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Jacobi collocation method for numerical solution of nonlinear weakly singular Volterra integro-differential equations: fractional and stochastic cases

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Abstract

This paper deals with the numerical solution of a class of nonlinear multi-term weakly singular fractional Volterra integro-differential equations by the Jacobi collocation method based on Jacobi orthogonal polynomials. Since the solution of the proposed equation is not smooth enough at the origin, the idea of a smoothing transformation is used to increase the smoothness of the solution. We represent an operator-based discussion of the smoothing transformation and Gauss-Jacobi quadrature for Riemann-Liouville integral operators and weakly singular integral operators using their similar constructions and extend it to the error analysis of the proposed method and obtain an error bound for the discrete collocation solution. In addition, we propose an improved stochastic method, based on the efficient sum-of-exponentials (SOE) approximation, to address the low computational efficiency of the proposed method. To test the efficiency and accuracy, various numerical examples are solved by the proposed method and the obtained error results are in accordance with the convergence analysis of the method. Finally, we present an example regarding the stochastic Volterra integro-differential equations with one singular kernel function.

Keywords. Fractional calculus, Multi-term Volterra integro-differential equations, Jacobi collocation method, Smoothing transformation, Sum-of-exponentials approximation, Stochastic.

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1. Introduction

In recent decades, much attention has been paid to fractional calculus (integrals and derivatives of non-integer order) and fractional differential equations due to their application in more accurate mathematical modeling of physical and engineering phenomena. Therefore, many researchers have studied the theoretical and numerical analysis of various definitions of fractional derivatives and their related fractional differential equations with initial or boundary conditions [3, 4, 8–10, 14, 21, 23].

There is an increasing interest in fractional integro-differential equations in the literature [1, 2, 24, 27], and some authors have investigated the numerical solution of their linear form with a weakly singular kernel [15, 18–20, 26]. However, there are very few studies for nonlinear equations with a weakly singular kernel in the literature. Our motivation in this paper is to consider the possibility of constructing a high order numerical method for solving the following initial value problem for nonlinear multi-term weakly singular fractional Volterra integro-differential equation:

$$(D_*^{\alpha}u)(t) = f(t,u(t)) + \sum_{q=1}^{Q} \frac{1}{\Gamma(1-\mu_q)} \int_0^t (t-s)^{-\mu_q} R_q\Big(t,s,u(s),(D_*^{\beta_q}u)(s)\Big)ds, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

for $t \in [0,1]$. In this equation, we suppose $0 < \beta_q < \alpha < 1$, $q = 1,...,Q, 1 \le Q \in \mathbb{N}$ and $0 < \mu_1 < \mu_2 < ... < \mu_Q < 1$. Furthermore, we suppose that $u(t) \in A^1[0,1]$ (the space of all absolutely continuous functions) is the unknown solution,

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and the function $f(t,u)$ and the kernels $R_q(t,s,u,v)$, $q=1,...,Q$ are all known and m times ($m \geq 0$) continuously differentiable with respect to all of their components.Thus, the summation in Eq. (1.1) can not be simplified for $m \geq 1$. The operator: $D_*^\gamma : L_1[0,1] \to L_1[0,1]$ for $0<\gamma<1$ denotes the fractional derivative of order γ in the Caputo sense, define by the following formula (see e.g. [9]):

$$(D_*^\gamma u)(t) := \frac{d}{dt}(J^{1-\gamma}[u-u(0)])(t), \tag{1.3}$$

in which we use the Riemann-Liouville fractional integral operator

$$(J^\gamma u)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}u(s)ds,\ t>0,\ J^0 := I, \tag{1.4}$$

with I the identity operator and Γ the Euler Gamma function. It is easy to check that

$$(J^\gamma D_*^\gamma u)(t) = u(t)-u(0),\ (D_*^\gamma J^\gamma u)(t) = u(t). \tag{1.5}$$

It is obvious from Eq. (1.1) that $(D_*^\alpha u)(t) \in C[0,1]$ which implies $(D_*^{\beta_q} u)(t) \in C[0,1], q=1,...,Q$ (see [9], p. 56). Using these results, simple calculations show that the solution of Eqs. (1.1) and (1.2) is not smooth. Indeed, by applying the operator J^α to both sides of Eq. (1.1) and using the mean-value theorem for integrals, we find that

$$\begin{aligned} u(t) &= u_0 + (J^\alpha f(.,u(.)))(t) + \sum_{q=1}^{Q} \frac{1}{\Gamma(1-\mu_q)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\int_0^\tau (\tau-s)^{-\mu_q} R_q\Big(\tau,s,u(s),(D_*^{\beta_q}u)(s)\Big)dsd\tau \\ &= u_0 + (J^\alpha f(.,u(.)))(t) + \sum_{q=1}^{Q} \frac{1}{\Gamma(1-\mu_q)\Gamma(\alpha)} R_q\Big(\xi,\eta,u(\eta),(D_*^{\beta_q}u)(\eta)\Big)\int_0^t\int_0^\tau (t-\tau)^{\alpha-1}(\tau-s)^{-\mu_q}dsd\tau \\ &= u_0 + (J^\alpha f(.,u(.)))(t) + \sum_{q=1}^{Q} \frac{1}{\Gamma(\alpha)} R_q\Big(\xi,\eta,u(\eta),(D_*^{\beta_q}u)(\eta)\Big)\frac{t^{1+\alpha-\mu_q}}{\Gamma(2+\alpha-\mu_q)}, \end{aligned}$$

where $0 \leq \xi,\eta \leq t$ are some constants. From this result, we conclude that $(D_*^\alpha u)(t) \notin C^1[0,1]$ and $u(t) \notin C^1[0,1]$ if $\alpha < \mu_M$. Also, when $t \to 0^+$, we have

$$(D_*^\alpha u)(t) \sim t^{1-\mu_Q}, \qquad u(t) \sim t^{1+\alpha-\mu_Q}.$$

Then employing classical numerical methods to solve Eqs. (1.1) and (1.2) without applying any smoothing transformation will yield low-order accuracy, as observed in works like [6]. This transformation is well-known and is employed to increase the smoothness of non-smooth functions [16, 17, 22]. The smoothing transformation that we apply on Eq. (1.1) is in the form of $t=x^r, s=y^r, r\in\mathbb{N}$ and this transformation will increase the smoothness of the solution $u(t)$ near $t=0$. Finally, we generalize the Volterra integral equations to the stochastic case with one singular kernel function and provide the error analysis, which is tabulated in Tables 5 and 6.

The remainder of this paper is organized as follows. In section 2, we reformulate the operator $(J^\alpha v)(t)$ using a variable transformation and apply Gauss-Jacobi quadrature. Section 3 extends these results to the main Equation (1.1), employing a smoothing transformation and the same quadrature approach. In section 4, we present the error analysis and discuss the convergence order of the proposed numerical method. Section 5 is dedicated to the efficient implementation of the method using the sum-of-exponentials (SOE) approximation, aiming to reduce both computational cost and memory usage. Finally, Section 6 provides numerical experiments to demonstrate the applicability and efficiency of the proposed method.

2. Gauss-Jacobi quadrature for Riemann-Liouville integral operator

Let $v(t) \in C[0,1]$ and $r \in \mathbb{N}$. By applying the change of variables

$$t=x^r,\ s=y^r,\ 0\leq y\leq x\leq 1, \tag{2.1}$$

and defining $w(x) := v(x^r)$, the Riemann-Liouville integral operator takes the form:

$$\begin{aligned}(J^{\alpha}v)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds\\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (x^r-y^r)^{\alpha-1}w(y)ry^{r-1}dy = (J^{\alpha}_{x,[x^r,1]}w)(x) := (J^{\alpha}_r w)(x). \end{aligned}\tag{2.2}$$

Here, $J^{\alpha}_{x,[x^r,1]}$ is the modified Erdelyi-Kober integral operator [10] with base function x^r and weight function 1, which we denote simply by J^{α}_r. To approximate $(J^{\alpha}_r w)(x)$ using Gauss-Jacobi quadrature, we define the kernel

$$\omega_{1-\alpha,r}(x,y) = \frac{ry^{r-1}}{\Gamma(\alpha)}\begin{cases}(\dfrac{x^r-y^r}{x-y})^{\alpha-1}, & x\neq y,\\ (rx^{r-1})^{\alpha-1}, & x=y,\end{cases}\quad 0\le y\le x\le 1,\tag{2.3}$$

and aim to map the integration interval in (2.2) to $[-1,1]$ using the transformations:

$$x=\frac{\xi+1}{2},\ y=\frac{\eta+1}{2},\ -1\le \xi,\eta\le 1,\tag{2.4}$$

$$\eta = \frac{\xi+1}{2}\theta + \frac{\xi-1}{2} := \eta_{\xi}(\theta),\ -1\le\theta\le 1.\tag{2.5}$$

Applying the first change of variables (2.4) to Eq. (2.2) yields:

$$(J^{\alpha}_r w)(\frac{\xi+1}{2}) = \frac{1}{2^{\alpha}}\int_{-1}^{\xi}(\xi-\eta)^{\alpha-1}\omega_{1-\alpha,r}(\frac{\xi+1}{2},\frac{\eta+1}{2})w(\frac{\eta+1}{2})d\eta.\tag{2.6}$$

Define $z(\xi) := w(\dfrac{\xi+1}{2})$. Substituting this into (2.6) and applying the second transformation (2.5) gives the final integral:

$$\begin{aligned}(\mathcal{J}^{\alpha}_r z)(\xi) :&= (J^{\alpha}_r w)(\frac{\xi+1}{2})\\ &= (\frac{\xi+1}{2})^{\alpha}\frac{1}{2^{\alpha}}\int_{-1}^{1}(1-\theta)^{\alpha-1}\omega_{1-\alpha,r}(\frac{\xi+1}{2},\frac{\eta_{\xi}(\theta)+1}{2})z(\eta_{\xi}(\theta))d\theta\\ &= \int_{-1}^{1}(1-\theta)^{\alpha-1}\bar{\omega}_{1-\alpha,r}(\xi,\eta_{\xi}(\theta))z(\eta_{\xi}(\theta))d\theta,\end{aligned}\tag{2.7}$$

where

$$\bar{\omega}_{1-\alpha,r}(\xi,\eta_{\xi}(\theta)) = (\frac{\xi+1}{4})^{\alpha}\omega_{1-\alpha,r}(\frac{\xi+1}{2},\frac{\eta_{\xi}(\theta)+1}{2}).\tag{2.8}$$

Note that for any fixed $\xi > -1$, the function $\bar{\omega}_{1-\alpha,r}(\xi,\eta_{\xi}(\theta))$ is smooth with respect to $\theta\in[-1,1]$. Now consider the weight function $\mathrm{w}^{a,b}(\theta) = (1-\theta)^a(1+\theta)^b$, with $a,b>-1$, associated with the orthogonal Jacobi polynomials $\{p^{a,b}_N(\theta)\}_{N=0}^{\infty}$ on the interval $[-1,1]$. We approximate the integral in (2.7) using Gauss-Jacobi quadrature as:

$$(\mathcal{J}^{\alpha}_r z)(\xi) \simeq \sum_{k=0}^{N}\bar{\omega}_{1-\alpha,r}(\xi,\eta_{\xi}(\bar{\theta}_k))z(\eta_{\xi}(\bar{\theta}_k))\bar{\mathrm{w}}_k := (\bar{\mathcal{J}}^{\alpha,N}_r z)(\xi),\ -1\le\xi\le 1,\tag{2.9}$$

where $\bar{\theta}_k$ and $\bar{\mathrm{w}}_k$, for $k=0,...,N$, are the quadrature nodes and weights associated with the weight function $\mathrm{w}^{\alpha-1,0}(\theta)$.

3. Jacobi collocation method with smoothing transformation

Consider integro-differential Equation (1.1) and define $v(t) := (D^{\alpha}_* u)(t)$. Then the solution $u(t)$ can be expressed as $u(t) = u_0 + (J^{\alpha}v)(t)$. Substituting this into Eq. (1.1) yields the integral equation

$$v(t) = f\big(t, u_0 + (J^{\alpha}v)(t)\big) + \sum_{q=1}^{Q}\frac{1}{\Gamma(1-\mu_q)}\int_0^t (t-s)^{-\mu_q}R_q\Big(t,s,u_0+(J^{\alpha}v)(s),(J^{\alpha-\beta_q}v)(s)\Big)ds.\tag{3.1}$$

After approximating the Riemann-Liouville integral operator via Gauss-Jacobi quadrature in section 2, here we apply the same numerical scheme to the integral terms of (3.1) and couple it with a collocation method. Throughout the computations, the operations performed on the terms $(J^{\alpha}v)(s)$ and $(J^{\alpha-\beta_q}v)(s)$ will follow the same pattern and notation introduced in section 2; hence, we omit the detailed repetition.

Since the solution $v(t)$ of Eq. (3.1) behaves like $t^{1-\mu_Q}$ near $t \to 0^+$, a direct numerical treatment may lack accuracy. To improve convergence and accuracy, we apply the smoothing change of variables (2.1) $t = x^r$, which transforms (3.1) into

$$w(x) = g\big(x, u_0 + (J_r^{\alpha}w)(x)\big) + \sum_{q=1}^{Q}\int_0^x (x-y)^{-\mu_q}\omega_{\mu_q,r}(x,y)R_{q,r}\Big(x, y, u_0 + (J_r^{\alpha}w)(y), (J_r^{\alpha-\beta_q}w)(y)\Big)dy, \tag{3.2}$$

where the functions are defined using the notation from Eqs. (2.2) and (2.3) as

$$\begin{aligned}
&g\big(x, u_0 + (J_r^{\alpha}w)(x)\big) = f\big(x^r, u_0 + (J^{\alpha}v)(x^r)\big),\\
&R_{q,r}\Big(x, y, u_0 + (J_r^{\alpha}w)(y), (J_r^{\alpha-\beta_q}w)(y)\Big) = R_q\Big(x^r, y^r, u_0 + (J^{\alpha}v)(y^r), (J^{\alpha-\beta_q}v)(y^r)\Big).
\end{aligned} \tag{3.3}$$

By applying the linear transformation (2.4), we can rewrite (3.2) as

$$\begin{aligned}
z(\xi) &= g\big(\frac{\xi+1}{2}, u_0 + (\mathcal{J}_r^{\alpha}z)(\xi)\big)\\
&\quad + \sum_{q=1}^{Q}\frac{1}{2^{1-\mu_q}}\int_{-1}^{\xi}(\xi-\eta)^{-\mu_q}\omega_{\mu_q,r}\big(\frac{\xi+1}{2}, \frac{\eta+1}{2}\big)\\
&\quad \times R_{q,r}\Big(\frac{\xi+1}{2}, \frac{\eta+1}{2}, u_0 + (\mathcal{J}_r^{\alpha}z)(\eta), (\mathcal{J}_r^{\alpha-\beta_q}z)(\eta)\Big)d\eta.
\end{aligned} \tag{3.4}$$

Next, applying the second change of variables (2.5), we transform Eq. (3.4)to the fixed interval $[-1,1]$, resulting in

$$\begin{aligned}
z(\xi) &= g\big(\frac{\xi+1}{2}, u_0 + (\mathcal{J}_r^{\alpha}z)(\xi)\big)\\
&\quad + \sum_{q=1}^{Q}\int_{-1}^{1}(1-\theta)^{-\mu_q}\bar{\omega}_{\mu_q,r}(\xi, \eta_\xi(\theta))\\
&\quad \times K_{q,r}\Big(\xi, \eta_\xi(\theta), u_0 + (\mathcal{J}_r^{\alpha}z)(\eta_\xi(\theta)), (\mathcal{J}_r^{\alpha-\beta_q}z)(\eta_\xi(\theta))\Big)d\theta\\
&:= g\big(\frac{\xi+1}{2}, u_0 + (\mathcal{J}_r^{\alpha}z)(\xi)\big) + \sum_{q=1}^{Q}\big(\mathcal{J}_r^{1-\mu_q}\mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha}z, \mathcal{J}_r^{\alpha-\beta_q}z)\big)(\xi),
\end{aligned} \tag{3.5}$$

where

$$K_{q,r}\Big(\xi, \eta_\xi(\theta), u_0 + (\mathcal{J}_r^{\alpha}z)(\eta_\xi(\theta)), (\mathcal{J}_r^{\alpha-\beta_q}z)(\eta_\xi(\theta))\Big) = R_{q,r}\Big(\frac{\xi+1}{2}, \frac{\eta_\xi(\theta)+1}{2}, u_0 + (\mathcal{J}_r^{\alpha}z)(\eta_\xi(\theta)), (\mathcal{J}_r^{\alpha-\beta_q}z)(\eta_\xi(\theta))\Big). \tag{3.6}$$

Let $\mathbb{P}_N$ be the space of all polynomials of degree at most N. Suppose $\{\xi_i\}_{i=0}^N$ are the roots of the Jacobi polynomial $p_{N+1}^{a,b}(\xi)$ of degree $N+1$, where $a, b > 1$. We construct the Lagrange fundamental polynomials $\{L_i(\xi)\}_{i=0}^N$ corresponding to nodes $\{\xi_i\}_{i=0}^N$ as a basis of $\mathbb{P}_N$. A collocation solution $z_N(\xi) \in \mathbb{P}_N$ of Eq. (4.2) is sought in the form $z_N(\xi) = \sum_{j=0}^N z_j L_j(\xi)$, such that it satisfies the collocation conditions

$$\begin{aligned}
z_N(\xi_i) &= g\big(\frac{\xi_i+1}{2}, u_0 + (\mathcal{J}_r^{\alpha}z_N)(\xi_i)\big)\\
&\quad + \sum_{q=1}^{Q}\int_{-1}^{1}(1-\theta)^{-\mu_q}\bar{\omega}_{\mu_q,r}(\xi_i, \eta_{\xi_i}(\theta))\\
&\quad \times K_{q,r}\Big(\xi_i, \eta_{\xi_i}(\theta), u_0 + (\mathcal{J}_r^{\alpha}z_N)(\eta_{\xi_i}(\theta)), (\mathcal{J}_r^{\alpha-\beta_q}z_N)(\eta_{\xi_i}(\theta))\Big)d\theta,
\end{aligned} \tag{3.7}$$

for $i=0,...,N$. We seek a discrete collocation solution $\bar{z}_N(\xi)\in\mathbb{P}_N$, where $\bar{z}_N(\xi)=\sum_{j=0}^{N}\bar{z}_jL_j(\xi)$. The integral terms in Eq. (3.7) are then approximated using the Gauss-Jacobi quadrature in two steps as follows:

$$\begin{aligned}
&\int_{-1}^{1}(1-\theta)^{-\mu_q}\bar{\omega}_{\mu_q,r}(\xi,\eta_\xi(\theta))K_{q,r}\Big(\xi,\eta,u_0+(\mathcal{J}_r^{\alpha}z)(\eta_\xi(\theta)),(\mathcal{J}_r^{\alpha-\beta_q}z)(\eta_\xi(\theta))\Big)d\theta\\
&\simeq\sum_{k=0}^{N}\bar{\omega}_{\mu_q,r}(\xi,\eta_\xi(\tilde{\theta}_k^q))K_{q,r}\Big(\xi,\eta_\xi(\tilde{\theta}_k^q),u_0+(\mathcal{J}_r^{\alpha}z)(\eta_\xi(\tilde{\theta}_k^q)),(\mathcal{J}_r^{\alpha-\beta_q}z)(\eta_\xi(\tilde{\theta}_k^q))\Big)\tilde{\mathrm{w}}_k^q\\
&\simeq\sum_{k=0}^{N}\bar{\omega}_{\mu_q,r}(\xi,\eta_\xi(\tilde{\theta}_k^q))K_{q,r}\Big(\xi,\eta_\xi(\tilde{\theta}_k^q),u_0+(\bar{\mathcal{J}}_r^{\alpha,N}z)(\eta_\xi(\tilde{\theta}_k^q)),(\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}z)(\eta_\xi(\tilde{\theta}_k^q))\Big)\tilde{\mathrm{w}}_k^q\\
&:=\big(\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}z,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}z)\big)(\xi),\qquad q=1,...,Q,
\end{aligned}\tag{3.8}$$

where $\tilde{\theta}_k^q$ and $\tilde{\mathrm{w}}_k^q$ denotes the nodes and weights associated with the weight function $\mathrm{w}^{-\mu_q,0}(\theta)$. Now, the discrete collocation solution $\bar{z}_N(\xi)$ is determined by satisfying the following discrete collocation equation:

$$\begin{aligned}
\bar{z}_N(\xi_i)&=g\big(\frac{\xi_i+1}{2},u_0+(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N)(\xi_i)\big)\\
&\quad+\sum_{q=1}^{Q}\sum_{k=0}^{N}K_{q,r}\Big(\xi_i,\eta_{\xi_i}(\tilde{\theta}_k^q)),u_0+(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N)(\eta_{\xi_i}(\tilde{\theta}_k^q)),(\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)(\eta_{\xi_i}(\tilde{\theta}_k^q))\Big)\tilde{\mathrm{w}}_k^q\\
&\quad=g\big(\frac{\xi_i+1}{2},u_0+(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N)(\xi_i)\big)+\sum_{q=1}^{Q}\big(\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)\big)(\xi_i),
\end{aligned}\tag{3.9}$$

for $i=0,...,N$. Substituting the representation $\bar{z}_N(\xi)=\sum_{j=0}^{N}\bar{z}_jL_j(\xi)$, yields

$$\begin{aligned}
\bar{z}_i&=g\big(\frac{\xi_i+1}{2},u_0+(\sum_{j=0}^{N}\bar{z}_j\bar{\mathcal{J}}_r^{\alpha,N}L_j)(\xi_i)\big)\\
&+\sum_{q=1}^{Q}\sum_{k=0}^{N}K_{q,r}\Big(\xi_i,\eta_{\xi_i}(\tilde{\theta}_k^q)),u_0+(\sum_{j=0}^{N}\bar{z}_j\bar{\mathcal{J}}_r^{\alpha,N}L_j)(\eta_{\xi_i}(\tilde{\theta}_k^q)),(\sum_{j=0}^{N}\bar{z}_j\bar{\mathcal{J}}_r^{\alpha-\beta_m,N}L_j)(\eta_{\xi_i}(\tilde{\theta}_k^q))\Big)\tilde{\mathrm{w}}_k^q\\
&=g\big(\frac{\xi_i+1}{2},u_0+(\sum_{j=0}^{N}\bar{z}_j\bar{\mathcal{J}}_r^{\alpha,N}L_j)(\xi_i)\big)\\
&+\sum_{q=1}^{Q}\big(\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\sum_{j=0}^{N}\bar{z}_j\bar{\mathcal{J}}_r^{\alpha,N}L_j,\sum_{j=0}^{N}\bar{z}_j\bar{\mathcal{J}}_r^{\alpha-\beta_m,N}L_j)\big)(\xi_i).
\end{aligned}\tag{3.10}$$

for $i=0,...,N$. Equation (3.10) represents a nonlinear system with unknowns $\bar{z}_i, i=0,...,N$. Solving this system yields the discrete collocation solution $\bar{z}_N(\xi)=\sum_{j=0}^{N}\bar{z}_jL_j(\xi)$. Based on the variable transformation, the relationship between the solution $v(t)$ of Eq. (3.1), the solution $w(x)$ of Eq. (3.2), and the solution $z(\xi)$ of Eq. (4.2) is given by

$$v(t)=w(\sqrt[r]{t})=z(2\sqrt[r]{t}-1),0\le t\le 1.$$

Accordingly, the numerical solution corresponding to each of these formulations is

$$\bar{v}_N(t)=\bar{w}_N(\sqrt[r]{t})=\bar{z}_N(2\sqrt[r]{t}-1).$$

Furthermore, the numerical solution $\bar{u}_N(t)$ of Eqs. (1.1) and (1.2) is obtained as:

$$\begin{aligned}
\bar{u}_N(t)&=u_0+(J^{\alpha}\bar{v}_N)(t)\\
&=u_0+(J_r^{\alpha}\bar{w}_N)(x)\\
&=u_0+(\mathcal{J}_r^{\alpha}\bar{z}_N)(\xi)
\end{aligned}$$

$$
\begin{aligned}
&\simeq u_0 + (\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N)(\xi) \\
&= u_0 + (\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N)(2\sqrt[r]{t}-1),\ 0 \le t \le 1.
\end{aligned} \tag{3.11}
$$

4. Convergence analysis

Assume that $\mathcal{P}_N^{a,b}$ is the interpolation operator that is defined on the nodes $\{\xi_i\}_{i=0}^N$, which are the roots of the Jacobi polynomial $p_{N+1}^{a,b}(\xi)$, and maps $C[-1,1]$ to $\mathbb{P}_N$. To analyze the convergence of the proposed method, we first present the following auxiliary lemmas.

Lemma 4.1. *[5] Suppose that $\{\xi_i\}_{i=0}^N$ are the roots of the orthogonal Jacobi polynomial of degree $N+1$, and let $L_i(\xi), i = 0,1,\ldots,N$, denote the Lagrange fundamental polynomials with respect to these nodes. Then*

$$
\|\mathcal{P}_N^{a,b}\|_\infty = \max_{-1\le\xi\le1}\sum_{i=0}^N |L_i(\xi)| = \begin{cases} \mathcal{O}(\log N), & -1 < a,b \le -\frac{1}{2}, \\ \mathcal{O}\big(N^{\gamma+\frac{1}{2}}\big), & -\frac{1}{2} < a,b < 1, \end{cases}
$$

where $\gamma = \max\{a,b\}$.

Lemma 4.2. *[25] Let $f(\xi) \in A^d[-1,1]$, the space of functions whose $(d-1)$-th derivative is absolutely continuous and satisfies*

$$
f^{(d-1)}(\xi) = f^{(d-1)}(-1) + \int_{-1}^{\xi} g(\eta)d\eta,
$$

where $g(\eta) \in L^1(-1,1)$ and is of bounded variation, i.e., $var(g(\eta)) < \infty$ on $[-1,1]$. Assume that $\{\xi_i\}_{i=0}^N$ are the roots of the Jacobi polynomial of degree $N+1$. Then for $N \ge d+1$

$$
\|f - \mathcal{P}_N^{a,b} f\|_\infty \le C N^{-d+\max\left\{0,\gamma-\frac{1}{2}\right\}}, \quad \gamma = \max\{a,b\}, \tag{4.1}
$$

where C is a constant independent of N.

Theorem 4.3. *Suppose that $u(t)$ is the exact solution of Eqs. (1.1) and (1.2), and $\bar{u}_N(t)$ defined in Eq. (3.11) is the numerical solution obtained by the method introduced in sections 2 and 3. Assume that $g(x,u)$ and $\mathcal{K}_{q,r}(u,v)$ are Fréchet differentiable with respect u and (u,v), respectively, and the homogeneous linear fractional integral equation*

$$
\phi(\xi) = \frac{\partial}{\partial u} g\Big(\frac{\xi+1}{2}, z_0\Big)(\mathcal{J}_r^{\alpha}\phi)(\xi) + \sum_{q=1}^{Q} \mathcal{J}_r^{1-\mu_q}\Big(\frac{\partial}{\partial u}\mathcal{K}_{q,r}(z_q,\zeta_q)\mathcal{J}_r^{\alpha} + \frac{\partial}{\partial v}\mathcal{K}_{q,r}(z_q,\zeta_q)\mathcal{J}_r^{\alpha-\beta_q}\Big)\phi(\xi), \tag{4.2}
$$

has only the trivial solution $\phi(\xi) = 0$ for all $z_0, z_q, \zeta_q \in C[-1,1]$. Then

$$
\|u - \bar{u}_N\|_\infty \le C' \begin{cases} N^{-d} + N^{-m}\log N, & -1 < a,b \le -\frac{1}{2}, \\ N^{-d+\max\{0,\gamma-\frac{1}{2}\}} + N^{-m+\gamma+\frac{1}{2}}, & -\frac{1}{2} < a,b \le 1. \end{cases} \tag{4.3}
$$

where $d = [r(1-\mu_Q)] + 1$.

Proof. Since $v(t) \sim t^{1-\mu_Q}$ as $t \to 0^+$, it follows that $z(\xi) \sim (\xi+1)^{r(1-\mu_Q)}$ as $\xi \to -1^+$ and $z(\xi)$ satisfies the assumptions of Lemma 4.2 with $d = [r(1-\mu_Q)] + 1$ and $g(\xi) = z^d(\xi) \in L^1(-1,1)$. Therefore, Lemma 4.2 yields

$$
\|z - \mathcal{P}_N^{a,b} z\|_\infty \le C_0 N^{-d+\max\left\{0,\gamma-\frac{1}{2}\right\}}, \quad \gamma = \max\{a,b\}, \tag{4.4}
$$

where C_0 is a constant independent of N. Multiplying both sides of Eq. (3.10) by $L_i(\xi)$ and summing over i, we obtain

$$
\bar{z}_N = \mathcal{P}_N^{a,b} g\big(x, u_0 + \bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\big) + \sum_{q=1}^{Q} \mathcal{P}_N^{a,b}\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N). \tag{4.5}
$$

Similarly, Eq. (4.2) can be rewritten as

$$z = g\big(x, u_0 + \mathcal{J}_r^{\alpha} z\big) + \sum_{q=1}^{Q} \mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z). \tag{4.6}$$

Applying the operator $\mathcal{P}_N^{a,b}$ to both sides and adding z, we get

$$z = z - \mathcal{P}_N^{a,b} z + \mathcal{P}_N^{a,b} g\big(x, u_0 + \mathcal{J}_r^{\alpha} z\big) + \sum_{q=1}^{Q} \mathcal{P}_N^{a,b} \mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z). \tag{4.7}$$

Subtracting both sides of Eq. (4.5) from the above, we arrive at

$$\begin{aligned} z - \bar{z}_N &= z - \mathcal{P}_N^{a,b} z + \mathcal{P}_N^{a,b} \Big(g\big(x, u_0 + \mathcal{J}_r^{\alpha} z\big) - g\big(x, u_0 + \bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N\big)\Big) \\ &\quad + \sum_{q=1}^{Q} \mathcal{P}_N^{a,b} \Big(\mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z) - \bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N)\Big). \end{aligned} \tag{4.8}$$

Since $g(x,u)$ is Fréchet differentiable with respect to u, the third term in the right hand side of Eq. (4.8) can be written as

$$\begin{aligned} \mathcal{P}_N^{a,b} \Big(g\big(x, u_0 + (\mathcal{J}_r^{\alpha} z)\big) - g\big(x, u_0 + (\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N)\big)\Big) &= \mathcal{P}_N^{a,b} \left(\frac{\partial g}{\partial u}(x, z_0) \right) \left(\mathcal{J}_r^{\alpha} z - \bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N \right) \\ = \mathcal{P}_N^{a,b} \left(\frac{\partial g}{\partial u}(x, z_0) \right) \left(\mathcal{J}_r^{\alpha} (z - \bar{z}_N) \right) &+ \mathcal{P}_N^{a,b} \left(\frac{\partial g}{\partial u}(x, z_0) \right) \left(\mathcal{J}_r^{\alpha} \bar{z}_N - \bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N \right), \end{aligned} \tag{4.9}$$

where $z_0 \in C[-1,1]$.

For the fourth term on the right-hand side of Eq. (4.8), we write

$$\begin{aligned} &\mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z) - \bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) \\ &\qquad = \mathcal{J}_r^{1-\mu_q} \left(\mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z) - \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) \right) \\ &\qquad + \mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) - \bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N), \end{aligned} \tag{4.10}$$

and since the operator $\mathcal{K}_{q,r}(u,v)$ is Fréchet differentiable with respect to u and v, then there exist $z_q, \zeta_q \in C[-1,1]$ such that

$$\begin{aligned} \mathcal{K}_{q,r}(\mathcal{J}_r^{\alpha} z, \mathcal{J}_r^{\alpha-\beta_q} z) - \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) &= \Big(\frac{\partial}{\partial u} \mathcal{K}_{q,r}\left(z_q, \zeta_q\right) \mathcal{J}_r^{\alpha} + \frac{\partial}{\partial v} \mathcal{K}_{q,r}\left(z_q, \zeta_q\right) \mathcal{J}_r^{\alpha-\beta_q} \Big)(z - \bar{z}_N) \\ &:= \mathcal{K}'_{q,r}(z_q, \zeta_q)(z - \bar{z}_N). \end{aligned} \tag{4.11}$$

Substituting Eqs. (4.9), (4.10), and (4.11) into Eq. (4.8) yields

$$\begin{aligned} \Big(\mathcal{I} - \mathcal{P}_N^{a,b} \big(\frac{\partial g}{\partial u}(x, z_0) \mathcal{J}_r^{\alpha} + \sum_{q=1}^{Q} \mathcal{J}_r^{1-\mu_q} \mathcal{K}'_{q,r}(z_q, \zeta_q) \big) \Big)(z - \bar{z}_N) &= (z - \mathcal{P}_N^{a,b} z) + \mathcal{P}_N^{a,b} \left(\frac{\partial g}{\partial u}(x, z_0) \left(\mathcal{J}_r^{\alpha} \bar{z}_N - \bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N \right) \right) \\ &+ \sum_{q=1}^{Q} \mathcal{P}_N^{a,b} \Big(\mathcal{J}_r^{1-\mu_q} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) \\ &- \bar{\mathcal{J}}_r^{1-\mu_q,N} \mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N} \bar{z}_N, \bar{\mathcal{J}}_r^{\alpha-\beta_q,N} \bar{z}_N) \Big). \end{aligned} \tag{4.12}$$

Since Eq. (4.2) has only the zero solution, the integral operator

$$\mathcal{I} - \big(\frac{\partial g}{\partial u}(x, z_0) \mathcal{J}_r^{\alpha} + \sum_{q=1}^{Q} \mathcal{J}_r^{1-\mu_q} \mathcal{K}'_{q,r}(z_q, \zeta_q) \big), \tag{4.13}$$

has a bounded inverse for all $z_0, z_q, \zeta_q \in C[-1,1]$. Moreover, by Lemma 4.2 we have

$$\|(\mathcal{I}-\mathcal{P}_N^{a,b})\big(\frac{\partial g}{\partial u}(x,z_0)\mathcal{J}_r^{\alpha}+\sum_{q=1}^{Q}\mathcal{J}_r^{1-\mu_q}\mathcal{K}'_{q,r}(z_q,\zeta_q)\big)\|_\infty \to 0 \text{ as } N\to\infty, \tag{4.14}$$

which implies that for sufficiently large N, the operator

$$\mathcal{I}-\mathcal{P}_N^{a,b}\big(\frac{\partial g}{\partial u}(x,z_0)\mathcal{J}_r^{\alpha}+\sum_{q=1}^{Q}\mathcal{J}_r^{1-\mu_q}\mathcal{K}'_{q,r}(z_q,\zeta_q)\big), \tag{4.15}$$

is also invertible with bounded inverse on $C[-1,1]$ as

$$\|\Big(\mathcal{I}-\mathcal{P}_N^{a,b}\big(\frac{\partial g}{\partial u}(x,z_0)\mathcal{J}_r^{\alpha}+\sum_{q=1}^{Q}\mathcal{J}_r^{1-\mu_q}\mathcal{K}'_{q,r}(z_q,\zeta_q)\big)\Big)^{-1}\|_\infty \le C_1. \tag{4.16}$$

Therefore, from Eq. (4.12), we obtain

$$\begin{aligned}\|z-\bar{z}_N\|_\infty \le C_1\Big(&\|z-\mathcal{P}_N^{a,b}z\|_\infty+\|\mathcal{P}_N^{a,b}\|_\infty\|\frac{\partial g}{\partial u}(x,z_0)\|_\infty\|\mathcal{J}_r^{\alpha}\bar{z}_N-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\|_\infty\\
&+\sum_{q=1}^{Q}\|\mathcal{P}_N^{a,b}\|_\infty\|\mathcal{J}_r^{1-\mu_q}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)\\
&-\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)\|_\infty\Big).\end{aligned} \tag{4.17}$$

Since $\bar{z}_N \in \mathbb{P}_N$, we know

$$\begin{aligned}&\left\|\mathcal{J}_r^{\alpha}\bar{z}_N-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\right\|_\infty \le C_2N^{-m},\\
&\|\mathcal{J}_r^{1-\mu_q}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)-\bar{\mathcal{J}}_r^{1-\mu_q,N}\mathcal{K}_{q,r}(\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N,\bar{\mathcal{J}}_r^{\alpha-\beta_q,N}\bar{z}_N)\|_\infty \le C_3N^{-m}, \quad q=1,2,\dots,Q.\end{aligned} \tag{4.18}$$

Finally, applying Lemma 4.1, Eq. (4.4), and bound (4.18) in Eq. (4.17), we obtain the following error estimate:

$$\|z-\bar{z}_N\|_\infty \le C\begin{cases}N^{-d}+N^{-m}\log N, & -1<a,b\le -\frac{1}{2},\\ N^{-d+\max\{0,\gamma-\frac{1}{2}\}}+N^{-m+\gamma+\frac{1}{2}}, & -\frac{1}{2}<a,b\le 1,\end{cases} \tag{4.19}$$

where C is a constant depending on C_0, C_1, C_2, and C_3 but independent of N, m and d. Using Eqs. (4.1), (4.18), and the error bound in (4.19), we obtain the following estimate for the numerical approximation of the solutions to Eqs. (1.1) and (1.2):

$$\begin{aligned}\|u-\bar{u}_N\|_\infty &= \left\|\mathcal{J}_r^{\alpha}z-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\right\|_\infty,\\
&\le \|\mathcal{J}_r^{\alpha}\|_\infty\|z-\bar{z}_N\|_\infty+\left\|\mathcal{J}_r^{\alpha}\bar{z}_N-\bar{\mathcal{J}}_r^{\alpha,N}\bar{z}_N\right\|_\infty\\
&\le C'\begin{cases}N^{-d}+N^{-m}\log N, & -1<a,b\le \frac{-1}{2},\\ N^{-d+\max\{0,\gamma-\frac{1}{2}\}}+N^{-m+\gamma+\frac{1}{2}}, & \frac{-1}{2}<a,b\le 1.\end{cases}\end{aligned}$$

□

5. The SOE approximation for the stochastic case

In this section, we address the challenges arising from the low-order accuracy of the proposed method. To improve computational efficiency, we employ an alternative approach known as the SOE approximation (see [[13], Theorem 2.1] or [11]).

Lemma 5.1. *For every $\gamma \in (0,1)$, there exist positive quadrature nodes τ_l and weights ω_l, $l = 1, 2, \cdots, M_{\exp}$, together with an absolute error tolerance $\epsilon \ll 1$ and a cut-off time point δ, such that*

$$|t^{-\gamma} - \sum_{l=1}^{M_{\exp}} \omega_l e^{-\tau_l t}| \leq \epsilon, \quad \forall t \in [\delta, T], \tag{5.1}$$

where $M_{\exp}$ satisfies

$$M_{\exp} = \mathcal{O}\Big(\log \frac{1}{\epsilon} \big(\log\log \frac{1}{\epsilon} + \log \frac{T}{\delta} \big) + \log \frac{1}{\delta} \big(\log\log \frac{1}{\epsilon} + \log \frac{1}{\delta} \big) \Big). \tag{5.2}$$

In other words, by setting the cut-off time as $\delta = h$ for a fixed precision ϵ, we obtain

$$M_{\exp} = \begin{cases} \mathcal{O}(\log N), & \text{if} \quad T \gg 1, \\ \mathcal{O}(\log^2 N), & \text{if} \quad T \approx 1, \end{cases}$$

where $h = \dfrac{T}{N}$.

By using the SOE approximation (5.1), the proposed method can be reformulated as

$$\bar{u}_N = u_0 + h \sum_{k=0}^{n-1} \sum_{l=1}^{M_{\exp,1}} w_{l,1} e^{-\tau_{l,1}(t_n - t_k)} f(\bar{u}_k) + \sum_{k=0}^{n-1} \sum_{l=1}^{M_{\exp,2}} w_{l,2} e^{-\tau_{l,2}(t_n - t_k)} g(\bar{u}_k) \Delta W_k, \tag{5.3}$$

where $n = 1, 2, \ldots, N$, and $\Delta W_k := W_{t_{k+1}} - W_{t_k}$ denotes the Wiener process increment. The singular kernels $(t_n - t_k)^{-\gamma_1}$ and $(t_n - t_k)^{-\gamma_2}$ are replaced by the exponential sums $\sum_{l=1}^{M_{\exp,j}} w_{l,j} e^{-\tau_{l,j}(t_n - t_k)}$, $j = 1, 2$, respectively. Furthermore, Eq. (5.3) can be compactly expressed as

$$\bar{u}_N = u_0 + \sum_{l=1}^{M_{\exp,1}} w_{l,1} P_{1,l}(t_n) + \sum_{l=1}^{M_{\exp,2}} w_{l,2} P_{2,l}(t_n), \tag{5.4}$$

where

$$\begin{aligned} P_{1,l}(t_n) &= h \sum_{k=0}^{n-1} e^{-\tau_{l,1}(t_n - t_k)} f(\bar{u}_k), \\ P_{2,l}(t_n) &= \sum_{k=0}^{n-1} e^{-\tau_{l,2}(t_n - t_k)} g(\bar{u}_k) \Delta W_k. \end{aligned} \tag{5.5}$$

This representation shows that the computational complexity is reduced from $\mathcal{O}(N^2)$ to $\mathcal{O}\big(N M_{\exp}\big)$.

Remark 5.2. For the proposed method based on the SOE approximation, the exponential structure leads to functions key recurrence relations

$$P_{1,l}(t_n) = e^{-\tau_{l,1} h} P_{1,l}(t_{n-1}) + e^{-\tau_{l,1} h} f(\bar{u}_{n-1}) h, \tag{5.6}$$

$$P_{2,l}(t_n) = e^{-\tau_{l,2} h} P_{2,l}(t_{n-1}) + e^{-\tau_{l,2} h} g(\bar{u}_{n-1}) \Delta W_{n-1}. \tag{5.7}$$

Combination Lemma 5.1 with the proposed scheme demonstrates that the computational cost and memory usage per sample are reduced from $\mathcal{O}(N^2)$ and $\mathcal{O}(N)$ to $\mathcal{O}(N \log N)$ and $\mathcal{O}(\log N)$, respectively, when $T \gg 1$. When $T \approx 1$, these complexities become $\mathcal{O}(N \log^2 N)$ and $\mathcal{O}(\log^2 N)$, respectively.

TABLE 1. The global root mean square errors, E_{RMS}, for varying values of N and r in Example 6.1.

N	$r=1$	$r=5$	$r=9$	$r=13$
8	1.55e − 04	8.38e − 07	2.42e − 05	4.20e − 04
16	2.88e − 05	1.34e − 08	2.13e − 10	3.19e − 11
24	6.60e − 06	1.28e − 09	6.35e − 12	5.81e − 14
32	3.07e − 06	2.44e − 10	4.66e − 13	2.02e − 15

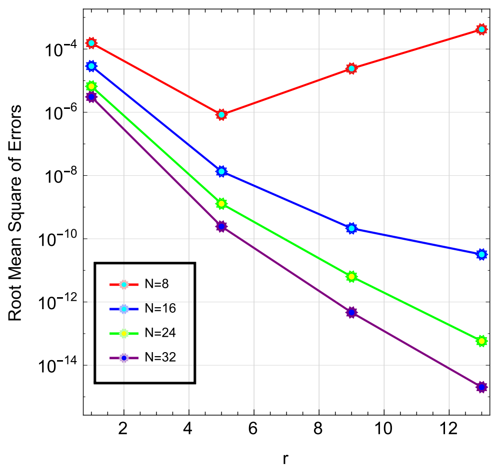

FIGURE 1. Graph of the root mean square of the absolute errors for Example 6.1.

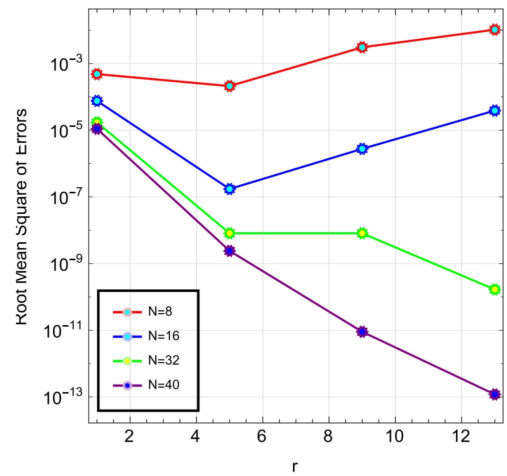

FIGURE 2. Graph of the root mean square of the absolute errors for Example 6.2.

6. Numerical experiments and results

To demonstrate the accuracy and the efficiency of the proposed Jacobi collocation method combined with the SOE approximation, we apply the scheme to several numerical examples involving Eqs. (1.1) and (1.2). In each example, the root mean square error (RMSE) of the absolute errors, denoted by E_{RMS}, is computed at equidistant points $t_i = \frac{i}{10},\ i = 1, 2, \dots, 10$.

Example 6.1. Consider the following nonlinear fractional Volterra integro-differential equation:

$$(D_*^{\alpha} u)(t) = f(t, u(t)) + \int_0^t (t-s)^{-\mu} t^{2-\mu} s^{1+2\alpha+\beta} u(s)(D_*^{\beta} u)(s) ds, \qquad t \in [0,1], \quad u(0) = 0, \tag{6.1}$$

where the nonlinear source term is given by

$$f(t,u) = \frac{\Gamma(2+\alpha-\mu)}{\Gamma(2-\mu)} t^{1-\mu} - \frac{\Gamma(4+4\alpha-2\mu)\Gamma(1-\mu)\Gamma(2+\alpha-\mu)}{\Gamma(5+4\alpha-3\mu)\Gamma(1-\mu)\Gamma(2+\alpha-\beta-\mu)} t^2 u^4.$$

The exact solution of this problem is $u(t) = t^{1+\alpha-\mu}$. For the specific choices $\alpha = \frac{\pi}{6}, \beta = \frac{\pi}{8}$, and $\mu = \frac{2}{\pi}$, we have $u(t) \in A^1[0,1]$.

Table 1 presents the root mean square error E_{RMS} for various values of N and $r = 1, 5, 9, 13$. As expected from the error bound in Eq. (4.2), increasing r (and thus $d = \left[r(1-\frac{2}{\pi})\right] + 1$) leads to a reduction in the numerical error, particularly for sufficiently large N. Figure 1 illustrates the error behavior reported in Table 1 for different values of N and r. The convergence trend observed in the figure confirms the high accuracy of the proposed method and validates the theoretical findings.

TABLE 2. The global root mean square errors E_{RMS} for increasing values of N and r in Example 6.2.

N	$r=1$	$r=5$	$r=9$	$r=13$
8	4.82e − 04	2.11e − 04	3.06e − 03	1.03e − 02
16	7.57e − 05	1.72e − 07	2.76e − 06	3.88e − 05
32	1.71e − 05	8.10e − 09	4.50e − 11	1.67e − 10
40	1.10e − 05	2.40e − 09	9.00e − 12	1.19e − 13

TABLE 3. The global root mean square errors E_{RMS} for increasing values of N and r in Example 6.3.

N	$r=1$	$r=5$	$r=9$	$r=13$	$r=17$	$r=21$
8	3.60e − 02	1.57e − 01	6.69e − 01	8.50e − 01	3.84e − 02	6.70e − 01
16	1.01e − 02	6.64e − 06	1.93e − 06	2.78e − 07	2.63e − 06	2.39e − 07
24	4.46e − 03	1.27e − 06	5.39e − 08	3.22e − 08	6.42e − 09	2.18e − 09
32	2.46e − 03	4.96e − 07	1.41e − 08	4.20e − 09	5.81e − 10	4.24e − 11

Example 6.2. We consider the following fractional integro-differential equation:

$$(D_*^{\frac{\sqrt{2}}{3}}u)(t) = f(t) + \int_0^t (t-s)^{-\frac{\sqrt{2}}{2}} \frac{(D_*^{\frac{\sqrt{2}}{5}}u)(s)}{1+u^2(s)} ds, \qquad t \in [0,1], \quad u(0) = 1. \tag{6.2}$$

In this equation, the function $f(t)$ is chosen such that the exact solution is the Mittag-Leffler function

$$u(t) = E_{\frac{\sqrt{2}}{2}}(t^{\frac{\sqrt{2}}{2}}) \in A^1[0,1],$$

defined by

$$E_b(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(bk+1)}.$$

We compute the errors E_{RMS} by applying the proposed Jacobi collocation method to Eq. (6.2) for various values of N and r. The results are presented in Table 2 and Figure 2.

Example 6.3. As the third example, consider the following nonlinear weakly singular fractional Volterra integro-differential equation:

$$(D_*^{\frac{3}{5}}u)(t) = f(t) + \int_0^t (t-s)^{-\frac{1}{3}} \big(u(s) + (D_*^{\frac{2}{5}}u)(s)\big)^2 ds, \qquad t \in [0,1], \quad u(0) = 0. \tag{6.3}$$

For this problem, the data function $f(t)$ is selected such that the exact solution is $u(t) = t^{\frac{7}{10}} \in A^1[0,1]$. We apply the proposed method to this equation and report the errors E_{RMS} for different values of N and r in Table 3 and Figure 3.

Example 6.4. As a multi-term equation, consider the following nonlinear two-term weakly singular fractional integro-differential equation:

$$(D_*^{\alpha}u)(t) = f(t,u(t)) + \sum_{q=1}^{2} \frac{1}{\Gamma(1-\mu_q)} \int_0^t (t-s)^{-\mu_q} R_q\Big(t,s,u(s),(D_*^{\beta_q}u)(s)\Big) ds, \quad u(0) = 0, \tag{6.4}$$

where $\mu_1 = \dfrac{1}{4}$, $\mu_2 = \dfrac{3}{4}$, $\alpha = \dfrac{1}{2}$, $\beta_1 = \dfrac{1}{3}$, $\beta_2 = \dfrac{1}{6}$, the kernel functions R_1 and R_2, and the forcing function f are given by

$$R_1\big(t,s,u,v\big) = u^2 v,$$
$$R_2\big(t,s,u,v\big) = uv^2,$$

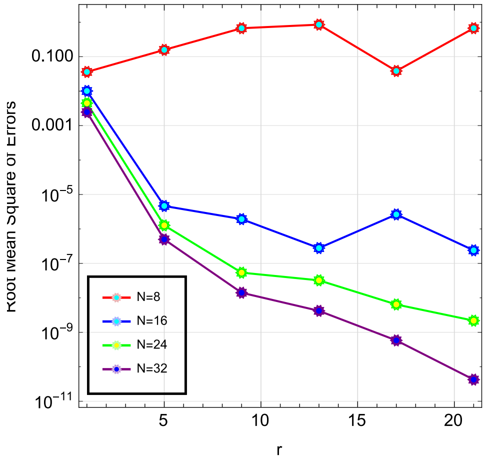

FIGURE 3. Graph of the root mean square of the absolute errors for Example 6.3.

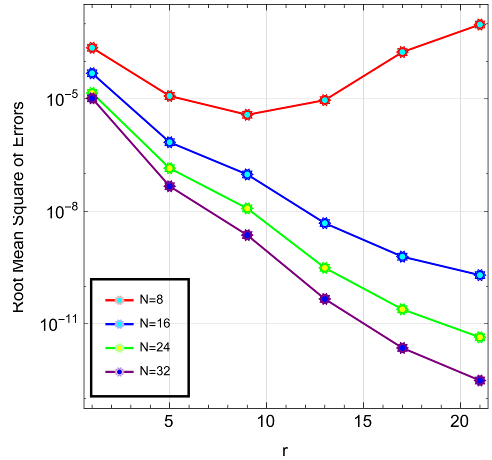

FIGURE 4. Graph of the root mean square of the absolute errors for Example 6.4.

TABLE 4. The global root mean square errors E_{RMS} for increasing values of N, r in Example 6.4.

N	$r=1$	$r=5$	$r=9$	$r=13$	$r=17$	$r=21$
8	2.30e − 04	1.19e − 05	3.76e − 06	9.22e − 06	1.77e − 04	9.59e − 04
16	4.81e − 05	6.95e − 07	9.61e − 08	4.83e − 09	6.16e − 10	1.98e − 10
24	1.43e − 05	1.41e − 07	1.19e − 08	3.11e − 10	2.43e − 11	4.36e − 12
32	1.03e − 05	4.65e − 08	2.32e − 09	4.59e − 11	2.26e − 12	3.06e − 13

$$f(t,u) = \frac{\Gamma(\frac{5}{3})}{\Gamma(\frac{7}{6})} t^{\frac{1}{6}} - \frac{4(\Gamma(\frac{5}{3}))^2\Gamma(\frac{8}{3})}{\pi\Gamma(\frac{35}{12})} t^{\frac{7}{12}} u^2 - \frac{\Gamma(\frac{5}{3})\Gamma(\frac{8}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{41}{12})} t^{\frac{5}{12}} u^3.$$

The exact solution of this problem is $u(t) = t^{\frac{2}{3}}$. Similar to the previous examples, we approximate the solution of Eq. (6.4) and report the errors for different values of N and r in Table 4 and Figure 4.

Example 6.5. Consider the following linear multi-term weakly singular integro-differential equation:

$$(D_*^{\alpha} u)(t) = f(t, u(t)) + \sum_{q=1}^{2} \int_0^t (t-s)^{-\mu_q} R_q\Big(t, s, u(s), (D_*^{\beta_q} u)(s)\Big) ds, \quad u(0) = 0, \tag{6.5}$$

with parameters $\mu_1 = \dfrac{1}{4}$, $\mu_2 = \dfrac{1}{3}$, $\alpha = \dfrac{1}{2}$, $\beta_2 = \dfrac{3}{8}$. The kernel functions R_1, R_2 and the forcing function f are given by

$$R_1(t,s,u,v) = -u,$$
$$R_2(t,s,u,v) = -v,$$
$$f(t,u) = 2t^{\frac{5}{4}} + \frac{2(\Gamma(\frac{3}{4}))^2}{\sqrt{\pi}} t^{\frac{3}{2}} + \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})} t^{\frac{1}{4}} + \frac{2\Gamma(\frac{2}{3})\Gamma(\frac{7}{4})}{\Gamma(\frac{49}{24})} t^{\frac{25}{24}} - t^{\frac{1}{2}} u.$$

The exact solution of this problem is $u(t) = 2t^{\frac{3}{4}}$. This problem was previously solved by the spline collocation method in [18]. The authors computed maximum absolute errors, ε_N, defined by

$$\varepsilon_N = \max_{j=1,\dots,N} \max_{k=0,\dots,10} |u(\tau_{jk}) - u_N(\tau_{jk})|,$$

TABLE 5. The global root mean square errors E_{RMS} for a range of increasing values of N, r in Example 6.5.

N	$r=1$	$r=5$	$r=9$	$r=13$	$r=17$	$r=21$
8	4.25e − 04	1.19e − 05	1.21e − 06	2.54e − 07	1.32e − 06	1.69e − 05
16	8.78e − 05	3.73e − 07	1.10e − 08	6.07e − 10	5.22e − 11	6.73e − 12
24	2.62e − 05	5.17e − 08	6.80e − 10	1.50e − 11	6.14e − 13	3.44e − 14
32	7.24e − 06	1.31e − 08	8.76e − 11	1.22e − 12	2.39e − 14	7.97e − 16

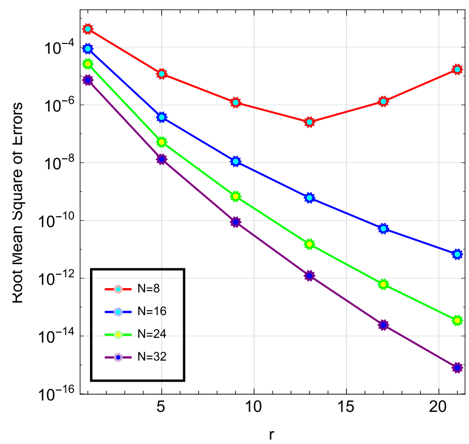

FIGURE 5. Graph of the root mean square of the absolute errors for Example 6.5.

where

$$\tau_{jk} = t_{j-1} + k\left(t_j - t_{j-1}\right)/10,\ k = 0,\dots,10,\ j = 1,\dots,N,$$

and the grid points t_j are defined by $t_j = b\Big(\dfrac{j}{N}\Big)^r$, $r \in \mathbb{R}, r \geq 1$, for various values of N, r, and spline orders $m = 2, 3$ (see Table 6). To compare with the spline collocation method, we compute the root mean square of absolute errors E_RMS for a range of N, r using the proposed method and present the results in Table 5 and Figure 5. The comparison shows that the proposed method achieves significantly smaller errors than the spline collocation method.

Example 6.6. As a stochastic case, consider the nonlinear weakly singular stochastic Volterra integral equation [11]:

$$du(t) = (1-\theta)\int_0^t (t-s)^{-\gamma} \sin\big(u(s)\big)ds + \theta\int_0^t (t-s)^{-\sigma} \cos^2\big(u(s)\big)dW(s), \quad t \in [0,1], \tag{6.6}$$

with initial condition $u(0) = 0$. Two cases are considered for positive parameters γ and σ:

- Case I: $\gamma = 0.4$, $\sigma = 0.4$,
- Case II: $\gamma = 0.7$, $\sigma = 0.1$.

The numerical results for this example, with step sizes $h = 2^{-7}, 2^{-8}, 2^{-9}$, and 2^{-10} are presented in Tables 7 and 8. Tables 7 and 8 demonstrate that the proposed method (5.4) achieves a convergence order of approximately 0.2 in both cases.

7. CONCLUSIONS

In this article, we applied the Jacobi collocation method to numerically solve nonlinear multi-term weakly singular fractional Volterra integro-differential equations. We introduced an operator-based framework to incorporate a smoothing transformation in conjunction with Gauss-Jacobi quadrature for the Riemann–Liouville fractional integral operator, which we employed to approximate the integral terms in the main equation. Increasing the smoothing

TABLE 6. Numerical results from [18] for Example 6.5.

ε_N with $m = 2$				
N	$r = 1$	$r = 2$	$r = \frac{10}{3}$	$r = 5$
8	2.64e − 02	6.70e − 03	2.65e − 03	5.75e − 03
16	1.69e − 02	2.27e − 03	4.05e − 04	1.03e − 03
32	1.04e − 02	8.19e − 04	7.01e − 05	1.81e − 04
64	6.32e − 03	2.85e − 04	1.18e − 05	3.20e − 05
128	3.80e − 03	1.01e − 04	1.90e − 06	4.94e − 06
256	2.27e − 03	3.57e − 05	2.77e − 07	8.02e − 07
512	1.36e − 03	1.26e − 05	6.37e − 08	1.54e − 07
1024	8.07e − 04	4.46e − 06	1.54e − 08	2.57e − 08
ε_N with $m = 3$				
N	$r = 1$	$r = 2$	$r = 4$	$r = 5$
8	1.04e − 02	2.49e − 03	2.49e − 04	4.51e − 04
16	6.65e − 03	9.01e − 04	4.80e − 05	4.10e − 05
32	4.11e − 03	3.20e − 04	4.64e − 06	3.65e − 06
64	2.50e − 03	1.13e − 04	5.59e − 07	2.92e − 07
128	1.50e − 03	4.00e − 05	7.76e − 08	2.50e − 08
256	8.99e − 04	1.42e − 05	1.02e − 08	2.21e − 09
512	5.37e − 04	5.01e − 06	1.26e − 09	1.82e − 10
1024	3.20e − 04	1.77e − 06	1.51e − 10	1.73e − 11

TABLE 7. Convergence orders and CPU times of the proposed method with $\epsilon = 10^{-12}$ for $u(t)$ in case I.

h	$\theta = 0.2$	$\theta = 0.5$	$\theta = 0.8$	CPU time
2^{-7}	0.4967	0.5247	0.5683	0.64
2^{-8}	0.3893	0.4102	0.4326	2.79
2^{-9}	0.2941	0.3098	0.3259	10.42
2^{-10}	0.2336	0.2468	0.2649	24.81
order	0.2148	0.2206	0.2433	53.26

TABLE 8. Convergence orders and CPU times of the proposed method with $\epsilon = 10^{-12}$ for $u(t)$ in case II.

h	$\theta = 0.2$	$\theta = 0.5$	$\theta = 0.8$	CPU time
2^{-7}	0.4737	0.4952	0.5180	0.58
2^{-8}	0.3692	0.3973	0.4214	2.33
2^{-9}	0.2741	0.2941	0.3143	9.14
2^{-10}	0.2318	0.2422	0.2516	22.56
order	0.2103	0.2158	0.2190	48.83

parameter r enhances the regularity of the solution as well as the regularity of the integrands within the integrals approximated by the Gauss-Jacobi quadrature.

Our error analysis shows that the error bound depends on N, the number of collocation nodes, and on r, the smoothing parameter. Numerical results indicate that for large values of N, increasing r reduces the error, in agreement with the theoretical analysis. However, for small values of N, using large values of r actually increases the error, which is contrary to the expected behavior. This phenomenon occurs because, when the number of nodes is small, applying a smoothing transformation with a large r clusters nodes near the origin and leads to larger step sizes elsewhere on the interval, thus reducing overall accuracy. This observation suggests the existence of an optimal relationship between r and N to minimize the error.

Additionally, our numerical experiments demonstrate the computational efficiency of the proposed scheme, particularly when combined with the sum-of-exponentials (SOE) approximation in the stochastic case.

Statement

A preprint of this paper has previously been published [12].

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