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Jacobi collocation method for numerical solution of nonlinear weakly singular Volterra integro-differential equations: fractional and stochastic cases

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Abstract

This paper deals with the numerical solution of a class of nonlinear multi-term weakly singular fractional Volterra integro-differential equations by Jacobi collocation method based on the orthogonal polynomials. Since the solution of the proposed equation is not smooth enough in the origin, thus the idea of the smoothing transformation is used on the equation to incearse the smoothness of the solution. We represent an operator-based discussion of smoothing transformation and Gauss-Jacobi quadrature for Riemann-Liouville integral operators and weakly singular integral operators using their similar constructions and extend it to the error analysis of the proposed method and obtain an error bound for the discrete collocation solution. In addition, we propose an improved stochastic method, based on the efficient sum-of-exponentials (SOE) approximation, to address the low computational efficiency of the proposed method. To test the efficiency and accuracy, various numerical examples are solved by the proposed method and the obtained error results are in accordance with the convergence analysis of the method. Finally we brought an example regarding the stochastic Volterra integro-differential equations of one singular kernel function.

Keywords. Fractional calculus, multi-term Volterra integro-differential equations, Jacobi collocation method, Smoothing transformation, SOE approximation, Stochastic.
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1. Introduction

In recent decades, much attention has been paid to fractional calculus (integrals and derivatives of non-integer order) and fractional differential equations due to their application in more accurate mathematical modeling of the physical and engineering phenomena. Therefore, many researchers have studied theoretical and numerical analysis of variant definitions of fractional derivatives and their related fractional differential equations with initial/boundary conditions [3, 4, 8–10, 14, 21, 23].

There is an increasing study for fractional integro-differential equations in the literature [1, 2, 24, 27] and some authors have investigated numerical solution of their linear form with a weakly singular kernel [15, 18–20, 26]. However, there are very little works for nonlinear equations with a weakly singular kernel in the literature. Our motivation in this paper is to consider the possibility of constructing a high order numerical method for solving the following initial value problem for nonlinear multi-term weakly singular fractional Volterra integro-differential equation:

$$(D_*^{\alpha}u)(t) = f(t,u(t)) + \sum_{q=1}^{Q} \frac{1}{\Gamma(1-\mu_q)} \int_0^t (t-s)^{-\mu_q} R_q\Big(t,s,u(s),(D_*^{\beta_q}u)(s)\Big)ds, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

for $t \in [0,1]$. In this equation we suppose $0 < \beta_q < \alpha < 1$, $q = 1,...,Q, 1 \leq Q \in \mathbb{N}$ and $0 < \mu_1 < \mu_2 < ... < \mu_Q < 1$. Furthermore, we suppose that $u(t) \in A^1[0,1]$ (the space of all absolutely continuous functions) is unknow solution and the function $f(t,u)$ and the kernels $R_q(t,s,u,v)$, $q = 1,...,Q$ are all known and m times ($m \geq 0$) continuously

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differentiable with respect to the all of their components and then the summation in Eq. (1.1) does not reduce if $m \geq 1$. The operator $D_*^\gamma : L_1[0,1] \to L_1[0,1]$ for $0 < \gamma < 1$ means the fractional derivative of order γ in the Caputo sense with the following formula (see e.g. [9]):

$$(D_*^\gamma u)(t) := \frac{d}{dt}(J^{1-\gamma}[u - u(0)])(t), \tag{1.3}$$

in which we used the Riemann-Liouville fractional integral operator

$$(J^\gamma u)(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(s) ds,\ t > 0,\ J^0 := I, \tag{1.4}$$

with I the identity operator and Γ the Euler Gamma function. It is easy to check that

$$(J^\gamma D_*^\gamma u)(t) = u(t) - u(0),\ (D_*^\gamma J^\gamma u)(t) = u(t). \tag{1.5}$$

It is obvious from Eq. (1.1) that $(D_*^\alpha u)(t) \in C[0,1]$ which deduces $(D_*^{\beta_q} u)(t) \in C[0,1], q = 1, ..., Q$ (see [9], p. 56). Using these results, simple calculations show that the solution of Eqs. (1.1) and (1.2) is not smooth. Indeed, by applying the operator J^α to both sides of Eq. (1.1) and using mean-value theorem for integrals we find that

$$\begin{aligned} u(t) &= u_0 + (J^\alpha f(., u(.)))(t) \\ &\quad + \sum_{q=1}^{Q} \frac{1}{\Gamma(1-\mu_q)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (\tau - s)^{-\mu_q} R_q\Big(\tau, s, u(s), (D_*^{\beta_q} u)(s)\Big) ds d\tau \\ &= u_0 + (J^\alpha f(., u(.)))(t) \\ &\quad + \sum_{q=1}^{Q} \frac{1}{\Gamma(1-\mu_q)\Gamma(\alpha)} R_q\Big(\xi, \eta, u(\eta), (D_*^{\beta_q} u)(\eta)\Big) \int_0^t \int_0^\tau (t-\tau)^{\alpha-1} (\tau - s)^{-\mu_q} ds d\tau \\ &= u_0 + (J^\alpha f(., u(.)))(t) + \sum_{q=1}^{Q} \frac{1}{\Gamma(\alpha)} R_q\Big(\xi, \eta, u(\eta), (D_*^{\beta_q} u)(\eta)\Big) \frac{t^{1+\alpha-\mu_q}}{\Gamma(2+\alpha-\mu_q)}, \end{aligned}$$

where $0 \leq \xi, \eta \leq t$ are some constants. From this result we imply that $(D_*^\alpha u)(t) \notin C^1[0,1]$ and $u(t) \notin C^1[0,1]$ if $\alpha < \mu_M$. Also when $t \to 0^+$ we have

$$(D_*^\alpha u)(t) \sim t^{1-\mu_Q},\ u(t) \sim t^{1+\alpha-\mu_Q}.$$

Then employing classical numerical methods to solve Eqs. (1.1) and (1.2) without applying any smoothing transformation will give low order results as we see in works like [6]. This transformation is well-known and is employed to increase the smoothness of the nonsmooth functions [16, 17, 22]. The smoothing transformation that we apply on Eq. (1.1) is in the form of $t = x^r, s = y^r, r \in \mathbb{N}$ and this transformation will increase the smoothness of the solution $u(t)$ near $t = 0$. Finally, we generalize the Volterra integral equations for one singular kernel function in stochastic sense, and brought the error analysis which are tabulated in Tabs. 5 and 6.

The rest of the paper is organized as follows: In Section 2 we reformulate the operator $(J^\alpha v)(t)$ with respect to the transformation and Gauss-Jacobi quadrature. Using the obtained results in Section 2, in Section 3 we apply smoothing trtansformation and Gauss-Jacobi quadrature for our main equation (1.1). In Section 4 we discuss on the error analysis and convergence order of the proposed numerical method. The main objective of Section 5 is to establish the proposed method, employing the SOE approximation, with the primary goal of reducing computational costs and storage requirements efficiently. finally Section 6 includes numerical experiments that show the applicability and efficiency of the suggested method.

2. Gauss-Jacobi quadrature for Riemann-Liouville integral operator

Let $v(t) \in C[0,1]$ and $r \in \mathbb{N}$. By applying change of variables

$$t = x^r,\ s = y^r,\ 0 \leq y \leq x \leq 1, \tag{2.1}$$

and defining $w(x) := v(x^r)$, the Riemann-Liouville integral operator will be as follows:

$$(J^\alpha v)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^x (x^r - y^r)^{\alpha-1} w(y) r y^{r-1} dy = (J^\alpha_{x,[x^r,1]} w)(x) := (J^\alpha_r w)(x). \tag{2.2}$$

In this equation, $J^\alpha_{x,[x^r,1]}$ is the modified Erdelyi-Kober integration operator [10] with base function x^r and weight function 1 that we simply represent it by J^α_r. Since we approximate $(J^\alpha_r w)(x)$ by Gauss-Jacobi quadrature, then we define

$$\omega_{1-\alpha,r}(x,y) = \frac{ry^{r-1}}{\Gamma(\alpha)} \begin{cases} (\frac{x^r - y^r}{x-y})^{\alpha-1} & x \neq y, \\ (rx^{r-1})^{\alpha-1} & x = y, \end{cases} \quad 0 \leq y \leq x \leq 1, \tag{2.3}$$

and then try to convert the interval of integration in (2.2) into the interval $[-1,1]$ by setting the following two other mappings:

$$x = \frac{\xi+1}{2}, \; y = \frac{\eta+1}{2}, \; -1 \leq \xi, \eta \leq 1, \tag{2.4}$$
$$\eta = \frac{\xi+1}{2}\theta + \frac{\xi-1}{2} := \eta_\xi(\theta), \; -1 \leq \theta \leq 1. \tag{2.5}$$

By applying change of variables (2.4) to Eq. (2.2) we will get

$$(J^\alpha_r w)(\frac{\xi+1}{2}) = \frac{1}{2^\alpha} \int_{-1}^{\xi} (\xi-\eta)^{\alpha-1} \omega_{1-\alpha,r}(\frac{\xi+1}{2}, \frac{\eta+1}{2}) w(\frac{\eta+1}{2}) d\eta, \tag{2.6}$$

and by defining $z(\xi) := w(\frac{\xi+1}{2})$ and then applying the second change of variable (2.5) to Eq. (2.6) we reach the final integral as follows:

$$(\mathcal{J}^\alpha_r z)(\xi) : = (J^\alpha_r w)(\frac{\xi+1}{2})$$
$$= (\frac{\xi+1}{2})^\alpha \frac{1}{2^\alpha} \int_{-1}^{1} (1-\theta)^{\alpha-1} \omega_{1-\alpha,r}(\frac{\xi+1}{2}, \frac{\eta_\xi(\theta)+1}{2}) z(\eta_\xi(\theta)) d\theta$$
$$= \int_{-1}^{1} (1-\theta)^{\alpha-1} \bar{\omega}_{1-\alpha,r}(\xi, \eta_\xi(\theta)) z(\eta_\xi(\theta)) d\theta, \tag{2.7}$$

where

$$\bar{\omega}_{1-\alpha,r}(\xi, \eta_\xi(\theta)) = (\frac{\xi+1}{4})^\alpha \omega_{1-\alpha,r}(\frac{\xi+1}{2}, \frac{\eta_\xi(\theta)+1}{2}). \tag{2.8}$$

Here we note that for $\xi > -1$ and fixed, $\bar{\omega}_{1-\alpha,r}(\xi, \eta_\xi(\theta))$ as a function of θ is smooth on $[-1,1]$. Now, assume that $\mathrm{w}^{a,b}(\theta) = (1-\theta)^a (1+\theta)^b, a, b > -1$ is the weight function of the orthogonal Jacobi polynomials $\{p^{a,b}_N(\theta)\}_{N=0}^{\infty}$ on the interval $[-1,1]$. Then we approximate the integral in (2.7) by the Gauss-Jacobi quadrature formula in the following form:

$$(\mathcal{J}^\alpha_r z)(\xi) \simeq \sum_{k=0}^{N} \bar{\omega}_{1-\alpha,r}(\xi, \eta_\xi(\bar{\theta}_k)) z(\eta_\xi(\bar{\theta}_k)) \bar{\mathrm{w}}_k := (\bar{\mathcal{J}}^{\alpha,N}_r z)(\xi), \; -1 \leq \xi \leq 1, \tag{2.9}$$

in which $\bar{\theta}_k$ and $\bar{\mathrm{w}}_k$ for $k = 0, ..., N$, are the nodes and weights of the quadrature corresponding to the weight function $\mathrm{w}^{\alpha-1,0}(\theta)$.

3. Jacobi collocation method with smoothing transformation

Consider integro-differential equation (1.1) and denote $v(t) := (D_*^\alpha u)(t)$, then $u(t) = u_0 + (J^\alpha v)(t)$. Replacing this result in Eq. (1.1), we obtain the following integral equation:

$$v(t) = f\big(t, u_0 + (J^\alpha v)(t)\big) + \sum_{q=1}^{Q} \frac{1}{\Gamma(1-\mu_q)} \int_0^t (t-s)^{-\mu_q} R_q\Big(t, s, u_0 + (J^\alpha v)(s), (J^{\alpha-\beta_q} v)(s)\Big) ds. \tag{3.1}$$

After approximating the Riemann-Liouville integral operator using the Gauss-Jacobi quadrature in Section 2, in this current section we try to apply the same scheme to the integral parts of integral equation (3.1) together with the collocation method. In each step of computations we will do the same operations on the terms $(J^\alpha v)(s)$ and $(J^{\alpha-\beta_q} v)(s)$ and we will show these operations just by employing the same symbols that we used in Section 2 and we will not discuss the details.

Here we note that $v(t)$, the solution of Eq. (3.1), behaves like $t^{1-\mu_Q}$ when $t \to 0^+$, thus to reach high order precision in the numerical solution we have to use change of variables (2.1) for Eq. (3.1) and we derive

$$\begin{aligned} w(x) &= g\big(x, u_0 + (J_r^\alpha w)(x)\big) \\ &+ \sum_{q=1}^{Q} \int_0^x (x-y)^{-\mu_q} \omega_{\mu_q, r}(x,y) R_{q,r}\Big(x, y, u_0 + (J_r^\alpha w)(y), (J_r^{\alpha-\beta_q} w)(y)\Big) dy, \end{aligned} \tag{3.2}$$

where we use the symbols introduced in Eqs. (2.2) and (2.3) and define

$$\begin{aligned} g\big(x, u_0 + (J_r^\alpha w)(x)\big) &= f\big(x^r, u_0 + (J^\alpha v)(x^r)\big), \\ R_{q,r}\Big(x, y, u_0 + (J_r^\alpha w)(y), (J_r^{\alpha-\beta_q} w)(y)\Big) &= R_q\Big(x^r, y^r, u_0 + (J^\alpha v)(y^r), (J^{\alpha-\beta_q} v)(y^r)\Big). \end{aligned} \tag{3.3}$$

Employing linear transformation (2.4) gives

$$\begin{aligned} z(\xi) &= g\big(\frac{\xi+1}{2}, u_0 + (\mathcal{J}_r^\alpha z)(\xi)\big) \\ &+ \sum_{q=1}^{Q} \frac{1}{2^{1-\mu_q}} \int_{-1}^{\xi} (\xi-\eta)^{-\mu_q} \omega_{\mu_q, r}\big(\frac{\xi+1}{2}, \frac{\eta+1}{2}\big) \\ &\times R_{q,r}\Big(\frac{\xi+1}{2}, \frac{\eta+1}{2}, u_0 + (\mathcal{J}_r^\alpha z)(\eta), (\mathcal{J}_r^{\alpha-\beta_q} z)(\eta)\Big) d\eta, \end{aligned} \tag{3.4}$$

and finally by applying change of variable (2.5), Eq. (3.4) will be in the following form:

$$\begin{aligned} z(\xi) &= g\big(\frac{\xi+1}{2}, u_0 + (\mathcal{J}_r^\alpha z)(\xi)\big) \\ &+ \sum_{q=1}^{Q} \int_{-1}^{1} (1-\theta)^{-\mu_q} \bar{\omega}_{\mu_q, r}(\xi, \eta_\xi(\theta)) \\ &\times K_{q,r}\Big(\xi, \eta_\xi(\theta), u_0 + (\mathcal{J}_r^\alpha z)(\eta_\xi(\theta)), (\mathcal{J}_r^{\alpha-\beta_q} z)(\eta_\xi(\theta))\Big) d\theta \\ &:= g\big(\frac{\xi+1}{2}, u_0 + (\mathcal{J}_r^\alpha z)(\xi)\big) + \sum_{q=1}^{Q} \big(\mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^\alpha z, \mathcal{J}_r^{\alpha-\beta_q} z)\big)(\xi), \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} &K_{q,r}\Big(\xi, \eta_\xi(\theta), u_0 + (\mathcal{J}_r^\alpha z)(\eta_\xi(\theta)), (\mathcal{J}_r^{\alpha-\beta_q} z)(\eta_\xi(\theta))\Big) = \\ &\quad R_{q,r}\Big(\frac{\xi+1}{2}, \frac{\eta_\xi(\theta)+1}{2}, u_0 + (\mathcal{J}_r^\alpha z)(\eta_\xi(\theta)), (\mathcal{J}_r^{\alpha-\beta_q} z)(\eta_\xi(\theta))\Big). \end{aligned} \tag{3.6}$$

Let $\mathbb{P}_N$ is the space of polynomials of degree at most N. Also, suppose that $\{\xi_i\}_{i=0}^N$ are the roots of the Jacobi polynomial $p_{N+1}^{a,b}(\xi)$ of degree $N+1$. We choose Lagrange fundamental polynomials $\{L_i(\xi)\}_{i=0}^N$ constructed on the points $\{\xi_i\}_{i=0}^N$ as a basis for $\mathbb{P}_N$. A collocation solution for Eq. (4.2) is a polynomial $z_N(\xi) \in \mathbb{P}_N$ with a representation $z_N(\xi) = \sum_{j=0}^N z_j L_j(\xi)$ that satisfies the following collocation conditions:

$$\begin{aligned} z_N(\xi_i) &= g\big(\frac{\xi_i+1}{2}, u_0 + (\mathcal{J}_r^{\alpha} z_N)(\xi_i)\big) \\ &+ \sum_{q=1}^{Q} \int_{-1}^{1} (1-\theta)^{-\mu_q} \bar{\omega}_{\mu_q,r}(\xi_i, \eta_{\xi_i}(\theta)) \\ &\times K_{q,r}\Big(\xi_i, \eta_{\xi_i}(\theta), u_0 + (\mathcal{J}_r^{\alpha} z_N)(\eta_{\xi_i}(\theta)), (\mathcal{J}_r^{\alpha-\beta_q} z_N)(\eta_{\xi_i}(\theta))\Big) d\theta, \end{aligned} \tag{3.7}$$

for $i = 0, ..., N$. We look for a discrete collocation solution $\bar{z}_N(\xi) \in \mathbb{P}_N, \bar{z}_N(\xi) = \sum_{j=0}^N \bar{z}_j L_j(\xi)$, then we approximate the integral terms in Eq. (3.7) using Gauss-Jacobi quadrature in two steps as follows:

$$\begin{aligned} &\int_{-1}^{1} (1-\theta)^{-\mu_q} \bar{\omega}_{\mu_q,r}(\xi, \eta_\xi(\theta)) K_{q,r}\Big(\xi, \eta, u_0 + (\mathcal{J}_r^{\alpha} z)(\eta_\xi(\theta)), (\mathcal{J}_r^{\alpha-\beta_q} z)(\eta_\xi(\theta))\Big) d\theta \\ &\simeq \sum_{k=0}^{N} \bar{\omega}_{\mu_q,r}(\xi, \eta_\xi(\tilde{\theta}_k^q)) K_{q,r}\Big(\xi, \eta_\xi(\tilde{\theta}_k^q), u_0 + (\mathcal{J}_r^{\alpha} z)(\eta_\xi(\tilde{\theta}_k^q)), (\mathcal{J}_r^{\alpha-\beta_q} z)(\eta_\xi(\tilde{\theta}_k^q))\Big) \tilde{\mathrm{w}}_k^q \\ &\simeq \sum_{k=0}^{N} \bar{\omega}_{\mu_q,r}(\xi, \eta_\xi(\tilde{\theta}_k^q)) K_{q,r}\Big(\xi, \eta_\xi(\tilde{\theta}_k^q), u_0 + (\bar{\mathcal{J}}_r^{\alpha,N} z)(\eta_\xi(\tilde{\theta}_k^q)), (\bar{\mathcal{J}}_r^{\alpha-\beta_q,N} z)(\eta_\xi(\tilde{\theta}_k^q))\Big) \tilde{\mathrm{w}}_k^q \\ &:= \big(\bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} z, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} z)\big)(\xi), \ q = 1, ..., Q, \end{aligned} \tag{3.8}$$

where $\tilde{\theta}_k^q$ and $\tilde{\mathrm{w}}_k^q$ are the nodes and weights with respect to the weight function $\mathrm{w}^{-\mu_q,0}(\theta)$. Now, discrete collocation solution $\bar{z}_N(\xi)$ is obtained by satisfying the following discrete collocation equation:

$$\begin{aligned} \bar{z}_N(\xi_i) &= g\big(\frac{\xi_i+1}{2}, u_0 + (\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N)(\xi_i)\big) \\ &+ \sum_{q=1}^{Q} \sum_{k=0}^{N} K_{q,r}\Big(\xi_i, \eta_{\xi_i}(\tilde{\theta}_k^q)), u_0 + (\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N)(\eta_{\xi_i}(\tilde{\theta}_k^q)), (\bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N)(\eta_{\xi_i}(\tilde{\theta}_k^q))\Big) \tilde{\mathrm{w}}_k^q \\ &= g\big(\frac{\xi_i+1}{2}, u_0 + (\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N)(\xi_i)\big) + \sum_{q=1}^{Q} \big(\bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N)\big)(\xi_i), \end{aligned} \tag{3.9}$$

for $i = 0, ..., N$. By replacing the representation $\bar{z}_N(\xi) = \sum_{j=0}^N \bar{z}_j L_j(\xi)$, we derive

$$\begin{aligned} \bar{z}_i &= g\big(\frac{\xi_i+1}{2}, u_0 + (\sum_{j=0}^{N} \bar{z}_j \bar{\mathcal{J}}_r^{\alpha,N} L_j)(\xi_i)\big) \\ &+ \sum_{q=1}^{Q} \sum_{k=0}^{N} K_{q,r}\Big(\xi_i, \eta_{\xi_i}(\tilde{\theta}_k^q)), u_0 + (\sum_{j=0}^{N} \bar{z}_j \bar{\mathcal{J}}_r^{\alpha,N} L_j)(\eta_{\xi_i}(\tilde{\theta}_k^q)), (\sum_{j=0}^{N} \bar{z}_j \bar{\mathcal{J}}_r^{\alpha-\beta_m,N} L_j)(\eta_{\xi_i}(\tilde{\theta}_k^q))\Big) \tilde{\mathrm{w}}_k^q \\ &= g\big(\frac{\xi_i+1}{2}, u_0 + (\sum_{j=0}^{N} \bar{z}_j \bar{\mathcal{J}}_r^{\alpha,N} L_j)(\xi_i)\big) \\ &+ \sum_{q=1}^{Q} \big(\bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\sum_{j=0}^{N} \bar{z}_j \bar{\mathcal{J}}_r^{\alpha,N} L_j, \sum_{j=0}^{N} \bar{z}_j \bar{\mathcal{J}}_r^{\alpha-\beta_m,N} L_j)\big)(\xi_i). \end{aligned} \tag{3.10}$$

for $i = 0, ..., N$. Eq. (3.10) is a nonlinear system with unknowns $\bar{z}_i, i = 0, ..., N$, and by solving it we obtain the discrete collocation solution $\bar{z}_N(\xi) = \sum_{j=0}^{N} \bar{z}_j L_j(\xi)$. According to the change of variables, the relation between the solution $v(t)$ of Eq. (3.1), the solution $w(x)$ of Eq. (3.2) and the solution $z(\xi)$ of Eq. (4.2) is as $v(t) = w(\sqrt[r]{t}) = z(2\sqrt[r]{t}-1), 0 \leq t \leq 1$, then the numerical solution for these equations will be as $\bar{v}_N(t) = \bar{w}_N(\sqrt[r]{t}) = \bar{z}_N(2\sqrt[r]{t}-1)$. Moreover, the numerical solution $\bar{u}_N(t)$ of Eqs. (1.1) and (1.2) can be computed as follows:

$$\begin{aligned}
\bar{u}_N(t) &= u_0 + (J^\alpha \bar{v}_N)(t) \\
&= u_0 + (J_r^\alpha \bar{w}_N)(x) \\
&= u_0 + (\mathcal{J}_r^\alpha \bar{z}_N)(\xi) \\
&\simeq u_0 + (\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N)(\xi) \\
&= u_0 + (\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N)(2\sqrt[r]{t}-1), \;\; 0 \leq t \leq 1. \qquad (3.11)
\end{aligned}$$

4. Convergence analysis

Assume that $\mathcal{P}_N^{a,b}$ is the interpolation operator that is defined on the nodes $\{\xi_i\}_{i=0}^N$, the roots of the Jacobi polynomial $p_{N+1}^{a,b}(\xi)$ and maps $C[-1,1]$ on $\mathbb{P}_N$. In order to discuss the convergence of the proposed method, we need the following lemmas and results.

Lemma 4.1. *[5] suppose that $\{\xi_i\}_{i=0}^N$ are the roots of the orthogonal Jacobi polynomial of degree $N+1$ and $L_i(\xi), i = 0, 1, \ldots, N$, are the Lagrange fundamental polynomials wih respect to these nodes. Then*

$$\|\mathcal{P}_N^{a,b}\|_\infty = \max_{-1\leq\xi\leq1} \sum_{i=0}^{N} |L_i(\xi)| = \begin{cases} \mathcal{O}(\log N), & -1 < a, b \leq -\frac{1}{2}, \\ \mathcal{O}\big(N^{\gamma+\frac{1}{2}}\big), & -\frac{1}{2} < a, b < 1, \end{cases}$$

where $\gamma = \max\{a, b\}$.

Lemma 4.2. *[25] Let $f(\xi) \in A^d[-1,1]$, the space of all functions with an absolutely continuous $(d-1)$st derivative,*

$$f^{(d-1)}(\xi) = f^{(d-1)}(-1) + \int_{-1}^{\xi} g(\eta) d\eta,$$

with $g(\eta) \in L^1(-1,1)$ and of bounded variation $var(g(\eta)) < \infty$ on $[-1,1]$. Furthermore, assume that $\{\xi_i\}_{i=0}^N$ are the roots of the Jacobi polynomial of degree $N+1$ on the interval $[-1,1]$. Then for $N \geq d+1$

$$\|f - \mathcal{P}_N^{a,b} f\|_\infty \leq C N^{-d+\max\left\{0,\gamma-\frac{1}{2}\right\}}, \quad \gamma = \max\{a, b\}, \tag{4.1}$$

with C a constant independent of N.

Theorem 4.3. *Suppose that $u(t)$ is the exact solution of Eqs. (1.1) and (1.2) and $\bar{u}_N(t)$ defined in Eq. (3.11) is the numerical solution of Eqs. (1.1) and (1.2) obtained by the presented method in Sections 2 and 3. Moreover, using the Fréchet differentiability of $g(x, u)$ and $\mathcal{K}_{q,r}(u, v)$ as operators on u and (u, v), respectively, let the homogeneous linear fractional integral equation*

$$\begin{aligned}
\phi(\xi) \;&=\; \frac{\partial}{\partial u} g\Big(\frac{\xi+1}{2}, z_0\Big)(\mathcal{J}_r^\alpha \phi)(\xi) \\
&+\; \sum_{q=1}^{Q} \mathcal{J}_r^{1-\mu_q}\Big(\frac{\partial}{\partial u}\mathcal{K}_{q,r}(z_q, \zeta_q)\mathcal{J}_r^\alpha + \frac{\partial}{\partial v}\mathcal{K}_{q,r}(z_q, \zeta_q)\mathcal{J}_r^{\alpha-\beta_q}\Big)\phi(\xi), \qquad (4.2)
\end{aligned}$$

has just the trivial solution $\phi(\xi) = 0$ for every $z_0, z_q, \zeta_q \in C[-1,1]$. Then we have

$$\|u - \bar{u}_N\|_\infty \leq C' \begin{cases} N^{-d} + N^{-m}\log N, & -1 < a, b \leq -\frac{1}{2}, \\ N^{-d+\max\{0,\gamma-\frac{1}{2}\}} + N^{-m+\gamma+\frac{1}{2}}, & -\frac{1}{2} < a, b \leq 1. \end{cases} \tag{4.3}$$

where $d = [r(1-\mu_Q)] + 1$.

Proof. Since $v(t) \sim t^{1-\mu_Q}$ as $t \to 0^+$, then $z(\xi) \sim (\xi+1)^{r(1-\mu_Q)}$ as $\xi \to -1^+$ and $z(\xi)$ satisfies the hypotheses of Lemma 4.2 with $d = [r(1-\mu_Q)] + 1$ and $g(\xi) = z^d(\xi) \in L^1(-1,1)$. Thus, Lemma 4.2 implies that

$$\|z - \mathcal{P}_N^{a,b} z\|_\infty \le C_0 N^{-d+\max\left\{0,\gamma-\frac{1}{2}\right\}}, \quad \gamma = \max\{a,b\}, \tag{4.4}$$

where C_0 is a constant that does not dependent on N. Multiplying both sides of Eq. (3.10) by $L_i(\xi)$ and getting summation over i, we derive

$$\bar{z}_N = \mathcal{P}_N^{a,b} g\big(x, u_0 + \bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N\big) + \sum_{q=1}^{Q} \mathcal{P}_N^{a,b} \bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N). \tag{4.5}$$

On the other hand, Eq. (4.2) can be represented as

$$z = g\big(x, u_0 + \mathcal{J}_r^{\alpha} z\big) + \sum_{q=1}^{Q} \mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z). \tag{4.6}$$

By employing operator $\mathcal{P}_N^{a,b}$ and adding z to the both sides of Eq. (4.6), we obtain

$$z = z - \mathcal{P}_N^{a,b} z + \mathcal{P}_N^{a,b} g\big(x, u_0 + \mathcal{J}_r^{\alpha} z\big) + \sum_{q=1}^{Q} \mathcal{P}_N^{a,b} \mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z), \tag{4.7}$$

and then by subtracting both sides of Eq. (4.5) from Eq. (4.7), we get

$$\begin{aligned} z - \bar{z}_N &= z - \mathcal{P}_N^{a,b} z + \mathcal{P}_N^{a,b}\Big(g\big(x, u_0 + \mathcal{J}_r^{\alpha} z\big) - g\big(x, u_0 + \bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N\big)\Big) \\ &+ \sum_{q=1}^{Q} \mathcal{P}_N^{a,b}\Big(\mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z) - \bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N)\Big). \end{aligned} \tag{4.8}$$

The function $g(x,u)$, as an operator on u is Fréchet differentiable with respect to u. Therefore the third term in the right hand side of Eq. (4.8) can be written as

$$\begin{aligned} &\mathcal{P}_N^{a,b}\Big(g\big(x, u_0 + (\mathcal{J}_r^{\alpha} z)\big) - g\big(x, u_0 + (\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N)\big)\Big) = \mathcal{P}_N^{a,b}\left(\frac{\partial g}{\partial u}(x, z_0)\right)\left(\mathcal{J}_r^{\alpha} z - \bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N\right) \\ &= \mathcal{P}_N^{a,b}\left(\frac{\partial g}{\partial u}(x, z_0)\right)\left(\mathcal{J}_r^{\alpha}(z - \bar{z}_N)\right) + \mathcal{P}_N^{a,b}\left(\frac{\partial g}{\partial u}(x, z_0)\right)\left(\mathcal{J}_r^{\alpha} \bar{z}_N - \bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N\right), \end{aligned} \tag{4.9}$$

where $z_0 \in C[-1,1]$.
For the fourth term in the right hand side of Eq. (4.8), we write

$$\begin{aligned} &\mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z) - \bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) \\ &= \mathcal{J}_r^{1-\mu_q}\left(\mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z) - \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N)\right) \\ &+ \mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) - \bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N), \end{aligned} \tag{4.10}$$

and since the operator $\mathcal{K}_{q,r}(u,v)$ is Fréchet differentiable with respect to u and v, then there exist $z_q, \zeta_q \in C[-1,1]$ such that

$$\begin{aligned} &\mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z) - \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) \\ &\quad = \Big(\frac{\partial}{\partial u} \mathcal{K}_{q,r}(z_q, \zeta_q)\, \mathcal{J}_r^{\alpha} + \frac{\partial}{\partial v} \mathcal{K}_{q,r}(z_q, \zeta_q)\, \mathcal{J}_r^{\alpha-\beta_q}\Big)(z - \bar{z}_N) \\ &\quad := \mathcal{K}'_{q,r}(z_q, \zeta_q)(z - \bar{z}_N). \end{aligned} \tag{4.11}$$

Substituting Eqs. (4.9), (4.10) and (4.11) in Eq. (4.8) deduces

$$\Big(\mathcal{I}-\mathcal{P}_N^{a,b}\big(\frac{\partial g}{\partial u}(x,z_0)\mathcal{J}_r^{\alpha}+\sum_{q=1}^{Q}\mathcal{J}_r^{1-\mu_q}\mathcal{K}'_{q,r}(z_q,\zeta_q)\big)\Big)(z-\bar{z}_N)$$
$$=(z-\mathcal{P}_N^{a,b}z)+\mathcal{P}_N^{a,b}\left(\frac{\partial g}{\partial u}(x,z_0)\left(\mathcal{J}_r^{\alpha}\bar{z}_N-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\right)\right)$$
$$+\sum_{q=1}^{Q}\mathcal{P}_N^{a,b}\Big(\mathcal{J}_r^{1-\mu_q}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)-\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)\Big). \tag{4.12}$$

Since Eq. (4.2) has just zero solution, then integral operator

$$\mathcal{I}-\big(\frac{\partial g}{\partial u}(x,z_0)\mathcal{J}_r^{\alpha}+\sum_{q=1}^{Q}\mathcal{J}_r^{1-\mu_q}\mathcal{K}'_{q,r}(z_q,\zeta_q)\big), \tag{4.13}$$

for every $z_0,z_q,\zeta_q\in C[-1,1]$ has a bounded inverse. On the other hand, using Lemma 4.2 it is easy to check that

$$\|(\mathcal{I}-\mathcal{P}_N^{a,b})\big(\frac{\partial g}{\partial u}(x,z_0)\mathcal{J}_r^{\alpha}+\sum_{q=1}^{Q}\mathcal{J}_r^{1-\mu_q}\mathcal{K}'_{q,r}(z_q,\zeta_q)\big)\|_\infty\to 0 \text{ as } N\to\infty, \tag{4.14}$$

then for sufficiently large N operator

$$\mathcal{I}-\mathcal{P}_N^{a,b}\big(\frac{\partial g}{\partial u}(x,z_0)\mathcal{J}_r^{\alpha}+\sum_{q=1}^{Q}\mathcal{J}_r^{1-\mu_q}\mathcal{K}'_{q,r}(z_q,\zeta_q)\big), \tag{4.15}$$

is invertible too and has a bounded inverse on $C[-1,1]$ as

$$\|\Big(\mathcal{I}-\mathcal{P}_N^{a,b}\big(\frac{\partial g}{\partial u}(x,z_0)\mathcal{J}_r^{\alpha}+\sum_{q=1}^{Q}\mathcal{J}_r^{1-\mu_q}\mathcal{K}'_{q,r}(z_q,\zeta_q)\big)\Big)^{-1}\|_\infty\le C_1, \tag{4.16}$$

thus from Eq. (4.12) we derive

$$\begin{aligned}\|z-\bar{z}_N\|_\infty &\le C_1\Big(\|z-\mathcal{P}_N^{a,b}z\|_\infty+\|\mathcal{P}_N^{a,b}\|_\infty\|\frac{\partial g}{\partial u}(x,z_0)\|_\infty\|\mathcal{J}_r^{\alpha}\bar{z}_N-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\|_\infty\\ &+\sum_{q=1}^{Q}\|\mathcal{P}_N^{a,b}\|_\infty\|\mathcal{J}_r^{1-\mu_q}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)\\ &-\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)\|_\infty\Big).\end{aligned} \tag{4.17}$$

Since $\bar{z}_N\in\mathbb{P}_N$, then

$$\begin{aligned}&\left\|\mathcal{J}_r^{\alpha}\bar{z}_N-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\right\|_\infty\le C_2N^{-m},\\ &\|\mathcal{J}_r^{1-\mu_q}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)-\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)\|_\infty\le C_3N^{-m},\end{aligned} \tag{4.18}$$

for $q=1,2,\dots,Q$. Now, using Lemma 4.1 and Eq. (4.4), we obtain from Eq. (4.17) that

$$\|z-\bar{z}_N\|_\infty\le C\begin{cases}N^{-d}+N^{-m}\log N, & -1<a,b\le-\frac{1}{2},\\ N^{-d+\max\{0,\gamma-\frac{1}{2}\}}+N^{-m+\gamma+\frac{1}{2}}, & -\frac{1}{2}<a,b\le 1,\end{cases} \tag{4.19}$$

where C is a constant dependent on the constants C_0,C_1,C_2 and C_3 and independent of N,m and d. Using Eqs. (4.1) and (4.18) and error bound (4.19), we yield the following error bound for the numerical solution of Eqs. (1.1) and

(1.2):

$$\begin{aligned}\|u-\bar{u}_N\|_\infty &= \left\|\mathcal{J}_r^\alpha z-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\right\|_\infty,\\ &\le \|\mathcal{J}_r^\alpha\|_\infty \|z-\bar{z}_N\|_\infty + \left\|\mathcal{J}_r^\alpha\bar{z}_N-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\right\|_\infty\\ &\le C'\begin{cases}N^{-d}+N^{-m}\log N, & -1<a,b\le\frac{-1}{2},\\ N^{-d+\max\{0,\gamma-\frac{1}{2}\}}+N^{-m+\gamma+\frac{1}{2}}, & \frac{-1}{2}<a,b\le 1.\end{cases}\end{aligned}$$

□

5. The SOE approximation for stochastic case

In this section, we focus on the challenges brought by the low order of the proposed method. To enhance the computational efficiency, we introduce an alternative approach known as the SOE approximation (see [[13], Theorem 2.1] or [11]).

Lemma 5.1. *For every $\gamma\in(0,1)$, there exist positive quadrature nodes τ_l and weights ω_l, $l=1,2,\cdots,M_{\exp}$, along with an absolute tolerance error $\epsilon\ll 1$ and a cut-off time point δ, such that*

$$|t^{-\gamma}-\sum_{l=1}^{M_{\exp}}\omega_l e^{-\tau_l t}|\le\epsilon,\quad \forall t\in[\delta,T], \tag{5.1}$$

where $M_{\exp}$ satisfies

$$M_{\exp}=\mathcal{O}\Big(\log\frac{1}{\epsilon}\big(\log\log\frac{1}{\epsilon}+\log\frac{T}{\delta}\big)+\log\frac{1}{\delta}\big(\log\log\frac{1}{\epsilon}+\log\frac{1}{\delta}\big)\Big). \tag{5.2}$$

In other words, if we use the cut-off time $\delta=h$ for a fixed precision ϵ, then we obtain

$$M_{\exp}=\begin{cases}\mathcal{O}(\log N), & \text{if}\quad T\gg 1,\\ \mathcal{O}(\log^2 N), & \text{if}\quad T\approx 1,\end{cases}$$

where $h=\dfrac{T}{N}$.

By using the SOE approximation (5.1), the proposed method can be restructured as

$$\bar{u}_N=u_0+h\sum_{k=0}^{n-1}\sum_{l=1}^{M_{\exp,1}}w_{l,1}e^{-\tau_{l,1}(t_n-t_k)}f(\bar{u}_k)+\sum_{k=0}^{n-1}\sum_{l=1}^{M_{\exp,2}}w_{l,2}e^{-\tau_{l,2}(t_n-t_k)}g(\bar{u}_k)\Delta W_k, \tag{5.3}$$

where $n=1,2,\ldots,N$, $\Delta W_k:=W_{t_{k+1}}-W_{t_k}$ is the increment of Wiener process, $(t_n-t_k)^{-\gamma_1}$ and $(t_n-t_k)^{-\gamma_2}$ are replaced by $\sum_{l=1}^{M_{\exp,j}}w_{l,j}e^{-\tau_{l,j}(t_n-t_k)}$, $j=1,2$, respectively. In addition, Eq. (5.3) can be rewritten as

$$\bar{u}_N=u_0+\sum_{l=1}^{M_{\exp,1}}w_{l,1}P_{1,l}(t_n)+\sum_{l=1}^{M_{\exp,2}}w_{l,2}P_{2,l}(t_n), \tag{5.4}$$

where

$$\begin{aligned}P_{1,l}(t_n)&=h\sum_{k=0}^{n-1}e^{-\tau_{l,1}(t_n-t_k)}f(\bar{u}_k),\\ P_{2,l}(t_n)&=\sum_{k=0}^{n-1}e^{-\tau_{l,2}(t_n-t_k)}g(\bar{u}_k)\Delta W_k.\end{aligned} \tag{5.5}$$

This suggests that the cost of calculation drops from $\mathcal{O}(N^2)$ to $\mathcal{O}\big(NM_{\exp}\big)$

Table 1. The global E_{RMS} errors for a range of increasing values of N and r for Example 6.1.

N	$r=1$	$r=5$	$r=9$	$r=13$
8	1.55e − 04	8.38e − 07	2.42e − 05	4.20e − 04
16	2.88e − 05	1.34e − 08	2.13e − 10	3.19e − 11
24	6.60e − 06	1.28e − 09	6.35e − 12	5.81e − 14
32	3.07e − 06	2.44e − 10	4.66e − 13	2.02e − 15

Remark 5.2. For the proposed method, based on the SOE approximation, the property of exponential functions contributes the key recurrence relations

$$P_{1,l}(t_n) = e^{-\tau_{l,1}h}P_{1,l}(t_{n-1}) + e^{-\tau_{l,1}h}f(\bar{u}_{n-1})h, \tag{5.6}$$

$$P_{2,l}(t_n) = e^{-\tau_{l,2}h}P_{2,l}(t_{n-1}) + e^{-\tau_{l,2}h}g(\bar{u}_{n-1})\Delta W_{n-1}. \tag{5.7}$$

Combination of Lemma 5.1 and the proposed method demonstrates that the computational cost and storage requirements for a single sample are reduced from $\mathcal{O}(N^2)$ and $\mathcal{O}(N)$ to $\mathcal{O}(N\log N)$ and $\mathcal{O}(\log N)$ respectively, when $T \gg 1$. For $T \approx 1$, the reductions are to $\mathcal{O}(N\log^2 N)$ and $\mathcal{O}(\log^2 N)$, respectively.

6. Numerical experiments and results

To show the accuracy and the efficiency of the proposed Jacobi collocation method and the SOE approximation, in this section we apply the proposed method on different numerical examples of Eqs. (1.1) and (1.2). The algorithm of the suggested method has been performed in Mathematica software. In each example, we report the root mean square of the absolute errors, E_{RMS}, at equidistant points, $t_i = \frac{i}{10}, i = 1, 2, \dots, 10$.

Example 6.1. Consider the following nonlinear fractional Volterra integro-differential equation:

$$\begin{aligned}&(D_*^{\alpha}u)(t) = f(t, u(t)) + \int_0^t (t-s)^{-\mu}t^{2-\mu}s^{1+2\alpha+\beta}u(s)(D_*^{\beta}u)(s)ds,\ t \in [0,1],\\ &u(0) = 0,\end{aligned} \tag{6.1}$$

where

$$f(t,u) = \frac{\Gamma(2+\alpha-\mu)}{\Gamma(2-\mu)}t^{1-\mu} - \frac{\Gamma(4+4\alpha-2\mu)\Gamma(1-\mu)\Gamma(2+\alpha-\mu)}{\Gamma(5+4\alpha-3\mu)\Gamma(1-\mu)\Gamma(2+\alpha-\beta-\mu)}t^2u^4.$$

The exact solution of this problem is $u(t) = t^{1+\alpha-\mu}$. For $\alpha = \frac{\pi}{6}, \beta = \frac{\pi}{8}$ and $\mu = \frac{2}{\pi}$, we have $u(t) \in A^1[0,1]$. The absolute error E_{RMS} of the proposed method for different values of N and $r = 1, 5, 9, 13$ are listed in Table 1. When N is big enough, increasing r and then $d = [r(1-\frac{2}{\pi})] + 1$ decreases the error as we expect from the error bound (4.2). In Figure 1 we represent the errors obtained in Table 1 for different values of N and r. From this figure we observe the accuracy of the proposed method which verifies the theoretical results.

Example 6.2. We examine the following fractional integro-differential equation:

$$\begin{aligned}&(D_*^{\frac{\sqrt{2}}{3}}u)(t) = f(t) + \int_0^t (t-s)^{-\frac{\sqrt{2}}{2}}\frac{(D_*^{\frac{\sqrt{2}}{5}}u)(s)}{1+u^2(s)}ds,\ t \in [0,1],\\ &u(0) = 1.\end{aligned} \tag{6.2}$$

In this equation, the function $f(t)$ is chosen such that the exact solution of this problem to be the Mittag-Leffler function $u(t) = E_{\frac{\sqrt{2}}{2}}(t^{\frac{\sqrt{2}}{2}}) \in A^1[0,1]$ defined by

$$E_b(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(bk+1)}.$$

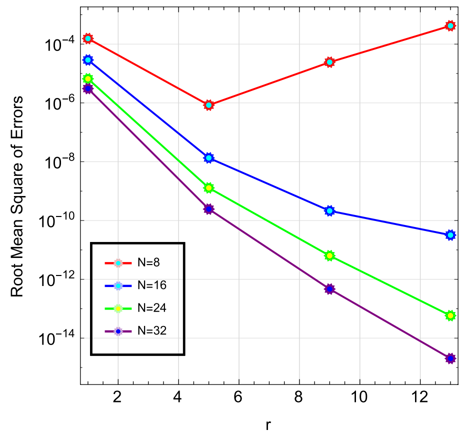

FIGURE 1. Graph of the root mean square of the absolute errors for Example 6.1.

TABLE 2. The global E_{RMS} errors for a range of increasing values of N and r for Example 6.2.

N	$r = 1$	$r = 5$	$r = 9$	$r = 13$
8	4.82e − 04	2.11e − 04	3.06e − 03	1.03e − 02
16	7.57e − 05	1.72e − 07	2.76e − 06	3.88e − 05
32	1.71e − 05	8.10e − 09	4.50e − 11	1.67e − 10
40	1.10e − 05	2.40e − 09	9.00e − 12	1.19e − 13

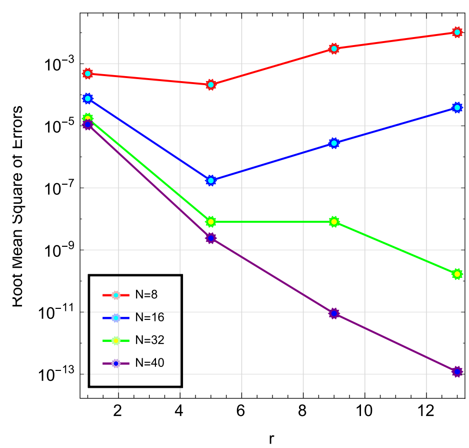

FIGURE 2. Graph of the root mean square of the absolute errors for Example 6.2.

We derive errors E_{RMS} by applying the proposed Jacobi collocation method on Eq. (6.2) for several values of N and r and list these errors in Table 2 and Figure 2.

Table 3. The global E_{RMS} errors for a range of increasing values of N and r for Example 6.3.

N	$r=1$	$r=5$	$r=9$	$r=13$	$r=17$	$r=21$
8	3.60e − 02	1.57e − 01	6.69e − 01	8.50e − 01	3.84e − 02	6.70e − 01
16	1.01e − 02	6.64e − 06	1.93e − 06	2.78e − 07	2.63e − 06	2.39e − 07
24	4.46e − 03	1.27e − 06	5.39e − 08	3.22e − 08	6.42e − 09	2.18e − 09
32	2.46e − 03	4.96e − 07	1.41e − 08	4.20e − 09	5.81e − 10	4.24e − 11

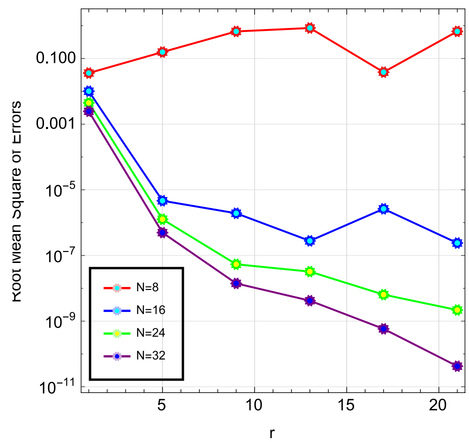

Figure 3. Graph of the root mean square of the absolute errors for Example 6.3.

Example 6.3. As the third example, consider the following nonlinear weakly singular fractional Volterra integro-differential equation:

$$\begin{aligned} (D_*^{\frac{3}{5}}u)(t) &= f(t) + \int_0^t (t-s)^{-\frac{1}{3}}\big(u(s) + (D_*^{\frac{2}{5}}u)(s)\big)^2 ds, \ \ t \in [0,1], \\ u(0) &= 0, \end{aligned} \tag{6.3}$$

For this problem we select the data function $f(t)$ such that the exact solution to be $u(t) = t^{\frac{7}{10}} \in A^1[0,1]$. We employ the presented method on this equation and report the errors E_{RMS} for different values of N, r in Table 3 and Figure 3.

Example 6.4. As a multi-term equation, consider the following nonlinear two-term weakly singular fractional integro-differential equation:

$$\begin{aligned} (D_*^{\alpha}u)(t) &= f(t,u(t)) + \sum_{q=1}^{2} \frac{1}{\Gamma(1-\mu_q)} \int_0^t (t-s)^{-\mu_q} R_q\Big(t,s,u(s),(D_*^{\beta_q}u)(s)\Big) ds, \\ u(0) &= 0, \end{aligned} \tag{6.4}$$

where $\mu_1 = \dfrac{1}{4}$, $\mu_2 = \dfrac{3}{4}$, $\alpha = \dfrac{1}{2}$, $\beta_1 = \dfrac{1}{3}$, $\beta_2 = \dfrac{1}{6}$, and the kernel functions R_1, R_2 and the forcing function f are given, respectively as

$$\begin{aligned} R_1\big(t,s,u,v\big) &= u^2 v, \\ R_2\big(t,s,u,v\big) &= uv^2, \\ f(t,u) &= \frac{\Gamma(\frac{5}{3})}{\Gamma(\frac{7}{6})} t^{\frac{1}{6}} - \frac{4(\Gamma(\frac{5}{3}))^2\Gamma(\frac{8}{3})}{\pi\Gamma(\frac{35}{12})} t^{\frac{7}{12}} u^2 - \frac{\Gamma(\frac{5}{3})\Gamma(\frac{8}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{41}{12})} t^{\frac{5}{12}} u^3. \end{aligned}$$

TABLE 4. The global E_{RMS} errors for a range of increasing values of N, r in Example 6.4.

N	$r=1$	$r=5$	$r=9$	$r=13$	$r=17$	$r=21$
8	2.30e − 04	1.19e − 05	3.76e − 06	9.22e − 06	1.77e − 04	9.59e − 04
16	4.81e − 05	6.95e − 07	9.61e − 08	4.83e − 09	6.16e − 10	1.98e − 10
24	1.43e − 05	1.41e − 07	1.19e − 08	3.11e − 10	2.43e − 11	4.36e − 12
32	1.03e − 05	4.65e − 08	2.32e − 09	4.59e − 11	2.26e − 12	3.06e − 13

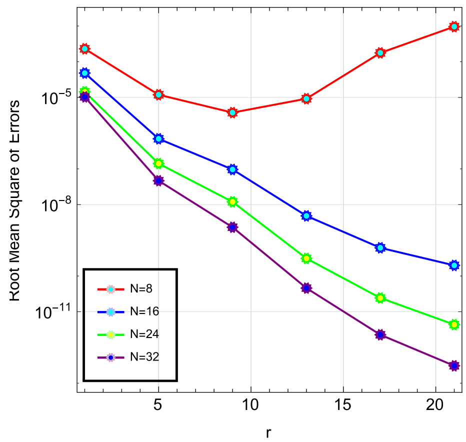

FIGURE 4. Graph of the root mean square of the absolute errors for Example 6.4.

The exact solution of this problem is $u(t) = t^{\frac{2}{3}}$. Similar to the previous examples, we approximate the solution of Eq. (6.4) and report the errors for different values of N, r in Table 4 and Figure 4.

Example 6.5. Consider the following linear multi-term weakly singular integro-differential equation:

$$\begin{aligned} (D_*^\alpha u)(t) &= f(t,u(t)) + \sum_{q=1}^{2} \int_0^t (t-s)^{-\mu_q} R_q\Big(t,s,u(s),(D_*^{\beta_q}u)(s)\Big)ds, \\ u(0) &= 0, \end{aligned} \tag{6.5}$$

with $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{1}{3}$, $\alpha = \frac{1}{2}$, $\beta_2 = \frac{3}{8}$. In this equation, the kernel functions R_1, R_2 and the forcing function f are given, respectively as

$$\begin{aligned} R_1(t,s,u,v) &= -u, \\ R_2(t,s,u,v) &= -v, \\ f(t,u) &= 2t^{\frac{5}{4}} + \frac{2(\Gamma(\frac{3}{4}))^2}{\sqrt{\pi}} t^{\frac{3}{2}} + \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})} t^{\frac{1}{4}} + \frac{2\Gamma(\frac{2}{3})\Gamma(\frac{7}{4})}{\Gamma(\frac{49}{24})} t^{\frac{25}{24}} - t^{\frac{1}{2}} u. \end{aligned}$$

The exact solution of this problem is $u(t) = 2t^{\frac{3}{4}}$. This problem has been solved by Spline collocation method in [18]. The authors computed maximum absolute errors, ε_N as

$$\varepsilon_N = \max_{j=1,\ldots,N} \max_{k=0,\ldots,10} |u(\tau_{jk}) - u_N(\tau_{jk})|,$$

where

$$\tau_{jk} = t_{j-1} + k\,(t_j - t_{j-1})/10,\ k = 0,\ldots,10,\ j = 1,\ldots,N,$$

TABLE 5. The global E_{RMS} errors for a range of increasing values of N, r in Example 6.5.

N	$r = 1$	$r = 5$	$r = 9$	$r = 13$	$r = 17$	$r = 21$
8	4.25e − 04	1.19e − 05	1.21e − 06	2.54e − 07	1.32e − 06	1.69e − 05
16	8.78e − 05	3.73e − 07	1.10e − 08	6.07e − 10	5.22e − 11	6.73e − 12
24	2.62e − 05	5.17e − 08	6.80e − 10	1.50e − 11	6.14e − 13	3.44e − 14
32	7.24e − 06	1.31e − 08	8.76e − 11	1.22e − 12	2.39e − 14	7.97e − 16

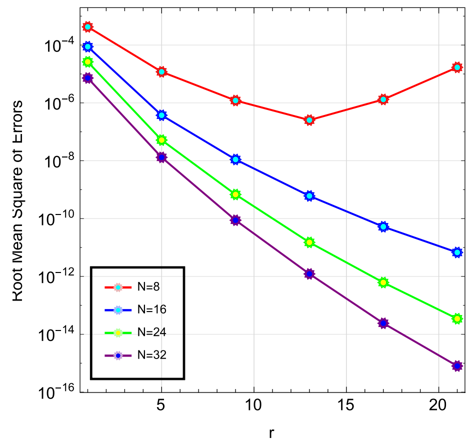

FIGURE 5. Graph of the root mean square of the absolute errors for Example 6.5.

with the grid points t_j defined by $t_j = b\left(\frac{j}{N}\right)^r$, $r \in \mathbb{R}, r \geq 1$, for a range of increasing values of N, r, and $m = 2, 3$ (see Table 6). To compare the errors by using the proposed method with those obtained by the Spline collocation method, we compute the root mean square of the absolute errors for a range of increasing values of N, r by the proposed method and report them in Table 5 and Figure 5. We can see from Tables 6 and 5 that the errors of the proposed method are very small than that of the Spline method.

Example 6.6. As a stochastic case, we examine the following nonlinear weakly singular stochastic volterra integral equations [11]:

$$du(t) = (1-\theta)\int_0^t (t-s)^{-\gamma}\sin\big(u(s)\big)ds + \theta\int_0^t (t-s)^{-\sigma}\cos^2\big(u(s)\big)dW(s), \quad t \in [0,1], \tag{6.6}$$

with the initial data $u(0) = 0$. In the following two cases, the positive arguments γ and σ are considered as:

- Case I: $\gamma = 0.4$, $\sigma = 0.4$,
- Case II: $\gamma = 0.7$, $\sigma = 0.1$.

The numerical results for this example are displayed in Tables 7 and 8, using step sizes $h = 2^{-7}, 2^{-8}, 2^{-9}$ and 2^{-10}. Tables 7 and 8 demonstrate that the proposed method (5.4) exhibits a significant convergence order of approximately 0.2 in both case I and case II.

7. CONCLUSIONS

In this article, we applied Jacobi collocation method to consider the numerical solution of the nonlinear multi-term weakly singular fractional Volterra integro-differential equations. We introduced an opeartor-based discussion for applying smoothing transformation along with Gauss-Jacobi quadrature for the Riemann-Liouville fractional integral operator and used it in the approximation of the integral terms in the main equation. Increasing the smoothing parameter r increases the regularity of the solution and also the regularity of all integrands inside integrals that are

TABLE 6. Numerical results for Example 6.5 in [18].

ε_N with $m=2$				
N	$r=1$	$r=2$	$r=\frac{10}{3}$	$r=5$
8	2.64e − 02	6.70e − 03	2.65e − 03	5.75e − 03
16	1.69e − 02	2.27e − 03	4.05e − 04	1.03e − 03
32	1.04e − 02	8.19e − 04	7.01e − 05	1.81e − 04
64	6.32e − 03	2.85e − 04	1.18e − 05	3.20e − 05
128	3.80e − 03	1.01e − 04	1.90e − 06	4.94e − 06
256	2.27e − 03	3.57e − 05	2.77e − 07	8.02e − 07
512	1.36e − 03	1.26e − 05	6.37e − 08	1.54e − 07
1024	8.07e − 04	4.46e − 06	1.54e − 08	2.57e − 08
ε_N with $m=3$				
N	$r=1$	$r=2$	$r=4$	$r=5$
8	1.04e − 02	2.49e − 03	2.49e − 04	4.51e − 04
16	6.65e − 03	9.01e − 04	4.80e − 05	4.10e − 05
32	4.11e − 03	3.20e − 04	4.64e − 06	3.65e − 06
64	2.50e − 03	1.13e − 04	5.59e − 07	2.92e − 07
128	1.50e − 03	4.00e − 05	7.76e − 08	2.50e − 08
256	8.99e − 04	1.42e − 05	1.02e − 08	2.21e − 09
512	5.37e − 04	5.01e − 06	1.26e − 09	1.82e − 10
1024	3.20e − 04	1.77e − 06	1.51e − 10	1.73e − 11

TABLE 7. Convergence orders and CPU times of the proposed method with $\epsilon = 10^{-12}$ for $u(t)$ in case I.

h	$\theta=0.2$	$\theta=0.5$	$\theta=0.8$	CPU time
2^{-7}	0.4967	0.5247	0.5683	0.64
2^{-8}	0.3893	0.4102	0.4326	2.79
2^{-9}	0.2941	0.3098	0.3259	10.42
2^{-10}	0.2336	0.2468	0.2649	24.81
order	0.2148	0.2206	0.2433	53.26

TABLE 8. Convergence orders and CPU times of the proposed method with $\epsilon = 10^{-12}$ for $u(t)$ in case II.

h	$\theta=0.2$	$\theta=0.5$	$\theta=0.8$	CPU time
2^{-7}	0.4737	0.4952	0.5180	0.58
2^{-8}	0.3692	0.3973	0.4214	2.33
2^{-9}	0.2741	0.2941	0.3143	9.14
2^{-10}	0.2318	0.2422	0.2516	22.56
order	0.2103	0.2158	0.2190	48.83

approximated by the Gauss-Jacobi quadrature. In the error analysis, we showed that the error bound is dependent on N, the number of the node points and also on r, the smoothing parameter. It is observed from the numerical results that for the large values of N, enhancing the value of r reduces the error which is in accordance with the analytic results. But on the other hand, for the small values of N using large values for r increases the error, an opposite result that we expect. In the later case, since the number of node points is small, applying smoothing transformation with a large smoothing parameter r collects node points near the origin and increases the step size on the other side of the interval which reduces the accuracy. This shows that there should be an optimal choice for r with respect to N to reach minimum error. In addition, through our numerical experiments, we have substantiated that the computational efficiency of the proposed scheme, utilizing the SOE approximation for the stochastic case.

STATEMENT

A preprint of this paper has previously been published [12].

References

[1] A. Aghajani, Y. Jalilian, and J. J. Trujillo, *On the existence of solutions of fractional integro-differential equations*, Fractional Calculus and Applied Analysis, *15*, 44–69.

[2] A. Arikoglu and I. Ozkol, *Solution of fractional integro-differential equations by using fractional differential transform method*, Chaos, Solitons & Fractals, *40*(2) (2009), 521–529.

[3] R. L. Bagley and P. J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol. *27* (1983), 201–210.

[4] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus. Models and Numerical Methods*, World Scientific Publishing Co. Pte. Ltd., Singapore, (2012).

[5] G. Ben-Yu and L. L. Wang, *Jacobi interpolation approximations and their applications to singular differential equations*, Advances in Computational Mathematics *14*(3) (2001), 227–276.

[6] J. Biazar and K. Sadri, *Solution of weakly singular fractional integro-differential equations by using a new operational approach*, Journal of Computational and Applied Mathematics, *352* (2019), 453-477.

[7] H. Brunner, *Volterra Integral Equations; An Introduction to Theory and Applications*, Cambridge University Press, (2017).

[8] A. Cardone M. Donatelli, F. Durastante, R. Garrappa, M. Mazza, and M. Popolizio, *Fractional differential equations: modeling, discretization, and numerical solvers*, Springer INdAM Series 50, Springer Nature Singapore Pte Ltd., (2023).

[9] K. Diethelm, *The Analysis of Fractional Differential Equations, an Application Oriented, Exposition Using Differential Operators of Caputo Type*, Lecture Notes in Mathematics, Springer, Berlin, (2010).

[10] S. Das, *Kindergarten of Fractional Calculus*, Cambridge Scholars Publishing, (2020).

[11] O. Farkhondeh Rouz, S. Shahmorad, and D. Ahmadian, *Double weakly singular kernels in stochastic Volterra integral equations with application to the rough Heston model*, Appl. Math. Comput. *475* (2024), 128720.

[12] Q. H. Haddam, E. Najafi, and S. Sohabi, *Numerical solution of nonlinear multi-term weakly singular fractional Volterra integro-differential equations using Jacobi collocation method*, (2024), PREPRINT (Version 1) available at Research Square.

[13] S. Jiang, J. Zhang, Q. Zhang, and Z. Zhang, *Fast evaluation of the Caputo fractional derivative and its applications to fractional diffusion equations*, Commun. Comput. Phys., *21* (2017), 650–678.

[14] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, (2010).

[15] P. Mokhtary, *Numerical analysis of an operational Jacobi Tau method for fractional weakly singular integro-differential equations*, Applied Numerical Mathematics, *121* (2017), 52–67.

[16] E. Najafi, *Nyström-quasilinearization method and smoothing transformation for the numerical solution of nonlinear weakly singular Fredholm integral equations*, Journal of Computational and Applied Mathematics,*368* (2020), 112538.

[17] E. Najafi, *Smoothing transformation for numerical solution of nonlinear weakly singular Volterra integral equations using quasilinearization and product integration methods*, Applied Numerical Mathematics *153* (2020), 540–557.

[18] A. Pedas, E. Tamme, and M. Vikerpuur, *Spline collocation for fractional weakly singular integro-differential equations*, Applied Numerical Mathematics, *110* (2016), 204–214.

[19] A. Pedas, E. Tamme, and M. Vikerpuur, *Numerical solution of linear fractional weakly singular integro-differential equations with integral boundary conditions*, Applied Numerical Mathematics, *149* (2020), 124-140.

[20] A. Pedas and M. Vikerpuur, *Spline collocation for multi-term fractional integro-differential equations with weakly singular kernels*, Fractal and Fractional, *5*(3) (2021), 90.

[21] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Ukraine, Elsevier Science, (1998).

[22] T. Rezazadeh and E. Najafi, *Jacobi collocation method and smoothing transformation for numerical solution of neutral nonlinear weakly singular Fredholm integro-differential equations*, Applied Numerical Mathematics, *181* (2022), 135–150.

[23] E. Shishkina and S. Sitnik, *Transmutations Singular and Fractional Differential Equations with Applications to Mathematical Physics*, London: Academic Press, (2020).

[24] M. Vikerpuur, *Two Collocation Type Methods for Fractional Differential Equations with Non-Local Boundary Conditions*, Mathematical Modelling and Analysis, *22*(5), 654-670.

[25] S. Xiang, *On interpolation approximation: convergence rates for polynomial interpolation for functions of limited regularity*, SIAM Journal on Numerical Analysis, *54*(4) (2016), 2081–2113.

[26] M. Yi, L. Wang, and J. Huang, *Legendre wavelets method for the numerical solution of fractional integro-differential equations with weakly singular kernel*, Applied Mathematical Modelling, *40*(4) (2016), 3422–3437.

[27] J. Zhao, J. Xiao, and N. J Ford, *Collocation methods for fractional integro-differential equations with weakly singular kernels*, Numer. Algor., *65* (2014), 723–743.