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An efficient numerical scheme for solving a competitive lotka-volterra system with two discrete delays

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Abstract

In this study, the Euler series solution method is developed to solve the Lotka–Volterra predator-prey model with two discrete delays. The improved method depends on a matrix-collocation method and Euler polynomials. While creating the method, all terms in the system are converted into matrix forms. Hence the fundamental matrix equation of the system is obtained. A nonlinear algebraic equation system is achieved by inserting the collocation points into the fundamental system. Then, the unknown coefficients that arise from Euler series expansion are calculated by solving the final system. Two different error estimation procedures are used to estimate the error of the approximation; the first one is the residual correction procedure and the second one is a technique similar to RK45. In numerical examples, the variations in the population of both species are presented by figures regarding time. Also, the method's validity is checked by using residual error analysis.

Keywords. Error estimation, Euler series solution method, Delayed prey-predator system.

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1. Introduction

In recent years, population models have been proposed and extensively studied in various fields of mathematical biology. Interactions amongst species are one of the fundamental issues of biology and ecology. Prey-prey interaction is a basic structure in population dynamics. It was helpful to investigate multi-species interactions to understand the dynamics of population models. Predator-prey interaction, which was first modelled by Lotka [13] and Volterra [15], is one of these relations. To reflect the dynamic behaviour of models clearly, it is often necessary to include time delays in models that depend on their history. These kinds of predator-prey models should be given by delayed differential equation systems [7],[18],[10],[5],[3].

In this work, we examine the Lotka-Volterra predator-prey model with two discrete delays described as:

$$\begin{cases} y_1'(t) = y_1(t)\left[r_1 - a_{11}y_1(t) - a_{12}y_2(t-\tau_1)\right], \\ y_2'(t) = y_2(t)\left[-r_2 + a_{21}y_1(t-\tau_2) - a_{22}y_2(t)\right], \end{cases} \tag{1.1}$$

where $y_1(t)$ and $y_2(t)$ denote the population density of prey and predator at time t, respectively. Here, a_{11} and a_{22} are non-negative constants, $a_{12}, a_{21} > 0$, r_1 and r_2 are the intrinsic growth rate and the death rate for prey and predator species, respectively. τ_1 and τ_2 are called the hunting delay and the maturation time of the prey species, respectively. Also, system (1.1) means that the population of prey species are modeled by

$$y_1'(t) = y_1(t)\left(r_1 - a_{11}y_1(t)\right),$$

which is the logistic equation in the absence of predator species. In the absence of prey species, the population of predatory species will decrease.

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Yan and Chu [16] considered system (1.1) and investigated the effects of delay on the solutions. They obtained the necessary conditions on the parameters so that the oscillations would not occur in the system. Faria [7] analyzed the dynamics of (1.1). She studied the stability of positive equilibrium and the existence of local Hopf bifurcation. In case of $a_{22} = 0$ and $\tau_1 = 0$ which is a special case of (1.1), Beretta and Kuang [2] provided a procedure for obtaining some regions of attraction for the positive steady state. Freedman and Rao [8] discussed a generalization of system (1.1). They determined the stability of equilibria and found the conditions that yield no changes in the stability, even if unbounded delays occurred. Yuzbasi [17] considered the following delayed Lotka–Volterra predator-prey system.

$$\begin{cases} y_1'(t) = y_1(t)\left[r_1 - a_{11}y_1(t-\tau_{11}) - a_{12}y_2(t-\tau_{12})\right], \\ y_2'(t) = y_2(t)\left[-r_2 + a_{21}y_1(t-\tau_{21}) - a_{22}y_2(t-\tau_{22})\right], \end{cases} \tag{1.2}$$

which is a generalization of (1.1). He used an operational matrix method based on the operational matrices of the standard basis functions and obtained the coefficients by using the least-squares method. Although the method was given for generalized problems, he gave two examples as the special cases of (1.2). He also applied a residual correction procedure for the method to estimate the error. In another study, the criteria for uniform persistence were examined by Freedman and Ruan [9] for $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = \tau$. They found that the conditions obtained for uniform persistence guaranteed the existence of an interior equilibrium. Some other matrix-collocation methods have been applied to many different kinds of equations and their systems until this day [11],[4],[12],[14],[1].

The aim of the study is to introduce the Euler series solution (ESS) method which is based on the Euler polynomials and the collocation method for system (1.1) with the initial conditions

$$y_1(0) = \xi_1 \text{ and } y_2(0) = \xi_2, \ \xi_1, \xi_2 \in \mathbb{R}^+. \tag{1.3}$$

The paper is organized as follows. In section 2, some necessary information about the Euler polynomials and the stability condition of the equilibrium point of system (1.1) are given. The ESS method is explained in section 3. Section 4 includes error estimation techniques. Owing to these procedures, the absolute error might be predicted even if the exact solution of the system is unknown. In section 5, two numerical examples are given to explain how the procedures are employed in practice. Additionally, the findings are presented with graphs and tables. In the last section, the results obtained are summarized.

2. Euler Polynomials and Some Stability Results for System (1.1)

In this section, the definition of the Euler polynomials is presented. Then, the condition regarding the asymptotic stability of the equilibrium points of the system (1.1) found by Yan and Chu [16] is given.

2.1. **Euler Polynomials.**

The Euler polynomials, which combine Bernoulli numbers and binomial coefficients are defined by the following generating function.

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \ |t| < \pi.$$

The recurrence relation of the Euler polynomials is

$$E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x) = 2x^n, \ n = 1, 2, \ldots \tag{2.1}$$

From relation (2.1), the first few Euler polynomials are specified by

$$E_0(x) = 1, E_1(x) = x - \frac{1}{2}, E_2(x) = x^2 - x, E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \ldots$$

2.2. **Stability of Positive Equilibrium.** Yan and Chu [16], studied the stability analysis of the positive equilibrium points for system (1.1). They determined under which circumstances system (1.1) has asymptotically stable equilibria. The system has three boundry equilibria as shown by $O(0,0), A = \left(\frac{r_1}{a_{11}}, 0\right), B = \left(0, \frac{r_2}{a_{22}}\right)$ and a unique positive equilibrium point $E^* = (y_1^*, y_2^*)$ as shown by the following:

$$y_1^* = \frac{r_1 a_{22} + r_2 a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \qquad y_2^* = \frac{r_1 a_{21} - r_2 a_{11}}{a_{11}a_{22} + a_{12}a_{21}}.$$

According to Theorem 2.5 in [16], if $q \geq r$, then $E^* = (y_1^*, y_2^*)$ is asymptotically stable for any $\tau \geq 0$ where

$$q = a_{11}a_{22}y_1^*y_2^* \text{ and } r = a_{12}a_{21}y_1^*y_2^*.$$

3. Method of Solutions

This part comprises how the method based on Euler polynomials and collocation points is created. This method's main principle is to convert all terms of the system (1.1) into matrix form. So, the first step would be to start with the procedure by converting the Euler series solutions into the matrix forms.

The Euler bases functions can be expressed in terms of the standart basis of polynomial space by using (2.1) as:

$$\mathbf{T}(t) = \mathbf{D}\mathbf{E}(t), \tag{3.1}$$

where

$$\mathbf{T}(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{n}{0} & \frac{1}{2}\binom{n}{1} & \frac{1}{2}\binom{n}{2} & & 1 \end{bmatrix}, \ \mathbf{E}(t) = \begin{bmatrix} E_0(t) \\ E_1(t) \\ E_2(t) \\ \vdots \\ E_n(t) \end{bmatrix}.$$

From relation (3.1) we get

$$\mathbf{E}(t) = \mathbf{A}\mathbf{T}(t), \tag{3.2}$$

where $\mathbf{A} = \mathbf{D}^{-1}$.

To acquire the approximate solutions of system (1.1), we first determine the matrix forms for the ESSs, which are given as:

$$y_{in}(t) = \sum_{k=0}^{n} y_{ik} E_k(t - c), \tag{3.3}$$

where $0 \leq t \leq b < \infty$ and $y_{ik},\ i = 1, 2,\ k = 0, 1, \ldots, n$ are the Euler coefficients which are determined. Thus, the solution in (3.3) can be written in the following form

$$y_{in}(t) = \mathbf{C}_i\mathbf{E}(t),\ i = 1, 2, \tag{3.4}$$

where

$$\mathbf{C}_i = \begin{bmatrix} y_{i0} & y_{i1} & \cdots & y_{in} \end{bmatrix},$$

By substituting relation (3.2) into relation (3.4), the matrix forms of ESSs are obtained as:

$$y_{in}(t) = \mathbf{C}_i\mathbf{A}\mathbf{T}(t), \qquad i = 1, 2. \tag{3.5}$$

The derivative of equation (3.4) is acquired as:

$$y'_{in}(t) = \mathbf{C}_i\mathbf{E}'(t), \qquad i = 1, 2. \tag{3.6}$$

Since $\mathbf{E}'(t) = \mathbf{A}\frac{d}{dt}\mathbf{T}(t)$, then we get

$$\mathbf{E}'(t) = \mathbf{A}\mathbf{T}^*(t), \tag{3.7}$$

where

$$\mathbf{T}^*(t) = \frac{d}{dt}\mathbf{T}(t).$$

If we substitute (3.7) into (3.6), then derivative of $y_{in}(t)$ is converted into a matrix form:

$$y'_{in}(t) = \mathbf{C}_i\mathbf{A}\mathbf{T}^*(t), \qquad i = 1, 2. \tag{3.8}$$

Similarly, the matrix forms of the terms including delays in (1.1) can be expressed as

$$y_{in}(t-\tau_i) = \mathbf{C}_i\mathbf{A}\bar{\mathbf{T}}(t), \qquad i = 1, 2, \tag{3.9}$$

where

$$\bar{\mathbf{T}}(\mathbf{t}) = \begin{bmatrix} 1 \\ t-\tau_i \\ (t-\tau_i)^2 \\ \vdots \\ (t-\tau_i)^n \end{bmatrix}.$$

If the relations (3.5),(3.8), and (3.9) are substituted into system (1.1), the fundamental matrix equations are written as:

$$\begin{aligned} \mathbf{C}_1\mathbf{A}\mathbf{T}^*(t) - r_1\mathbf{C}_1\mathbf{A}\mathbf{T}(t) + a_{11}\left(\mathbf{C}_1\mathbf{A}\mathbf{T}(t)\right)^2 + a_{12}\left(\mathbf{C}_1\mathbf{A}\mathbf{T}(t)\right)\left(\mathbf{C}_2\mathbf{A}\bar{\mathbf{T}}(t)\right) &= 0, \\ \mathbf{C}_2\mathbf{A}\mathbf{T}^*(t) - r_2\mathbf{C}_2\mathbf{A}\mathbf{T}(t) - a_{21}\left(\mathbf{C}_2\mathbf{A}\mathbf{T}(t)\right)\left(\mathbf{C}_1\mathbf{A}\bar{\mathbf{T}}(t)\right) + a_{22}\left(\mathbf{C}_2\mathbf{A}\mathbf{T}(t)\right)^2 &= 0. \end{aligned} \tag{3.10}$$

The same process is applied for the initial conditions, whereby the following matrix relations are acquired.

$$\mathbf{C}_i\mathbf{A}\mathbf{T}(0) = \xi_i, \qquad i = 1, 2. \tag{3.11}$$

Finally, if the equidistant collocation nodes

$$t_i = \frac{b}{n}, \qquad i = 0, 1, \ldots, n, \tag{3.12}$$

are substituted into (3.10), we obtain a system which is constituted by $2(n+1)$ equations. Solving this system yields the coefficients y_{1k} and y_{2k}, $k = 0, 1, \ldots, n$ and hence, we get the ESSs $y_{1n}(t)$ and $y_{2n}(t)$.

4. Error Estimation Procedures

In this part of the work, we give two processes to estimate the errors based on the residue terms, one of which is called the residual correction procedure. To create the residual correction procedure, we first substitute the ESSs $y_{1n}(t)$ and $y_{2n}(t)$ into the system (1.1) and define the residue terms R_{1n} and R_{2n} as

$$R_{1n} = y'_{1n}(t) - y_{1n}(t)\left[r_1 - a_{11}y_{1n}(t) - a_{12}y_{2n}(t-\tau_1)\right], \tag{4.1}$$

$$R_{2n} = y'_{2n}(t) - y_{2n}(t)\left[-r_2 + a_{21}y_{1n}(t-\tau_2) - a_{22}y_{2n}(t)\right]. \tag{4.2}$$

Subtracting R_{1n} from the both sides of the first equation in (1.1) gives as

$$\begin{aligned} &e'_{1n}(t) - r_1 e_{1n}(t) + a_{11}e_{1n}^2(t) + a_{12}e_{1n}(t)e_{2n}(t-\tau_1) + 2a_{11}y_{1n}(t)e_{1n}(t) \\ &+ a_{12}\left(y_{2n}(t-\tau_1)e_{1n}(t) + y_{1n}(t)e_{2n}(t-\tau_1)\right) = -R_{1n}, \end{aligned} \tag{4.3}$$

where $e_{in}(t) = y_i(t) - y_{in}(t), i = 1, 2$.

Accordingly, we apply the same process for R_{2n} and the second equation in (1.1), then we get the other relation.The initial conditions are satisfied for both the exact solution and the ESS. Hence the conditions for the system (1.3) are converted into

$$\begin{aligned} e_{1n}(0) &= 0, \\ e_{2n}(0) &= 0. \end{aligned} \tag{4.4}$$

If the ESS method is applied to system (4.3)-(4.4) for m ,which is not necessarily different from n, the ESSs of the error equation system are acquired. These solutions are indicated as e_{1n}^m and e_{2n}^m for the first error equation and the second error equation, respectively. With this process, we get another approximate solutions whose names are the corrected ESSs represented as

$$y_{in}^m(t) = y_{in}(t) + e_{in}^m(t), \qquad i = 1, 2.$$

Thus, $y_{in}^m(t)$ is a better approximation than $y_{in}(t)$ for $i = 1, 2$ provided that

$$\|e_{in}(t) - e_{in}^m(t)\| < \|y_i(t) - y_{in}(t)\| .$$

As a result, the error $e_{in}(t)$ may be estimated by $e_{in}^m(t)$ in case of

$$\|e_{in}(t) - e_{in}^m(t)\| < \varepsilon.$$

Practically, the absolute errors can be estimated by $e_{in}^m(t)$ for $m > n$ in general.

Regarding the second error estimation process, let $y_{in}(t)$ and $y_{iz}(t)$ be any two ESSs of (1.1) for $n, z \in \mathbb{Z}^+$, $n \neq z$, $i = 1, 2$.Suppose

$$\|y_i(t) - y_{iz}(t)\| < \|y_i(t) - y_{in}(t)\| ,$$

and

$$\|y_i(t) - y_{in}(t)\| = C\, \|y_i(t) - y_{iz}(t)\| , \quad C > 1.$$

Then, using triangle inequality yields as presented in

$$\|y_i(t) - y_{iz}(t)\| < \frac{1}{C-1}\, \|y_{iz}(t) - y_{in}(t)\| .$$

Note that $\|e_{iz}(t)\|$ can be bounded by difference between any ESSs in case of $C \geq 2$. Hence, the error $\|e_{iz}(t)\|$ can be bounded by $\left\|y_{i(z+1)}(t) - y_{iz}(t)\right\|$when $\langle \|e_{iz}(t)\| \rangle$ is a monotone sequence and $C \gg 1$.

5. Numerical Experiments

In this section, two different competitive Lotka-Volterra models are examined and described in (1.1) under initial conditions (1.3). A useful and efficient code is written on the Maple 15 computer algebraic system to solve the problems. The ∞-norm ,defined below ,is used to measure the error.

$$\|f\|_\infty = \sup_{0 \leq t \leq b} |f(t)| .$$

Example 5.1. We first consider the model:

$$\begin{aligned} y_1^{'}(t) &= y_1(t)\left[1 - y_1(t) - 0.5 y_2\left(t - 0.5\right), \right] \\ y_2^{'}(t) &= y_2(t)\left[-1 + 1.5 y_1\left(t - 0.5\right) - y_2(t)\right], \end{aligned}$$

where $0 \leq t \leq 40$ under initial conditions $y_1(0) = 0.2, y_2(0) = 0.15$.This system has the positive equilibrium point $E^* = (\frac{6}{7}, \frac{2}{7})$ which is locally asymtotically stable under conditions mentioned in section 2.2. In Figure 1, the ESSs $y_{1n}(t)$ and $y_{2n}(t)$ are plotted for some n values. Accuracies of ESSs and corrected ESSs are presented in Figure 2 for $n = 5$ and $m = 7$. By calculating the consecutive ESSs for $n = 4, 5, 6$, we obtain the estimations of the absolute error using the second procedure. The results are presented as follows:

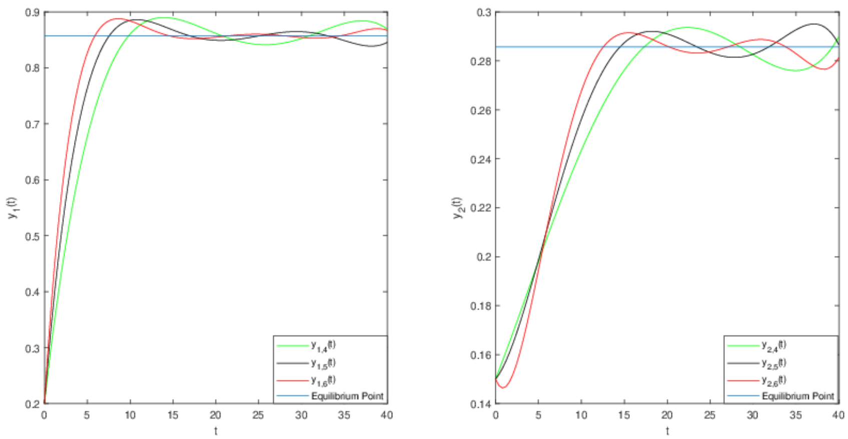

FIGURE 1. The ESSs for the prey population $y_1(t)$ and predator population $y_2(t)$ for Example 5.1.

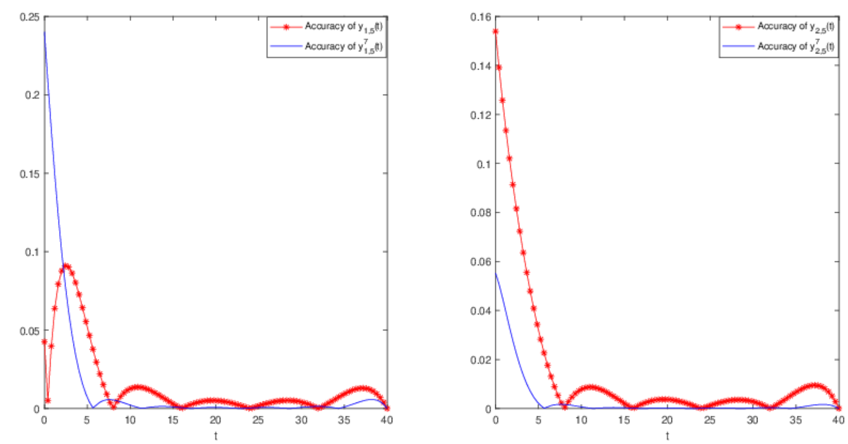

FIGURE 2. Comparison of the accuracies of ESSs and corrected ESSs for Example 5.1.

$$\begin{aligned} \|y_{15}(t) - y_{14}(t)\|_\infty &= 0.86383 \times 10^{-1}, \|y_{25}(t) - y_{24}(t)\|_\infty = 0.17780 \times 10^{-1}, \\ \|y_{16}(t) - y_{15}(t)\|_\infty &= 0.72715 \times 10^{-1}, \|y_{26}(t) - y_{25}(t)\|_\infty = 0.12549 \times 10^{-1}. \end{aligned} \tag{5.1}$$

For the same problem we calculate the residue terms in (4.3) -(4.4) and find the estimations of the errors as follows:

$$\begin{aligned} \left\|e_{14}^7(t)\right\|_\infty &= 0.84450 \times 10^{-1}, \left\|e_{24}^7(t)\right\|_\infty = 0.17682 \times 10^{-1}, \\ \left\|e_{15}^7(t)\right\|_\infty &= 0.65325 \times 10^{-1}, \left\|e_{25}^7(t)\right\|_\infty = 0.11592 \times 10^{-1}. \end{aligned} \tag{5.2}$$

It can be seen from (5.1)-(5.2) that the errors can be bounded by the errors obtained by consequtive approximations.

Example 5.2. [12] For the second model, we examine the following system for $0 \le t \le 60$

Table 1. The accuracies of the ESSs of Example 5.2 for $n = 5, 7$.

t_i	$\|R_{15}(t)\|$	$\|R_{17}(t)\|$	$\|R_{25}(t)\|$	$\|R_{27}(t)\|$
10	0.1392×10^{-1}	0.4432×10^{-2}	0.1206×10^{-1}	0.5770×10^{-2}
17	0.9198×10^{-2}	0.1115×10^{-3}	0.8247×10^{-2}	0.1491×10^{-3}
25	0.1156×10^{-2}	0.2278×10^{-3}	0.1054×10^{-2}	0.2991×10^{-3}
30	0.3489×10^{-2}	0.6787×10^{-3}	0.3154×10^{-2}	0.8939×10^{-3}
38	0.1621×10^{-2}	0.6473×10^{-3}	0.1439×10^{-2}	0.8626×10^{-3}
45	0.2863×10^{-2}	0.6978×10^{-3}	0.2572×10^{-2}	0.9144×10^{-3}
50	0.2791×10^{-2}	0.8609×10^{-3}	0.2567×10^{-2}	0.1124×10^{-2}
55	0.8764×10^{-2}	0.3746×10^{-2}	0.8186×10^{-2}	0.5065×10^{-2}

Table 2. The norms of the difference between consecutive ESSs and the estimations of the absolute errors by residual corrections for m=10 on [0,60] for Example 5.2.

n	$\|y_{1,n+1} - y_{1n}\|_\infty$	$\|e_{1n}^{10}\|_\infty$	$\|y_{2,n+1} - y_{2n}\|_\infty$	$\|e_{2n}^{10}\|_\infty$
5	0.8446×10^{-1}	0.1392×10^{-1}	0.5031×10^{-1}	0.1812×10^{-1}
6	0.4732×10^{-1}	0.2814×10^{-1}	0.3219×10^{-1}	0.1664×10^{-1}
7	0.1918×10^{-1}	0.1918×10^{-1}	0.1555×10^{-1}	0.1555×10^{-1}
8	0.1065×10^{-1}	0.1064×10^{-1}	0.1412×10^{-1}	0.1411×10^{-1}

$$
\begin{aligned}
y_1'(t) &= y_1(t)\left[1 - y_1(t) - y_2(t-1.7)\right], \\
y_2'(t) &= y_2(t)\left[-1 + 2y_1(t-1.8) - y_2(t)\right],
\end{aligned}
$$

under initial conditions $y_1(0) = 0.35$, $y_2(0) = 0.15$ where $r_1 = r_2 = 1$, $a_{21} = 2$, $a_{11} = a_{22} = 1$, $\tau_1 = 1.7, \tau_2 = 1.8$.This system has an equilibrium point $E^* = \left(\frac{2}{3}, \frac{1}{3}\right)$. By using the ESS method for $n = 5, 6, 8$, we get the approximate solutions. The obtained ESSs for different n values are shown in Figure 3. As it can be seen from the figure, the ESSs oscillate around the equilibrium point. In Table 1, the accuracies of solutions $|R_{in}(t)|$, $i = 1, 2$ are calculated at some points for $n = 5, 7$. From this table it can be stated that the results are better as the value of n increases. In Figure 4, the comparison of accuracies of the ESSs and the corrected ESSs are presented. From these images, it is inferred that the corrected solutions present a better approach. In Table 2, the results acquired by error estimation techniques are demonstrated. One can deduce from this table that the results obtained by these two techniques are consistent with each other on estimating the error. We investigate the stability of the problem. A perturbation of $\varepsilon = 1.0 \times 10^{-6}$ in the initial conditions causes an approximately 10^{-5} change as 10^{-5} in the solutions.

Conclusion

This paper proposes a numerical solution method to solve the Lotka–Volterra predator-prey model with two discrete delays. This solution method, the ESS method, is costituted by using the Euler polynomials and collocation method. Two different procedures are given to estimate the absolute errors. The first technique is the residual error correction and the second one is the difference between any consecutive solutions. These techniques can be performed to ESSs of any system (1.1) even if the exact solutions are unknown. The method is applied to some test examples. It has been observed that each ESS oscillates around the equilibrium point and gradually approaches the stability point. Moreover, both error estimation techniques are applied to the problem. The results obtained by error estimation procedures are consistent with each other. The residuals obtained by the corrected ESSs are smaller than the ESSs' residuals for both problems. The initial conditions are perturbed, and the perturbed equations are solved. It can be concluded that small perturbations can cause small changes in the solutions for the test examples.

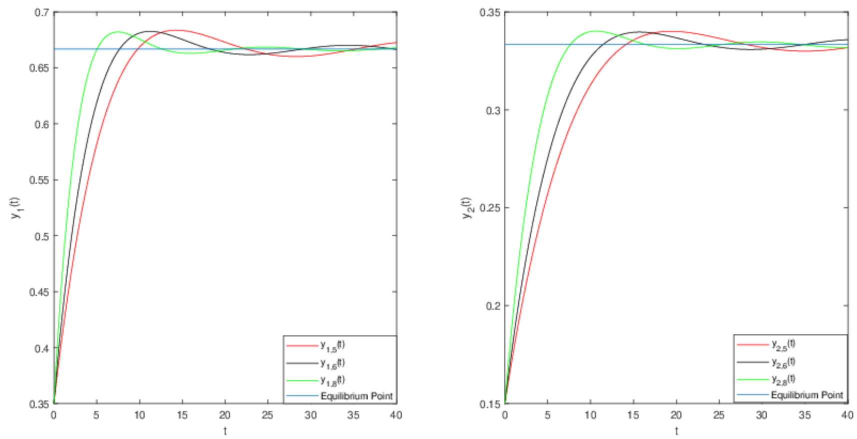

FIGURE 3. The ESSs for the prey population $y_1(t)$ and predator population $y_2(t)$ for Example 5.2.

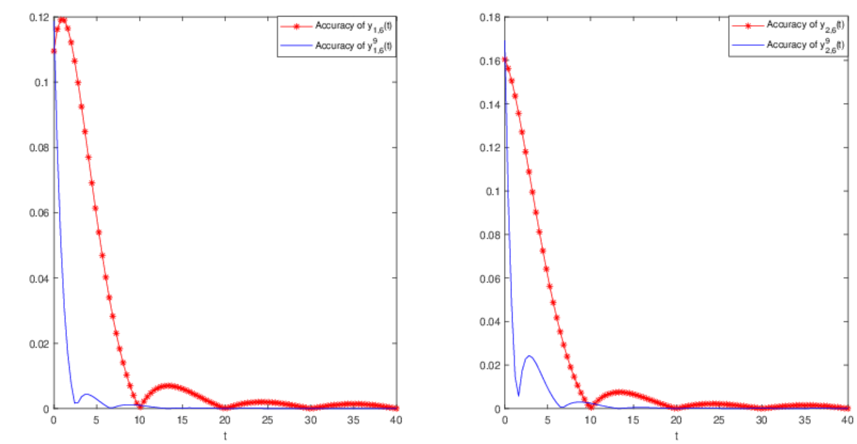

FIGURE 4. Comparison of the accuracies of ESSs and corrected ESSs for Example 5.2.

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