proposed in this paper for various step size $h = 2^{-2}, \cdots, 2^{-8}$. The computed $L^\infty-$ and L^2-norm errors of our method are compared with those obtained by the multiscale Galerkin methods presented in [7, 8], and the results are summarized in Table 2. From this, we can conclude that the results obtained confirm the computational convergence order for the proposed method (2.20), i.e., $\frac{7}{2}$ (in L^2-norm) is in a good agreement with theoretical one provided in Theorem 3.3. Moreover, from the above table, it can be observed that the multiscale Galerkin approach [8] converges with order 2, while for the our presented method is $\frac{7}{2}$ which both of them evaluated by L^2-norm. In addition, the earned convergence order of the our proposed method w.r.t L^∞-norm is 4, whereas this amount for multiscale Galerkin method [7] is 3. On the other hand, for the current method in the case of $h = 2^{-9}$, and for the multiscale Galerkin method [8] in the case of $h = 2^{-11}$, we



(a) Our proposed method with $h = 2^{-9}$.



(b) Reproducing kernel method [8] with $h = 2^{-10}$.

FIGURE 2. Absolute errors of the methods for Example 4.2.

have computed the absolute errors and then displayed them in Figure 2. Numerical results in Figure 2 indicate that for the presented method maximum absolute errors is of order $O(10^{-12})$, while for the multiscale Galerkin method expressed in [8] is $O(10^{-6})$.

TABLE 2. Computational results obtained by our proposed method and those obtained by the methods provided in [7, 8] for Example 4.2.

	Our proposed method				Method of [7]		Method of [8]	
N	L_∞ error	Order	L_2 error	Order	L_∞ error	Order	L_2 error	Order
4	6.5415e-04	–	9.3101e-04	–	–	–	–	–
8	4.7219e-05	3.7922	9.4865e-05	3.2949	–	–	–	–
16	3.0455e-06	3.9546	8.6493e-06	3.4552	5.307e-4	–	1.6002e-2	–
32	1.9216e-07	3.9863	7.7028e-07	3.4891	6.214e-5	3.0942	4.0613e-3	1.9782
64	1.2036e-08	3.9969	6.8211e-08	3.4973	7.632e-6	3.0254	1.0192e-3	1.9945
128	7.5299e-10	3.9985	6.0343e-09	3.4988	9.504e-7	3.0056	2.5504e-4	1.9986
256	4.7358e-11	3.9909	5.3362e-10	3.4993	1.186e-7	3.0022	6.3776e-5	1.9997

Example 4.3. As third example, we consider the following FIDE of the second kind [6, 13]

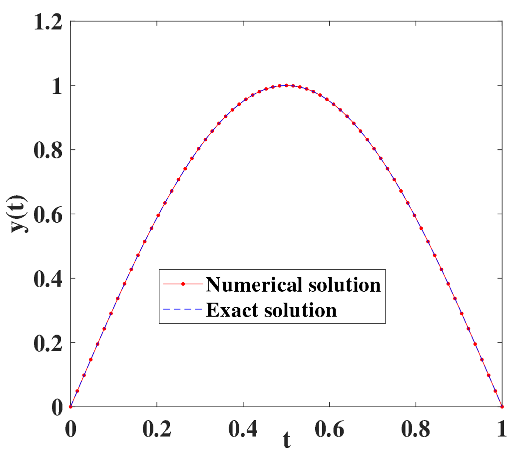
$$y''(t) - \int_0^1 v(t,s)y(s)ds = f(t), \quad t \in [0,1],$$
$$y(0) = y(1) = 0,$$

where $v(t,s) = s + t$, $f(t) = -\pi^2 \sin(\pi t) - \frac{2t+1}{\pi}$. The exact solution of this problem is $y(t) = \sin(\pi t)$.

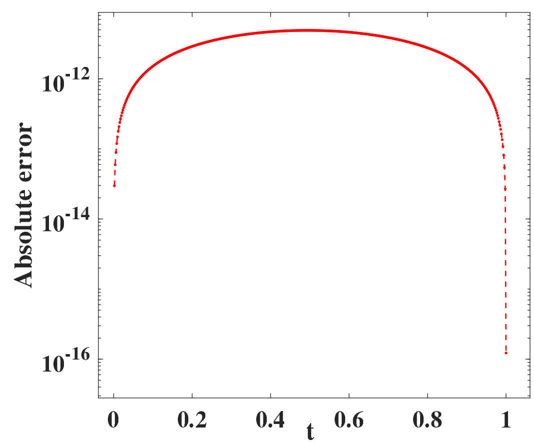
In this example, we compare the results of the fast multiscale Galerkin method [6] and exponential spline method [13]. Table 3 displays the numerical results obtained using different step sizes. These results demonstrate that the proposed method (2.20) achieves convergence orders of 3, 3.5, and 4 with respect to the L_1, L_2 and L_∞ norms, respectively. In contrast, the fast multiscale Galerkin method [6] and exponential spline method [13] achieve convergence orders of approximately 1 and 2 in the L_1 and L_∞ norms, respectively. Therefore, the proposed method is more

TABLE 3. Computational results of the fast multiscale Galerkin method [6], exponential spline method [13] and present method for Example 4.3.

	Present method						Method of [6]		Method of [13]	
N	L_∞ error	Order	L_1 error	Order	L_2 error	Order	L_1 error	Order	L_∞ error	Order
4	1.3383e-03	–	3.2064e-03	–	1.8806e-03	–	–	–	–	–
8	8.2511e-05	4.0197	4.1098e-04	2.9638	1.6394e-04	3.5199	2.5525e-1	–	–	–
16	5.1390e-06	4.0050	5.1675e-05	2.9915	1.4441e-05	3.5050	1.2791e-1	0.9968	2.4213e-4	–
32	3.2091e-07	4.0013	6.4687e-06	2.9979	1.2753e-06	3.5012	6.3991e-2	0.9992	5.9642e-5	2.0214
64	2.0052e-08	4.0003	8.0888e-07	2.9995	1.1270e-07	3.5003	3.2001e-2	0.9998	1.4851e-5	2.0058
128	1.2535e-09	3.9998	1.0112e-07	2.9999	9.9604e-09	3.5001	1.6001e-2	0.9999	3.7093e-6	2.0013
256	7.8342e-11	4.0000	1.2640e-08	2.9999	8.8039e-10	3.5000	8.0006e-3	1.0000	–	–



(a) Exact and approximate solutions for $h = 2^{-9}$.



(b) Absolute error.

FIGURE 3. Numerical results of the proposed method to solve the Example 4.3 with $h = 2^{-9}$.

accurate than the existing techniques reported in the literature. Also, in In Figure 3(a), we plot the numerical and exact solution for $h = 2^{-9}$. The absolute errors shown in Figure 3(b) clearly confirm the effectiveness of the proposed method.

Example 4.4. Consider the following nonlinear FIDE [1]

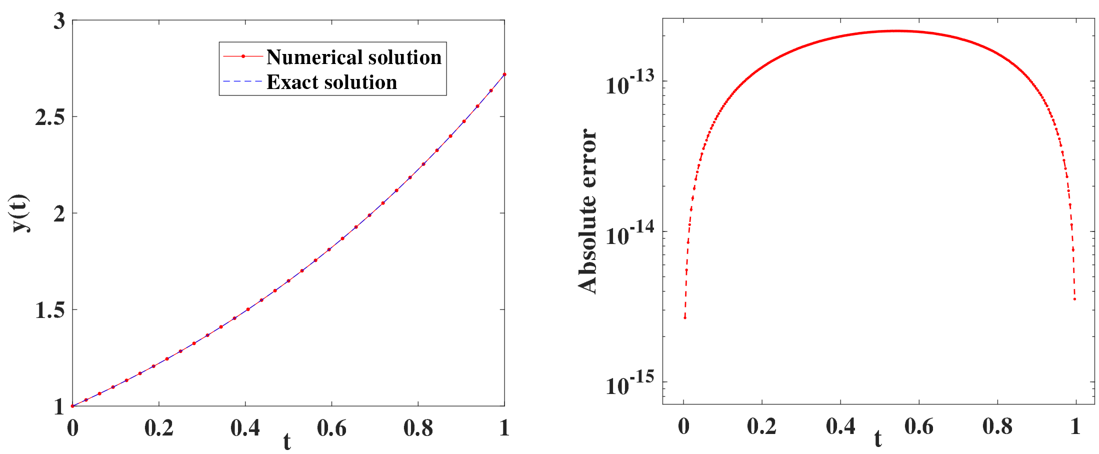
$$y''(t) = \exp(t) + \frac{1}{4}(\exp(2) - 2)t + \frac{1}{2}\int_0^1 t(s - y^2(s))ds, \quad t \in [0, 1],$$
$$y(0) = 1, y(1) = \exp(1),$$

while its real solution is $y(t) = \exp(t)$.

In this example, a comparison between the haar wavelet method [1] and proposed method (2.20) is performed. The computed L^2-norm of errors obtained by our method and method [1] are reported in Table 4. The reported numerical results in Table 4 illustrate that the proposed method (2.20) has more accurate than haar wavelet method [1]. Furthermore, one can observe that the maximum convergence order of the haar wavelet method [1] will be reached 2, while for the presented method maximum order 3.5 will be achieved. Moreover, the computed numerical results and real solution for this problem are plotted in the Figure 3(a) for the case $h = 2^{-8}$. Again, according to the illustrated absolute errors in Figure 3(b), we can observe the accuracy and efficiency of the presented method, clearly.

TABLE 4. Computational results of the methods for Example 4.4.

	Present method		Haar wavelet method [1] with collocation points		Haar wavelet method [1] with Guass points	
N	L_2 error	Order	L_2 error	Order	L_2 error	Order
4	4.8916e-05	–	1.9468e-3	–	7.7106e-4	–
8	3.7499e-06	3.7053	5.1227e-4	1.9261	1.9580e-4	1.9774
16	3.2990e-07	3.5067	1.3141e-4	1.9627	4.9353e-5	1.9881
32	2.9110e-08	3.5024	3.3281e-5	1.9813	1.2390e-5	1.9939
64	2.5412e-09	3.5179	8.3724e-6	1.9910	3.1039e-6	1.9970
128	2.2154e-10	3.5198	2.0983e-6	1.9964	7.7662e-7	1.9988



(a) Exact and approximate solutions for $h = 2^{-8}$. (b) Absolute error.

FIGURE 4. Numerical results of the proposed method to solve the Example 4.4 with $h = 2^{-8}$.

Example 4.5. Consider the following nonlinear FIDE

$$y''(t) + 4y'(t) + \ln(t^2+1)y(t) = f(t) + \int_0^1 s^8(t+\sinh(t))(y^2(s) + \exp(y^2(s)))ds, \quad t \in [0,1],$$
$$y(0) = 0, y(1) = 1,$$

in which $f(t) = \sqrt{t}\left(\frac{63}{4}t^2 + 18t^3 + t^4\ln(t^2+1)\right) - \frac{1}{18}(2\exp(1)-1)(t+\sinh(t))$ and its exact solution is $y(t) = t^4\sqrt{t}$.

This nonlinear problem is solved numerically, utilizing the proposed mentioned method in (2.20). The computed L^1-,L^2- and maximum-errors and the order of convergence of the presented technique are demonstrated in Table 5. Although, the function $f(t)$ as well as the exact solution $y(t)$ are non-smooth, the claimed orders $3, 3.5$ and 4 are achieved with respect to the L^1-,L^2- and maximum-norms, respectively.

Example 4.6. Consider the following nonlinear FIDE

$$y''(t) + y'(t) + t^3\cos(t^3)y(t) = f(t) + \int_0^1 s\sqrt{t}\left(y^4(s) + \frac{1}{6}y^6(s)\right)ds, \quad t \in [0,1],$$
$$y(0) = 0, y(1) = 1,$$

in which $f(t) = -\frac{221}{3036}\sqrt{t} + \frac{70}{9}t^{4/3} + \frac{10}{3}t^{7/3} + t^{19/3}\cos(t^3)$ and its exact solution is $y(t) = t^3\sqrt[3]{t}$.

TABLE 5. Errors and convergence order of the results obtained by the method proposed in (2.20) for Example 4.5.

N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
4	1.1413e-02	–	7.0522e-03	–	5.0705e-03	–
8	7.2369e-03	0.6572	2.8742e-03	1.2949	1.4142e-03	1.8421
16	1.7853e-03	2.0192	4.9238e-04	2.5453	1.6918e-04	3.0634
32	2.8578e-04	2.6432	5.5488e-05	3.1495	1.3468e-05	3.6510
64	3.8475e-05	2.8929	5.2767e-06	3.3945	9.0545e-07	3.8948
128	4.9051e-06	2.9716	4.7555e-07	3.4720	5.7692e-08	3.9722
256	6.1620e-07	2.9928	4.2240e-08	3.4929	3.6235e-09	3.9929

TABLE 6. Errors and convergence order of the results obtained by the method proposed in (2.20) for Example 4.6.

N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
4	6.7495e-03	–	3.9339e-03	–	2.6863e-03	–
8	1.7574e-03	1.9413	6.8927e-04	2.5128	3.3115e-04	3.0200
16	2.8796e-04	2.6095	7.8521e-05	3.1339	2.6476e-05	3.6447
32	4.0748e-05	2.8210	7.7744e-06	3.3363	1.8330e-06	3.8524
64	5.6754e-06	2.8439	8.5875e-07	3.1784	1.2418e-07	3.8837
128	7.1425e-07	2.9902	7.6432e-08	3.4900	8.6377e-09	3.8457
256	8.9989e-08	2.9886	6.7942e-09	3.4918	5.4524e-10	3.9857

The above nonlinear FIDE has been solved numerically, by the presented method for different values of step size $h = 2^{-2}, \cdots, 2^{-8}$. As we can see, the functions $f(t), v(t,s)$, and the exact solution $y(t)$ are not smooth, because $f(t) \notin C^1[0,1], v(t,s) \notin C^1([0,1] \times [0,1])$ and $y(t) \notin C^4[0,1]$. The reported computational results in Table 6 show that the proposed method has also good numerical approximation for this problem. Again, we can observe that the demanded order of convergence can be obtained by the proposed method (2.20).

5. CONCLUSIONS

This paper presented a new numerical method for solving a class of nonlinear second-order FIDEs. First, the matrix formulation of the proposed method was derived, and some results from linear algebra were employed. It was then proved that the established method achieves a convergence order of $O(h^{\min\{\frac{7}{2},\gamma-\frac{1}{2}\}})$, with respect to the L^2-norm, where γ denotes the order of convergence of the selected numerical integration method.

To demonstrate the accuracy and efficiency of the proposed approach, several numerical examples were provided. Furthermore, the theoretical convergence order was verified through these numerical problems.

Finally, it is important that implementation the proposed approach for high-dimensional FIDEs or systems of FIDEs presents certain challenges. In such cases, increasing the dimensionality of the associated vectors and matrices inevitably leads to higher computational costs.

To mitigate this, one must reduce the size of the vectors and matrices, which in turn results in a decrease in the approximation accuracy, as smaller step sizes cannot be used. Therefore, the authors intend to extend this work by developing variants of the method that maintain high accuracy while reducing computational cost.

REFERENCES

[1] R. Amin, I. Mahariq, K. Shah, M. Awais, and F. Elsayed, *Numerical solution of the second order linear and nonlinear integro-differential equations using Haar wavelet method*, Arab Journal of Basic and Applied Sciences, *28*(1) (2021), 11–19.

[2] S. Amiri, *Effective numerical methods for nonlinear singular two-point boundary value Fredholm integro-differential equations*, Iranian Journal of Numerical Analysis and Optimization, *13*(3) (2023), 444–459.

[3] S. H. Behiry and H. Hashish, *Wavelet methods for the numerical solution of Fredholm integro-differential equations*, Int. J. Appl. Math., *11*(1) (2002), 27–35.

[4] S. H. Behiry and S. I. Mohamed, *Solving high-order nonlinear Volterra–Fredholm integro-differential equations by differential trasform method*, Nat. Sci., *4*(8) (2012), 581–587.

[5] A. H. Bhrawy, E. Tohidi, and F. Soleymani, *A new Bernoulii matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals*, Appl. Math. Comput., *219* (2012), 482–497.

[6] J. Chen, M. F. He, and Y. Huang, *A fast multiscale Galerkin method for solving second order linear fredholm integro-differential equation with Dirichlet boundary conditions*, J. Comput. Appl. Math., *364* (2020), 112352.

[7] J. Chen, M. He, and T. Zeng, *A multiscale Galerkin method for second-order boundary value problems of Fredholm integro-differential equation II: Effcient algorithm for the discrete linear system*, J. Vis. Commun. Image Represent., *58* (2019), 112–118.

[8] J. Chen, Y. Huang, H. Rong, T. Wu, and T. Zeng, *A multiscale Galerkin method for second-order boundary value problems of Fredholm integro-differential equation*, J. Comput. Appl. Math., *290* (2015), 633–640.

[9] M. Dehghan and A. Saadatmandi, *Chebyshev finite difference method for Fredholm integro-differential equation*, Int. J. Comput. Math. *85* (2008), 123–130.

[10] D. S. Dzhumabaev, *On one approach to solve the linear boundary value problems for Fredholm integro-differential equations*, J. Comput. Appl. Math., *294* (2016), 342–357.

[11] O. A. Gegele, O. P. Evans, and D. Akoh, *Numerical solution of higher order linear Fredholm integro-differential equations*, Applied Journal of Engineering, *8* (2014), 243–247.

[12] S. Islam, I. Aziz, and M. Fayyaz, *A new approach for numerical solution of integro-differential equations via Harr wavelets*, Int. J. Comput. Math., *90*(9) (2013), 1971–1989.

[13] R. Jalilian and T. Tahernezhad, *Exponential spline method for approximation solution of Fredholm integro-differential equation*, Int. J. Comput. Math., *97*(4) (2020), 791–801.

[14] P. Kanwal, *Linear integral equations theory and technique*, London, Academic Press, 1971.

[15] G. Long, G. Nelakanti, and X. Zhang, *Iterated fast multiscale Galerkin methods for Fredholm integral equations of second kind with weakly singular kernels*, Appl. Numer. Math., *62* (2012), 201–211.

[16] F. Mirzaee, *Numerical solution of nonlinear Fredholm-Volterra integral equations via Bell polynomials*, Comput. Methods Differ. Equ., *5*(2) (2017), 88–102.

[17] A. Molabahrami, *Direct computation method for solving a general nonlinear Fredholm integro-differential equation under the mixed conditions: Degenerate and non-degenerate kernels*, J. Comput. Appl. Math., *282* (2015), 34–43.

[18] Y. Ordokhani, *An application of Walsh functions for Fredholm–Hammerstein integro-differential equations*, Int. J. Contemp. Math. Sci., *5*(22) (2010), 1055–1063.

[19] P. K. Pandey, *Non-standard difference method for numerical solution of linear Fredholm integro-differential type two-point boundary value problems*, Open Access Lib. J., *2* (2015), 1–10.

[20] M. A. L. Ramadan, K. M. Raslan, and M. A. E. G. Nassear, *A rational Chebyshev functions approach for Fredholm-Volterra integro-differential equations*, Comput. Methods Differ. Equ., *3*(4) (2015), 284-297.

[21] M. Razzaghi and S. Yousefi, *Legendre wavelets method for the nonlinear Volterra Fredholm integral equations*, Math. Comput. Simul., *70* (2005), 1–8.

[22] A. Saadatmandi and M. Dehghan, *Numerical solution of high-order linear Fredholm integro-differential–difference equation with variable coefficients*, Comput. Math. Appl., *59* (2010), 2996–3004.

[23] T. L. Saaty, *Modern nonlinear equations*, New York, Dover publications., 1981.

[24] A. Shidfar, A. Molabahrami, A. Babaei, and A. Yazdanian, *A series solution of the nonlinear Volterra and Fredholm integro-differential equations*, Commun. Nonlinear Sci. Numer. Simul., *15* (2010), 205–215.

[25] H. R. Thiem, *A model for spatio spead of an epidemic*, J. Math. Biol., *4* (1977), 337–351.

[26] Q. Xue , J. Niu, D. Yu, and C. Ran, *An improved reproducing kernel method for fredholm integro-differential type two-point boundary value problems*, Int. J. Comput. Math., *95* (2017), 1015–1023.

[27] S. Yalcinbas, *Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations*, Appl. Math. Comput., *127* (2002), 195–206.

[28] S. Yalcinbas and M. Sezer, *The approximate solution of high-order linear Volterra-Fredholm integro-differential equations in terms of Taylor polynomials*, Appl. Math. Comput., *112* (2000), 291–308.

[29] S. Yang, X. Luo, F. Li, and G. Long, *A fast multiscale Galerkin method for the first kind ill-posed integral equations via iterated regularization*, Appl. Math. Comput., *219* (2013), 10527–10537.

[30] S. Yeganeh, Y. Ordokhani, and A. Saadatmandi, *A Sinc-collocation method for second-order boundary value problems of nonlinear integro-differential equation*, J. Inf. Comput. Sci., *7*(2) (2012), 151–160.

[31] W. Yulan, T. Chaolu, and P. Jing, *New algorithm for second-order boundary value problems of integro-differential equation*, J. Comput. Appl. Math., *229* (2009), 1–6.

[32] F. Zhang, *Matrix Theory*, Springer, New York, 2011.