Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-19
DOI:10.22034/cmde.2024.61871.2696

Dynamics and bifurcation control of a fractional-order delayed predator-prey model with an omnivore

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Abstract

In this study, we propose a novel fractional delayed predator-prey model that includes an omnivorous species and explore bifurcation control through a state feedback control strategy. We begin by deriving the characteristic polynomial using the Laplace transform and establish new sufficient conditions for stability analysis and Hopf bifurcation, treating the time delay τ as a bifurcation parameter. To address Hopf bifurcation in the uncontrolled system, we design a state feedback controller with time delay. Our results indicate that the time delay τ significantly affects the onset of Hopf bifurcation. Additionally, the inclusion of fractional order $0 < \alpha \leq 1$ enhances solution stability while adding complexity to dynamics of the model. We find that judicious selection of the feedback gain can delay bifurcation, highlighting the critical role of control effort. To validate our theoretical findings, we present numerical simulations conducted using a modified Adams-Bashforth-Moulton predictor-corrector method. These simulations support our theoretical results and demonstrate the efficacy of our proposed control strategy in managing dynamical behaviors of the model

Keywords. Fractional order, Bifurcation control, Predator–prey model, Time–delay, Hopf bifurcation..
1991 Mathematics Subject Classification.

1. Introduction

In ecological systems, the predator-prey models which describe the relationships between different species play an important role in connection with complex food chains and food networks [22, 29, 43]. Due to the universal existence and importance, population models, especially predator–prey systems become popular from the traditional and pioneering work of Lotka [26] and Volterra [39] and the qualitative properties of Lotka-Volterra like predator–prey models, such as stability, chaos, bifurcations and periodic oscillations are usually depending on the parameters of the models [9]. In the fields of population dynamics, there are many applied mathematicians and ecologists who have been extensively investigated the dynamical behaviors of predator–prey systems from different aspects [8, 9, 24, 35]. For example, the study in [15] investigated a predator-prey model that includes the Allee effect and alternative food sources for the predator. Their findings indicate that a weak Allee effect can enhance stability and support population survival. In contrast, a strong Allee effect tends to destabilize the system, leading to an unstable limit cycle and heightening the risk of extinction for the populations involved. Therefore, the dynamics of predator–prey model is a major theme in ecology and evolutionary biology which can be considered as the basic block in order to constructs the complicated dynamics of food web, food chain and some other biochemical network structure [13, 24, 43].

Amongst the different ecological models, there is a food web model with an omnivore wherein a population feed on resources at more than one trophic level [1]. The role and prevalence of omnivore is critical to understand food web structure and its dynamics. In fact, the presence of an omnivore can lead to stabilization or destabilization of any given food webs which indicate its important role in the investigations of the food chain models [1, 25]. The presence of omnivores introduces additional complexity to predator-prey relationships. Research by Huang et al. [12]

Received: 29 May 2024 ; Accepted: 06 October 2024.
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indicated that omnivores could either promote stability or lead to destabilization within ecosystems, highlighting the need for models that consider multiple trophic levels. Tanabe and Namba [30, 37] found a simple food web model with linear functional responses consisting of twelve parameters on which the omnivore leads to chaos in their models. They proposed a three species evolution model that one of the species was an omnivore, which can consume both the predator and prey.

In this research work, we consider a predator-prey model introduced by Previte and Hoffman [33] which represents a food web model with an omnivore (or a scavenger) given by

$$\begin{aligned} \frac{dx}{dt} &= x(1-bx) - xy - xz, \\ \frac{dy}{dt} &= y(-c+x), \\ \frac{dz}{dt} &= (-e-\beta z)z + fxz + gyz, \end{aligned} \tag{1.1}$$

where x, y, z indicate the densities of prey, predator, and omnivore population as functions of time, respectively. All parameters of the system (1.1) are positive and their biological meanings can be found in [20, 33]. System (1.1) is further studied by Y. Li. and V. G. Romanovski in [20] by employing a new approach which was based on using of the commutative computational algebra algorithms. They have shown that under variation of some parameters, the system undergoes Hopf and degenerate Hopf bifurcations.

The study of predator-prey dynamics has evolved significantly with the incorporation of fractional calculus and time delays, providing deeper insights into ecological interactions. Traditional models often fail to capture the complexities inherent in biological systems, necessitating more sophisticated approaches. For example, time-delayed nonlinear dynamical systems, which incorporate time delays to account for past system states, are commonly used to model various natural systems in fields such as epidemiology, physiology, control theory, and biological processes [3, 21, 28, 36]. Time delays, also known as time lags, are essential for accurately reflecting the dynamics of biological systems that depend on their past history [9, 23]. These delays can represent various biological processes such as gestation periods, maturation periods, reaction times, resource regeneration times, and feeding times [8, 9, 14, 41]. Furthermore, fractional-order derivatives have emerged as a powerful tool for modeling ecological processes characterized by memory effects. These derivatives offer a more accurate representation of real-world systems, accounting for the influence of past events on present dynamics. Zhang et al. [44] illustrated that incorporating delays can lead to oscillatory behavior and potential instability within populations. The combination of fractional derivatives and time delays introduces layers of complexity, where adjustments in delay parameters can significantly alter system dynamics.

Fractional-order differential equations have proven instrumental in modeling ecological processes characterized by memory effects, offering a more comprehensive representation of real-world systems than traditional integer-order models [18]. While numerous predator-prey models have been analyzed using delay differential equations (DDEs) with classical integer-order derivatives [13, 17, 24, 27, 29, 42], a growing body of research has explored the fractional dynamics of predator-prey systems, both with and without time delays [7, 11, 19, 22, 34, 43]. The presence of time delays and fractional-order derivatives in population models share a common characteristic: they both account for memory effects. The inclusion of these two types of memory significantly increases model complexity and enriches the dynamics of the system [34]. The memory inherent in these models significantly impacts the current and future development of the modeled processes. The history of the process is more accurately described when the model incorporates memory effects, making fractional-order differential equations more suitable for capturing real-world phenomena compared to classical integer-order differential equations [43].
From a different perspective, fractional-order derivatives are inherently non-local and capable of modeling memory effects, while time delays in the model represent the history of an earlier state [16]. Investigating the dynamical behaviors of these models in a fractional-order setting holds significant promise for achieving improved results compared to traditional integer-order approaches. The development of both theory and applications of fractional-order systems has attracted considerable attention from mathematicians, physicists, and engineers [2, 7, 16, 19, 38]. This research is motivated by the need for a deeper understanding of fractional-order delayed population models, particularly

predator-prey systems incorporating both fractional order and time delay. While the theory of delay differential equations (DDEs) is well established, the study of fractional differential systems with delay is still in its early stages [31]. Limited research has focused on analyzing and numerically solving delayed fractional-order models [38].
In recent years, control strategies have become increasingly important in mathematical biology and control theory, particularly for effective ecosystem management. Hopf bifurcation analysis has proven valuable for gaining insights into the behavior of complex dynamical systems near equilibrium points. Bifurcation control, as a necessary and efficient method, has been extensively developed and applied [11, 41, 43]. For instance, C-Huang et al. [11] successfully controlled the Hopf bifurcation of a fractional delayed predator-prey system using an active extended delayed feedback control strategy. Additionally, Li et al. [22] employed delay feedback control to control the Hopf bifurcation in a fractional-order delayed Lotka-Volterra predator-prey system, utilizing time delay as a free parameter. Zhao et al. [45] proposed feedback control methods for stabilizing populations within a delayed fractional-order framework, highlighting their potential for practical applications in ecological conservation. Among various bifurcation control approaches, time-delayed feedback control has been widely used to mitigate bifurcation phenomena in nonlinear systems [11]. However, limited research has focused on bifurcation control in delayed fractional-order predator-prey systems using this technique [7, 19]. This gap in knowledge motivates our investigation into the dynamics, stability, Hopf bifurcation, and bifurcation control of a fractional-order delayed predator-prey system utilizing time-delayed feedback control.

This paper extends the work of E. Bonyah et al. [5], by incorporating delay into the model, enabling a more realistic representation of system dynamics. While their study explored different fractional derivatives without delay, our research focuses on the Caputo derivative and investigates the impact of delay on the model's behavior. Specifically, we examine the emergence of Hopf bifurcations, a critical phenomenon that arises due to the presence of delay. To effectively manage these bifurcations, we introduce a novel state feedback control mechanism with time delay. Our findings demonstrate the ability of this control strategy to postpone the onset of bifurcation and highlight the importance of delay in accurately capturing complex system dynamics.

Stimulated by the previous discussions, we extend model (1.1) by combining the two types of memory, i.e, fractional order α and the delay τ in the model in order to investigate their impacts and describe the predator–prey interactions with memory effects. We also discuss the qualitative behaviors of the model by means of stability and Hopf bifurcation analysis. Then we use a theoretical analysis to realize the bifurcation control in the considered delayed fractional-order predator–prey system based on time delayed feedback approach. Therefore, the fractional delayed mathematical model in the sense of Caputo fractional derivative can be presented as follows:

$$\begin{cases} D_t^{\alpha} x(t) = x(1-bx) - xy - xz, \\ D_t^{\alpha} y(t) = -cy + xy, \\ D_t^{\alpha} z(t) = fx(t-\tau)z(t-\tau) + gy(t-\tau)z(t-\tau) - (e+\beta z)z, \end{cases} \tag{1.2}$$

where $\alpha \in (0,1]$ is fractional order, and $\tau \geq 0$ is a constant time delay due to gestation. The initial conditions are $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$ and for $s \in [-\tau, 0]$, the smooth functions can be selected as $x(s) = \psi_1(s) \geq 0$, $y(s) = \psi_2(s) \geq 0$, $z(s) = \psi_3(s) \geq 0$, where $\psi_i(s) \in C([-\tau, 0], \mathbb{R})$ for $i = 1, 2, 3$, in which $C([-\tau, 0], \mathbb{R})$ denotes the space of continuous mappings from $[-\tau, 0]$ to $\mathbb{R}$.

To control the dynamics of model (1.2), a time delayed feedback controller is designed which can be added in the second equation of the fractional delayed system and has the following form:

$$u(t) = K(y(t) - y(t-\tau)), \tag{1.3}$$

where τ stands for feedback time and the constant K denotes a negative feedback gain, which can be smartly manipulated for controlling the Hopf bifurcation in order to achieve the desirable dynamical behaviors.
The main highlights of this paper is to investigate the influence and relation of the two important types of memories, one is the fractional order α and the other is time delay τ on the dynamic behavior of the fractional delayed predator–prey system. An efficient time-delayed feedback controller is also elaborately introduced to stabilize the unstable orbit and control the creation of Hopf bifurcation for the proposed fractional-order delayed differential system, and the desirable dynamical behaviors are obtained. Ultimately, the effect of feedback gain on the bifurcation point is further studied.

The rest framework of the current paper is arranged as follows: Some basic materials regarding fractional calculus are presented in Section 2. In Section 3, the main results are discussed including of a detailed analysis on the local stability of the interior equilibrium point and Hopf bifurcation conditions for the uncontrolled model as well as bifurcation control results for the controlled model, respectively. In Section 4, by employing Adams–Bashforth–Moulton predictor-corrector scheme(ABM) for solving fractional-order delay differential equation where the main steps of its algorithm is given in [4, 34], we carry out some numerical simulations to verify the obtained theoretical results. In Section 5, the paper is concluded with a brief conclusion.

2. Basic tools and definitions

It is recognized that among the several definitions of fractional derivatives, the three definitions, such as, Grünwald–Letnikov (G-L), the Riemann–Liouville (R-L) and the Caputo's definitions are the extensively used ones in the field of fractional-order nonlinear dynamical systems. The Caputo derivative has more advantage then other fractional-order derivatives, since it only requires the initial conditions in the form of integer-order derivatives, which is more appropriate for mathematical modeling of many real world problems. Moreover, the operator of fractional-order model can be provided more flexibility to the various physical and biological models in order to investigate their considerable dynamical behaviors. Therefore, we use the definition of Caputo derivative and utilize some helpful lemmas and theorems for fractional systems introduced in this section.

Definition 2.1. ([10]) The fractional-order integral of non-integer order α for a function $f(t)$ can be defined as follows:

$$I^{\alpha} f(t) = D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \qquad \alpha > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. ([40, 41]) The Caputo fractional-order derivative of order α for a function $f(t) \in ([0,\infty), \mathbb{R})$ is given by

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, & n-1 < \alpha < n, \;\; n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{2.2}$$

Particularly, when $0 < \alpha < 1$, the Caputo fractional-order derivative takes the form

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds. \tag{2.3}$$

Definition 2.3. ([22, 43]) The Laplace transform of the fractional-order derivatives in the sense of Caputo can be defined as

$$\mathcal{L}\{D_t^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \tag{2.4}$$

where $n-1 < \alpha < n$ is a positive integer and $F(s) = \mathcal{L}\{f(t)\}$.

Definition 2.4. ([10, 19]) Consider a high-dimension fractional-order system as in the following form

$$D_t^{\alpha} x(t) = f\big(x(t)\big), \tag{2.5}$$

where $x(t) = \big(x_1(t), x_2(t), \cdots, x_n(t)\big)^T \in \mathbb{R}^n$, $f(t) = \big(f_1(t), f_2(t), \cdots, f_n(t)\big)^T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. The equilibrium solutions of (2.5) can be defined by $f(x^*) = 0$. Hence, the corresponding equilibrium points can be obtained as $(x_1^*, x_2^*, \cdots, x_n^*)$.

Lemma 2.5. *(see [6, 32]). Consider the fractional-order system*

$$\frac{d^\alpha x}{dt^\alpha} = f(x), \tag{2.6}$$

where $\alpha \in (0,1)$, $\frac{d^\alpha}{dt^\alpha} = D_t^\alpha$ *and* $x \in \mathbb{R}^n$. *The equilibrium points of system* (2.6) *can be computed via solutions of the equation* $f(x) = 0$. *It also can be concluded that an equilibrium point* x^* *of system* (2.6) *is locally asymptotically stable if all the eigenvalues of the Jacobian matrix* $J = \frac{\partial f}{\partial x}\Big|_{x^*}$ *satisfy* $\left|\arg(\lambda)\right| > \frac{\alpha\pi}{2}$.

Lemma 2.6. *(see [11, 14]). Consider the following n-dimensional autonomous linear fractional-order system*

$$\begin{cases} D^\alpha \xi_1(t) = \kappa_{11}\xi_1(t) + \kappa_{12}\xi_2(t) + \cdots + \kappa_{1n}\xi_n(t), \\ D^\alpha \xi_2(t) = \kappa_{21}\xi_1(t) + \kappa_{22}\xi_2(t) + \cdots + \kappa_{2n}\xi_n(t), \\ \quad \vdots \\ D^\alpha \xi_n(t) = \kappa_{n1}\xi_1(t) + \kappa_{n2}\xi_2(t) + \cdots + \kappa_{nn}\xi_n(t), \end{cases} \tag{2.7}$$

where $0 < \alpha < 1$. *By taking the Laplace transform on both sides of Eq.* (2.7) *and after doing some simplification, we eventually can obtain the following determinant:*

$$\det\big(\triangle(\lambda)\big) = \det \begin{bmatrix} \lambda^\alpha - \kappa_{11} & -\kappa_{12} & \cdots & -\kappa_{1n} \\ -\kappa_{21} & \lambda^\alpha - \kappa_{22} & \cdots & -\kappa_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa_{n1} & -\kappa_{n2} & \cdots & \lambda^\alpha - \kappa_{nn} \end{bmatrix}. \tag{2.8}$$

Thus the zero solution $(\xi = 0)$ *of system* (2.7) *is globally asymptotically stable in the Lyapunov sense if all roots of the equation* $\det\big(\triangle(\lambda)\big) = 0$ *in* (2.8) *satisfy* $|\arg(\lambda)| > \frac{\alpha\pi}{2}$.

Lemma 2.7. *(see [11, 14]). Consider the following n-dimensional linear fractional-order delayed system*

$$\begin{cases} D^\alpha \xi_1(t) = b_{11}\xi_1(t-\tau_{11}) + b_{12}\xi_2(t-\tau_{12}) + \cdots + b_{1n}\xi_n(t-\tau_{1n}), \\ D^\alpha \xi_2(t) = b_{21}\xi_1(t-\tau_{21}) + b_{22}\xi_2(t-\tau_{22}) + \cdots + b_{2n}\xi_n(t-\tau_{2n}), \\ \quad \vdots \\ D^\alpha \xi_n(t) = b_{n1}\xi_1(t-\tau_{n1}) + b_{n2}\xi_2(t-\tau_{n2}) + \cdots + b_{nn}\xi_n(t-\tau_{nn}), \end{cases} \tag{2.9}$$

where $\alpha \in \mathbb{Q}$, *i.e,* $0 < \alpha < 1$. *For system* (2.9), $\tau_{ij} \geq 0$, $i,j = 1,2,\cdots,n$ *and* $B = (b_{i,j})_{n\times n}$ *is indicating the coefficient matrix.*
Next, to study the stability of system (2.9) *we take the Laplace transform on both sides of* (2.9) *which gives the characteristic equation* $\det\big(\triangle(s)\big)$ *as follows*

$$\det\big(\triangle(s)\big) = \det \begin{bmatrix} s^\alpha - b_{11}\mathrm{e}^{-s\tau_{11}} & -b_{12}\mathrm{e}^{-s\tau_{12}} & \cdots & -b_{1n}\mathrm{e}^{-s\tau_{1n}} \\ -b_{21}\mathrm{e}^{-s\tau_{21}} & s^\alpha - b_{22}\mathrm{e}^{-s\tau_{22}} & \cdots & -b_{2n}\mathrm{e}^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1}\mathrm{e}^{-s\tau_{n1}} & -b_{n2}\mathrm{e}^{-s\tau_{n2}} & \cdots & s^\alpha - b_{nn}\mathrm{e}^{-s\tau_{nn}} \end{bmatrix} = 0. \tag{2.10}$$

Since the stability of the zero solution of system (2.9) *exactly depends on the roots distribution of the associated characteristic equation of* (2.10), *therefore, if all the roots of the characteristic equation* (2.10) *have negative real parts, then the zero solution of system* (2.9) *is asymptotically stable.*

3. Main results

In this section, we first investigate the effects of time delay and fractional-order on the dynamics of uncontrolled system (1.2) by taking time delay as a bifurcation parameter. The conditions of bifurcations will be further discussed.

Then for controlling the Hopf bifurcation, a proper controller based on time delayed feedback method is designed and the impact of the corresponding feedback gain on bifurcation is studied. Morevere, comparative analysis is performed by means of numerical simulations.

3.1. **Stability and Hopf bifurcation analysis for uncontrolled system (1.2).** In this subsection, we analyze the local stability of the positive steady state based on the linearization technique. To obtain the linearized system around the positive steady state $E^*(x^*, y^*, z^*)$, let us to employe the transformation $X(t) = x(t)-x^*, Y(t) = y(t)-y^*, Z(t) = z(t) - z^*$. Then the linearized form of (1.2) can be described as

$$\begin{cases} D_t^\alpha X(t) = -bx^*X(t) - x^*Y(t) - x^*Z(t), \\ D_t^\alpha Y(t) = y^*X(t), \\ D_t^\alpha Z(t) = (-e-2\beta z^*)Z(t) + fz^*X(t-\tau) + gz^*Y(t-\tau) + (fx^*+gy^*)Z(t-\tau). \end{cases} \tag{3.1}$$

By applying the Laplace transform on both sides of (3.1), we have

$$\begin{cases} s^\alpha \mathcal{L}\{X(t)\} - s^{\alpha-1}x(0) = -bx^*\mathcal{L}\{X(t)\} - x^*\mathcal{L}\{Y(t)\} - x^*\mathcal{L}\{Z(t)\}, \\ s^\alpha \mathcal{L}\{Y(t)\} - s^{\alpha-1}y(0) = y^*\mathcal{L}\{X(t)\}, \\ s^\alpha \mathcal{L}\{Z(t)\} - s^{\alpha-1}z(0) = (-e-2\beta z^*)\mathcal{L}\{Z(t)\} + fz^*\mathrm{e}^{-s\tau}\Big(\mathcal{L}\{X(t)\} + \int_{-\tau}^0 \mathrm{e}^{-st}\psi_1(t)dt\Big) \\ \qquad +gz^*\mathrm{e}^{-s\tau}\Big(\mathcal{L}\{Y(t)\} + \int_{-\tau}^0 \mathrm{e}^{-st}\psi_2(t)dt\Big) + (fx^*+gy^*)\mathrm{e}^{-s\tau}\Big(\mathcal{L}\{Z(t)\} + \int_{-\tau}^0 \mathrm{e}^{-st}\psi_3(t)dt\Big). \end{cases} \tag{3.2}$$

Let $\mathcal{L}\{f(t)\} = F(s)$. Then, the above system (3.2) can be written as in the following matrix form

$$\Delta(s)\cdot\begin{pmatrix} X(s) \\ Y(s) \\ Z(s) \end{pmatrix} = \begin{pmatrix} \delta_1(s) \\ \delta_2(s) \\ \delta_3(s) \end{pmatrix}, \tag{3.3}$$

in which

$$\Delta(s) = \begin{bmatrix} s^\alpha + bx^* & x^* & x^* \\ -y^* & s^\alpha & 0 \\ -fz^*\mathrm{e}^{-s\tau} & -gz^*\mathrm{e}^{-s\tau} & s^\alpha + e + 2\beta z^* - (fx^*+gy^*)\mathrm{e}^{-s\tau} \end{bmatrix},$$

and

$$\begin{cases} \delta_1(s) = s^{\alpha-1}x(0), \\ \delta_2(s) = s^{\alpha-1}y(0), \\ \delta_3(s) = s^{\alpha-1}z(0) + fz^*\mathrm{e}^{-s\tau}\int_{-\tau}^0 \mathrm{e}^{-st}\psi_1(t)dt + gz^*\mathrm{e}^{-s\tau}\int_{-\tau}^0 \mathrm{e}^{-st}\psi_2(t)dt \\ \qquad +(fx^*+gy^*)\mathrm{e}^{-s\tau}\int_{-\tau}^0 \mathrm{e}^{-st}\psi_3(t)dt. \end{cases}$$

In Eq. (3.3), $\Delta(s)$ represents the characteristic matrix of system (3.1). The stability of system (3.1) can be entirely determined via the properties of eigenvalues of the associated characteristic equation $\det(\Delta(s)) = 0$, given by

$$s^{3\alpha} + R_2 s^{2\alpha} + R_1 s^\alpha + R_0 + \big[P_2 s^{2\alpha} + P_1 s^\alpha + P_0\big]\mathrm{e}^{-s\tau} = 0, \tag{3.4}$$

where

$$R_2 = bx^* + 2\beta z^* + e, \quad R_1 = (2b\beta z^* + be + y^*)x^*, \quad R_0 = (2\beta z^* + e)x^*y^*,$$
$$P_2 = -fx^* - gy^*, \quad P_1 = -(bfx^* + bgy^* - fz^*)x^*, \quad P_0 = (fx^* + g(y^* - z^*))x^*y^*.$$

Now, we consider the following two cases:

Case I:: When $\tau = 0$, we consider the following conditions

$$(\mathbf{H_1})\ h_2 > 0,\ h_0 > 0,\ h_2 h_1 > h_0,$$

in which

$$h_2 = P_2 + R_2, \quad h_1 = P_1 + R_1, \quad h_0 = P_0 + R_0.$$

In order to discuss the stability of system (1.2) in the absence of delay, we establish the following lemma.

Lemma 3.1. *If the condition* (H_1) *holds, then* E^* *is locally asymptotically stable when* $\tau = 0$.

Proof. In the absence of delay, the characteristic polynomial (3.4) becomes

$$\lambda^3 + h_2\lambda^2 + h_1\lambda + h_0 = 0,\ (\lambda = s^\alpha). \tag{3.5}$$

When the assumption (H_1) holds, then it is apparent that all the roots λ_i $(i = 1, 2, 3)$ of (3.5) have negative real parts based on the Routh–Hurwitz criterion which satisfy $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$, $i = 1, 2, 3$. Therefore, we can conclude that E^* is locally asymptotically stable and the proof is completed. □

Case II:: When $\tau \neq 0$, the Eq. (3.4) can be reduced to

$$N_1^\star(s) + N_2^\star(s)\mathrm{e}^{-s\tau} = 0, \tag{3.6}$$

in which

$$N_1^\star = s^{3\alpha} + R_2 s^{2\alpha} + R_1 s^\alpha + R_0, \quad N_2^\star = P_2 s^{2\alpha} + P_1 s^\alpha + P_0.$$

In this case, following [10, 11], we assume that $s = \mathrm{i}\omega = \omega\big(\cos(\frac{\pi}{2}) + \mathrm{i}\sin(\frac{\pi}{2})\big)$, $\omega > 0$ be a purely imaginary root of Eq. (3.6). Then by substituting it into the characteristic Eq. (3.6), we obtain

$$\mathcal{C}_1 + \mathrm{i}\mathcal{D}_1 + (\mathcal{C}_2 + \mathrm{i}\mathcal{D}_2)\big(\cos(\omega\tau) - \mathrm{i}\sin(\omega\tau)\big) = 0, \tag{3.7}$$

where

$$\begin{aligned}
\mathcal{C}_1 &= \omega^{3\alpha}\cos\left(\frac{3\alpha\pi}{2}\right) + R_2\omega^{2\alpha}\cos(\alpha\pi) + R_1\omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right) + R_0,\\
\mathcal{D}_1 &= \omega^{3\alpha}\sin\left(\frac{3\alpha\pi}{2}\right) + R_2\omega^{2\alpha}\sin(\alpha\pi) + R_1\omega^\alpha\sin\left(\frac{\alpha\pi}{2}\right),\\
\mathcal{C}_2 &= P_2\omega^{2\alpha}\cos(\alpha\pi) + P_1\omega^\alpha\cos\big(\frac{\alpha\pi}{2}\big) + P_0,\\
\mathcal{D}_2 &= P_2\omega^{2\alpha}\sin(\alpha\pi) + P_1\omega^\alpha\sin(\frac{\alpha\pi}{2}).
\end{aligned}$$

Using Eq. (3.7), we get the corresponding real and imaginary parts as follows

$$\begin{aligned}
&\mathcal{C}_2\cos(\omega\tau) + \mathcal{D}_2\sin(\omega\tau) = -\mathcal{C}_1,\\
&D_2\cos(\omega\tau) - \mathcal{C}_2\sin(\omega\tau) = -\mathcal{D}_1.
\end{aligned} \tag{3.8}$$

From (3.8) and by labelling of

$$\Theta_1(\omega) = -\mathcal{C}_1\mathcal{C}_2 - \mathcal{D}_1\mathcal{D}_2, \quad \Theta_2(\omega) = \mathcal{D}_1\mathcal{C}_2 - \mathcal{D}_2\mathcal{C}_1, \quad \Theta(\omega) = \mathcal{C}_2^2 + \mathcal{D}_2^2, \tag{3.9}$$

we get

$$\begin{cases}
\cos(\omega\tau) = \dfrac{\Theta_1(\omega)}{\Theta(\omega)},\\
\sin(\omega\tau) = \dfrac{\Theta_2(\omega)}{\Theta(\omega)},
\end{cases} \tag{3.10}$$

which leads to

$$\Theta^2(\omega) = \Theta_1^2(\omega) + \Theta_2^2(\omega). \tag{3.11}$$

According to Eq. (3.11), we can define

$$F(\omega) = \Theta^2(\omega) - \Theta_1^2(\omega) - \Theta_2^2(\omega) = 0. \tag{3.12}$$

Then by (3.12), we obtain

$$\begin{aligned} F(\omega) = m_{10}\omega^{10\alpha} + m_9\omega^{9\alpha} + m_8\omega^{8\alpha} + m_7\omega^{7\alpha} + m_6\omega^{6\alpha} \\ + m_5\omega^{5\alpha} + m_4\omega^{4\alpha} + m_3\omega^{3\alpha} + m_2\omega^{2\alpha} + m_1\omega^{\alpha} + m_0 = 0, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
m_{10} =& -P_2^2, \qquad m_9 = -2P_2(P_1 + P_2R_2)\cos\left(\frac{\alpha\pi}{2}\right),\\
m_8 =& P_2^4 - P_1^2 - P_2^2R_2^2 - 2P_1P_2R_2 - 2P_2(P_1R_2 + P_2R_1 + P_0)\cos(\alpha\pi)\\
m_7 =& -2P_2(P_0R_2 + P_1R_1 + P_2R_0)\cos\left(\frac{3\alpha\pi}{2}\right) - \big[2P_1^2R_2 - (4P_2^3 - (-2R_1 - 2R_2^2)P_2 - 2P_0)P_1\\
& + 2P_2R_2(P_2R_1 + P_0)\big]\cos\left(\frac{\alpha\pi}{2}\right)\\
m_6 =& -2P_2\big(P_0R_1 + P_1R_0\big)\cos(2\alpha\pi) + \Big[4P_0P_2^3 + \big[(2P_1^2 - 2R_0R_2)P_2^2 + \big((-2R_1R_2 - 2R_0)P_1 - 2P_0R_2^2\big)\big)P_2\big]\\
& - 2P_0P_1R_2 - 2P_1^2R_1\Big]\cos(\alpha\pi) + 4P_1^2P_2^2 - P_1^2R_2^2 - 2P_1P_2R_1R_2 - P_2^2R_1^2 - 2P_0P_1R_2 - 2P_0P_2R_1 - P_0^2,\\
m_5 =& -2P_0P_2R_0\cos\left(\frac{5\alpha\pi}{2}\right) - 2P_1^2R_0 + \big(4P_0P_2^2 - 2R_0R_2P_2 - 2P_0R_1\big)P_1 - 2P_0P_2R_1R_2\Big]\cos\left(\frac{3\alpha\pi}{2}\right)\\
& + \Big[4P_2P_1^3 - 2P_1^2R_1R_2 + \big(8P_0P_2^2 - (2R_1^2 + 2R_0R_2)P_2 - 2P_0(R_1 + R_2^2)\big)P_1\\
& - 2(P_0 + P_2R_1)(P_0R_2 + R_0P_2)\Big]\cos\left(\frac{\alpha\pi}{2}\right)\\
m_4 =& 2\Big[(-P_1 - P_2R_2)R_0 + P_0P_2^2\Big]P_0\cos(2\alpha\pi) + \Big[(8P_0P_2 - 2R_0R_2)P_1^2 + \big[(-2R_0 - 2R_1R_2)P_0\\
& - 2R_0R_1P_2\big)P_1 - 2P_0P_2R_1^2 - 2R_1P_0^2\Big]\cos(\alpha\pi) + 4P_0^2P_2^2 - P_0^2R_2^2 + 4P_0P_1^2P_2 - 2R_1R_2P_0P_1\\
& - 2P_0P_2R_0R_2 + P_1^4 - P_1^2R_1^2 - 2P_1P_2R_0R_1 - P_2^2R_0^2,\\
m_3 =& -2P_0\big[(P_1R_2 + P_2R_1 + P_0)R_0 - 2P_0P_1P_2\big]\cos\left(\frac{3\alpha\pi}{2}\right) + \Big[4P_0P_1^3 - 2P_1^2R_0R_1 + \big(8P_0^2P_2\\
& - 2(R_1^2 + R_0R_2)P_0 - 2P_2R_0^2\big)P_1 - 2P_0R_1(P_0R_2 + P_2R_0)\Big]\cos\left(\frac{\alpha\pi}{2}\right)\\
m_2 =& \big[4P_0^3P_2 + 2(P_1^2 - R_0R_2)P_0^2 - 2(P_1R_0R_1 + R_0^2P_2)P_0\big]\cos(\alpha\pi) + (4P_1^2 - R_1^2)P_0^2 - 2P_0P_1R_0R_1 - P_1^2R_0^2,\\
m_1 =& 2\big[2P_0^3P_1 - P_0^2R_0R_1 - P_0P_1R_0^2\big]\cos\left(\frac{\alpha\pi}{2}\right),\\
m_0 =& -P_0^2R_0^2 + P_0^4.
\end{aligned}$$

Eq. (3.13) can also be rewritten as

$$\begin{aligned} F(\omega) = \omega^{10\alpha} + M_9\omega^{9\alpha} + M_8\omega^{8\alpha} + M_7\omega^{7\alpha} + M_6\omega^{6\alpha} \\ + M_5\omega^{5\alpha} + M_4\omega^{4\alpha} + M_3\omega^{3\alpha} + M_2\omega^{2\alpha} + M_1\omega^{\alpha} + M_0 = 0, \end{aligned} \tag{3.14}$$

in which

$$M_i = \frac{m_i}{m_{10}}, \ (i = 0, 1, 2, \cdots, 9).$$

Now, we present the following lemma that guarantee the occurrence of Hopf bifurcation.

Lemma 3.2. *Considering the bifurcation point as* $\tau_0 = \min\{\tau^{(k)}\}$, $(k = 0, 1, 2, \cdots)$, *we can conclude the following results:*

(i): *If* $M_i > 0\,(i = 0, 1, 2, \cdots, 9)$, *then Eq. (3.6) has no positive roots for all* $\tau \geq 0$.
(ii): *If* $M_i < 0\,(i = 1, 2, \cdots, 9)$ *and* $M_0 > 0$, *then Eq. (3.6) has a pair of complex conjugate roots* $\pm \mathrm{i}\omega_0$ *when*

$$\tau^{(k)} = \frac{1}{\omega_0}\Big\{ \arccos\Big[\frac{\Theta_1(\omega)}{\Theta(\omega)}\Big] + 2k\pi\Big\},\ (k = 0, 1, 2, \cdots), \tag{3.15}$$

where ω_0 *is the unique positive root of* (3.13).

Proof. (**i**) Since $M_i > 0\,(i = 0, 1, 2, \cdots, 9)$, we have $F(0) = M_0 > 0$. By combining $\alpha > 0$ and $M_i > 0$, we deduce that Eq. (3.14) has no positive real root, hence Eq. (3.6) has no purely imaginary root.
(**ii**) Since $M_0 > 0$ and $F(0) = M_0 > 0$, we have $F'(\omega) > 0$, $\lim_{\omega\to+\infty} F(\omega) = +\infty$. Then there exists a unique positive number ω_0 , i.e, $F(\omega_0) = 0$. So, ω_0 is a root of Eq. (3.14) and for Eq. (3.15), (ω_0, τ_0) is a root of (3.8). Therefore, Eq. (3.6) has a pair of complex conjugate roots $\pm \mathrm{i}\omega_0$, when $\tau = \tau^{(k)}$, $k = 0, 1, 2, \cdots$. This completes the proof. ☐

To investigate the transversality condition for the occurrence of Hopf bifurcation, we further provide the following necessary assumption:

$$(\mathbf{H_2}) \qquad \widehat{l}_R\widehat{\mu}_R + \widehat{l}_I\widehat{\mu}_I > 0,$$

where

$$\begin{aligned}
\widehat{l}_R =&\ -P_2\omega_0^{2\alpha+1}\sin(\alpha\pi - \omega_0\tau_0) - P_1\omega_0^{\alpha+1}\sin\Big(\frac{\alpha\pi}{2} - \omega_0\tau_0\Big) + P_0\omega_0\sin(\omega_0\tau_0),\\
\widehat{l}_I =&\ P_2\omega_0^{2\alpha+1}\cos(\alpha\pi - \omega_0\tau_0) + P_1\omega_0^{\alpha+1}\cos\Big(\frac{\alpha\pi}{2} - \omega_0\tau_0\Big) + P_0\omega_0\cos(\omega_0\tau_0),\\
\widehat{\mu}_R =&\ 3\alpha\omega_0^{3\alpha-1}\sin\Big(\frac{3\alpha\pi}{2}\Big) + 2\alpha\Big[R_2\sin(\alpha\pi) + P_2\sin(\alpha\pi - \omega_0\tau_0)\Big]\omega_0^{2\alpha-1}\\
&+ \alpha\Big[R_1\sin\Big(\frac{\alpha\pi}{2}\Big) + P_1\sin\Big(\frac{\alpha\pi}{2} - \omega_0\tau_0\Big)\Big]\omega_0^{\alpha-1},\\
\widehat{\mu}_I =&\ -3\alpha\omega_0^{3\alpha-1}\cos\Big(\frac{3\alpha\pi}{2}\Big) - 2\alpha\Big[R_2\cos(\alpha\pi) + P_2\cos(\alpha\pi - \omega_0\tau_0)\Big]\omega_0^{2\alpha-1}\\
&- \alpha\Big[R_1\cos\Big(\frac{\alpha\pi}{2}\Big) + P_1\cos\Big(\frac{\alpha\pi}{2} - \omega_0\tau_0\Big)\Big]\omega_0^{\alpha-1}.
\end{aligned}$$

According to the above hypothesis, the following lemma can be achieved.

Lemma 3.3. *Suppose that* $s(\tau) = \zeta(\tau) + \mathrm{i}\omega(\tau)$ *is a root of Eq.* (3.6) *near* $\tau = \tau^{(k)}$ *satisfying* $\zeta(\tau^{(k)}) = 0$, $\omega(\tau^{(k)}) = \omega_0$, *then the transversality condition*

$$\Re\Big(\frac{ds}{d\tau}\Big)\Big|_{(\omega=\omega_0,\,\tau=\tau_0)} > 0,$$

holds.

Proof. By implicit differentiating of (3.6) with respect to τ, we obtain

$$N_1^{\star\prime}(s)\frac{ds}{d\tau} + N_2^{\star\prime}(s)\mathrm{e}^{-s\tau}\frac{ds}{d\tau} + N_2^{\star}(s)\mathrm{e}^{-s\tau}\Big(-\tau\frac{ds}{d\tau} - s\Big) = 0,$$

where $N_i^{\star\prime}(s)$ indicate the derivatives of $N_i^{\star}(s)$ $(i = 1, 2)$. It follows that

$$\frac{ds}{d\tau} = \frac{sN_2^{\star}(s)\mathrm{e}^{-s\tau}}{N_1^{\star\prime}(s) + N_2^{\star\prime}(s)\mathrm{e}^{-s\tau} - \tau N_2^{\star}(s)\mathrm{e}^{-s\tau}} = \frac{l(s)}{\mu(s)}, \tag{3.16}$$

where $l(s) = sN_2^{\star}(s)\mathrm{e}^{-s\tau}$ and $\mu(s) = N_1^{\star\prime}(s) + N_2^{\star\prime}(s)\mathrm{e}^{-s\tau} - \tau N_2^{\star}(s)\mathrm{e}^{-s\tau}$.
To check $\Re\big(\frac{ds}{d\tau}\big) > 0$, we can use $\big(\frac{ds}{d\tau}\big)^{-1}$ instead of $\big(\frac{ds}{d\tau}\big)$. In such a way, we have

$$\left[\frac{ds}{d\tau}\right]^{-1} = \frac{\widehat{\mu}(s)}{\widehat{l}(s)} - \frac{\tau}{s}, \tag{3.17}$$

where

$$\begin{aligned}\widehat{\mu}(s) = & -\alpha\big[P_2 s^{5\alpha} + 2P_1 s^{4\alpha} + (3P_0 + P_1R_2 - P_2R_1)s^{3\alpha} + 2(P_0R_2 - P_1R_0)s^{2\alpha} \\ & + (P_0R_1 - P_1R_0)s^{\alpha}\big],\end{aligned}$$

and

$$\begin{aligned}\widehat{l}(s) = s^2\big[& P_2 s^{5\alpha} + (P_1 + P_2R_2)s^{4\alpha} + (P_1R_2 + P_2R_1 + P_0)s^{3\alpha} + (P_0R_2 + P_1R_1 + P_2R_0)s^{2\alpha} \\ & + (P_0R_1 + P_1R_0)s^{\alpha} + P_0R_0\big].\end{aligned}$$

By Eq. (3.17), we have

$$\Re\left[\left(\frac{ds}{d\tau}\right)^{-1}\right]_{(\omega=\omega_0,\,\tau=\tau_0)} = \Re\left[\frac{\widehat{\mu}(s)}{\widehat{l}(s)}\right]_{(\omega=\omega_0,\,\tau=\tau_0)} = \frac{\widehat{l}_R\widehat{\mu}_R + \widehat{l}_I\widehat{\mu}_I}{\widehat{l}_R^2 + \widehat{l}_I^2}.$$

Applying (H_2), we deduce

$$\Re\left[\left(\frac{ds}{d\tau}\right)^{-1}\right]_{(\omega=\omega_0,\,\tau=\tau_0)} > 0,$$

which completes the proof. □

By the previous analysis and combining the Lemmas 3.1, 3.2 and 3.3, we can establish the following theorem.

Theorem 3.4. *For the uncontrolled system (1.2) if the assumptions* (H_1) *and* (H_2) *hold, then we deduce the following results.*

- **(i):** *E^* is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$.*
- **(ii):** *When $\tau = \tau_0$, the uncontrolled system (1.2) undergoes a Hopf bifurcation at E^* i.e., a periodic solution bifurcates from E^* near $\tau = \tau_0$.*

3.2. **Influence of the controller** (1.3) **on bifurcation for uncontrolled system** (1.2)**.** In this subsection, by employing the time-delayed force (1.3), we focus on the problem of bifurcation control for stabilizing the creation of bifurcation in uncontrolled system (1.2). The considered controlled system takes the form

$$\begin{cases} D_t^{\alpha}x(t) = x(1-bx) - xy - xz, \\ D_t^{\alpha}y(t) = -cy + xy + u(t), \\ D_t^{\alpha}z(t) = fx(t-\tau)z(t-\tau) + gy(t-\tau)z(t-\tau) - (e+\beta z)z, \end{cases} \tag{3.18}$$

where $u(t)$ is stated in (1.3).

Analogously, by performing a transformation $W_1(t) = x(t) - x^*$, $W_2(t) = y(t) - y^*$, $W_3(t) = z(t) - z^*$ and using the linearized scheme, the linear equation of the controlled system (3.18) around E^* can be derived as follows:

$$\begin{cases} D_t^{\alpha}W_1(t) = -bx^*W_1(t) - x^*W_2(t) - x^*W_3(t), \\ D_t^{\alpha}W_2(t) = y^*W_1(t) + K\big(W_2(t) - W_2(t-\tau)\big), \\ D_t^{\alpha}W_3(t) = fz^*W_1(t-\tau) + gz^*W_2(t-\tau) + (fx^* + gy^*)W_3(t-\tau) - (e+2\beta z^*)W_3(t). \end{cases} \tag{3.19}$$

The corresponding characteristic equation can be formulated as

$$\det \begin{bmatrix} s^\alpha + bx^* & x^* & x^* \\ -y^* & s^\alpha - K + K\mathrm{e}^{-s\tau} & 0 \\ -fz^*\mathrm{e}^{-s\tau} & -gz^*\mathrm{e}^{-s\tau} & s^\alpha + e + 2\beta z^* - (fx^* + gy^*)\mathrm{e}^{-s\tau} \end{bmatrix} = 0,$$

which is equivalent to

$$s^{3\alpha} + R_{22}s^{2\alpha} + R_{21}s^\alpha + R_{20} + (P_{22}s^{2\alpha} + P_{21}s^\alpha + P_{20})\mathrm{e}^{-s\tau} + (Q_{11}s^\alpha + Q_{10})\mathrm{e}^{-2s\tau} = 0, \tag{3.20}$$

where

$$\begin{aligned}
R_{22} &= bx^* + 2\beta z^* - K + e,\\
R_{21} &= (-bK + (e + 2\beta z)b + y^*)x^* + (-e - 2\beta z^*)K,\\
R_{20} &= \big((-e - 2\beta z^*)bK + (e + 2\beta z^*)y^*\big)x^*,\\
P_{22} &= K - fx^* - gy^*,\\
P_{21} &= -bfx^* + \big((b + f)K - bgy^* + fz^*\big)x^* + (2\beta z^* + gy^* + e)K,\\
P_{20} &= \big(f(bK - y^*)x^* + \big((2\beta z^* + gy^* + e)b - fz^*\big)K - g(y^* - z^*)y^*\big)x^*,\\
Q_{11} &= -K(fx^* + gy^*),\\
Q_{10} &= K(bfx^* + bgy^* - fz^*)x^*.
\end{aligned}$$

We further simplify the Eq. (3.20) as

$$\xi_{01}(s) + \xi_{11}(s)\mathrm{e}^{-s\tau} + \xi_{21}(s)\mathrm{e}^{-2s\tau} = 0, \tag{3.21}$$

where

$$\begin{aligned}
\xi_{01} &= s^{3\alpha} + R_{22}s^{2\alpha} + R_{21}s^\alpha + R_{20},\\
\xi_{11} &= P_{22}s^{2\alpha} + P_{21}s^\alpha + P_{20},\\
\xi_{21} &= Q_{11}s^\alpha + Q_{10}.
\end{aligned}$$

By multiplying $\mathrm{e}^{s\tau}$ on both side of (3.21), we get

$$\xi_{01}(s)\mathrm{e}^{s\tau} + \xi_{11}(s) + \xi_{21}(s)\mathrm{e}^{-s\tau} = 0. \tag{3.22}$$

Let $s = \mathrm{i}\varpi = \varpi\big(\cos(\frac{\pi}{2}) + \mathrm{i}\sin(\frac{\pi}{2})\big)$ $(\varpi > 0)$ be a root of Eq. (3.22). Then by substituting s into Eq. (3.22) and separating the real and imaginary parts, we obtain

$$\begin{cases} (\beta_1 + \beta_3)\cos(\varpi\tau) - \gamma_1\sin(\varpi\tau) = -\beta_2,\\ \gamma_1\cos(\varpi\tau) + (\beta_1 - \beta_3)\sin(\varpi\tau) = -\gamma_2, \end{cases} \tag{3.23}$$

in which

$$\begin{aligned}
\beta_1 &= \varpi^{3\alpha}\cos(\frac{3\alpha\pi}{2}) + R_{22}\varpi^{2\alpha}\cos(\alpha\pi) + R_{20},\\
\beta_2 &= P_{22}\varpi^{2\alpha}\cos(\alpha\pi) + P_{21}\varpi^\alpha\cos(\frac{\alpha\pi}{2}) + P_{20},\\
\beta_3 &= (R_{21} + Q_{11})\varpi^\alpha\cos(\frac{\alpha\pi}{2}) + Q_{10},\\
\gamma_1 &= \varpi^{3\alpha}\sin(\frac{3\alpha\pi}{2}) + R_{22}\varpi^{2\alpha}\sin(\alpha\pi) + (R_{21} + Q_{11})\varpi^\alpha\sin(\frac{\alpha\pi}{2}),\\
\gamma_2 &= P_{22}\varpi^{2\alpha}\sin(\alpha\pi) + P_{21}\varpi^\alpha\sin(\frac{\alpha\pi}{2}).
\end{aligned}$$

Then we can derive

$$\begin{cases} \cos(\varpi\tau) = -\dfrac{(\beta_1-\beta_3)\beta_2+\gamma_1\gamma_2}{\beta_1^2+\gamma_1^2+(\beta_2-\beta_3)\beta_1-\beta_2\beta_3} = \dfrac{\overline{\Theta}_1(\varpi)}{\overline{\Theta}(\varpi)}, \\ \sin(\varpi\tau) = \dfrac{(\gamma_1-\gamma_2)\beta_2-\beta_1\gamma_2}{\beta_1^2+\gamma_1^2+(\beta_2-\beta_3)\beta_1-\beta_2\beta_3} = \dfrac{\overline{\Theta}_2(\varpi)}{\overline{\Theta}(\varpi)}. \end{cases} \tag{3.24}$$

Since $\sin^2(\omega\tau)+\cos^2(\omega\tau)=1$, we have

$$\overline{\Theta}^2(\varpi) = \overline{\Theta}_1^2(\varpi) + \overline{\Theta}_2^2(\varpi). \tag{3.25}$$

According to $\cos(\omega\tau) = \frac{\overline{\Theta}_1(\varpi)}{\overline{\Theta}(\varpi)}$, we deduce

$$\tau^{(l)} = \frac{1}{\varpi}\left[\arccos\left(\frac{\overline{\Theta}_1(\varpi)}{\overline{\Theta}(\varpi)}\right)+2l\pi\right],\ l=0,1,2,\cdots. \tag{3.26}$$

The bifurcation point can be presented by

$$\bar{\tau}_{10} = \min\{\tau^{(l)}\},\ l=0,1,2,\cdots,$$

where $\tau^{(l)}$ is already defined by Eq. (3.26).
Now based on (3.25), we further suppose

$$\bar{F}(\varpi) = \overline{\Theta}^2(\varpi) - \overline{\Theta}_1^2(\varpi) - \overline{\Theta}_2^2(\varpi) = 0, \tag{3.27}$$

which can be deduced that Eq. (3.27) has at least one nonnegative real root.
In the following, we further looking for the conditions for the emergence of Hopf bifurcation. Hence, the additional hypothesis can be introduced as follows:

$$(\mathbf{H_3}) \qquad \frac{\bar{l}_1\bar{\mu}_1+\bar{l}_2\bar{\mu}_2}{\bar{\mu}_1^2+\bar{\mu}_2^2} \neq 0,$$

where

$$
\begin{aligned}
\bar{l}_1 =& P_{22}\sin(\alpha\pi - \varpi_{10}\bar{\tau}_{10})\varpi_{10}^{2\alpha+1} + \Big(2Q_{11}\sin\Big(\frac{\alpha\pi}{2} - 2\varpi_{10}\bar{\tau}_{10}\Big) \\
&+ P_{21}\sin\Big(\frac{\alpha\pi}{2} - \varpi_{10}\bar{\tau}_{10}\Big)\Big)\varpi_{10}^{\alpha+1} - \varpi_{10}\big(P_{20}\sin(\varpi_{10}\bar{\tau}_{10}) + 2Q_{10}\sin(2\varpi_{10}\bar{\tau}_{10})\big),\\
\bar{l}_2 =& - P_{22}\cos(\alpha\pi - \varpi_{10}\bar{\tau}_{10})\varpi_{10}^{2\alpha+1} - \Big[2Q_{11}\cos\Big(\frac{\alpha\pi}{2} - 2\varpi_{10}\bar{\tau}_{10}\Big) \\
&+ P_{21}\cos\Big(\frac{\alpha\pi}{2} - \varpi_{10}\bar{\tau}_{10}\Big)\Big]\varpi_{10}^{\alpha+1} - \varpi_{10}\big(P_{20}\cos(\varpi_{10}\bar{\tau}_{10}) + 2Q_{10}\cos(2\varpi_{10}\bar{\tau}_{10})\big),\\
\bar{\mu}_1 =& - 3\alpha\sin\Big(\frac{3\alpha\pi}{2}\Big)\varpi_{10}^{3\alpha-1} - 2\alpha\Big(R_{22}\sin(\alpha\pi) + P_{22}\sin(\alpha\pi - \varpi_{10}\bar{\tau}_{10})\Big)\varpi_{10}^{2\alpha-1}\\
&- \alpha\Big(P_{21}\sin\Big(\frac{\alpha\pi}{2} - \varpi_{10}\bar{\tau}_{10}\Big) + Q_{11}\sin\Big(\frac{\alpha\pi}{2} - 2\varpi_{10}\bar{\tau}_{10}\Big) + R_{21}\sin\Big(\frac{\alpha\pi}{2}\Big)\Big)\varpi_{10}^{\alpha-1}\\
&+ \bar{\tau}_{10}\Big[P_{22}\cos(\alpha\pi - \varpi_{10}\bar{\tau}_{10})\varpi_{10}^{2\alpha} + \Big(2Q_{11}\cos\Big(\frac{\alpha\pi}{2} - 2\varpi_{10}\bar{\tau}_{10}\Big) + P_{21}\cos\Big(\frac{\alpha\pi}{2} - \varpi_{10}\bar{\tau}_{10}\Big)\Big)\varpi_{10}^{\alpha}\\
&+ P_{20}\cos(\varpi_{10}\bar{\tau}_{10}) + 2Q_{10}\cos(2\varpi_{10}\bar{\tau}_{10})\Big],\\
\bar{\mu}_2 =& 3\alpha\cos\Big(\frac{3\alpha\pi}{2}\Big)\varpi_{10}^{3\alpha-1} + 2\alpha\big(P_{22}\cos(\alpha\pi - \varpi_{10}\bar{\tau}_{10}) + R_{22}\cos(\alpha\pi)\big)\varpi_{10}^{2\alpha-1}\\
&+ \alpha\Big[Q_{11}\cos\Big(\frac{\alpha\pi}{2} - 2\varpi_{10}\bar{\tau}_{10}\Big) + P_{21}\cos\Big(\frac{\alpha\pi}{2} - \varpi_{10}\bar{\tau}_{10}\Big) + R_{21}\cos\Big(\frac{\alpha\pi}{2}\Big)\Big]\varpi_{10}^{\alpha-1}\\
&+ \bar{\tau}_{10}\Big[P_{22}\sin(\alpha\pi - \varpi_{10}\bar{\tau}_{10})\varpi_{10}^{2\alpha} + \Big(P_{21}\sin\Big(\frac{\alpha\pi}{2} - \varpi_{10}\bar{\tau}_{10}\Big) + 2Q_{11}\sin\Big(\frac{\alpha\pi}{2} - 2\varpi_{10}\bar{\tau}_{10}\Big)\Big)\varpi_{10}^{\alpha}\\
&- P_{20}\sin(\varpi_{10}\bar{\tau}_{10}) - 2Q_{10}\sin(2\varpi_{10}\bar{\tau}_{10})\Big].
\end{aligned}
$$

Lemma 3.5. *Suppose that $s(\tau) = \bar{\zeta}(\tau) + \mathrm{i}\varpi(\tau)$ be the root of Eq. (3.21) near $\tau = \tau^{(l)}$ satisfying $\bar{\zeta}(\tau^{(l)}) = 0$, $\varpi(\tau^{(l)}) = \varpi_{10}$, then the following transversality condition holds*

$$
\Re\Big(\frac{ds}{d\tau}\Big)\Big|_{(\omega=\varpi_{10},\, \tau=\bar{\tau}_{10})} \neq 0.
$$

Proof. Differentiating Eq. (3.21) with regard to τ in the light of implicit function theorem, we acquire that

$$
\xi_{01}'(s)\frac{ds}{d\tau} + \xi_{11}'(s)\mathrm{e}^{-s\tau}\frac{ds}{d\tau} + \xi_{11}(s)\mathrm{e}^{-s\tau}\Big(-\tau\frac{ds}{d\tau} - s\Big) + \xi_{21}'(s)\mathrm{e}^{-2s\tau}\frac{ds}{d\tau} + \xi_{21}(s)\mathrm{e}^{-2s\tau}\Big(-2\tau\frac{ds}{d\tau} - 2s\Big) = 0, \tag{3.28}
$$

where $\xi_{i1}'(s)$ is the derivatives of $\xi_{i1}(s)$ $(i = 0, 1, 2)$.
Thus we have

$$
\frac{ds}{d\tau} = \frac{\bar{l}(s)}{\bar{\mu}(s)}, \tag{3.29}
$$

where

$$
\begin{aligned}
\bar{l}(s) &= s\big[\xi_{11}(s)\mathrm{e}^{-s\tau} + 2\xi_{21}(s)\mathrm{e}^{-2s\tau}\big],\\
\bar{\mu}(s) &= \xi_{01}'(s) + \big[\xi_{11}'(s) - \tau\xi_{11}(s)\big]\mathrm{e}^{-s\tau} + \big[\xi_{21}'(s) - 2\tau\xi_{21}(s)\big]\mathrm{e}^{-2s\tau}.
\end{aligned}
$$

Let $\bar{l}_i$ $(i = 1, 2)$ be the real and imaginary parts of $\bar{l}(s)$ and $\bar{\mu}_j$ $(j = 1, 2)$ be the real and imaginary parts of $\bar{\mu}(s)$, respectively. Then it follows from Eq. (3.29) that

$$
\Re\Big[\frac{ds}{d\tau}\Big]\Big|_{(\omega=\varpi_{10},\, \tau=\bar{\tau}_{10})} = \frac{\bar{l}_1\bar{\mu}_1 + \bar{l}_2\bar{\mu}_2}{\bar{\mu}_1^2 + \bar{\mu}_2^2}. \tag{3.30}
$$

The assumption $(\mathbf{H_3})$ intimates that transversality condition is met. Hence, the proof is accomplished. □

Based on the previous analysis, we have the following theorem.

Theorem 3.6. *Under the assumptions* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$*, the following results are deduced.*

(i): *If* $\tau \in [0, \bar{\tau}_{10})$*, then the positive equilibrium* E^* *of the controlled system* (3.18) *is asymptotically stable.*

(ii): *If the condition* $(\mathbf{H_3})$ *of Lemma 3.5 holds, then the controlled system* (3.18) *exhibits a Hopf bifurcation at* E^* *when* $\tau = \bar{\tau}_{10}$.

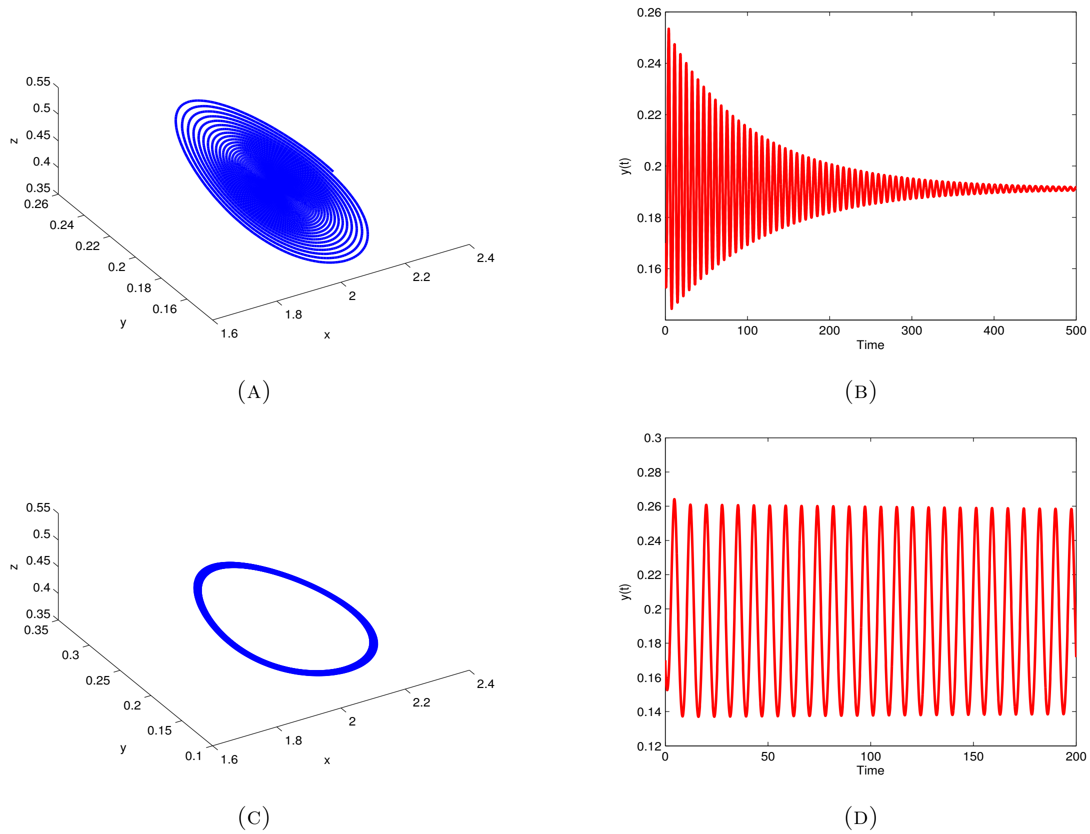


FIGURE 1. Phase portraits and time series solutions of system (4.1) when $\alpha = 0.97$. (A) and (B) indicate that for $\tau = 0.45 < \tau_0 = 0.6122$, E^* is locally asymptotically stable. (C) and D) show that for $\tau = 0.6322 > \tau_0$, a periodic oscillation emerges.

4. Numerical examples and simulations

In this section, we present two numerical examples to verify the feasibility of the theoretical analysis regarding the stability and bifurcation of the fractional-order delayed model (1.2), as well as the controlled system with a time-delayed feedback controller. Additionally, we aim to reveal more dynamical behavior of the system through numerical simulations. To carry out these simulations, we employ the Adams–Bashforth–Moulton (ABM) predictor-corrector scheme, as discussed in previous works [4, 34]. The simulations are conducted using MATLAB 2013a and Maple 2017 as computational tools. The ABM predictor-corrector scheme combines two approaches, namely, the explicit Adams-Bashforth method and the implicit Adams-Moulton method. The explicit component utilizes past values to estimate future values, providing a quick estimate, while the implicit component refines this estimate by incorporating future values, thus enhancing stability and accuracy. This combination allows for higher precision compared to using

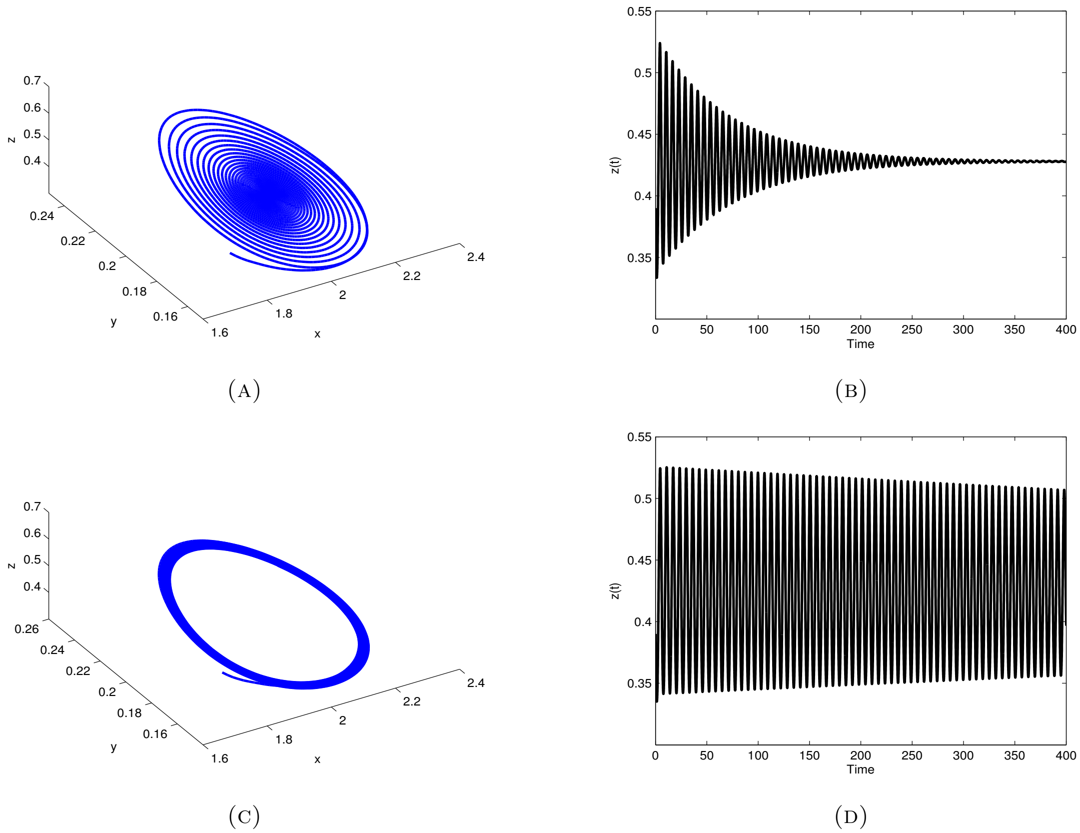


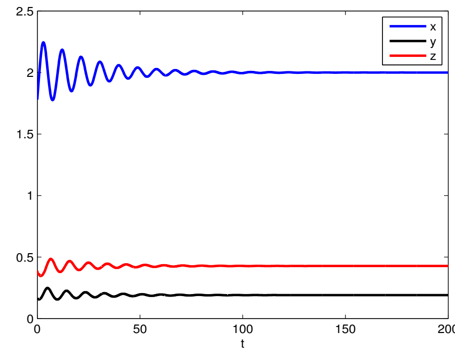
FIGURE 2. Phase portraits and time series solutions of system (4.1) when $\alpha = 1$. (A) and (B) indicate that E^* is locally asymptotically stable when $\tau = 0.221 < \tau_0^* = 0.2508$, while (C) and (D) show that E^* becomes unstable and a periodic oscillation emerges when $\tau = 0.265 > \tau_0^*$.

either method independently. The implicit part of the scheme is particularly advantageous for stiff ordinary differential equations, where traditional explicit methods may encounter difficulties. By utilizing past values, the ABM method reduces the number of function evaluations needed, making it computationally efficient for long-term integration problems. Furthermore, the ABM method can be easily adapted to various types of ODEs and boundary conditions, making it a versatile tool in numerical analysis and computational applications.
In the simulation, the step-length is designated as $h = 0.05$, and the initial values are taken as $(1.78, 0.17, 0.39)$ for all the examples. Moreover, we consider the same parameter values as in [20] which is listed in Table 1. For these biologically feasible parameter set, the fractional-order time delay system (1.2) has only one positive equilibrium computed as $E^*(x^*, y^*, z^*) = (2, \frac{3211}{16800}, \frac{1027}{2400})$.

TABLE 1. The set of parameter and their values.

Parameter	b	β	c	e	f	g
Value	$\frac{4}{21}$	2	2	3	$\frac{907}{700}$	$\frac{86}{13}$

(A)



(B)

FIGURE 3. (A) Phase portrait of the system (4.2) when $\alpha = 0.97$ and $\tau = 0.896$. (B) Time series evolution of species for $\alpha = 0.97$ and $\tau = 0.896$.

Example 4.1. Consider the following fractional-order predator–prey system with time delay:

$$\begin{cases} D_t^\alpha x(t) = x(1 - \dfrac{4}{21}x) - xy - xz, \\ D_t^\alpha y(t) = -2y + xy, \\ D_t^\alpha z(t) = \dfrac{907}{700}x(t-\tau)z(t-\tau) + \dfrac{86}{13}y(t-\tau)z(t-\tau) - (3+2z)z, \end{cases} \tag{4.1}$$

where $\alpha \in (0, 1]$.

Choosing $\alpha = 0.97$, we compute $\omega_0 \approx 0.8267$, $\tau_0 \approx 0.6122$. If we consider the time delay τ as a bifurcation parameter, then by the aid of Theorem 3.4 we can observe that E^* is asymptotically stable and their state trajectories converges to E^* when $\tau = 0.45 < \tau_0$. The corresponding dynamics behavior is depicted by the subfigures (a) and (b) of Figure 1. The subfigures (c) and (d) in Figure 1 indicate that E^* of uncontrolled system (4.1) is unstable, i.e, a Hopf bifurcation occurs when $\tau = 0.6322 > \tau_0$. When $\alpha = 1$, the uncontrolled system (4.1) coincides with the integer order system and for this case, the critical frequency and bifurcation value are acquired as $\omega_0^* \approx 0.9918$, $\tau_0^* \approx 0.2508$, respectively. For $\tau = 0.221$ and $\tau = 0.265$, the associated stable and unstable behaviours are depicted in Figure 2. We can see that fractional-order is capable to postpone the onset and appearance of Hopf bifurcation which leads to amplify the stability domain of the corresponding fractional-order system.

Example 4.2. Herein, for controlling the onset of Hopf bifurcation of uncontrolled system (1.2), the controller (1.3) is imposed on uncontrolled system (1.2) to better display the effects of bifurcation control in the system. The same set of parameters and the fractional-orders as in Example 4.1 are taken, then the fractional-order controlled system can be described as

$$\begin{cases} D_t^\alpha x(t) = x(1 - \dfrac{4}{21}x) - xy - xz, \\ D_t^\alpha y(t) = -2y + xy + K\big(y(t) - y(t-\tau)\big), \\ D_t^\alpha z(t) = \dfrac{907}{700}x(t-\tau)z(t-\tau) + \dfrac{86}{13}y(t-\tau)z(t-\tau) - (3+2z)z. \end{cases} \tag{4.2}$$

We numerically investigate the impacts of feedback gain K for postponing the Hopf bifurcation onset of system (4.2). Thus by choosing $K = -0.48$ and applying Eq. (3.26) and Eq. (3.27), we obtain $\overline{\omega}_0 \approx 0.16858$, and $\bar{\tau}_0 \approx 0.9032$. Figure 3 depicts the numerical simulations of the controlled model (4.2) for $\tau = 0.896 < \bar{\tau}_0 \approx 0.9032$, which clearly reveals that the stability performance of the system is ameliorated and the considered uncontrolled system is successfully controlled.

Also for varying the feedback gain K, some comparative numerical results versus the associated critical frequency ϖ_0 and the bifurcation point $\bar{\tau}_0$ are carefully computated which are listed in Table 2.
Moreover, it has been shown that the impact of the feedback gain plays an important role on the stabilizing bifurcation

Table 2. The impact of K on the values of ϖ_0 for controlled system (4.2) with $\alpha = 0.97$.

Feedback gain K	**Critical frequency ϖ_0**	**Bifurcation point $\bar{\tau}_0$**
-1.98	0.60045	0.47716
-1.85	0.57980	0.47796
-1.48	0.51383	0.54986
-1.15	0.44199	0.63538
-0.95	0.38854	0.70080
-0.65	0.27837	0.82250
-0.48	0.16858	0.90318
-0.35	0.87732	0.56016
-0.25	0.80251	0.84443
-0.15	0.72102	1.05859
-0.095	0.67589	1.43534

and it can be observed that the performance of bifurcation control are effectively improved when the feedback gain gradually decreases.

5. Conclusion

In this study, we present a novel fractional-order delayed predator-prey model that incorporates three species, with one species functioning as an omnivore. We derive the characteristic equation associated with the model utilizing the Laplace transform and subsequently compute the critical frequencies alongside the critical time delay values that facilitate the emergence of Hopf bifurcation. Our analysis yields significant results, including a comprehensive set of sufficient conditions that guarantee the local asymptotic stability of the proposed model. The findings indicate that the incorporation of time delays exerts a profound influence on both stability and the occurrence of Hopf bifurcation; specifically, the fractional-order model with delay demonstrates the emergence of Hopf bifurcation at the positive equilibrium point when the time delay exceeds its critical threshold. A comparative analysis between integer-order and fractional-order models reveals that the fractional-order system possesses a more extensive stability domain than its integer-order counterpart. Furthermore, both fractional-order dynamics and time delays are pivotal in shaping the dynamical behaviors of the fractional delay model, thereby enriching its complexity and enhancing stability. To address the potential instability arising from Hopf bifurcation in an uncontrolled system, we implement state feedback control with time delay. Our results illustrate that through judicious adjustment of the feedback gain and time delay parameters, it is possible to effectively postpone the onset of Hopf bifurcation within the fractional-order system. Numerical analyses indicate that this control strategy, which incorporates time delays, demonstrates superior efficacy in stabilizing bifurcations. For the numerical approximation of the model's solutions, we employ a modified Adams-Bashforth-Moulton predictor-corrector scheme. The simulations conducted validate the accuracy and effectiveness of our theoretical findings.

Key insights from our research include:

- The control strategy can stabilize populations and minimize oscillations.
- Understanding time delays and fractional-order dynamics enhances conservation efforts for vulnerable species.
- The model aids in ecosystem monitoring, allowing for early detection of instability.

Future research should focus on addressing the following key areas:

- Expanding the model: The model's predictive capabilities can be significantly improved by incorporating spatial heterogeneity, environmental variability, and additional ecological interactions.

- Optimizing the control effectiveness: Determining the optimal feedback gain and time delay for different ecological scenarios is essential for maximizing the effectiveness of control strategies.

6. Acknowledgments

We are grateful to the reviewers for their thoughtful and constructive feedback, which has greatly enhanced our manuscript.

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