Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-15
DOI:10.22034/cmde.2024.58818.2489

High-order numerical solution for a class of nonlinear Fredholm integro-differential equations

Sadegh Amiri* **and Mohammad Eshaghnezhad**

Department of Basic Sciences, Shahid Sattari Aeronautical University of Science and Technology P.O. Box: 13846-63113, Tehran, Iran.

Abstract

The main objective of this work is to provide a high-order numerical method to solve a class of integro-differential equations. By multiplying some efficient factors and constructing an appropriate approximate function, as well as utilizing a numerical integration method with order γ, the above-mentioned problem can be simplified to a nonlinear system of algebraic equations. Furthermore, we discuss the convergence analysis of the presented method, in detail, and demonstrate that it converges with order $\mathcal{O}(h^{3.5})$ in L^2-norm. Some test examples are provided to demonstrate that the claimed order of convergence is obtained.

Keywords. Efficient factors, Approximate function, Nonlinear algebraic system, Convergence analysis, Order of convergence.

2010 Mathematics Subject Classification. 45A05, 45C05, 65R20, 45J05, 65L10, 45B05.

1. Introduction

There are numerous applications of integro-differential equations in the fields of natural sciences and engineering. Some of these equations can be seen in the mathematical modeling of spatiotemporal developments, epidemic modeling [25], heat and mass transfer [14], scattering in quantum physics and population models [23].

Since, the extraction of exact solutions of integro-differential equations is too hard, researchers have compelled to solve them numerically, and so the various numerical procedures are extended to approximate their solutions, widely. Recently, several approaches have been extended for solving integral equation and specially integro-differential equations, numerically or approximately. Some of the well-known methods that have been recently used for these equations are Taylor collocation method [27, 28], iterative method [31], rational Chebyshev functions approach [20], sinc-collocation method [30], wavelet method [3, 12], Legendre wavelets method [21], Walsh function method [18], improved reproducing kernel method [26], differential transform method [4], Bell polynomials method [16], homotopy analysis method [24], Haar wavelet method [1], Chebshyev series and power series technique [11], Chebyshev finite difference method [9], direct computation method [17], non-standard difference method [19], exponential spline method [13], Legendre polynomial method [22], parametrization method [10], and Bernoulli matrix method [5]. It should be pointed out that the convergence rate of these above mentioned methods is less than or equal two.

The multiscale Galerkin method is one of the practical tools that researchers have used in many mathematical applications. For example, the authors of [6–8, 15, 29] have used the multiscale Galerkin method for various kinds of integral/integro-differential equations. We remind that except for the method used in [7], the convergence order of other applied methods is maximum 2. In fact, the convergence order of the method utilized in [7] is 3.

Due to the lack of numerical methods with a high convergence order for two-point boundary value integro-differential equations, in this research, a suitable high-order numerical method will be introduced to solve a class of nonlinear second-order Fredholm integro-differential equations (FIDEs). The proposed method is based on multiplying some appropriate exponential functions on both sides of the problem under consideration, applying some numerical integration methods and employing an approximate function for solving the second-order boundary value problem governed by

Received: 13 October 2023 ; Accepted: 28 October 2024.
∗ Corresponding author. Email: s.amiri@ssau.ac.ir, amirimath@yahoo.com.

the nonlinear FIDEs. The established method is straightforward to fully discretize the considered FIDEs. By implementing this method, the problem under consideration is transformed into a system of nonlinear algebraic equations. The convergence analysis of the proposed method is established. It will be shown that the convergence order of the developed method is $\mathcal{O}(h^{\min\{\frac{7}{2},\gamma-\frac{1}{2}\}})$ in L^2-norm.

This paper is structured in the following way. In section 2, a class of FIDEs will be considered and then an effective numerical method will be presented. Error analysis of the new method is discussed in section 3 in L^2-norm. In addition, we will demonstrate in this section that the order of convergence of the proposed method is achievable up to $\mathcal{O}(h^{\frac{7}{2}})$ in L^2-norm. The efficiency of the method presented in section 2 is verified through the use of comparative numerical examples in section 4.

2. Description of the proposed method

In light of the significance of integro-differential equations mentioned in the preceding section, the following class of the second-order two-point boundary value problems of FIDEs are considered:

$$\begin{cases} y''(t) + p(t)y'(t) + q(t)y(t) = f(t) + \int_0^1 v(t,s)u(y(s))ds, \quad t \in (0,1], & \text{(2.1a)} \\ y(0) = a, \quad y(1) = b, & \text{(2.1b)} \end{cases}$$

where p, q, f, v and u are given functions, and y is an unknown function, and all of them are in $C^4([0,1])$.
Consider the partition $\{t_k = kh : \ k = 0, 1, \dots, N\}$ for the closed interval $[0,1]$, with $h = \frac{1}{N}$ and $t_0 = 0$, $t_N = 1$. Assume that Y_k and $V_{k,n}$ be the approximation of $y_k := y(t_k)$ and $v_{k,n} := v(t_k, t_n)$ for $k = 0, 1, \dots, N$,respectively. Also, let $\mathcal{I}_k^{\pm} = [\min\{t_k, t_{k\pm1}\}, \max\{t_k, t_{k\pm1}\}]$. In the next section, a new numerical approach is presented to find the solution to the above-mentioned problem. Before stating the main result, we provide some requirements. First, suppose that $p(t) \geq 0$ is integrable and there exists a function $\phi(t)$ such that

$$\begin{cases} \phi'(t) = p(t), \qquad t \in (0,1], \\ \phi(0) = 0. \end{cases}$$

Then, with the aid of the following lemma, we try to find an appropriate approximation for the integral forms $\varphi_k^+(s) := \int_s^{t_{k+1}} \phi^-(t)dt$ and $\varphi_k^-(s) := -\int_{t_{k-1}}^s \phi^-(t)dt$ for $k = 1, \cdots, N-1$, where $\phi^-(t) = e^{-\phi(t)}$.

Lemma 2.1. *If $g \in L^2([0,1])$ is a differentiable function, and its derivatives up to order κ be continuous, then we have*

$$\begin{cases} \int_s^{t_{k\pm1}} g(t)dt = \sum_{m=0}^{\kappa-1} \frac{(-1)^m (t_k - s)^{m+1} + (\pm h)^{m+1}}{(m+1)!} \ g_k^{(m)} + \frac{(-1)^\kappa (t_k - s)^{\kappa+1} + (\pm h)^{\kappa+1}}{(\kappa+1)!} \ g^{(\kappa)}(\eta_k^{\pm}), & \text{(2.2a)} \\ g_k^{(m)} = g^{(m)}(t_k), \eta_k^{\pm} \in [0,1], \qquad m = 0, 1, \cdots, \kappa, \qquad k = 1, \cdots, N-1. & \text{(2.2b)} \end{cases}$$

Proof. According to the expressed assumptions, if the function g is expanded in the grid t_k, then there exists $\eta_k(t) \in [0,1]$, such that

$$g(t) = \sum_{m=0}^{\kappa-1} \frac{(t-t_k)^m}{m!} \ g_k^{(m)} + \frac{(t-t_k)^\kappa}{\kappa!} \ g^{(\kappa)}(\eta_k(t)).$$

So, by employing the mean value theorem for definite integrals, there exists constants $\eta_k^+, \eta_k^- \in [0,1]$ such that

$$\begin{cases} \int_s^{t_{k+1}} g(t)dt &= \sum_{m=0}^{\kappa-1} \int_s^{t_{k+1}} \frac{(t-t_k)^m}{m!} g_k^{(m)} dt + \int_s^{t_{k+1}} \frac{(t-t_k)^\kappa}{\kappa!} g^{(\kappa)}(\eta_k(t))dt \\ &= \sum_{m=0}^{\kappa-1} \frac{h^{m+1}-(s-t_k)^{m+1}}{(m+1)!} g_k^{(m)} + \frac{h^{\kappa+1}-(s-t_k)^{\kappa+1}}{(\kappa+1)!} g^{(\kappa)}(\eta_k^+) \\ &= \sum_{m=0}^{\kappa-1} \frac{(-1)^m(t_k-s)^{m+1}+h^{m+1}}{(m+1)!} g_k^{(m)} + \frac{(-1)^\kappa(t_k-s)^{\kappa+1}+h^{\kappa+1}}{(\kappa+1)!} g^{(\kappa)}(\eta_k^+), \\ \\ \int_{t_{k-1}}^s g(t)dt &= \sum_{m=0}^{\kappa-1} \int_{t_{k-1}}^s \frac{(t-t_k)^m}{m!} g_k^{(m)} dt + \int_{t_{k-1}}^s \frac{(t-t_k)^\kappa}{\kappa!} g^{(\kappa)}(\eta_k(t))dt \\ &= \sum_{m=0}^{\kappa-1} \frac{(s-t_k)^{m+1}-(-h)^{m+1}}{(m+1)!} g_k^{(m)} + \frac{(s-t_k)^{\kappa+1}-(-h)^{\kappa+1}}{(\kappa+1)!} g^{(\kappa)}(\eta_k^-) \\ &= \sum_{m=0}^{\kappa-1} \frac{(-1)^{m+1}(t_k-s)^{m+1}-(-h)^{\kappa+1}}{(m+1)!} g_k^{(m)} + \frac{(-1)^{\kappa+1}(t_k-s)^{\kappa+1}-(-h)^{\kappa+1}}{(\kappa+1)!} g^{(\kappa)}(\eta_k^-), \end{cases}$$

that completes the proof. □

Therefore, if we apply Lemma 2.1 with $\kappa = 4$ for $\varphi_k^\pm(s)$, then we conclude that

$$\varphi_k^\pm(s) = \sum_{m=0}^{3} \frac{(-1)^m(t_k-s)^{m+1}+(\pm h)^{m+1}}{(m+1)!} \phi_k^{-(m)} + \frac{(t_k-s)^5 \pm h^5}{5!} \phi^{-(4)}(\eta_k^\pm), \eta_k^\pm \in [0,1], \tag{2.3}$$

in which $\phi_k^{-(m)} = \frac{d^m}{dt^m}\phi^-(t)|_{t=t_k}$. Here, the main aim of these calculations is to discretize FIDE (2.1) and formulate the proposed method. Since, we will confront to integral forms $\int_{t_k}^{t_{k\pm1}} g(t)\varphi_k^\pm(t)dt$ for some functions g, therefore, we first propose a pair of approximate functions of $g(t)$ on $\mathcal{I}_k^-$ and $\mathcal{I}_k^+$ as follow:

$$\begin{cases} \mathcal{L}_k^-(t) := g_{k-1} + \frac{g_k - g_{k-1}}{h}(t-t_{k-1}) + \frac{g_{k-1}-2g_k+g_{k+1}}{2h^2}(t-t_k)(t-t_{k-1}), & t \in \mathcal{I}_k^-, \\ \mathcal{L}_k^+(t) := g_{k+1} + \frac{g_{k+1} - g_k}{h}(t-t_{k+1}) + \frac{g_{k-1}-2g_k+g_{k+1}}{2h^2}(t-t_k)(t-t_{k+1}), & t \in \mathcal{I}_k^+. \end{cases} \tag{2.4}$$

Error of the approximate functions $\mathcal{L}_k^\pm(t)$ for $g(t)$ on $\mathcal{I}_k^-$ and $\mathcal{I}_k^+$ is stated and proved in the following Lemma.

Lemma 2.2. *Let $g \in \mathbb{C}^4([0,1])$. Then, the error of $\mathcal{L}_k^\pm(t)$ for approximating $g(t)$ is as follows:*

$$\begin{cases} g(t) - \mathcal{L}_k^-(t) = (t-t_k)(t-t_{k-1})\left(\frac{1}{3!}(t-t_{k+1})g^{(3)}(\xi_{0,k}^-) - \frac{h^2}{4!}g^{(4)}(\xi_{1,k}^-)\right), & t \in \mathcal{I}_k^-, \\ g(t) - \mathcal{L}_k^+(t) = (t-t_k)(t-t_{k+1})\left(\frac{1}{3!}(t-t_{k-1})g^{(3)}(\xi_{0,k}^+) - \frac{h^2}{4!}g^{(4)}(\xi_{1,k}^+)\right), & t \in \mathcal{I}_k^+, \end{cases} \tag{2.5}$$

where $\xi_{0,k}^\pm$, $\xi_{1,k}^\pm \in (t_{k-1}, t_{k+1})$ and $k = 1, \dots, N-1$.

Proof. The proof is left to the reader. □

In the next step, we try to find an appropriate approximation for the integral forms $I_k^\pm(g) := \int_{t_k}^{t_{k\pm1}} g(t)\varphi_k^\pm(t)dt$ where $g \in C^4([0,1])$.

Lemma 2.3. *Let $g \in C^4([0,1])$. If one applies (2.3) for $\varphi_k^\pm(t)$ and then utilises approximate method (2.4) for integrand of $I_k^\pm(g) = \int_{t_k}^{t_{k\pm1}} g(t)\varphi_k^\pm(t)dt$, then there exists constants $C_{\mathfrak{g}_k}^\pm \in \mathbb{R}$ such that an appropriate approximation for $I_k^\pm(g)$ can be obtained as follows:*

$$I_k^\pm(g) = h^2\mathbf{M}_k^\pm\mathbf{g}_\mathbf{k}^\pm + C_{\mathfrak{g}_k}^\pm h^5, \tag{2.6}$$

in which $\mathbf{M}_k^{\pm} = (H.\widehat{\phi}_k)\ A^{\pm\top}$, $\mathbf{g}_\mathbf{k}^{\pm} = \left[g_0{}_k^{\pm}, g_1{}_k^{\pm}, g_2{}_k^{\pm}\right]^\top$, *and* $C_{\mathfrak{g}_k}^{\pm}$ *depends on the forth-order derivatives of function* g, *and*

$$\begin{cases} H = \left[1, h, h^2, h^3\right], \widehat{\phi}_k = \left[\phi_k^-, \phi_k^{-(1)}, \phi_k^{-(2)}, \phi_k^{-(3)}\right], \\ A^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & \frac{1}{30} \\ \frac{1}{3} & \frac{5}{24} & \frac{3}{40} & \frac{7}{360} \\ \frac{-1}{24} & \frac{-7}{240} & \frac{-1}{90} & \frac{-1}{336} \end{bmatrix}, A^- = \begin{bmatrix} \frac{1}{2} & \frac{-1}{3} & \frac{1}{8} & \frac{1}{30} \\ \frac{1}{3} & \frac{-5}{24} & \frac{3}{40} & \frac{7}{360} \\ \frac{-1}{24} & \frac{7}{240} & \frac{-1}{90} & \frac{-1}{336} \end{bmatrix}, \\ g_0{}_k^{\pm} = g_{k\pm1}, \quad g_1{}_k^{\pm} = g_k - g_{k\pm1}, \quad g_2{}_k = g_{k-1} - 2g_k + g_{k+1}. \end{cases}$$

Proof. By applying (2.3) together with (2.5) and then rearrange the coefficients, the proof is straightforward. □

Now, we are ready to solve FIDE (2.1). At first, we multiply the function $\phi^+(t) = e^{\phi(t)}$ on both sides of (2.1). Thus, the following equation can be obtained:

$$\left(\phi^+(t)y'(t)\right)' + \phi^q(t)y(t) = \phi^f(t) + \phi^+(t)\int_0^1 v(t,s)u(y(s))ds, \tag{2.7}$$

in which $\phi^q(t) = \phi^+(t)q(t), \phi^f(t) = \phi^+(t)f(t)$. Then, by integration of (2.7) over $[t_k, t]$ and employing integration by parts, result that

$$\begin{aligned} \phi^+(t)y'(t) - \phi_k^+ y_k' + \int_{t_k}^t \phi^q(\xi)y(\xi)d\xi &= \int_{t_k}^t \phi^f(\xi)d\xi \\ &+ \int_{t_k}^t\int_0^1 \phi^+(\xi)v(\xi,s)u(y(s))dsd\xi, \quad k = 1,\ldots,N-1. \end{aligned} \tag{2.8}$$

Now, by dividing both sides of (2.8) on $\phi^+(t)$, and then integration over intervals $\mathcal{I}_k^{\pm}$, the subsequent system can be achieved:

$$\begin{cases} y_{k+1} - y_k - \phi_k^+\vartheta_k y_k' + \int_{t_k}^{t_{k+1}}\int_{t_k}^t \phi^-(t)\phi^q(\xi)y(\xi)d\xi dt = \int_{t_k}^{t_{k+1}}\int_{t_k}^t \phi^-(t)\phi^f(\xi)d\xi dt \\ \qquad + \int_{t_k}^{t_{k+1}}\int_{t_k}^t \phi^-(t)\int_0^1 \phi^+(\xi)v(\xi,s)u(y(s))dsd\xi dt, & (2.9a) \\ y_{k-1} - y_k + \phi_k^+\vartheta_{k-1} y_k' + \int_{t_k}^{t_{k-1}}\int_{t_k}^t \phi^-(t)\phi^q(\xi)y(\xi)d\xi dt = \int_{t_k}^{t_{k-1}}\int_{t_k}^t \phi^-(t)\phi^f(\xi)d\xi dt \\ \qquad + \int_{t_k}^{t_{k-1}}\int_{t_k}^t \phi^-(t)\int_0^1 \phi^+(\xi)v(\xi,s)u(y(s))dsd\xi dt, \ k = 1,\ldots,N-1, & (2.9b) \end{cases}$$

where $\vartheta_k := \int_{t_k}^{t_{k+1}} \phi^-(t)dt$. Next, using the Fubini's theorem, the system of Equations (2.9) can be expressed as follows:

$$\begin{cases} y_{k+1} - y_k - \phi_k^+\vartheta_k y_k' + I_k^+(\phi^q y) = I_k^+(\phi^f) + \int_0^1 u(y(s))I_k^+(\phi^+(.)v(.,s))ds, & (2.10a) \\ y_{k-1} - y_k - \phi_k^+\vartheta_{k-1} y_k' + I_k^-(\phi^q y) = I_k^-(\phi^f) + \int_0^1 u(y(s))I_k^-(\phi^+(.)v(.,s))ds, & (2.10b) \end{cases}$$

in which $k = 1,\ldots,N-1$, and

$$\begin{aligned} &I_k^{\pm}(\phi^q y) = \int_{t_k}^{t_{k\pm1}} \phi^q(t)y(t)\varphi_k^{\pm}(t)dt, \quad I_k^{\pm}(\phi^f) = \int_{t_k}^{t_{k\pm1}} f(t)\varphi_k^{\pm}(t)dt, \\ &I_k^{\pm}(\phi^+(.)v(.,s)) = \int_{t_k}^{t_{k\pm1}} \phi^+(t)v(t,s)\varphi_k^{\pm}(t)dt. \end{aligned}$$

By adding the equations available in (2.10) together, the following system of integral equations is obtained

$$\tau_k y_{k+1} - (\tau_{k-1} + \tau_k) y_k + \tau_{k-1} y_{k-1} + I_k(\phi^q y) = I_k(\phi^f) + \int_0^1 u(y(s)) I_k(\phi^+(.)v(.,s))ds, \tag{2.11}$$

where $\tau_k = \frac{1}{\vartheta_k}, k = 1, \dots, N-1$, and

$$\begin{cases} I_k(\phi^q y) := \tau_{k-1} I_k^-(\phi^q y) + \tau_k I_k^+(\phi^q y), & (2.12\text{a}) \\ I_k(\phi^f) := \tau_{k-1} I_k^-(\phi^f) + \tau_k I_k^+(\phi^f), & (2.12\text{b}) \\ I_k(\phi^+(.)v(.,s)) := \tau_{k-1} I_k^-(\phi^+(.)v(.,s)) + \tau_k I_k^+(\phi^+(.)v(.,s)), \quad k = 1, \dots, N-1. & (2.12\text{c}) \end{cases}$$

According to the boundary conditions of the original problem (2.1), we have $y_0 = a$ and $y_N = b$. Therefore, to find the unknown values $y_1, \dots, y_{N-1}$, we must solve system (2.11). It should be mentioned that this nonlinear system is difficult to solve by the analytical methods, and it needs to be used some approximate and numerical methods. Therefore, to reach the approximate solution of the system (2.11), let's use some numerical integration techniques for the integrals in (2.12) and the integral term in the right side of (2.11). For this purpose, we utilize the numerical integration method (2.6) to approximate the integrals given by (2.12). Furthermore, if we use a numerical integration method of order γ to approximate the integral term $\int_0^1 u(y(s)) I_k(\phi^+(.)v(.,s))ds$ given in the right-hand side of (2.11), with the integration points $\{t_j\}_{j=0}^N$ and weights $\mathbf{w} = (\ell_0, \ell_1, \dots, \ell_N)^\top$, then it is inferred that there are constants $c_{0_k}^{\pm} \in \mathbb{R}$ where

$$\begin{aligned} \int_0^1 u(y(s)) I_k(\phi^+(.)v(.,s))ds &= \tau_{k-1} \int_0^1 u(y(s)) I_k^-(\phi^+(.)v(.,s))ds + \tau_k \int_0^1 u(y(s)) I_k^+(\phi^+(.)v(.,s))ds \\ &= h\tau_{k-1} \sum_{j=0}^N \ell_j u(y_j) I_k^-(\phi^+(.)v(.,t_j)) + h\tau_k \sum_{j=0}^N \ell_j u(y_j) I_k^+(\phi^+(.)v(.,t_j)) \\ &+ (\tau_{k-1} c_{0_k}^- + \tau_k c_{0_k}^+) h^{\gamma+2}. \end{aligned}$$

So, according to (2.6), we conclude that there exists constants $c_{l,k}^{\pm} \in \mathbb{R}, k = 1, \dots, N-1, l = 0, \dots, 3$, such that

$$\begin{cases} I_k(\phi^q y) := h^2 \Phi_{0_k} + c_{1_k} h^5, \qquad I_k(\phi^f) := h^2 \Phi_{1_k} + c_{2_k} h^5, \\ \displaystyle\int_0^1 u(y(s)) I_k(\phi^+(.)v(.,s))ds := h^3 \sum_{j=0}^N \ell_j u_j \Phi_{2_{k,j}} + h^6 c_{3_k} \sum_{j=0}^N \ell_j u_j + c_{0_k} h^{\gamma+2}, \quad k = 1, \dots, N-1, \end{cases}$$

where for $k = 1, \dots, N-1, i = 0, 1, l = 0, \dots, 3$,

$$\begin{cases} \Phi_{i_k} = \tau_{k-1} \mathbf{M}_k^- \Phi_{i_k}^- + \tau_k \mathbf{M}_k^+ \Phi_{i_k}^+, \\ \Phi_{2_{k,j}} = \tau_{k-1} \mathbf{M}_k^- \Phi_{2_{k,j}}^- + \tau_k \mathbf{M}_k^+ \Phi_{2_{k,j}}^+, \\ c_{l_k} = \tau_{k-1} c_{l,k}^- + \tau_k c_{l,k}^+, \end{cases}$$

and for $k = 1, \dots, N-1, j = 0, \dots, N$,

$$\begin{cases} \Phi_{0_k}^{\pm} = \mathcal{C}_0^{\pm} \phi_{k-1}^q y_{k-1} + \mathcal{C}_1 \phi_k^q y_k + \mathcal{C}_0^{\mp} \phi_{k+1}^q y_{k+1}, \\ \Phi_{1_k}^{\pm} = \mathcal{C}_0^{\pm} f_{k-1} + \mathcal{C}_1 f_k + \mathcal{C}_0^{\mp} f_{k+1}, \\ \Phi_{2_{k,j}}^{\pm} = \mathcal{C}_0^{\pm} \phi_{k-1}^+ v_{k-1,j} + \mathcal{C}_1 \phi_k^+ v_{k,j} + \mathcal{C}_0^{\mp} \phi_{k+1}^+ v_{k+1,j}, \\ \mathcal{C}_0^+ = [0,0,1]^\top, \quad \mathcal{C}_0^- = [1,-1,1]^\top, \quad \mathcal{C}_1 = [0,1,-2]^\top. \end{cases}$$

Hence, the following fully discrete method is developed to solve the two-point boundary value FIDE (2.1):

$$a_k y_{k-1} + b_k y_k + c_k y_{k+1} + h^3 \sum_{j=0}^N \ell_j u_j \Phi_{2_{k,j}} = -h^2 \Phi_{1_k} + C_k h^5 + c_{0_k} h^{\gamma+2}, \tag{2.13}$$

where

$$\begin{cases} a_k = -\tau_{k-1} - h^2\left(\tau_{k-1}\mathbf{M}_k^-\mathcal{C}_0^- + \tau_k\mathbf{M}_k^+\mathcal{C}_0^+\right)\phi_{k-1}^q, \\ b_k = (\tau_{k-1}+\tau_k) - h^2\left(\tau_{k-1}\mathbf{M}_k^- + \tau_k\mathbf{M}_k^+\right)\mathcal{C}_1\phi_k^q, \\ c_k = -\tau_k - h^2\left(\tau_{k-1}\mathbf{M}_k^-\mathcal{C}_0^+ + \tau_k\mathbf{M}_k^+\mathcal{C}_0^-\right)\phi_{k+1}^q, \\ C_k = c_{1_k} - c_{2_k} - hc_{3_k}\sum_{j=0}^{N}\ell_j u_j. \end{cases}$$

Thus, if we set

$$\begin{cases} \widehat{\Phi}_2^k = \left[\Phi_{2_{1,k}}, \Phi_{2_{2,k}}, \dots, \Phi_{2_{N-1,k}}\right]^\top, \widehat{\Phi}_2 = \left[\widehat{\Phi}_2^1, \dots, \widehat{\Phi}_2^{N-1}\right], \widehat{\Phi}_1 = \left[\Phi_{1_1}, \dots, \Phi_{1_{N-1}}\right]^\top, \\ \mathbf{W} = \mathrm{diag}(\ell_1, \dots, \ell_{N-1}), \mathbf{L} = \widehat{\Phi}_2\mathbf{W}, \\ \bar{\mathbf{C}} = (C_1, \dots, C_{N-1})^\top, \bar{\mathbf{c}}_0 = (c_{0_1}, \dots, c_{0_{N-1}})^\top, \quad \mathbf{y}^h = (y_1, \dots, y_{N-1})^\top, \end{cases}$$

then a matrix formulation of the proposed method (2.13) can be demonstrated as follows:

$$\Omega^{N-1}\ \mathbf{y}^h + h^3\ \mathbf{L}\ u(\mathbf{y}^h) = -h^2\widehat{\Phi}_1 + \mathbf{b_0} + h^5\bar{\mathbf{C}} + h^{\gamma+2}\bar{\mathbf{c}}_0, \tag{2.14}$$

in which

$$\mathbf{b_0} = -a_1 y_0 \mathbf{I}_1 - c_{N-1}y_N\mathbf{I}_{N-1} - h^3\ell_0 u_0\widehat{\Phi}_2^0 - h^3\ell_N u_N\widehat{\Phi}_2^N,$$

and $\Omega^{N-1} = \mathrm{tridiag}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an $(N-1)\times(N-1)$ tridiagonal matrix, where

$$\mathbf{a} = (a_2, \dots, a_{N-1})^\top, \mathbf{b} = (b_1, \dots, b_{N-1})^\top, \mathbf{c} = (c_1, \dots, c_{N-2})^\top,$$

and $\mathbf{I}_i, i = 1, N-1$ is a column vector composed of $(N-1)$ elements, all of which are zero, except for one element at position i that is equal to 1. So, the corresponding proposed numerical method can be expressed as form:

$$\Omega^{N-1}\ \mathbf{Y}^h + h^3\ \mathbf{L}\ u(\mathbf{Y}^h) = \mathbf{b}, \tag{2.15}$$

where $\mathbf{Y}^h = (Y_1, \dots, Y_{N-1})^\top$ and $\mathbf{b} = \mathbf{b_0} - h^2\widehat{\Phi}_1$. Note that the derived nonlinear equations can be solved utilizing the Newton iteration solver or some other iteration techniques.

3. Error analysis and convergence rate of the proposed method

This section examines the error analysis of the numerical method for FIDE (2.1), as well as its convergence properties, as outlined in the previous section. To this aim, we focus on the matrix formulations (2.14) and (2.15) that are real and approximate solutions of FIDE (2.1). According to (2.14) and (2.15), one can extract the equation of error as follows:

$$\Omega^{N-1}(\mathbf{y}^h - \mathbf{Y}^h) + h^3\ \mathbf{L}\big(u(\mathbf{y}^h) - u(\mathbf{Y}^h)\big) = \mathbf{E_0}, \tag{3.1}$$

where $\mathbf{E_0} = h^5\bar{\mathbf{C}} + h^{\gamma+2}\bar{\mathbf{c}}_0$. Therefore, if the jacobian of kernel $u(y)$ is denoted by $\mathbf{J}_U$ then a linearization form of the error system can be obtained as

$$(\Omega^{N-1} + h^3\ \mathbf{L}\mathbf{J}_U)\mathbf{E}(h) = \mathbf{E_0}, \tag{3.2}$$

where $\mathbf{E}(h) = \mathbf{y}^h - \mathbf{Y}^h$. It should be mentioned that, for the jacobian $\mathbf{J}_U$ we have a diagonal matrix $\mathbf{J}_U = \mathrm{diag}(\big[\frac{\partial}{\partial y}u(y)\big|_{y=y_k}\big]_{k=1}^{N-1})$. Using the fixed-point theorem, the iterative solutions of the linear system (3.2) tend to the solution of nonlinear system (3.1). It should be mentioned that we can consider $\Omega^{N-1} = \mathbf{T} - h^2\mathbf{Q}$, in which

$$\begin{cases} \mathbf{T} = \mathrm{tridiag}(-\mathbf{T}_{N-2}^1, \mathbf{T}_{N-2}^0 + \mathbf{T}_{N-1}^1, -\mathbf{T}_{N-2}^1), & (3.3a) \\ \mathbf{Q} = \mathrm{tridiag}(\mathbf{Q}_{N-1}^{2^-}, \mathbf{Q}_{N-1}^1, \mathbf{Q}_{N-2}^{1^+}), & (3.3b) \end{cases}$$

are tridiagonal matrices with

$$\mathbf{T}_n^k = [\tau_k, \tau_{k+1}, \dots, \tau_n]^\top, \mathbf{Q}_n^{k^\pm} = \left[\alpha_k^\pm, \alpha_{k+1}^\pm, \dots, \alpha_n^\pm\right]^\top, \mathbf{Q}_n^k = [\alpha_k, \alpha_{k+1}, \dots, \alpha_n]^\top,$$

while for $k = 1, \ldots, N-1$,

$$\alpha_k^{\pm} = \left(\tau_{k-1}\mathbf{M}_k^{-}\mathcal{C}_0^{\pm} + \tau_k\mathbf{M}_k^{+}\mathcal{C}_0^{\mp}\right)\phi_{k\pm1}^{q}, \alpha_k = \left(\tau_{k-1}\mathbf{M}_k^{-} + \tau_k\mathbf{M}_k^{+}\right)\mathcal{C}_1\phi_k^{q}.$$

Now, we will survey some of the characteristics of matrix $\mathbf{T}$ as defined in (3.3a). At first, we know that the matrix $\mathbf{T}$ satisfies the weak diagonally dominant symmetric property. For this reason, it can be concluded that it is a semiidefinite matrix due to the positive values of its diagonal elements. We can observe that, the elements of $\mathbf{T}_n^k$ are not vanish. For this matrix, considering an LU-decomposition in [2] (p:7, Eqs: (16)-(17)) and utilizing Lemma 3 of [2], yields that $|\mathbf{T}| \neq 0$. This results in positive singular values σ_k and positive eigenvalues λ_k for matrix $\mathbf{T}$. In the next section, we just need to prove that $\lambda_{min}(\mathbf{T}) > h$ where $\lambda_{min}(\mathbf{T})$ is the smallest eigenvalue of $\mathbf{T}$. In order to do so, first, we consider $\mathbf{M} = \mathbf{T} - h\mathbf{I}$, where $\mathbf{I}$ is an identity matrix with dimension $(N-1)\times(N-1)$. Taking account λ as an eigenvalue of $\mathbf{T}$ implies that $\lambda - h$ be the eigenvalue of $\mathbf{M}$. Therefore, all we have to do is indicate that $\lambda_{min}(\mathbf{M}) > 0$. If we consider $\bar{a}_k, \bar{c}_k$ as lower- and upper-diagonal elements and $\bar{d}_k$ as diagonal elements of the matrix $\mathbf{M}$ then we have,

$$\begin{cases} \bar{d}_k = \tau_{k-1} + \tau_k - h, & k = 1, 2, \cdots, N-1, \\ \bar{a}_k = \bar{c}_k = -\tau_k, & k = 1, 2, \cdots, N-2. \end{cases}$$

If there exist a matrix decomposition, according to the corresponding LU-factorization [2] (p:7, Eqs: (16)-(17)), for the tridiagonal matrix $\mathbf{M}$, then we get

$$\underbrace{\begin{bmatrix} \bar{d}_1 & \bar{c}_1 & & & \\ \bar{a}_1 & \bar{d}_2 & \bar{c}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \bar{a}_{N-2} & \bar{d}_{N-1} & \bar{c}_{N-1} \\ & & & \bar{a}_{N-1} & \bar{d}_N \end{bmatrix}}_{\mathbf{M}} = \underbrace{\begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & \ddots & \ddots & \\ & & l_{N-1} & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} p_1 & \bar{c}_1 & & \\ & p_2 & \ddots & \\ & & \ddots & \bar{c}_{N-1} \\ & & & p_N \end{bmatrix}}_{U_1}, \tag{3.4}$$

and also we obtain,

$$p_1 = \bar{d}_1, \quad l_k = \frac{\bar{a}_k}{p_k}, \quad p_{k+1} = \bar{d}_{k+1} - l_k\bar{c}_k, \quad k = 1, 2, \cdots, N-1. \tag{3.5}$$

It should be pointed out that according to (2.2) with $s = t_k$ we can obtain an approximation for τ_k as

$$\tau_k \sim \frac{1}{h\sum_{m=0}^{\kappa}\frac{h^m}{(m+1)!}\bar{\phi}_k^{(m)}}. \tag{3.6}$$

Remark 3.1. Since, $h = \frac{1}{N}$, from the definition of τ_k and according to (3.6), we conclude that

$$\tau_k \sim \mathcal{O}\left(\frac{1}{h}\right),$$

as $h \to 0$. Therefore, according to the property $p(t) \geq 0$, it is easy to indicate that $\tau_k > 1 + h$.

Lemma 3.2. *Suppose for the matrix* $\mathbf{M}$*, there exist a matrix decomposition according to LU-factorization* (3.4)*, then we can conclude that* $|\mathbf{M}| > 0$.

Proof. Taking $p_1 = \bar{d}_1, \quad l_k = \frac{\bar{a}_k}{p_k}, \quad p_{k+1} = \bar{d}_{k+1} - l_k\bar{c}_k, \quad k = 1, 2, \cdots, N-1$, yields $p_{k+1} = \tau_{k+1} + \tau_k - h - \tau_k\bar{l}_k$, in which $\bar{l}_k = -l_k$. Therefore, we get

$$\bar{l}_{k+1} = \frac{\gamma_{k+1}}{\gamma_{k+1} + 1 - \bar{l}_k - h/\tau_k},$$

in which $\gamma_{k+1} = \frac{\tau_{k+1}}{\tau_k}, k = 1, \cdots, N-1$. Since $\tau_k \sim \mathcal{O}\left(\frac{1}{h}\right)$, we conclude that $\bar{l}_1 = \frac{\tau_1}{\tau_0+\tau_1-h} < 1$ and $\lim_{k\to\infty}\bar{l}_k = 1$. So, form the results expressed in Remark 3.1, we conclude that $h\left(\frac{1}{\tau_0-h} + \frac{1}{\tau_1}\right) < 1$ which in turn infers $\bar{l}_1 < 1 - \frac{h}{\tau_1} < 1$.

Thus, while $h \to 0$ assuming $\bar{l}_k < 1 - \frac{h}{\tau_k} < 1$ concludes $\bar{l}_{k+1} < 1 - \frac{h}{\tau_{k+1}} < 1$. Finally, we get $0 < \bar{l}_k < 1, k = 1, \cdots, N-1$. Hence, we obtain

$$p_{k+1} > \tau_{k+1} - h > 1.$$

Therefore, the proof is complete through the decomposition (3.4). □

It follows from the Theorem 7.2 in [32] that matrix $\mathbf{M}$ is positive definite and $\lambda_{min}(\mathbf{M}) > 0$. This implies that eigenvalues of $\mathbf{T}$ satisfy $\lambda > h$. Totaly, setting $A = \mathbf{T} - h^2\mathbf{Q} + h^3\ \mathbf{LJ}_U$ and $\mathbf{x} = \mathbf{y}^h - \mathbf{Y}^h$ and then apply Lemma 1 in [2] for (3.2), concludes that

$$\|\mathbf{E}(h)\|_2 \leq \frac{\|\mathbf{E_0}\|_2}{\sigma_{min}\left(\mathbf{T} - h^2\mathbf{Q} + h^3\ \mathbf{LJ}_U\right)}. \tag{3.7}$$

Because, matrix $\mathbf{T}$ is tridiagonal, non-singular and positive semidefinite, then for sufficiently small h, we can easily deduce that

$$\sigma_{min}\left(\mathbf{T} - h^2\mathbf{Q} + h^3\ \mathbf{LJ}_U\right) \sim \sigma_{min}\left(\mathbf{T}\right), \quad as \quad h \to 0.$$

Theorem 3.3. *Suppose that functions p, q, f, v, u has continuous fourth-order derivatives. If $\boldsymbol{E}(h)$ be the solution of the error system (3.2), then we have $\|\mathbf{E}(h)\|_2 = \mathcal{O}(h^{\min\{\frac{7}{2}, \gamma - \frac{1}{2}\}})$.*

Proof. Since, the mentioned functions are continuously differentiable up to order 4, then from the error Equations (3.1) and (3.2), one can observe that there are constants $\mathbf{d}, \bar{\mathbf{d}} \in \mathbb{R}$ where

$$\|\mathbf{E_0}\|_2 \leq \bar{\mathbf{d}} h^{\frac{9}{2}} + \mathbf{d} h^{\gamma + \frac{1}{2}},$$

and so, from (3.7) we get

$$\|\mathbf{E}(h)\|_2 \leq \frac{\|\mathbf{E_0}\|_2}{\lambda_{min}\left(\mathbf{T}\right)} \leq \frac{\bar{\mathbf{d}} h^{\frac{9}{2}} + \mathbf{d} h^{\gamma + \frac{1}{2}}}{h} = \bar{\mathbf{d}} h^{\frac{7}{2}} + \mathbf{d} h^{\gamma - \frac{1}{2}}.$$

□

Theorem 3.3 states that the maximum order of convergence is attained for $\gamma \geq 4$. To make sure the method converges with the highest order, all that is required is to apply a quadrature rule with $\gamma \geq 4$ for integral parts of (2.1).

4. Numerical results

This section includes some numerical examples to illustrate the efficiency of the presented technique in section 2. For the numerical simulations, we consider step sizes $h = 2^{-k}, k = 2, 3, \ldots$, to calculate error $\|\mathbf{E}(h)\|_2$ or $\|\mathbf{E}(h)\|_\infty$ (maximum norm of error). Also, in the numerical results, the order $\log_2\left(\frac{\|\mathbf{E}(h)\|}{\|\mathbf{E}(\frac{h}{2})\|}\right)$ is computed where it is required.

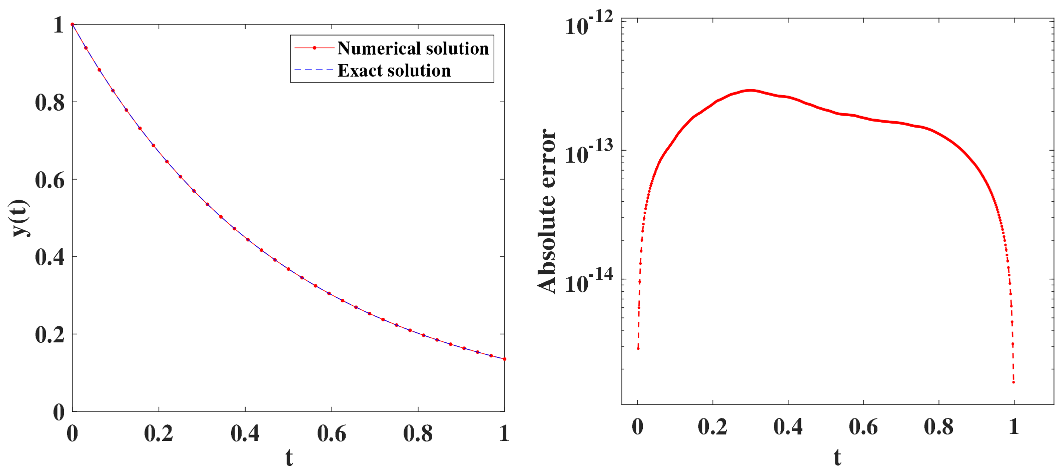
Example 4.1. In the first example, let's take the FIDE:

$$y''(t) + t^2 y'(t) + \sin(\pi t) y(t) = f(t) + \int_0^1 s(t + 2s) y(s) ds, \quad y(0) = 1, \quad y(1) = e^{-2}, \quad t \in [0, 1],$$

in which $f(t) = \frac{1}{4}\left(-8e^{-2t}\left(t^2 - 2\right) - t + \frac{3t+10}{e^2} - 2\right) + e^{-2t}\sin(\pi t)$, and the exact solution is $y(t) = e^{-2t}$. Here, $p(t) = t^2$ and consequently we get $\bar{\phi}(t) = e^{-\frac{t^3}{3}}$. We solved this problem numerically using the presented mentioned method in (2.15). In Table 1, the obtained L^2- and maximum-errors and the order of convergence of the proposed technique are exhibited. The computational results of this table show that the proposed method can obtain highly accurate solutions of orders $\frac{7}{2}$ and 4, respectively with L^2- and maximum-norms. This table indicates that the expected convergence order for the proposed technique has been achieved. So, it is in well agreement with the theoretical order claimed in Theorem 3.3. Also, a comparison between the real and numerical solution of the problem with $h = 2^{-9}$ is displayed in Figure 1(a). The absolute error of the obtained solution for this example is plotted in Figure 1(b).

TABLE 1. Errors and convergence order of the results obtained by the method proposed in (2.15) for Example 4.1.

N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
4	1.9400e-04	–	1.1428e-04	–	8.1978e-05	–
8	2.5192e-05	2.9450	1.0125e-05	3.4965	5.1354e-06	3.9967
16	3.1811e-06	2.9853	8.9647e-07	3.4976	3.2176e-07	3.9964
32	3.9867e-07	2.9963	7.9276e-08	3.4993	2.0217e-08	3.9923
64	4.9866e-08	2.9991	7.0080e-09	3.4998	1.2637e-09	3.9998
128	6.2334e-09	2.9999	6.1936e-10	3.5001	7.8975e-11	4.0001
256	7.8274e-10	2.9934	5.4735e-11	3.5002	4.9301e-12	4.0017



(a) Exact and approximate solutions with $h = 2^{-9}$. (b) Absolute error.

FIGURE 1. Numerical results obtained of the proposed method for Example 4.1 with $h = 2^{-9}$.

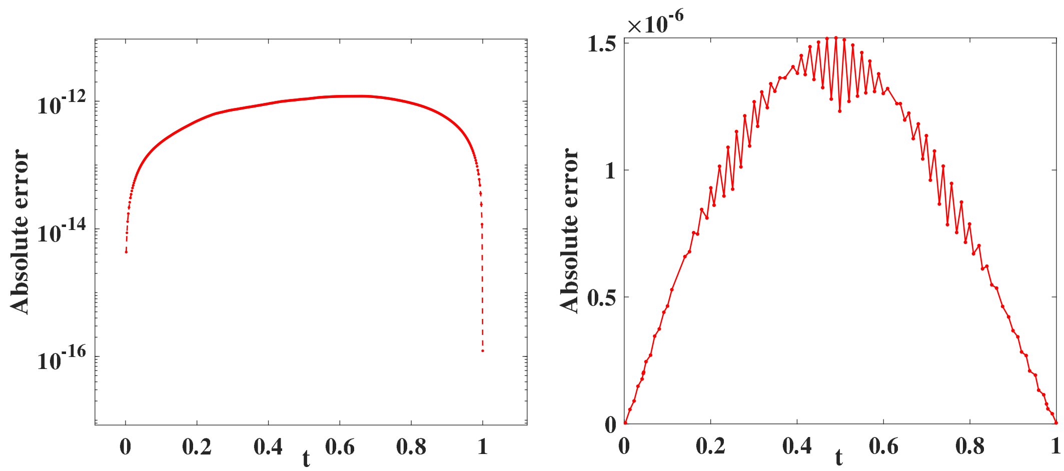
Example 4.2. ([6, 8]) In the second example, consider the test problem

$$y''(t) + ty'(t) + \pi^2 y(t) = \pi t \cos(\pi t) - \frac{2t+1}{\pi} + \int_0^1 (s+t)y(s)ds, \quad t \in [0,1],$$
$$y(0) = y(1) = 0,$$

where its real solution is $y(t) = \sin(\pi t)$. The above FIDE has been solved by the method established in this paper for some selected values of step size $h = 2^{-2}, \cdots, 2^{-8}$. The computed L^∞- and L^2-norm of errors obtained by our method with those obtained by the multiscale Galerkin methods [7, 8] are reported in Table 2. From this, we can conclude that the results obtained confirm the computational convergence order for the proposed method (2.15), i.e., $\frac{7}{2}$ (in L^2-norm) is in a good agreement with theoretical one provided in Theorem 3.3. Moreover, from the above table, it can be observed that the multiscale Galerkin approach [8] converges with order 2, while for the our presented method is $\frac{7}{2}$ which both of them evaluated by L^2-norm. In addition, the earned convergence order of the our proposed method w.r.t L^∞-norm is 4, whereas this amount for multiscale Galerkin method [7] is 3. On the other hand, for the current method in the case of $h = 2^{-9}$, and for the multiscale Galerkin method [8] in the case of $h = 2^{-11}$, we have computed the absolute errors and then displayed them in Figure 2. Numerical results in Figure 2 indicate that for the presented method maximum absolute errors is of order $O(10^{-12})$, while for the multiscale Galerkin method expressed in [8] is $O(10^{-6})$.

Table 2. Computational results obtained by our proposed method and those obtained by the methods provided in [7, 8] for Example 4.2.

	Our proposed method				Method of [7]		Method of [8]	
N	L_∞ error	Order	L_2 error	Order	L_∞ error	Order	L_2 error	Order
4	6.5415e-04	–	9.3101e-04	–	–	–	–	–
8	4.7219e-05	3.7922	9.4865e-05	3.2949	–	–	–	–
16	3.0455e-06	3.9546	8.6493e-06	3.4552	5.307e-4	–	1.6002e-2	–
32	1.9216e-07	3.9863	7.7028e-07	3.4891	6.214e-5	3.0942	4.0613e-3	1.9782
64	1.2036e-08	3.9969	6.8211e-08	3.4973	7.632e-6	3.0254	1.0192e-3	1.9945
128	7.5299e-10	3.9985	6.0343e-09	3.4988	9.504e-7	3.0056	2.5504e-4	1.9986
256	4.7358e-11	3.9909	5.3362e-10	3.4993	1.186e-7	3.0022	6.3776e-5	1.9997



(a) Our proposed method with $h = 2^{-9}$. (b) Reproducing kernel method [8] with $h = 2^{-10}$.

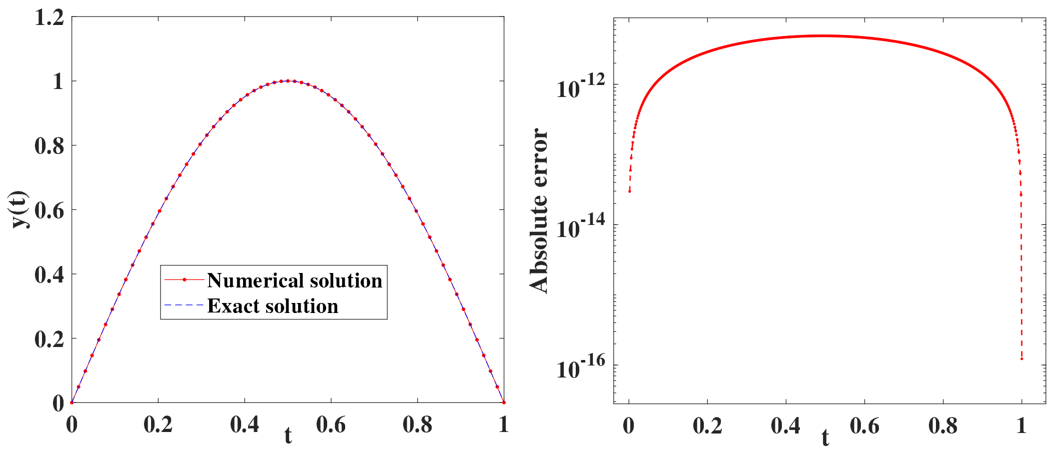
Figure 2. Absolute errors of the methods for Example 4.2.

Table 3. Computational results of the fast multiscale Galerkin method [6], exponential spline method [13] and present method for Example 4.3.

	Present method						Method of [6]		Method of [13]	
N	L_∞ error	Order	L_1 error	Order	L_2 error	Order	L_1 error	Order	L_∞ error	Order
4	1.3383e-03	–	3.2064e-03	–	1.8806e-03	–	–	–	–	–
8	8.2511e-05	4.0197	4.1098e-04	2.9638	1.6394e-04	3.5199	2.5525e-1	–	–	–
16	5.1390e-06	4.0050	5.1675e-05	2.9915	1.4441e-05	3.5050	1.2791e-1	0.9968	2.4213e-4	–
32	3.2091e-07	4.0013	6.4687e-06	2.9979	1.2753e-06	3.5012	6.3991e-2	0.9992	5.9642e-5	2.0214
64	2.0052e-08	4.0003	8.0888e-07	2.9995	1.1270e-07	3.5003	3.2001e-2	0.9998	1.4851e-5	2.0058
128	1.2535e-09	3.9998	1.0112e-07	2.9999	9.9604e-09	3.5001	1.6001e-2	0.9999	3.7093e-6	2.0013
256	7.8342e-11	4.0000	1.2640e-08	2.9999	8.8039e-10	3.5000	8.0006e-3	1.0000	–	–

Example 4.3. As third example we take the following FIDE of the second kind [6, 13]

$$y''(t) - \int_0^1 v(t,s)y(s)ds = f(t), \quad t \in [0,1],$$
$$y(0) = y(1) = 0,$$



(a) Exact and approximate solutions for $h = 2^{-9}$.

(b) Absolute error.

FIGURE 3. Numerical results of the proposed method to solve the Example 4.3 with $h = 2^{-9}$.

where $v(t,s) = s + t$, $f(t) = -\pi^2 \sin(\pi t) - \frac{2t+1}{\pi}$. The exact solution of this problem is $y(t) = \sin(\pi t)$.

In this example, we consider the fast multiscale Galerkin method [6] and exponential spline method [13] to compare the results. Table 3 displays the numerical results reported for the methods with different step sizes. The numerical results of the Table 3 demonstrate that the proposed method (2.15) converges with order 3, 3.5 and 4 with respect to L_1, L_2 and L_∞ norms, respectively. While the fast multiscale Galerkin method [6] and exponential spline method [13] have the accuracy-order one and two with respect to L_1 and L_∞ norms, respectively. Consequently the present method is more accurate than those techniques given in the literature. Also, we plot the computed numerical results and real solution in the Figure 3(a) for the case $h = 2^{-9}$. From the displayed absolute errors in Figure 3(b), the effectiveness of the present techniques observe, obviously.

Example 4.4. Consider the following nonlinear FIDE [1]

$$y''(t) = \exp(t) + \frac{1}{4}(\exp(2) - 2)t + \frac{1}{2}\int_0^1 t(s - y^2(s))ds, \quad t \in [0,1],$$
$$y(0) = 1, y(1) = \exp(1),$$

while its real solution is $y(t) = \exp(t)$.

In this example, a comparison between the haar wavelet method [1] and proposed method (2.15) is performed. The computed L^2-norm of errors obtained by our method and method [1] are reported in Table 4. The reported numerical results in Table 4 illustrate that the proposed method (2.15) has more accurate than haar wavelet method [1]. Furthermore, one can observe that the maximum convergence order of the haar wavelet method [1] will be reached 2, while for the presented method maximum order 3.5 will be achieved. Moreover, the computed numerical results and real solution for this problem are plotted in the Figure 3(a) for the case $h = 2^{-8}$. Again, according to the illustrated absolute errors in Figure 3(b), we can observe the accuracy and efficiency of the presented method, clearly.

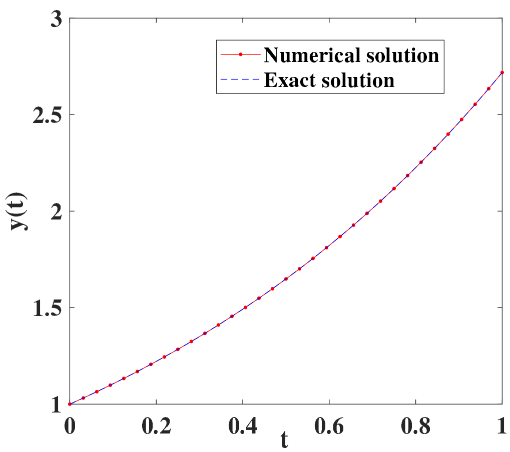
Example 4.5. Consider the following nonlinear FIDE

$$y''(t) + 4y'(t) + \ln(t^2+1)y(t) = f(t) + \int_0^1 s^8(t + \sinh(t))(y^2(s) + \exp(y^2(s)))ds, \quad t \in [0,1],$$
$$y(0) = 0, y(1) = 1,$$

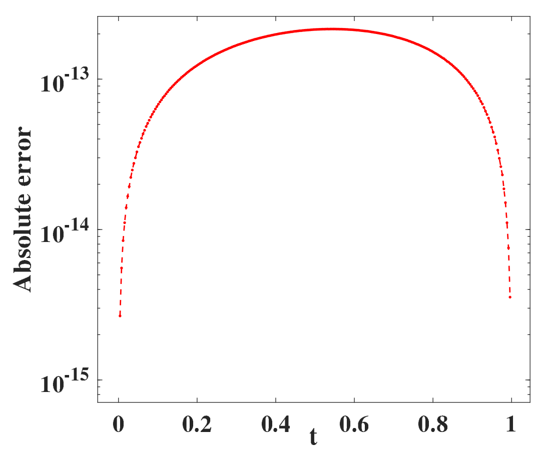
in which $f(t) = \sqrt{t}\left(\frac{63}{4}t^2 + 18t^3 + t^4\ln(t^2+1)\right) - \frac{1}{18}(2\exp(1) - 1)(t + \sinh(t))$ and its exact solution is $y(t) = t^4\sqrt{t}$.

TABLE 4. Computational results of the methods for Example 4.4.

	Present method		Haar wavelet method [1] with collocation points		Haar wavelet method [1] with Guass points	
N	L_2 error	Order	L_2 error	Order	L_2 error	Order
4	4.8916e-05	–	1.9468e-3	–	7.7106e-4	–
8	3.7499e-06	3.7053	5.1227e-4	1.9261	1.9580e-4	1.9774
16	3.2990e-07	3.5067	1.3141e-4	1.9627	4.9353e-5	1.9881
32	2.9110e-08	3.5024	3.3281e-5	1.9813	1.2390e-5	1.9939
64	2.5412e-09	3.5179	8.3724e-6	1.9910	3.1039e-6	1.9970
128	2.2154e-10	3.5198	2.0983e-6	1.9964	7.7662e-7	1.9988



(a) Exact and approximate solutions for $h = 2^{-8}$.



(b) Absolute error.

FIGURE 4. Numerical results of the proposed method to solve the Example 4.4 with $h = 2^{-8}$.

TABLE 5. Errors and convergence order of the results obtained by the method proposed in (2.15) for Example 4.5.

N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
4	1.1413e-02	–	7.0522e-03	–	5.0705e-03	–
8	7.2369e-03	0.6572	2.8742e-03	1.2949	1.4142e-03	1.8421
16	1.7853e-03	2.0192	4.9238e-04	2.5453	1.6918e-04	3.0634
32	2.8578e-04	2.6432	5.5488e-05	3.1495	1.3468e-05	3.6510
64	3.8475e-05	2.8929	5.2767e-06	3.3945	9.0545e-07	3.8948
128	4.9051e-06	2.9716	4.7555e-07	3.4720	5.7692e-08	3.9722
256	6.1620e-07	2.9928	4.2240e-08	3.4929	3.6235e-09	3.9929

This nonlinear problem is solved numerically, utilizing the proposed mentioned method in (2.15). The computed L^1-,L^2- and maximum-errors and the order of convergence of the presented technique are demonstrated in Table 5. Although, the function $f(t)$ as well as the exact solution $y(t)$ are non-smooth, the claimed orders $3, 3.5$ and 4 are achieved with respect to the L^1-,L^2- and maximum-norms, respectively.

TABLE 6. Errors and convergence order of the results obtained by the method proposed in (2.15) for Example 4.6.

N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
4	6.7495e-03	–	3.9339e-03	–	2.6863e-03	–
8	1.7574e-03	1.9413	6.8927e-04	2.5128	3.3115e-04	3.0200
16	2.8796e-04	2.6095	7.8521e-05	3.1339	2.6476e-05	3.6447
32	4.0748e-05	2.8210	7.7744e-06	3.3363	1.8330e-06	3.8524
64	5.6754e-06	2.8439	8.5875e-07	3.1784	1.2418e-07	3.8837
128	7.1425e-07	2.9902	7.6432e-08	3.4900	8.6377e-09	3.8457
256	8.9989e-08	2.9886	6.7942e-09	3.4918	5.4524e-10	3.9857

Example 4.6. Consider the following nonlinear FIDE

$$\begin{aligned} &y''(t) + y'(t) + t^3\cos(t^3)y(t) = f(t) + \int_0^1 s\sqrt{t}\left(y^4(s) + \frac{1}{6}y^6(s)\right)ds, \quad t \in [0,1], \\ &y(0) = 0, y(1) = 1, \end{aligned}$$

in which $f(t) = -\frac{221}{3036}\sqrt{t} + \frac{70}{9}t^{4/3} + \frac{10}{3}t^{7/3} + t^{19/3}\cos(t^3)$ and its exact solution is $y(t) = t^3\sqrt[3]{t}$.

The above nonlinear FIDE has been solved numerically, by the presented method for different values of step size $h = 2^{-2}, \cdots, 2^{-8}$. As we can see, the functions $f(t), v(t,s)$, and the exact solution $y(t)$ are not smooth, because $f(t) \notin C^1[0,1], v(t,s) \notin C^1([0,1]\times[0,1])$ and $y(t) \notin C^4[0,1]$. The reported computational results in Table 6 show that the proposed method has also good numerical approximation for this problem. Again, we can observe that the demanded order of convergence can be obtained by the proposed method (2.15).

5. Conclusions

This paper provided a new numerical solution for a class of nonlinear second-order FIDEs. At first, matrix formulation of the proposed method is performed and some results of linear algebra rules is employed, then we proved that the established method possesses the order of convergence $O(h^{\min\{\frac{7}{2},\gamma-\frac{1}{2}\}})$, w.r.t L^2-norm, in which γ is the order of convergence of the selected numerical integration. In order to demonstrate the accuracy and effectiveness of the proposed method, several numerical examples are provided. Also, the theoretical order of convergence have been verified on the relevant numerical problems.

Finally, it should be noted that, when considering the implementation of the suggested approach in addressing high-dimensional FIDEs as well as systems of FIDEs, some challenges arise. Indeed, when faced with these cases, expanding the dimensionality of the vectors and matrices inevitably leads to a higher computational cost for the proposed technique. In order to reduce the computational cost associated with the method, it is necessary to decrease the size of the vectors and matrices, which leads to obtain an approximation with lower accuracy. Because, for this case, the step size should not be selected too small. Therefore, the proposed study investigators intend to extend it to approaches that will lead to low computational cost without losing accuracy.

References

[1] R. Amin, I. Mahariq, K. Shah, M. Awais, and F. Elsayed, *Numerical solution of the second order linear and nonlinear integro-differential equations using Haar wavelet method*, Arab Journal of Basic and Applied Sciences, *28*(1) (2021), 11–19.

[2] S. Amiri, *Effective numerical methods for nonlinear singular two-point boundary value Fredholm integro-differential equations*, Iranian Journal of Numerical Analysis and Optimization, *13*(3) (2023), 444–459.

[3] S. H. Behiry and H. Hashish, *Wavelet methods for the numerical solution of Fredholm integro-differential equations*, Int. J. Appl. Math., *11*(1) (2002) 27–35.

[4] S. H. Behiry and S. I. Mohamed, *Solving high-order nonlinear Volterra–Fredholm integro-differential equations by differential trasform method*, Nat. Sci. *4* (8) (2012) 581–587.

[5] A. H. Bhrawy, E. Tohidi, and F. Soleymani, *A new Bernoulii matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals*, Appl. Math. Comput., *219* (2012), 482–497.

[6] J. Chen, M. F. He, and Y. Huang, *A fast multiscale Galerkin method for solving second order linear fredholm integro-differential equation with Dirichlet boundary conditions*, J. Comput. Appl. Math., *364* (2020), 112352.

[7] J. Chen, M. He, and T. Zeng, *A multiscale Galerkin method for second-order boundary value problems of Fredholm integro-differential equation II: Effcient algorithm for the discrete linear system*, J. Vis. Commun. Image Represent., *58* (2019) 112–118.

[8] J. Chen, Y. Huang, H. Rong, T. Wu, and T. Zeng, *A multiscale Galerkin method for second-order boundary value problems of Fredholm integro-differential equation*, J. Comput. Appl. Math., *290* (2015), 633–640.

[9] M. Dehghan and A. Saadatmandi, *Chebyshev finite difference method for Fredholm integro-differential equation*, Int. J. Comput. Math. *85* (2008), 123–130.

[10] D. S. Dzhumabaev, *On one approach to solve the linear boundary value problems for Fredholm integro-differential equations*, J. Comput. Appl. Math., *294* (2016,) 342–357.

[11] O. A. Gegele, O. P. Evans, and D. Akoh, *Numerical solution of higher order linear Fredholm integro-differential equations*, Applied Journal of Engineering, *8* (2014), 243–247.

[12] S. Islam, I. Aziz, and M. Fayyaz, *A new approach for numerical solution of integro-differential equations via Harr wavelets*, Int. J. Comput. Math., *90*(9) (2013), 1971–1989.

[13] R. Jalilian and T. Tahernezhad, *Exponential spline method for approximation solution of Fredholm integro-differential equation*, Int. J. Comput. Math., *97*(4) (2020), 791–801.

[14] P. Kanwal, *Linear integral equations theory and technique*, London, Academic Press. 1971.

[15] G. Long, G. Nelakanti, and X. Zhang, *Iterated fast multiscale Galerkin methods for Fredholm integral equations of second kind with weakly singular kernels*, Appl. Numer. Math., *62* (2012), 201–211.

[16] F. Mirzaee, *Numerical solution of nonlinear Fredholm-Volterra integral equations via Bell polynomials*, Comput. Methods Differ. Equ. *5*(2) (2017), 88–102.

[17] A. Molabahrami, *Direct computation method for solving a general nonlinear Fredholm integro-differential equation under the mixed conditions: Degenerate and non-degenerate kernels*, J. Comput. Appl. Math., *282* (2015), 34–43.

[18] Y. Ordokhani, *An application of Walsh functions for Fredholm–Hammerstein integro-differential equations*, Int. J. Contemp. Math. Sci., *5*(22) (2010) 1055–1063.

[19] P. K. Pandey, *Non-standard difference method for numerical solution of linear Fredholm integro-differential type two-point boundary value problems*, Open Access Lib. J., *2* (2015), 1–10.

[20] M. A. L. Ramadan, K. M. Raslan, and M. A. E. G. Nassear, *A rational Chebyshev functions approach for Fredholm-Volterra integro-differential equations*, Comput. Methods Differ. Equ., *3*(4) (2015), 284-297.

[21] M. Razzaghi and S. Yousefi, *Legendre wavelets method for the nonlinear Volterra Fredholm integral equations*, Math. Comput. Simul., *70* (2005), 1–8.

[22] A. Saadatmandi and M. Dehghan, *Numerical solution of high-order linear Fredholm integro-differential–difference equation with variable coefficients*, Comput. Math. Appl., *59* (2010), 2996–3004.

[23] T. L. Saaty, *Modern nonlinear equations*, New York, Dover publications. 1981.

[24] A. Shidfar, A. Molabahrami, A. Babaei, and A. Yazdanian, *A series solution of the nonlinear Volterra and Fredholm integro-differential equations*, Commun. Nonlinear Sci. Numer. Simul., *15* (2010), 205–215.

[25] H. R. Thiem, *A model for spatio spead of an epidemic*, J. Math. Biol., *4* (1977), 337–351.

[26] Q. Xue , J. Niu, D. Yu, and C. Ran, *An improved reproducing kernel method for fredholm integro-differential type two-point boundary value problems*, Int. J. Comput. Math. *95* (2017), 1015–1023.

[27] S. Yalcinbas, *Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations*, Appl. Math. Comput. *127* (2002) 195–206.

[28] S. Yalcinbas and M. Sezer, *The approximate solution of high-order linear Volterra-Fredholm integro-differential equations in terms of Taylor polynomials*, Appl. Math. Comput., *112* (2000), 291–308.

[29] S. Yang, X. Luo, F. Li, and G. Long, *A fast multiscale Galerkin method for the first kind ill-posed integral equations via iterated regularization*, Appl. Math. Comput., *219* (2013), 10527–10537.

[30] S. Yeganeh, Y. Ordokhani, and A. Saadatmandi, *A Sinc-collocation method for second-order boundary value problems of nonlinear integro-differential equation*, J. Inf. Comput. Sci., *7*(2) (2012), 151–160.

[31] W. Yulan, T. Chaolu, and P. Jing, *New algorithm for second-order boundary value problems of integro-differential equation*, J. Comput. Appl. Math., *229* (2009), 1–6.

[32] F. Zhang, *Matrix Theory*, Springer New York, 2011.