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Zero-Hopf Bifurcation in a Four-dimensional Quartic Polynomial Differential System via Averaging Theory of Third Order

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Abstract

The averaging theory of third order shows that for a 4-dimensional Quartic Polynomial Differential System at most 36 limit cycles can bifurcate from one singularity with eigenvalues of the form $\pm\omega i$, 0 and 0.

Keywords. averaging theory, zero-Hopf bifurcation, Quartic polynomial differential systems.

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Uncorrected Proof

1. Introduction

The averaging approach is a well-established and sophisticated technique that enables us to study the behaviour of nonlinear differential systems under periodic stimulation. The extensive chronicle of the averaging method originates from the classical contributions of Lagrange and Laplace, who offered a rational rationale for the method. In 1928, Fatou provided the first formalisation of this theory [5].Significant practical and theoretical advancements in the field of averaging theory were accomplished in the 1930s by Bogoliubov and Krylov [7], and in 1945 by Bogoliubov [2]. In 2004, Buica and Llibre utilized the Brouwer degree to expand the application of the averaging theory in the analysis of periodic orbits in continuous differential systems [3]. For a comprehensive introduction to this theory, we refer to Sanders, Verhulst, and Murdock's book [12].

A zero-Hopf equilibrium in $\mathbb{R}^n$ with $n \geq 3$ is an isolated equilibrium point in the set of all equilibrium points of the differential system, having the eigenvalues $\pm bi$ with $b \neq 0$ and the eigenvalue 0 with multiplicity $n-2$.

In the paper [10] it was proved that 2^{n-3} periodic solutions can bifurcate from a zero–Hopf equilibrium of a polynomial differential system in $\mathbb{R}^n$ for $n \geq 3$ using the averaging theory of first order.

The authors investigated the zero–Hopf bifurcation in polynomial differential systems in $\mathbb{R}^3$ with quadratic homogeneous non-linearities. They utilized the averaging theory of third and second order in their respective studies, as indicated by references [1] and [8]. It has been demonstrated that a singular point with eigenvalues of the form $\pm bi$ and 0 can give rise to a maximum of 10 limit cycles and a minimum of 3 limit cycles through bifurcation.

The authors in [6] investigated the zero-Hopf bifurcation in polynomial differential systems in $\mathbb{R}^4$ with cubic homogeneous non-linearities, using the second-order averaging theory. It has been demonstrated that a maximum of 9 limit cycles can emerge from a singular point with eigenvalues in the form of $\pm bi$ and two zeros.

The authors in [4] examined the zero-Hopf bifurcation of a quadratic polynomial differential system in four dimensions using the third-order averaging theory. This paper aims to analyze the quantity of limit cycles that can bifurcate from a zero-Hopf equilibrium point of a polynomial differential system in $\mathbb{R}^4$ with **quartic** homogeneous non-linearities. The

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analysis will be conducted using the third-order averaging theory. To be more exact, we are examining the differential systems that have the following form

$$\begin{aligned}
\dot{x} &= (a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3)x - (b + b_1\epsilon + b_2\epsilon^2 + b_3\epsilon^3)y + \sum_{j=0}^{2} \epsilon^j X_j(x,y,z,w),\\
\dot{y} &= (b + b_1\epsilon + b_2\epsilon^2 + b_3\epsilon^3)x + (a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3)y + \sum_{j=0}^{2} \epsilon^j Y_j(x,y,z,w),\\
\dot{z} &= (c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3)z + \sum_{j=0}^{2} \epsilon^j Z_j(x,y,z,w),\\
\dot{w} &= (d_1\epsilon + d_2\epsilon^2 + d_3\epsilon^3)w + \sum_{j=0}^{2} \epsilon^j W_j(x,y,z,w),
\end{aligned} \tag{1.1}$$

where

$$\begin{aligned}
X_j(x,y,z,w) &= a_{j0}x^4 + a_{j1}x^3y + a_{j2}x^3z + a_{j3}x^3w + a_{j4}x^2y^2 + a_{j5}x^2z^2 + a_{j6}x^2w^2 + a_{j7}x^2yz + a_{j8}x^2yw\\
&+ a_{j9}x^2zw + a_{j10}xy^3 + a_{j11}xz^3 + a_{j12}xw^3 + a_{j13}xy^2z + a_{j14}xy^2w + a_{j15}xz^2y + a_{j16}xz^2w\\
&+ a_{j17}xw^2y + a_{j18}xw^2z + a_{j19}xyzw + a_{j20}y^4 + a_{j21}y^3z + a_{j22}wy^3 + a_{j23}y^2z^2 + a_{j24}w^2y^2\\
&+ a_{j25}wy^2z + a_{j26}yz^3 + a_{j27}w^3y + a_{j28}wyz^2 + a_{j29}w^2yz + a_{j30}z^4 + a_{j31}wz^3 + a_{j32}w^2z^2\\
&+ a_{j33}w^3z + a_{j34}w^4,
\end{aligned}$$

$Y_j(x,y,z,w)$, $Z_j(x,y,z,w)$ and $W_j(x,y,z,w)$ have the same expression as $X_j(x,y,z,w)$ by replacing a_{ji} by b_{ji}, c_{ji} and d_{ji} for $j = 0,1,2$ and $i = 0,1,\dots,34$, respectively. The coefficients $a_{ij}, b_{ij}, c_{ij}, d_{ij}, a_1, a_2, a_3, b, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3$ are real parameters with $b \neq 0$. Note that system (1.1) for $\epsilon = 0$ at the origin has eigenvalues $\pm bi, 0, 0$. So for $\epsilon = 0$ the origin is a *zero-Hopf equilibrium*.

Our main result is the following one.

Theorem 1.1. *By applying the averaging theory of the third order system* (1.1) *has at most* 36 *periodic solutions bifurcating from the origin when* $\epsilon = 0$, *if and only if the following condition is satisfied* $b((a_{013} + 3b_{021} + 3a_{02} + b_{07})\beta + \xi(a_{014} + 3b_{022} + 3a_{03} + b_{08}))^2 \neq 0$.

Theorem 1.1 is proved in section 3 using the averaging theory for computing periodic orbits. A summary of the results from the averaging theory that we need for proving Theorem 1.1 is presented in section 2. We will use the Bezout's theorem. This theorem gives the maximum number of zeros of a system of polynomial functions.

Theorem 1.2. *(Bezout's theorem). Let P_i be polynomials in the variables $(y_1, \cdots, y_n) \in \mathbb{R}^n$ of degree d_i for $i = 1, \cdots, n$. Consider the following polynomial system*

$$P_i(y_1, \cdots, y_n) = 0, \quad i = 1, \cdots, n,$$

If the number of solutions of this system is finite, then it is bounded by $d_1 \times d_2 \times \cdots \times d_n$.

See [14] for more details on the Bezout's theorem.

Uncorrected Proof

Corollary 1.3. *Consider the system*

$$
\begin{aligned}
\dot{x} = \;& -x\epsilon^3 - (\epsilon^3+\epsilon^2+\epsilon+1)y - \frac{5x^3w}{3} - xw^3 - xz^3 + w^4 + w^2xy - w^2xz + wxy^2 - 3wxz^2 + \\
& x^2yz + xy^2z + xyz^2 + y^4, \\
\dot{y} = \;& (\epsilon^3+\epsilon^2+\epsilon+1)x - y\epsilon^3 + 2w^4 + w^3y + wy^3 + 3wz^3 + x^3y + 2y^4 - yz^3 - w^2yz + wx^2y + \\
& wyz^2 + x^2yz, \\
\dot{z} = \;& -\frac{41}{4}\epsilon^3 z + y^2z^2 + w^2y^2 + x^2y^2 + x^2z^2 - 4wz^3 + 4w^3z - 3x^2w^2 - 2w^2z^2 + 2w^4 - \frac{23}{4}z^4 + \\
& y^4 - x^4 - wx^2z - wy^2z, \\
\dot{w} = \;& -\frac{206}{7}\epsilon^3 w + wy^2z + x^2zw + y^2z^2 + w^2y^2 + \frac{1331}{28}w^2z^2 + x^2y^2 - \frac{60}{7}w^3z - \frac{1377}{56}x^2z^2 - \\
& \frac{213}{14}x^2w^2 - \frac{347}{28}wz^3 + \frac{209}{14}w^4 + \frac{705}{14}z^4 + y^4 + x^4.
\end{aligned}
\tag{1.2}
$$

In this system
$a_{07} = a_{013} = a_{014} = a_{015} = a_{017} = a_{020} = a_{034} = b = b_1 = b_2 = b_3 = b_{01} = b_{07} = b_{08} = b_{022} = b_{027} = b_{028} = c_{04} = c_{05} = c_{020} = c_{023} = c_{024} = d_{00} = d_{04} = d_{09} = d_{020} = d_{023} = d_{024} = d_{025} = 1$,
$a_3 = a_{011} = a_{012} = a_{018} = b_{026} = b_{029} = c_{00} = c_{09} = c_{025} = -1$,
$b_{020} = b_{034} = c_{034} = 2$,
$a_{016} = c_{06} = -3$,
$c_{033} = -c_{031} = 4$,
$a_{03} = -5/3,\ b_{031} = 3,\ c_3 = -41/4,\ c_{030} = -23/4,\ c_{032} = -2,\ d_3 = -206/7,\ d_{05} = -1377/56$,
$d_{06} = -213/14,\ d_{030} = 705/14,\ d_{031} = -347/28,\ d_{032} = 1331/28,\ d_{033} = -60/7$,
$d_{034} = 209/14$, *and for the remaining coefficients, they are all zero.*

This system has eight periodic solutions bifurcating from the zero-Hopf equilibrium localized at the origin of coordinates when $\epsilon = 0$.

Corollary 1.3 is proved in section 3.

2. The averaging theory

In this section we recall the averaging theory of first, second, and third order as it was developed in [3] and [9]. This will be the main tool for proving Theorem 1.1.

Theorem 2.1. *Consider the differential system*

$$x'(t) = \epsilon G_1(t,x) + \epsilon^2 G_2(t,x) + \epsilon^3 G_3(t,x) + \epsilon^4 R(t,x,\epsilon), \tag{2.1}$$

where $G_3, G_2, G_1 : \mathbb{R}\times D \to \mathbb{R}^n, R : \mathbb{R}\times D \times (-\epsilon_f, \epsilon_f) \to \mathbb{R}^n$ *are* T*-periodic in the first variable, continuous functions and* D *is an open subset of* $\mathbb{R}^n$*. Assume that the following hypotheses (i) and (ii) hold.*

(i) $G_1(t,.) \in C^2(D), G_2(t,.) \in C^1(D)$ *for all* $t \in \mathbb{R}$, $G_3, G_2, G_1, R, D_x^2 G_1, D_x G_2$ *are locally lipschitz with respect to* x*, and* R *is twice differentiable with respect to* ϵ.
We define $G_{k0} : D \longrightarrow \mathbb{R}^n$ *for* $k = 1,2,3$ *as*

$$G_{10}(z) = \frac{1}{T}\int_0^T G_1(s,z)ds,$$

$$G_{20}(z) = \frac{1}{T}\int_0^T \left[D_z G_1(s,z).y_1(s,z) + G_2(s,z)\right] ds,$$

$$G_{30}(z) = \frac{1}{T}\int_0^T [\frac{1}{2}y_1(s,z)^T \frac{\partial^2 G_1}{\partial z^2}(s,z)y_1(s,z) + \frac{1}{2}\frac{\partial G_1}{\partial z}(s,z)y_2(s,z) + \frac{\partial G_2}{\partial z}(s,z)(y_1(s,z)) + G_3(s,z)]ds,$$

Uncorrected Proof

where

$$y_1(s,z) = \int_0^s G_1(t,z)dt,$$

$$y_2(s,z) = \int_0^s \left[\frac{\partial G_1}{\partial z}(t,z)\int_0^t G_1(r,z)dr + G_2(t,z)\right]dt.$$

(ii) *For $I \subset D$ an open and bounded set and for each $\epsilon \in (-\epsilon_f, \epsilon_f) \setminus \{0\}$, there exists $a_\epsilon \in I$ such that $G_{10}(a_\epsilon) + \epsilon G_{20}(a_\epsilon) + \epsilon^2 G_{30}(a_\epsilon) = 0$ and $d_B(G_{10} + \epsilon G_{20} + \epsilon^2 G_{30}, I, a_\epsilon) \neq 0$.*

Then, for $|\epsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(\cdot, \epsilon)$ of the system (2.1) such that $\varphi(0, \epsilon) = a_\epsilon$.

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The expression $d_B(G_{10}+\epsilon G_{20}+\epsilon^2 G_{30}, I, a_\epsilon) \neq 0$ means that the Brouwer degree of the function $G_{10}+\epsilon G_{20}+\epsilon^2 G_{30} : I \to \mathbb{R}^n$ at the fixed point a_ϵ is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $G_{10} + \epsilon G_{20} + \epsilon^2 G_{30}$ at a_ϵ is not zero.

If G_{10} is not identically zero, then the zeros of $G_{10} + \epsilon G_{20} + \epsilon^2 G_{30}$ are mainly the zeros of G_{10} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of first order.*

If G_{10} is identically zero and G_{20} is not identically zero, then the zeros of $G_{10} + \epsilon G_{20} + \epsilon^2 G_{30}$ are mainly the zeros of G_{20} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of second order.*

If G_{10} and G_{20} is identically zero and G_{30} is not identically zero, then the zeros of $G_{10} + \epsilon G_{20} + \epsilon^2 G_{30}$ are mainly the zeros of G_{30} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of third order.*

For more information about the averaging theory see [13] and [11].

3. Proofs

Proof of Theorem 1.1. For proving Theorem 1.1, we should write system (1.1) into the normal form for applying the averaging theory of Section 2. First, we rescale the variables, setting $(x, y, z, w) = (\epsilon X, \epsilon Y, \epsilon Z, \epsilon W)$. Second, changing to cylindrical coordinates $(X, Y, Z, W) = (\alpha \cos\theta, \alpha \sin\theta, \beta, \xi)$. Finally, we take the angle θ as the new independent variable. Thus in the variables (α, β, ξ) system (1.1) writes

$$\begin{aligned}
\frac{d\alpha}{d\theta} &= \epsilon G_{11}(\theta,\alpha,\beta,\xi) + \epsilon^2 G_{21}(\theta,\alpha,\beta,\xi) + \epsilon^3 G_{31}(\theta,\alpha,\beta,\xi) + O(\epsilon^4),\\
\frac{d\beta}{d\theta} &= \epsilon G_{12}(\theta,\alpha,\beta,\xi) + \epsilon^2 G_{22}(\theta,\alpha,\beta,\xi) + \epsilon^3 G_{32}(\theta,\alpha,\beta,\xi) + O(\epsilon^4),\\
\frac{d\xi}{d\theta} &= \epsilon G_{13}(\theta,\alpha,\beta,\xi) + \epsilon^2 G_{23}(\theta,\alpha,\beta,\xi) + \epsilon^3 G_{33}(\theta,\alpha,\beta,\xi) + O(\epsilon^4).
\end{aligned} \tag{3.1}$$

where

$$
\begin{aligned}
G_{11}(\theta,\alpha,\beta,\xi) &= \frac{a_1\alpha}{b}, & G_{21}(\theta,\alpha,\beta,\xi) &= \frac{(ba_2-a_1b_1)\alpha}{b^2},\\
G_{12}(\theta,\alpha,\beta,\xi) &= \frac{c_1\beta}{b}, & G_{22}(\theta,\alpha,\beta,\xi) &= \frac{(bc_2-b_1c_1)\beta}{b^2},\\
G_{13}(\theta,\alpha,\beta,\xi) &= \frac{d_1\xi}{b}, & G_{23}(\theta,\alpha,\beta,\xi) &= \frac{(bd_2-b_1d_1)\xi}{b^2},
\end{aligned}
$$

$$
\begin{aligned}
&G_{31}(\theta,\alpha,\beta,\xi) = \frac{1}{b^3}(-b^2\alpha^4(a_{04}+b_{01}-a_{020}-b_{010}-a_{00})\cos(\theta)^5+(\alpha(a_{01}-a_{010}+b_{00}-b_{04}+\\
&b_{020})\sin(\theta)+(a_{02}-b_{07}-a_{013}+b_{021})\beta+\xi(a_{03}-b_{08}-a_{014}+b_{022}))\alpha^3b^2\cos(\theta)^4+(\alpha((a_{07}+b_{02}\\
&-a_{021}-b_{013})\beta+\xi(a_{08}+b_{03}-a_{022}-b_{014}))\sin(\theta)+(a_{04}+b_{01}-2a_{020}-2b_{010})\alpha^2+(a_{05}-a_{023}\\
&-b_{015})\beta^2+\xi(a_{09}-a_{025}-b_{019})\beta+\xi^2(a_{06}-a_{024}-b_{017}))\alpha^2b^2\cos(\theta)^3+\alpha(\alpha((b_{04}+a_{010}-2b_{020})\\
&\alpha^2+(b_{05}+a_{015}-b_{023})\beta^2+\xi(b_{09}+a_{019}-b_{025})\beta+\xi^2(b_{06}+a_{017}-b_{024}))\sin(\theta)+((b_{07}+a_{013}-\\
&2b_{021})\beta+\xi(b_{08}+a_{014}-2b_{022}))\alpha^2+(a_{011}-b_{026})\beta^3+\xi(a_{016}-b_{028})\beta^2+\xi^2(a_{018}-b_{029})\beta+\xi^3(\\
&a_{012}-b_{027}))b^2\cos(\theta)^2+(\alpha(((a_{021}+b_{013})\beta+\xi(a_{022}+b_{014}))\alpha^2+(a_{026}+b_{011})\beta^3+\xi(a_{028}+b_{016})\\
&\beta^2+\xi^2(a_{029}+b_{018})\beta+\xi^3(a_{027}+b_{012}))\sin(\theta)+(a_{020}+b_{010})\alpha^4+((a_{023}+b_{015})\beta^2+\xi(a_{025}+\\
&b_{019})\beta+\xi^2(a_{024}+b_{017}))\alpha^2+a_{034}\xi^4+a_{033}\xi^3\beta+\beta^2\xi^2a_{032}+\beta^3\xi a_{031}+\beta^4a_{030})b^2\cos(\theta)+b^2(\alpha^4\\
&b_{020}+(\beta^2b_{023}+\beta\xi b_{025}+\xi^2b_{024})\alpha^2+\xi^4b_{034}+\xi^3b_{033}\beta+\xi^2b_{032}\beta^2+\xi b_{031}\beta^3+b_{030}\beta^4)\sin(\theta)+\alpha(\\
&((\beta b_{021}+\xi b_{022})\alpha^2+\xi^3b_{027}+\beta\xi^2b_{029}+\beta^2\xi b_{028}+\beta^3b_{026}+a_3)b^2+(-a_1b_2-a_2b_1)b+a_1b_1^2)),
\end{aligned}
$$

$$
\begin{aligned}
&G_{32}(\theta,\alpha,\beta,\xi) = \frac{1}{b^3}(b^2\alpha^4(c_{00}-c_{04}+c_{020})\cos(\theta)^4+\alpha^3b^2(\alpha(c_{01}-c_{010})\sin(\theta)+(c_{02}-c_{013})\beta+\xi\\
&(c_{03}-c_{014}))\cos(\theta)^3+(\alpha((c_{07}-c_{021})\beta+\xi(c_{08}-c_{022}))\sin(\theta)+(c_{04}-2c_{020})\alpha^2+(c_{05}-c_{023})\beta^2\\
&+\xi(c_{09}-c_{025})\beta+\xi^2(c_{06}-c_{024}))\alpha^2b^2\cos(\theta)^2+(\alpha(\beta^2c_{015}+\beta\xi c_{019}+\alpha^2c_{010}+\xi^2c_{017})\sin(\theta)+(\\
&\beta c_{013}+\xi c_{014})\alpha^2+\xi^3c_{012}+\xi^2c_{018}\beta+\xi c_{016}\beta^2+c_{011}\beta^3)\alpha b^2\cos(\theta)+\alpha((\beta c_{021}+\xi c_{022})\alpha^2+\xi^3c_{027}\\
&+\xi^2c_{029}\beta+\xi c_{028}\beta^2+c_{026}\beta^3)b^2\sin(\theta)+(\alpha^4c_{020}+(\beta^2c_{023}+\beta\xi c_{025}+\xi^2c_{024})\alpha^2+\beta^4c_{030}+\beta^3\xi\\
&c_{031}+\beta^2\xi^2c_{032}+(\xi^3c_{033}+c_3)\beta+\xi^4c_{034})b^2-\beta(b_1c_2+b_2c_1)b+\beta b_1^2c_1),
\end{aligned}
$$

$$
\begin{aligned}
&G_{33}(\theta,\alpha,\beta,\xi) = \frac{1}{b^3}(b^2\alpha^4(d_{00}-d_{04}+d_{020})\cos(\theta)^4+\alpha^3b^2(\alpha(d_{01}-d_{010})\sin(\theta)+\xi(d_{03}-d_{014})+\\
&\beta(d_{02}-d_{013}))\cos(\theta)^3+(\alpha(\xi(d_{08}-d_{022})+\beta(d_{07}-d_{021}))\sin(\theta)+(d_{04}-2d_{020})\alpha^2+\xi^2(d_{06}-\\
&d_{024})+\beta(d_{09}-d_{025})\xi+\beta^2(d_{05}-d_{023}))\alpha^2b^2\cos(\theta)^2+(\alpha(\beta^2d_{015}+\beta\xi d_{019}+\alpha^2d_{010}+\xi^2d_{017})\\
&\sin(\theta)+(\beta d_{013}+\xi d_{014})\alpha^2+\xi^3d_{012}+d_{018}\beta\xi^2+d_{016}\beta^2\xi+d_{011}\beta^3)\alpha b^2\cos(\theta)+\alpha((\beta d_{021}+\xi d_{022})\\
&\alpha^2+\xi^3d_{027}+\beta d_{029}\xi^2+\beta^2d_{028}\xi+\beta^3d_{026})b^2\sin(\theta)+(\alpha^4d_{020}+(\beta^2d_{023}+\beta\xi d_{025}+\xi^2d_{024})\alpha^2+\\
&\xi^4d_{034}+\beta\xi^3d_{033}+\beta^2\xi^2d_{032}+(\beta^3d_{031}+d_3)\xi+\beta^4d_{030})b^2-\xi(b_1d_2+b_2d_1)b+\xi b_1^2d_1).
\end{aligned}
$$

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Taking

$$\begin{aligned} x = z &= (\alpha, \beta, \xi), \\ t &= \theta, \\ G_1(t, x) &= (G_{11}(\theta, \alpha, \beta, \xi), G_{12}(\theta, \alpha, \beta, \xi), G_{13}(\theta, \alpha, \beta, \xi)), \\ G_2(t, x) &= (G_{21}(\theta, \alpha, \beta, \xi), G_{22}(\theta, \alpha, \beta, \xi), G_{23}(\theta, \alpha, \beta, \xi)), \\ G_3(t, x) &= (G_{31}(\theta, \alpha, \beta, \xi), G_{32}(\theta, \alpha, \beta, \xi), G_{33}(\theta, \alpha, \beta, \xi)), \\ T &= 2\pi. \end{aligned}$$

. System (3.1) is equivalent to system (2.1). Applying the averaging theory of first order to the system (3.1). We have that $g_1 = (g_{11}, g_{12}, g_{13})$, where for $i = 1, 2, 3$

$$g_{1i}(\alpha, \beta, \xi) = \frac{1}{2\pi} \int_0^{2\pi} G_{1i}(\theta, \alpha, \beta, \xi) d\theta.$$

Doing these computations we get that

$$\begin{cases} g_{11}(\alpha, \beta, \xi) = \dfrac{\alpha a_1}{b}, \\ g_{12}(\alpha, \beta, \xi) = \dfrac{\beta c_1}{b}, \\ g_{13}(\alpha, \beta, \xi) = \dfrac{\xi d_1}{b}. \end{cases} \tag{3.2}$$

Since we look for solutions $(\alpha^*, \beta^*, \xi^*)$ of $g_1(\alpha, \beta, \xi) = 0$ with $\alpha^* > 0$, the first averaging function does not provide any information on the periodic solutions of the differential system (3.1). In order that the second averaging function can give information on the periodic solutions of the differential system (3.1) the first averaging system must be identically zero. So we take $a_1 = c_1 = d_1 = 0$, and compute the second averaging function. Then from (3.1) we have that $g_2 = (g_{21}, g_{22}, g_{23}) = (g_{21}(\alpha, \beta, \xi), g_{22}(\alpha, \beta, \xi),\ g_{23}(\alpha, \beta, \xi))$ is given by

$$\begin{cases} g_{21}(\alpha, \beta, \xi) = \dfrac{\alpha a_2}{b}, \\ g_{22}(\alpha, \beta, \xi) = \dfrac{\beta c_2}{b} \\ g_{23}(\alpha, \beta, \xi) = \dfrac{\xi d_2}{b}. \end{cases} \tag{3.3}$$

Therefore, the second averaging system (3.3) does not provide any information on the limit cycles of the differential system (3.1). Since the third averaging system can give information on the limit cycles of the differential (3.1), we need the second average system to be identically zero. So we take $a_2 = c_2 = d_2 = 0$, we compute the third averaging

Uncorrected Proof

function $g_3 = (g_{31}, g_{32}, g_{33}) = (g_{31}(\alpha, \beta, \xi), g_{32}(\alpha, \beta, \xi),\ g_{33}(\alpha, \beta, \xi))$ and we obtain

$$\begin{cases} g_{31}(\alpha,\beta,\xi) = \dfrac{1}{8b}(\alpha(4\beta^3 a_{011} + 4\beta^3 b_{026} + 4\beta^2\xi a_{016} + 4\beta^2\xi b_{028} + 3\beta\alpha^2 a_{02} + \beta\alpha^2 a_{013} + \beta\alpha^2 b_{07} + \\ \qquad 3\beta\alpha^2 b_{021} + 4\beta\xi^2 a_{018} + 4\beta\xi^2 b_{029} + 3\alpha^2\xi a_{03} + \alpha^2\xi a_{014} + \alpha^2\xi b_{08} + 3\alpha^2\xi b_{022} + 4\xi^3 \\ \qquad a_{012} + 4\xi^3 b_{027} + 8a_3)), \\ g_{32}(\alpha,\beta,\xi) = \dfrac{1}{8b}(8\beta^4 c_{030} + 8\beta^3\xi c_{031} + 4\beta^2\alpha^2 c_{05} + 4\beta^2\alpha^2 c_{023} + 8\beta^2\xi^2 c_{032} + 4\beta\alpha^2\xi c_{09} + 4\beta\alpha^2\xi \\ \qquad c_{025} + 8\beta\xi^3 c_{033} + 3\alpha^4 c_{00} + \alpha^4 c_{04} + 3\alpha^4 c_{020} + 4\alpha^2\xi^2 c_{06} + 4\alpha^2\xi^2 c_{024} + 8\xi^4 c_{034} + \\ \qquad 8\beta c_3), \\ g_{33}(\alpha,\beta,\xi) = \dfrac{1}{8b}(8\beta^4 d_{030} + 8\beta^3\xi d_{031} + 4\beta^2\alpha^2 d_{05} + 4\beta^2\alpha^2 d_{023} + 8\beta^2\xi^2 d_{032} + 4\beta\alpha^2\xi d_{09} + 4\beta\alpha^2 \\ \qquad \xi d_{025} + 8\beta\xi^3 d_{033} + 3\alpha^4 d_{00} + \alpha^4 d_{04} + 3\alpha^4 d_{020} + 4\alpha^2\xi^2 d_{06} + 4\alpha^2\xi^2 d_{024} + 8\xi^4 d_{034} \\ \qquad + 8\xi d_3). \end{cases} \tag{3.4}$$

In order for looking for the limit cycles of system (1.1) by the averaging theory we need to compute the isolated real roots of the averaged system (3.4) with $\alpha > 0$.

We isolate α^2 from the equation$g_{31}(\alpha,\beta,\xi) = 0$, and we substitute it in $g_{3i}(\alpha,\beta,\xi) = 0$ for $i = 2, 3$. Then we get two functions $(h_{32}, h_{33}) = (h_{32}(\beta,\xi), h_{33}(\beta,\xi))$ given by

$$\begin{aligned} h_{32} = {} & \frac{1}{S}\big(C_0 + C_1\beta\xi^2 + C_2\beta\xi^5 + C_3\beta^2\xi + C_4\beta^2\xi^4 + C_5\beta^3 + C_6\beta^3\xi^3 + C_7\beta^4\xi^2 + C_8\beta^5\xi + C_9\beta^6 \\ & + C_{10}\xi^3 + C_{11}\xi^6\big), \\ h_{33} = {} & \frac{1}{S}\big(E_0 + E_1\beta\xi^2 + E_2\beta\xi^5 + E_3\beta^2\xi + E_4\beta^2\xi^4 + E_5\beta^3 + E_6\beta^3\xi^3 + E_7\beta^4\xi^2 + E_8\beta^5\xi + E_9\beta^6 \\ & + E_{10}\xi^3 + E_{11}\xi^6\big), \end{aligned}$$

where

$$\begin{aligned} S = {} & b((a_{013} + 3b_{021} + 3a_{02} + b_{07})\beta + \xi(a_{014} + 3b_{022} + 3a_{03} + b_{08}))^2 \neq 0, \\ C_0 = {} & 8a_3^2(3c_{020} + 3c_{00} + c_{04}), \\ C_1 = {} & ((-4c_{09} - 4c_{025})a_{014} + (-12c_{09} - 12c_{025})b_{022} + (-12c_{09} - 12c_{025})a_{03} + (-4c_{09} - 4c_{025}) \\ & b_{08} + (-4a_{013} - 12b_{021} - 12a_{02} - 4b_{07})c_{024} + (-4a_{013} - 12b_{021} - 12a_{02} - 4b_{07})c_{06} + 8(\\ & 3c_{020} + 3c_{00} + c_{04})(a_{018} + b_{029}))a_3 + c_3(a_{014} + 3b_{022} + 3a_{03} + b_{08})^2, \\ C_2 = {} & ((-2c_{09} - 2c_{025})a_{014} + (-6c_{09} - 6c_{025})b_{022} + (-2a_{013} - 6b_{021} - 6a_{02} - 2b_{07})c_{024} + (- \\ & 6c_{09} - 6c_{025})a_{03} + (-2c_{09} - 2c_{025})b_{08} + (-2a_{013} - 6b_{021} - 6a_{02} - 2b_{07})c_{06} + 4(3c_{020} + \\ & 3c_{00} + c_{04})(a_{018} + b_{029}))a_{012} + ((-2c_{09} - 2c_{025})a_{014} + (-6c_{09} - 6c_{025})b_{022} + (-2a_{013} - \\ & 6b_{021} - 6a_{02} - 2b_{07})c_{024} + (-6c_{09} - 6c_{025})a_{03} + (-2c_{09} - 2c_{025})b_{08} + (-2a_{013} - 6b_{021} - \\ & 6a_{02} - 2b_{07})c_{06} + 4(3c_{020} + 3c_{00} + c_{04})(a_{018} + b_{029}))b_{027} + (a_{014} + 3b_{022} + 3a_{03} + b_{08})(c_{033} \end{aligned}$$

Uncorrected Proof

$$a_{014} + 3c_{033}b_{022} + 2(-a_{018} - b_{029})c_{024} + 2(a_{013} + 3b_{021} + 3a_{02} + b_{07})c_{034} + 3c_{033}a_{03} + b_{08}$$
$$c_{033} - 2c_{06}(a_{018} + b_{029})),$$

$$C_3 = ((-4c_{09} - 4c_{025})a_{013} + (-4c_{023} - 4c_{05})a_{014} + (-12c_{09} - 12c_{025})b_{021} + (-12c_{023} - 12c_{05})$$
$$b_{022} + (-12c_{09} - 12c_{025})a_{02} + (-12c_{023} - 12c_{05})a_{03} + (-4c_{09} - 4c_{025})b_{07} + (-4c_{023} -$$
$$4c_{05})b_{08} + 8(3c_{020} + 3c_{00} + c_{04})(a_{016} + b_{028}))a_3 + 2c_3(a_{014} + 3b_{022} + 3a_{03} + b_{08})(a_{013} +$$
$$3b_{021} + 3a_{02} + b_{07}),$$

$$C_4 = ((-2c_{05} - 2c_{023})a_{014} + (-6c_{05} - 6c_{023})b_{022} + (-6c_{05} - 6c_{023})a_{03} + (-2c_{05} - 2c_{023})b_{08} + ($$
$$-2c_{09} - 2c_{025})a_{013} + (-6c_{09} - 6c_{025})b_{021} + (-6a_{02} - 2b_{07})c_{025} - 6a_{02}c_{09} - 2b_{07}c_{09} + 4($$
$$3c_{020} + 3c_{00} + c_{04})(a_{016} + b_{028}))a_{012} + (6c_{00} + 6c_{020} + 2c_{04})a_{018}^2 + ((12c_{020} + 12c_{00} + 4c_{04}$$
$$)b_{029} + (-2c_{09} - 2c_{025})a_{014} + (-6c_{09} - 6c_{025})b_{022} + (-6c_{09} - 6c_{025})a_{03} + (-2c_{09} - 2c_{025})$$
$$b_{08} - 2(c_{024} + c_{06})(a_{013} + 3b_{021} + 3a_{02} + b_{07}))a_{018} + ((-2c_{05} - 2c_{023})a_{014} + (-6c_{05} -$$
$$6c_{023})b_{022} + (-6c_{05} - 6c_{023})a_{03} + (-2c_{05} - 2c_{023})b_{08} + (-2c_{09} - 2c_{025})a_{013} + (-6c_{09} -$$
$$6c_{025})b_{021} + (-6a_{02} - 2b_{07})c_{025} - 6a_{02}c_{09} - 2b_{07}c_{09} + 4(3c_{020} + 3c_{00} + c_{04})(a_{016} + b_{028}))$$
$$b_{027} + (6c_{00} + 6c_{020} + 2c_{04})b_{029}^2 + ((-2c_{09} - 2c_{025})a_{014} + (-6c_{09} - 6c_{025})b_{022} + (-6c_{09} -$$
$$6c_{025})a_{03} + (-2c_{09} - 2c_{025})b_{08} - 2(c_{024} + c_{06})(a_{013} + 3b_{021} + 3a_{02} + b_{07}))b_{029} + c_{032}a_{014}^2 +$$
$$(6c_{032}a_{03} + 2a_{013}c_{033} + 2b_{08}c_{032} + 6c_{033}b_{021} + 6c_{032}b_{022} + (-2a_{016} - 2b_{028})c_{024} + (6a_{02} +$$
$$2b_{07})c_{033} - 2c_{06}(a_{016} + b_{028}))a_{014} + 9c_{032}b_{022}^2 + (18c_{032}a_{03} + 6a_{013}c_{033} + 6b_{08}c_{032} + 18c_{033}$$
$$b_{021} + (-6a_{016} - 6b_{028})c_{024} + (18a_{02} + 6b_{07})c_{033} - 6c_{06}(a_{016} + b_{028}))b_{022} + 9c_{032}a_{03}^2 + ($$
$$6a_{013}c_{033} + 6b_{08}c_{032} + 18c_{033}b_{021} + (-6a_{016} - 6b_{028})c_{024} + (18a_{02} + 6b_{07})c_{033} - 6c_{06}(a_{016}$$
$$+b_{028}))a_{03} + c_{032}b_{08}^2 + (2a_{013}c_{033} + 6c_{033}b_{021} + (-2a_{016} - 2b_{028})c_{024} + (6a_{02} + 2b_{07})c_{033} -$$
$$2c_{06}(a_{016} + b_{028}))b_{08} + c_{034}(a_{013} + 3b_{021} + 3a_{02} + b_{07})^2,$$

$$C_5 = ((-4c_{023} - 4c_{05})a_{013} + (-12c_{023} - 12c_{05})b_{021} + (-12c_{023} - 12c_{05})a_{02} + (-4c_{023} - 4c_{05})$$
$$b_{07} + 8(3c_{020} + 3c_{00} + c_{04})(a_{011} + b_{026}))a_3 + (a_{013} + 3b_{021} + 3a_{02} + b_{07})^2c_3,$$

$$C_6 = ((12c_{020} + 12c_{00} + 4c_{04})a_{018} + (12c_{020} + 12c_{00} + 4c_{04})b_{029} + (-2c_{06} - 2c_{024})a_{013} + (-2c_{09}$$
$$-2c_{025})a_{014} + (-6c_{06} - 6c_{024})b_{021} + (-6c_{09} - 6c_{025})b_{022} + (-6c_{06} - 6c_{024})a_{02} + (-6c_{09} -$$
$$6c_{025})a_{03} + (-2c_{06} - 2c_{024})b_{07} - 2b_{08}(c_{09} + c_{025}))a_{016} + ((12c_{020} + 12c_{00} + 4c_{04})b_{028} + (-$$
$$2c_{09} - 2c_{025})a_{013} + (-2c_{05} - 2c_{023})a_{014} + (-6c_{09} - 6c_{025})b_{021} + (-6c_{05} - 6c_{023})b_{022} + (-$$
$$6c_{09} - 6c_{025})a_{02} + (-6c_{05} - 6c_{023})a_{03} + (-2c_{09} - 2c_{025})b_{07} - 2(c_{05} + c_{023})b_{08})a_{018} + (($$
$$12c_{020} + 12c_{00} + 4c_{04})b_{029} + (-2c_{06} - 2c_{024})a_{013} + (-2c_{09} - 2c_{025})a_{014} + (-6c_{06} - 6c_{024})$$
$$b_{021} + (-6c_{09} - 6c_{025})b_{022} + (-6c_{06} - 6c_{024})a_{02} + (-6c_{09} - 6c_{025})a_{03} + (-2c_{06} - 2c_{024})$$
$$b_{07} - 2b_{08}(c_{09} + c_{025}))b_{028} + ((-2c_{09} - 2c_{025})a_{013} + (-2c_{05} - 2c_{023})a_{014} + (-6c_{09} - 6c_{025})$$
$$b_{021} + (-6c_{05} - 6c_{023})b_{022} + (-6c_{09} - 6c_{025})a_{02} + (-6c_{05} - 6c_{023})a_{03} + (-2c_{09} - 2c_{025})$$

Uncorrected Proof

$$
\begin{aligned}
& b_{07}-2(c_{05}+c_{023})b_{08})b_{029}+c_{033}a_{013}^2+(2c_{032}a_{014}+6c_{033}b_{021}+6c_{032}b_{022}+6c_{033}a_{02}+ \\
& 6c_{032}a_{03}+2c_{033}b_{07}+2b_{08}c_{032}-2(c_{05}+c_{023})(a_{012}+b_{027}))a_{013}+c_{031}a_{014}^2+(6c_{032}b_{021}+ \\
& 6c_{031}b_{022}+6c_{032}a_{02}+6c_{031}a_{03}+2c_{032}b_{07}+2b_{08}c_{031}-2(c_{024}+c_{06})(a_{011}+b_{026}))a_{014}+ \\
& 9c_{033}b_{021}^2+(18c_{032}b_{022}+18c_{033}a_{02}+18c_{032}a_{03}+6c_{033}b_{07}+6b_{08}c_{032}-6(c_{05}+c_{023})(a_{012}+ \\
& b_{027}))b_{021}+9c_{031}b_{022}^2+(18c_{032}a_{02}+18c_{031}a_{03}+6c_{032}b_{07}+6b_{08}c_{031}-6(c_{024}+c_{06})(a_{011}+ \\
& b_{026}))b_{022}+9c_{033}a_{02}^2+(18c_{032}a_{03}+6c_{033}b_{07}+6b_{08}c_{032}-6(c_{05}+c_{023})(a_{012}+b_{027}))a_{02}+ \\
& 9c_{031}a_{03}^2+(6c_{032}b_{07}+6b_{08}c_{031}-6(c_{024}+c_{06})(a_{011}+b_{026}))a_{03}+c_{033}b_{07}^2+(2b_{08}c_{032}-2(\\
& c_{05}+c_{023})(a_{012}+b_{027}))b_{07}+c_{031}b_{08}^2-2(c_{024}+c_{06})(a_{011}+b_{026})b_{08}+4(3c_{020}+3c_{00}+c_{04}) \\
& (a_{011}+b_{026})(a_{012}+b_{027}), \\
C_7 = & ((-2c_{06}-2c_{024})a_{013}+(-6c_{06}-6c_{024})b_{021}+(-6c_{06}-6c_{024})a_{02}+(-2c_{06}-2c_{024})b_{07}+ \\
& (-2c_{09}-2c_{025})a_{014}+(-6c_{09}-6c_{025})b_{022}+(-6a_{03}-2b_{08})c_{025}-6a_{03}c_{09}-2b_{08}c_{09}+4(\\
& a_{018}+b_{029})(3c_{020}+3c_{00}+c_{04}))a_{011}+(6c_{00}+6c_{020}+2c_{04})a_{016}^2+((12c_{00}+12c_{020}+4c_{04} \\
&)b_{028}+(-2c_{09}-2c_{025})a_{013}+(-6c_{09}-6c_{025})b_{021}+(-6c_{09}-6c_{025})a_{02}+(-2c_{09}-2c_{025}) \\
& b_{07}-2(c_{05}+c_{023})(a_{014}+3b_{022}+3a_{03}+b_{08}))a_{016}+((-2c_{06}-2c_{024})a_{013}+(-6c_{06}- \\
& 6c_{024})b_{021}+(-6c_{06}-6c_{024})a_{02}+(-2c_{06}-2c_{024})b_{07}+(-2c_{09}-2c_{025})a_{014}+(-6c_{09}- \\
& 6c_{025})b_{022}+(-6a_{03}-2b_{08})c_{025}-6a_{03}c_{09}-2b_{08}c_{09}+4(a_{018}+b_{029})(3c_{020}+3c_{00}+c_{04})) \\
& b_{026}+(6c_{00}+6c_{020}+2c_{04})b_{028}^2+((-2c_{09}-2c_{025})a_{013}+(-6c_{09}-6c_{025})b_{021}+(-6c_{09}- \\
& 6c_{025})a_{02}+(-2c_{09}-2c_{025})b_{07}-2(c_{05}+c_{023})(a_{014}+3b_{022}+3a_{03}+b_{08}))b_{028}+c_{032}a_{013}^2 \\
& +(6c_{032}a_{02}+2c_{031}a_{014}+2c_{032}b_{07}+6c_{032}b_{021}+6c_{031}b_{022}+(-2a_{018}-2b_{029})c_{023}+(6a_{03} \\
& +2b_{08})c_{031}-2c_{05}(a_{018}+b_{029}))a_{013}+9c_{032}b_{021}^2+(18c_{032}a_{02}+6c_{031}a_{014}+6c_{032}b_{07}+ \\
& 18c_{031}b_{022}+(-6a_{018}-6b_{029})c_{023}+(18a_{03}+6b_{08})c_{031}-6c_{05}(a_{018}+b_{029}))b_{021}+9c_{032}a_{02}^2 \\
& +(6c_{031}a_{014}+6c_{032}b_{07}+18c_{031}b_{022}+(-6a_{018}-6b_{029})c_{023}+(18a_{03}+6b_{08})c_{031}-6c_{05}(\\
& a_{018}+b_{029}))a_{02}+c_{032}b_{07}^2+(2c_{031}a_{014}+6c_{031}b_{022}+(-2a_{018}-2b_{029})c_{023}+(6a_{03}+2b_{08}) \\
& c_{031}-2c_{05}(a_{018}+b_{029}))b_{07}+c_{030}(a_{014}+3b_{022}+3a_{03}+b_{08})^2, \\
C_8 = & ((-2c_{09}-2c_{025})a_{013}+(-6c_{09}-6c_{025})b_{021}+(-2a_{014}-6b_{022}-6a_{03}-2b_{08})c_{023}+(- \\
& 6c_{09}-6c_{025})a_{02}+(-2c_{09}-2c_{025})b_{07}+(-2a_{014}-6b_{022}-6a_{03}-2b_{08})c_{05}+4(a_{016}+ \\
& b_{028})(3c_{020}+3c_{00}+c_{04}))a_{011}+((-2c_{09}-2c_{025})a_{013}+(-6c_{09}-6c_{025})b_{021}+(-2a_{014}- \\
& 6b_{022}-6a_{03}-2b_{08})c_{023}+(-6c_{09}-6c_{025})a_{02}+(-2c_{09}-2c_{025})b_{07}+(-2a_{014}-6b_{022}- \\
& 6a_{03}-2b_{08})c_{05}+4(a_{016}+b_{028})(3c_{020}+3c_{00}+c_{04}))b_{026}+(c_{031}a_{013}+3c_{031}b_{021}+(- \\
& 2a_{016}-2b_{028})c_{023}+(2a_{014}+6b_{022}+6a_{03}+2b_{08})c_{030}+3c_{031}a_{02}+c_{031}b_{07}-2c_{05}(a_{016}+ \\
& b_{028}))(a_{013}+3b_{021}+3a_{02}+b_{07}), \\
C_9 = & (6c_{00}+6c_{020}+2c_{04})a_{011}^2+((12c_{00}+12c_{020}+4c_{04})b_{026}-2(a_{013}+3b_{021}+3a_{02}+b_{07})(
\end{aligned}
$$

Uncorrected Proof

$$
\begin{aligned}
& c_{05}+c_{023}))a_{011}+6c_{00}+6c_{020}+2c_{04})b_{026}^2-2(a_{013}+3b_{021}+3a_{02}+b_{07})(c_{05}+c_{023})b_{026} \\
& +(a_{013}+3b_{021}+3a_{02}+b_{07})^2c_{030}, \\
C_{10} = \; & -4a_3((3b_{022}+3a_{03}+b_{08}+a_{014})c_{024}+(3b_{022}+3a_{03}+b_{08}+a_{014})c_{06}-2(a_{012}+b_{027})(\\
& 3c_{020}+3c_{00}+c_{04})), \\
C_{11} = \; & (6c_{020}+6c_{00}+2c_{04})a_{012}^2+((12c_{020}+12c_{00}+4c_{04})b_{027}-(6b_{022}+6a_{03}+2b_{08}+2a_{014})(\\
& c_{024}+c_{06}))a_{012}+(6c_{020}+6c_{00}+2c_{04})b_{027}^2-(6b_{022}+6a_{03}+2b_{08}+2a_{014})(c_{024}+c_{06}) \\
& b_{027}+c_{034}(3b_{022}+3a_{03}+b_{08}+a_{014})^2, \\
E_0 = \; & 8a_3^2(3d_{020}+3d_{00}+d_{04}), \\
E_1 = \; & ((-4d_{024}-4d_{06})a_{013}+(-4d_{09}-4d_{025})a_{014}+(-12d_{024}-12d_{06})b_{021}+(-12d_{09}-12d_{025} \\
&)b_{022}+(-12d_{024}-12d_{06})a_{02}+(-12d_{09}-12d_{025})a_{03}+(-4d_{024}-4d_{06})b_{07}+(-4d_{09}- \\
& 4d_{025})b_{08}+8(3d_{020}+3d_{00}+d_{04})(a_{018}+b_{029}))a_3+2d_3(3a_{03}+b_{08}+a_{014}+3b_{022})(a_{013}+ \\
& 3b_{021}+3a_{02}+b_{07}), \\
E_2 = \; & ((-2d_{09}-2d_{025})a_{014}+(-6d_{09}-6d_{025})b_{022}+(-2a_{013}-6b_{021}-6a_{02}-2b_{07})d_{024}+(- \\
& 6d_{09}-6d_{025})a_{03}+(-2d_{09}-2d_{025})b_{08}+(-2a_{013}-6b_{021}-6a_{02}-2b_{07})d_{06}+4(3d_{020}+ \\
& 3d_{00}+d_{04})(a_{018}+b_{029}))a_{012}+((-2d_{09}-2d_{025})a_{014}+(-6d_{09}-6d_{025})b_{022}+(-2a_{013}- \\
& 6b_{021}-6a_{02}-2b_{07})d_{024}+(-6d_{09}-6d_{025})a_{03}+(-2d_{09}-2d_{025})b_{08}+(-2a_{013}-6b_{021}- \\
& 6a_{02}-2b_{07})d_{06}+4(3d_{020}+3d_{00}+d_{04})(a_{018}+b_{029}))b_{027}+(a_{014}+3b_{022}+3a_{03}+b_{08})(\\
& d_{033}a_{014}+3d_{033}b_{022}+2(-a_{018}-b_{029})d_{024}+2(a_{013}+3b_{021}+3a_{02}+b_{07})d_{034}+3d_{033}a_{03}+ \\
& b_{08}d_{033}-2d_{06}(a_{018}+b_{029})), \\
E_3 = \; & ((-4d_{09}-4d_{025})a_{013}+(-12d_{09}-12d_{025})b_{021}+(-12d_{09}-12d_{025})a_{02}+(-4d_{09}-4d_{025}) \\
& b_{07}+(-4a_{014}-12b_{022}-12a_{03}-4b_{08})d_{023}+(-4a_{014}-12b_{022}-12a_{03}-4b_{08})d_{05}+8(\\
& a_{016}+b_{028})(3d_{020}+3d_{00}+d_{04}))a_3+d_3(a_{013}+3b_{021}+3a_{02}+b_{07})^2, \\
E_4 = \; & ((-2d_{05}-2d_{023})a_{014}+(-6d_{05}-6d_{023})b_{022}+(-6d_{05}-6d_{023})a_{03}+(-2d_{05}-2d_{023})b_{08}+ \\
& (-2d_{09}-2d_{025})a_{013}+(-6d_{09}-6d_{025})b_{021}+(-6a_{02}-2b_{07})d_{025}-6a_{02}d_{09}-2b_{07}d_{09}+4(\\
& a_{016}+b_{028})(3d_{020}+3d_{00}+d_{04}))a_{012}+(6d_{00}+6d_{020}+2d_{04})a_{018}^2+((12d_{00}+12d_{020}+ \\
& 4d_{04})b_{029}+(-2d_{09}-2d_{025})a_{014}+(-6d_{09}-6d_{025})b_{022}+(-6d_{09}-6d_{025})a_{03}+(-2d_{09}- \\
& 2d_{025})b_{08}-2(d_{06}+d_{024})(a_{013}+3b_{021}+3a_{02}+b_{07}))a_{018}+((-2d_{05}-2d_{023})a_{014}+(-6d_{05} \\
& -6d_{023})b_{022}+(-6d_{05}-6d_{023})a_{03}+(-2d_{05}-2d_{023})b_{08}+(-2d_{09}-2d_{025})a_{013}+(-6d_{09} \\
& -6d_{025})b_{021}+(-6a_{02}-2b_{07})d_{025}-6a_{02}d_{09}-2b_{07}d_{09}+4(a_{016}+b_{028})(3d_{020}+3d_{00}+d_{04} \\
&))b_{027}+(6d_{00}+6d_{020}+2d_{04})b_{029}^2+((-2d_{09}-2d_{025})a_{014}+(-6d_{09}-6d_{025})b_{022}+(-6d_{09} \\
& -6d_{025})a_{03}+(-2d_{09}-2d_{025})b_{08}-2(d_{06}+d_{024})(a_{013}+3b_{021}+3a_{02}+b_{07}))b_{029}+d_{032} \\
& a_{014}^2+(6d_{032}a_{03}+2a_{013}d_{033}+2b_{08}d_{032}+6d_{033}b_{021}+6d_{032}b_{022}+(-2a_{016}-2b_{028})d_{024}+(
\end{aligned}
$$

Uncorrected Proof

$$6a_{02} + 2b_{07})d_{033} - 2d_{06}(a_{016} + b_{028}))a_{014} + 9d_{032}b_{022}^2 + (18d_{032}a_{03} + 6a_{013}d_{033} + 6b_{08}d_{032}$$
$$+18d_{033}b_{021} + (-6a_{016} - 6b_{028})d_{024} + (18a_{02} + 6b_{07})d_{033} - 6d_{06}(a_{016} + b_{028}))b_{022} + 9d_{032}$$
$$a_{03}^2 + (6a_{013}d_{033} + 6b_{08}d_{032} + 18d_{033}b_{021} + (-6a_{016} - 6b_{028})d_{024} + (18a_{02} + 6b_{07})d_{033} -$$
$$6d_{06}(a_{016} + b_{028}))a_{03} + d_{032}b_{08}^2 + (2a_{013}d_{033} + 6d_{033}b_{021} + (-2a_{016} - 2b_{028})d_{024} + (6a_{02} +$$
$$2b_{07})d_{033} - 2d_{06}(a_{016} + b_{028}))b_{08} + d_{034}(a_{013} + 3b_{021} + 3a_{02} + b_{07})^2,$$

$$E_5 = 4a_3((6d_{00} + 6d_{020} + 2d_{04})(a_{011} + b_{026}) - (a_{013} + 3b_{021} + 3a_{02} + b_{07})d_{023} - (a_{013} + 3b_{021} +$$
$$3a_{02} + b_{07})d_{05}),$$

$$E_6 = ((12d_{00} + 12d_{020} + 4d_{04})a_{018} + (12d_{00} + 12d_{020} + 4d_{04})b_{029} + (-2d_{06} - 2d_{024})a_{013} + (-$$
$$2d_{09} - 2d_{025})a_{014} + (-6d_{06} - 6d_{024})b_{021} + (-6d_{09} - 6d_{025})b_{022} + (-6d_{06} - 6d_{024})a_{02} + (-$$
$$6d_{09} - 6d_{025})a_{03} + (-2d_{06} - 2d_{024})b_{07} - 2b_{08}(d_{09} + d_{025}))a_{016} + ((12d_{00} + 12d_{020} + 4d_{04})$$
$$b_{028} + (-2d_{09} - 2d_{025})a_{013} + (-2d_{05} - 2d_{023})a_{014} + (-6d_{09} - 6d_{025})b_{021} + (-6d_{05} - 6d_{023}$$
$$)b_{022} + (-6d_{09} - 6d_{025})a_{02} + (-6d_{05} - 6d_{023})a_{03} + (-2d_{09} - 2d_{025})b_{07} - 2b_{08}(d_{023} + d_{05}))$$
$$a_{018} + ((12d_{00} + 12d_{020} + 4d_{04})b_{029} + (-2d_{06} - 2d_{024})a_{013} + (-2d_{09} - 2d_{025})a_{014} + (-$$
$$6d_{06} - 6d_{024})b_{021} + (-6d_{09} - 6d_{025})b_{022} + (-6d_{06} - 6d_{024})a_{02} + (-6d_{09} - 6d_{025})a_{03} + (-$$
$$2d_{06} - 2d_{024})b_{07} - 2b_{08}(d_{09} + d_{025}))b_{028} + ((-2d_{09} - 2d_{025})a_{013} + (-2d_{05} - 2d_{023})a_{014} + ($$
$$-6d_{09} - 6d_{025})b_{021} + (-6d_{05} - 6d_{023})b_{022} + (-6d_{09} - 6d_{025})a_{02} + (-6d_{05} - 6d_{023})a_{03} + ($$
$$-2d_{09} - 2d_{025})b_{07} - 2b_{08}(d_{023} + d_{05}))b_{029} + d_{033}a_{013}^2 + (2d_{032}a_{014} + 6d_{033}b_{021} + 6d_{032}b_{022}$$
$$+6d_{033}a_{02} + 6d_{032}a_{03} + 2d_{033}b_{07} + 2b_{08}d_{032} - 2(d_{023} + d_{05})(a_{012} + b_{027}))a_{013} + d_{031}a_{014}^2 +$$
$$(6d_{032}b_{021} + 6d_{031}b_{022} + 6d_{032}a_{02} + 6d_{031}a_{03} + 2d_{032}b_{07} + 2b_{08}d_{031} - 2(d_{06} + d_{024})(a_{011} +$$
$$b_{026}))a_{014} + 9d_{033}b_{021}^2 + (18d_{032}b_{022} + 18d_{033}a_{02} + 18d_{032}a_{03} + 6d_{033}b_{07} + 6b_{08}d_{032} - 6($$
$$d_{023} + d_{05})(a_{012} + b_{027}))b_{021} + 9d_{031}b_{022}^2 + (18d_{032}a_{02} + 18d_{031}a_{03} + 6d_{032}b_{07} + 6b_{08}d_{031} -$$
$$6(d_{06} + d_{024})(a_{011} + b_{026}))b_{022} + 9d_{033}a_{02}^2 + (18d_{032}a_{03} + 6d_{033}b_{07} + 6b_{08}d_{032} - 6(d_{023} +$$
$$d_{05})(a_{012} + b_{027}))a_{02} + 9d_{031}a_{03}^2 + (6d_{032}b_{07} + 6b_{08}d_{031} - 6(d_{06} + d_{024})(a_{011} + b_{026}))a_{03} +$$
$$d_{033}b_{07}^2 + (2b_{08}d_{032} - 2(d_{023} + d_{05})(a_{012} + b_{027}))b_{07} + d_{031}b_{08}^2 - 2(d_{06} + d_{024})(a_{011} + b_{026})$$
$$b_{08} + 4(a_{012} + b_{027})(a_{011} + b_{026})(3d_{020} + 3d_{00} + d_{04}),$$

$$E_7 = ((-2d_{06} - 2d_{024})a_{013} + (-6d_{06} - 6d_{024})b_{021} + (-6d_{06} - 6d_{024})a_{02} + (-2d_{06} - 2d_{024})b_{07}$$
$$+(-2d_{09} - 2d_{025})a_{014} + (-6d_{09} - 6d_{025})b_{022} + (-6a_{03} - 2b_{08})d_{025} - 6d_{09}a_{03} - 2d_{09}b_{08} +$$
$$4(3d_{020} + 3d_{00} + d_{04})(a_{018} + b_{029}))a_{011} + (6d_{00} + 6d_{020} + 2d_{04})a_{016}^2 + ((12d_{00} + 12d_{020} +$$
$$4d_{04})b_{028} + (-2d_{09} - 2d_{025})a_{013} + (-6d_{09} - 6d_{025})b_{021} + (-6d_{09} - 6d_{025})a_{02} + (-2d_{09} -$$
$$2d_{025})b_{07} - 2(d_{023} + d_{05})(a_{014} + 3b_{022} + 3a_{03} + b_{08}))a_{016} + ((-2d_{06} - 2d_{024})a_{013} + (-$$
$$6d_{06} - 6d_{024})b_{021} + (-6d_{06} - 6d_{024})a_{02} + (-2d_{06} - 2d_{024})b_{07} + (-2d_{09} - 2d_{025})a_{014} + (-$$
$$6d_{09} - 6d_{025})b_{022} + (-6a_{03} - 2b_{08})d_{025} - 6d_{09}a_{03} - 2d_{09}b_{08} + 4(3d_{020} + 3d_{00} + d_{04})(a_{018}$$

Uncorrected Proof

$$
\begin{aligned}
& +b_{029}))b_{026} + (6d_{00} + 6d_{020} + 2d_{04})b_{028}^2 + ((-2d_{09} - 2d_{025})a_{013} + (-6d_{09} - 6d_{025})b_{021} + (\\
& -6d_{09} - 6d_{025})a_{02} + (-2d_{09} - 2d_{025})b_{07} - 2(d_{023} + d_{05})(a_{014} + 3b_{022} + 3a_{03} + b_{08}))b_{028} + \\
& d_{032}a_{013}^2 + (6d_{032}a_{02} + 2d_{031}a_{014} + 2d_{032}b_{07} + 6d_{032}b_{021} + 6d_{031}b_{022} + (-2a_{018} - 2b_{029}) \\
& d_{023} + (6a_{03} + 2b_{08})d_{031} - 2d_{05}(a_{018} + b_{029}))a_{013} + 9d_{032}b_{021}^2 + (18d_{032}a_{02} + 6d_{031}a_{014} + \\
& 6d_{032}b_{07} + 18d_{031}b_{022} + (-6a_{018} - 6b_{029})d_{023} + (18a_{03} + 6b_{08})d_{031} - 6d_{05}(a_{018} + b_{029})) \\
& b_{021} + 9d_{032}a_{02}^2 + (6d_{031}a_{014} + 6d_{032}b_{07} + 18d_{031}b_{022} + (-6a_{018} - 6b_{029})d_{023} + (18a_{03} + \\
& 6b_{08})d_{031} - 6d_{05}(a_{018} + b_{029}))a_{02} + d_{032}b_{07}^2 + (2d_{031}a_{014} + 6d_{031}b_{022} + (-2a_{018} - 2b_{029}) \\
& d_{023} + (6a_{03} + 2b_{08})d_{031} - 2d_{05}(a_{018} + b_{029}))b_{07} + d_{030}(a_{014} + 3b_{022} + 3a_{03} + b_{08})^2, \\
E_8 = \; & ((-2d_{09} - 2d_{025})a_{013} + (-6d_{09} - 6d_{025})b_{021} + (-2a_{014} - 6b_{022} - 6a_{03} - 2b_{08})d_{023} + (- \\
& 6d_{09} - 6d_{025})a_{02} + (-2d_{09} - 2d_{025})b_{07} + (-2a_{014} - 6b_{022} - 6a_{03} - 2b_{08})d_{05} + 4(a_{016} + \\
& b_{028})(3d_{020} + 3d_{00} + d_{04}))a_{011} + ((-2d_{09} - 2d_{025})a_{013} + (-6d_{09} - 6d_{025})b_{021} + (-2a_{014} - \\
& 6b_{022} - 6a_{03} - 2b_{08})d_{023} + (-6d_{09} - 6d_{025})a_{02} + (-2d_{09} - 2d_{025})b_{07} + (-2a_{014} - 6b_{022} - \\
& 6a_{03} - 2b_{08})d_{05} + 4(a_{016} + b_{028})(3d_{020} + 3d_{00} + d_{04}))b_{026} + (a_{013} + 3b_{021} + 3a_{02} + b_{07})(\\
& d_{031}a_{013} + 3d_{031}b_{021} + (-2a_{016} - 2b_{028})d_{023} + (2a_{014} + 6b_{022} + 6a_{03} + 2b_{08})d_{030} + 3d_{031} \\
& a_{02} + d_{031}b_{07} - 2d_{05}(a_{016} + b_{028})), \\
E_9 = \; & (6d_{00} + 6d_{020} + 2d_{04})a_{011}^2 + ((12d_{00} + 12d_{020} + 4d_{04})b_{026} - 2(a_{013} + 3b_{021} + 3a_{02} + b_{07})(\\
& d_{023} + d_{05}))a_{011} + (6d_{00} + 6d_{020} + 2d_{04})b_{026}^2 - 2(a_{013} + 3b_{021} + 3a_{02} + b_{07})(d_{023} + d_{05}) \\
& b_{026} + (a_{013} + 3b_{021} + 3a_{02} + b_{07})^2 d_{030}, \\
E_{10} = \; & ((-4d_{024} - 4d_{06})a_{014} + (-12d_{024} - 12d_{06})b_{022} + (-12d_{024} - 12d_{06})a_{03} + (-4d_{024} - 4d_{06}) \\
& b_{08} + 8(3d_{020} + 3d_{00} + d_{04})(a_{012} + b_{027}))a_3 + (a_{014} + 3b_{022} + 3a_{03} + b_{08})^2 d_3, \\
E_{11} = \; & (6d_{00} + 6d_{020} + 2d_{04})a_{012}^2 + ((12d_{00} + 12d_{020} + 4d_{04})b_{027} - 2(a_{014} + 3b_{022} + 3a_{03} + b_{08})(\\
& d_{06} + d_{024}))a_{012} + (6d_{00} + 6d_{020} + 2d_{04})b_{027}^2 - 2(a_{014} + 3b_{022} + 3a_{03} + b_{08})(d_{06} + d_{024})b_{027} \\
& +(a_{014} + 3b_{022} + 3a_{03} + b_{08})^2 d_{034}.
\end{aligned}
$$

Hence, It is easy to verify that the system $(h_{32}(\beta, \xi), h_{33}(\beta, \xi)) = (0, 0)$ has at most 36 real solutions by the Bezout's theorem. So, the coefficients of system (3.4) can be taken in such a way that this system has 36 real solutions different from zero for $\alpha > 0$.

Let $(\bar{\alpha}, \bar{\beta}, \bar{\xi})$ be a solution of system (3.4). In order to have a limit cycle according to the averaging theory in Section 2, we must have

$$
D(\bar{\alpha}, \bar{\beta}, \bar{\xi}) = det \begin{pmatrix} \dfrac{\partial g_{31}}{\partial \alpha} & \dfrac{\partial g_{31}}{\partial \beta} & \dfrac{\partial g_{31}}{\partial \xi} \\ \dfrac{\partial g_{32}}{\partial \alpha} & \dfrac{\partial g_{32}}{\partial \beta} & \dfrac{\partial g_{32}}{\partial \xi} \\ \dfrac{\partial g_{33}}{\partial \alpha} & \dfrac{\partial g_{33}}{\partial \beta} & \dfrac{\partial g_{33}}{\partial \xi} \end{pmatrix} \Bigg|_{(\alpha,\beta,\xi)=(\bar{\alpha},\bar{\beta},\bar{\xi})} \neq 0. \tag{3.5}
$$

In short, we deduce that system (1.1) has at most 36 limit cycles in a zero-Hopf bifurcation at the origin, using the averaging theory of third order. This completes the proof of Theorem 1.1. □

Proof of Corollary 3. By doing the change of variables $(x, y, z, w) = (\epsilon X, \epsilon Y, \epsilon Z, \epsilon W)$ system (1.2) becomes

Uncorrected Proof

$$
\begin{aligned}
\dot{X} = \;& \Big(-\frac{5}{3}X^3W + YZX^2 - XW^3 + YXW^2 - ZXW^2 + Y^2XW - 3Z^2XW + Y^2ZX + Y \\
& Z^2X - Z^3X - X + W^4 + Y^4 - Y \Big)\epsilon^3 - Y\epsilon^2 - \epsilon Y - Y, \\
\dot{Y} = \;& (2W^4 + W^3Y - W^2YZ + WX^2Y + WY^3 + WYZ^2 + 3WZ^3 + X^3Y + X^2YZ + 2Y^4 - \\
& YZ^3 + X - Y)\epsilon^3 + X\epsilon^2 + \epsilon X + X, \\
\dot{Z} = \;& \Big(Y^2Z^2 + W^2Y^2 + X^2Y^2 + X^2Z^2 - 4Z^3W + 4W^3Z - 3X^2W^2 - 2W^2Z^2 + 2W^4 - \\
& \frac{23}{4}Z^4 + Y^4 - X^4 - \frac{41}{4}Z - WY^2Z - X^2ZW \Big)\epsilon^3, \\
\dot{W} = \;& \Big(WY^2Z + X^2ZW + Y^2Z^2 + W^2Y^2 + \frac{1331}{28}W^2Z^2 + X^2Y^2 - \frac{60}{7}W^3Z - \frac{1377}{56}X^2Z^2 - \\
& \frac{213}{14}X^2W^2 - \frac{347}{28}Z^3W - \frac{206}{7}W + \frac{209}{14}W^4 + \frac{705}{14}Z^4 + Y^4 + X^4 \Big)\epsilon^3.
\end{aligned}
$$

We write this system in the cylindrical coordinates $(X, Y, Z, W) = (\alpha\cos(\theta), \alpha\sin(\theta), \beta, \xi)$ and we get

$$
\begin{aligned}
\dot{\alpha} = \;& \frac{1}{3}(2\alpha^3(3\alpha\sin(\theta) - 3\beta - 4\xi)\cos(\theta)^4 + 3\alpha^3(\beta\sin(\theta) - \alpha)\cos(\theta)^3 + 3(\alpha(\beta^2 - 4\alpha^2 + \xi^2) \\
& \sin(\theta) - 4\xi\beta^2 + 2\alpha^2\beta - 2\xi^3)\alpha\cos(\theta)^2 + 3(\alpha^4 + \xi^4)\cos(\theta) + 3(3\beta^3\xi + 2\alpha^4 + 2\xi^4)\sin(\theta) + \\
& 3\alpha(-\beta^3 + \beta^2\xi - \beta\xi^2 + \alpha^2\xi + \xi^3 - 1))\epsilon^3, \\
\dot{\theta} = \;& \frac{1}{3\alpha}((6\cos(\theta)^5\alpha^4 + 3\cos(\theta)^4\beta\alpha^3 + ((8\xi + 6\beta)\alpha\sin(\theta) + 3\xi^2 + 3\beta^2 - 12\alpha^2)\alpha^2\cos(\theta)^3 - \\
& 3\alpha^3(-2\alpha\sin(\theta) + \beta)\cos(\theta)^2 + (\alpha(12\beta^2\xi - 3\beta\alpha^2 + 6\xi^3)\sin(\theta) + 6\alpha^4 - (3\beta^2 + 3\xi^2)\alpha^2 + \\
& 6\xi^4 + 9\xi\beta^3)\cos(\theta) - 3(\alpha^4 + \xi^4)\sin(\theta) + 3\alpha)\epsilon^3 + 3\alpha\epsilon^2 + 3\alpha\epsilon + 3\alpha), \\
\dot{\beta} = \;& \frac{\epsilon^3}{4}(-4\alpha^4\cos(\theta)^4 - 4\alpha^4\cos(\theta)^2 - 16\alpha^2\cos(\theta)^2\xi^2 - 23\beta^4 - 16\xi\beta^3 + 4\alpha^2\beta^2 - 8\xi^2\beta^2 - 4\alpha^2\beta \\
& \xi + 16\xi^3\beta + 4\alpha^4 + 4\xi^2\alpha^2 + 8\xi^4 - 41\beta), \\
\dot{\xi} = \;& \frac{\epsilon^3}{56}(56\alpha^4\cos(\theta)^4 + (-1433\beta^2\alpha^2 - 56\alpha^4 - 908\alpha^2\xi^2)\cos(\theta)^2 + 56\alpha^4 + 56(\beta^2 + \beta\xi + \xi^2)\alpha^2 \\
& + 2820\beta^4 - 694\xi\beta^3 + 2662\xi^2\beta^2 - 480\xi^3\beta + 836\xi^4 - 1648\xi).
\end{aligned}
$$

We change the independent variable from t to θ, and denoting the derivative with respect to θ by a dot, then we get the system

$$
\begin{aligned}
\dot{\alpha} = \;& \frac{\epsilon^3}{3}(2\alpha^3(3\alpha\sin(\theta) - 3\beta - 4\xi)\cos(\theta)^4 + (3\alpha^3\beta\sin(\theta) - 3\alpha^4)\cos(\theta)^3 - 3(-\alpha(\beta^2 - 4\alpha^2 + \xi^2 \\
&)\sin(\theta) + 2\xi^3 + 4\xi\beta^2 - 2\alpha^2\beta)\alpha\cos(\theta)^2 + (3\alpha^4 + 3\xi^4)\cos(\theta) + (9\beta^3\xi + 6\alpha^4 + 6\xi^4)\sin(\theta) \\
& + 3\alpha(-\beta^3 + \beta^2\xi - \beta\xi^2 + \alpha^2\xi + \xi^3 - 1)) + O(\epsilon^4), \\
\dot{\beta} = \;& \frac{\epsilon^3}{4}(-4\alpha^4\cos(\theta)^4 + (-4\alpha^4 - 16\alpha^2\xi^2)\cos(\theta)^2 - 23\beta^4 - 16\xi\beta^3 + (4\alpha^2 - 8\xi^2)\beta^2 + (-4\alpha^2\xi + \\
& 16\xi^3 - 41)\beta + 8\xi^4 + 4\xi^2\alpha^2 + 4\alpha^4) + O(\epsilon^4), \\
\dot{\xi} = \;& \frac{\epsilon^3}{56}(56\alpha^4\cos(\theta)^4 - 1433\alpha^2\cos(\theta)^2\beta^2 - 56\cos(\theta)^2\alpha^4 - 908\alpha^2\cos(\theta)^2\xi^2 + 2820\beta^4 - 694\xi\beta^3 \\
& + 56\alpha^2\beta^2 + 2662\xi^2\beta^2 + 56\xi\alpha^2\beta - 480\xi^3\beta + 56\alpha^4 + 56\xi^2\alpha^2 + 836\xi^4 - 1648\xi) + O(\epsilon^4).
\end{aligned}
$$

Uncorrected Proof

Now we apply the averaging theory of third order and we get that the third averaging function $g_3(\alpha,\beta,\xi) = (g_{31}(\alpha,\beta,\xi), g_{32}(\alpha,\beta,\xi),$ is

$$\begin{cases} g_{31}(\alpha,\beta,\xi) = \Big(\Big(\dfrac{\alpha^2}{4} - \xi^2\Big)\beta - \beta^3 - \xi\beta^2 - 1\Big)\alpha, \\ g_{32}(\alpha,\beta,\xi) = \alpha^2\beta^2 - \xi^2\alpha^2 - 4\xi\beta^3 - 2\xi^2\beta^2 + 4\xi^3\beta - \xi\alpha^2\beta + \dfrac{1}{8}\alpha^4 - \dfrac{41}{4}\beta - \dfrac{23}{4}\beta^4 + 2\xi^4, \\ g_{33}(\alpha,\beta,\xi) = \dfrac{209\xi^4}{14} - \dfrac{60\xi^3\beta}{7} + \dfrac{(5324\beta^2 - 796\alpha^2)\xi^2}{112} + \dfrac{(-1388\beta^3 + 112\beta\alpha^2 - 3296)\xi}{112} + \dfrac{705\beta^4}{14} \\ \qquad\qquad - \dfrac{1321\alpha^2\beta^2}{112} + \dfrac{7\alpha^4}{8}. \end{cases} \tag{3.6}$$

In order for looking for the limit cycles of system (1.2) by the averaging theory we need to compute the isolated real roots of the averaged system (3.6) with $\alpha > 0$.

We isolate α^2 from the equation $g_{31}(\alpha,\beta,\xi) = 0$, and we substitute it in $g_{3i}(\alpha,\beta,\xi) = 0$ for $i = 2,3$. Then we get two functions $(h_{32}, h_{33}) = (h_{32}(\beta,\xi), h_{33}(\beta,\xi))$ given by

$$h_{32} = \frac{\beta^6 - 9\beta^3 + 8}{4\beta^2},$$

$$h_{33} = \frac{481\beta^6 - 772\beta^5\xi + 502\beta^4\xi^2 - 140\beta^3\xi^3 + 14\beta^2\xi^4 - 537\beta^3 + 72\beta^2\xi - 12\beta\xi^2 + 392}{28\beta^2},$$

where $\alpha^2 = \dfrac{4(\beta^3 + \beta^2\xi + \beta\xi^2 + 1)}{\beta}$.

Solving the system $(h_{32}(\beta,\xi), h_{33}(\beta,\xi)) = (0,0)$ for (α,β,ξ) with $\beta \neq 0$, we obtain the following eight real solutions $\big\{(4,1,1),\ (4\sqrt{2},1,2),\ (2\sqrt{14},1,3),\ (2\sqrt{22},1,4),\ (\sqrt{78},2,3),\ (\sqrt{114},2,4),\ (\sqrt{158},2,5),\ (13\sqrt{2},2,8)\big\}$. We can easily verify that these solutions are roots of the averaged system (3.6).

Since the determinants (3.5) at these eight solutions are $\{126,\ -84,\ 147,\ -693,\ (-4095/2),\ 1197,$

$(-4977/2),\ 53235\}$, respectively and thus non-zero, so the system (1.2) has eight limit cycle bifurcating from the origin provided by the averaging theory of third order. This completes the proof of **corollary** 1.3. □

Uncorrected Proof

4. Conclusions

First, we identified the system we are working on, which is a polynomial differential system in $\mathbb{R}^4$ with quartic homogeneous nonlinearities. We proved that this system can exhibit at most 36 limit cycles bifurcating from a singularity with eigenvalues of the form $\pm\omega i$, 0, and 0, using third-order averaging theory and Bezout's theorem. Finally, we applied an example to this case and obtained a system containing eight periodic solutions bifurcating from the zero-Hopf equilibrium localized at the origin of coordinates when $\epsilon = 0$, as demonstrated in the proof of corollary 3.

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Uncorrected Proof