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On the stability analysis and the solitonic wave structures for the Fordy-Gibbons-Jimbo-Miwa equation

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Abstract

In this article, the Fordy-Gibbons-Jimbo-Miwa equation is analyzed, a special form of the Kadomtsev-Petviashvili hierarchy equation, which is one of the most prominent nonlinear dynamical models with two spatial and a temporal coordinate that represents the evolution of long, nonlinear, small-amplitude waves with a gradual dependence on the transverse coordinate. The governing model is investigated analytically by employing the extended generalized Riccati equation mapping approach (GREM). Furthermore, the dynamics of several wave structures are visualized in 3D, 2D, and contour forms for a given set of parameters using Mathematica 13.0 to demonstrate their characteristics, which has been achieved by selecting appropriate values of the relevant parameters. These solutions exhibit the characteristics of v-shaped, singular, and multi-bell-shaped, singular periodic, and multi-periodic solitons. Additionally, it has been confirmed that the model under consideration is a stable nonlinear structure by validating the established results. A range of dynamic and static nonlinear equations governing evolutionary phenomena in computational physics and other relevant domains and research areas can be solved using these approaches, as demonstrated by their simplicity, clarity, and effectiveness, as well as the computational complexities and results.

Keywords. Fordy-Gibbons-Jimbo-Miwa equation, Soliton solutions, Kadomtsev-Petviashvili equation, Nonlinear dynamics, Stability analysis, Analytical solutions.

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1. Introduction

Over the past few decades, researchers have recognized interesting properties of nonlinear partial differential equations (NPDEs) in science and engineering. The dynamical study of nonlinear evaluation equations is one of the most exciting areas of research, and have received considerable attention from eminent scientists and experts in spectroscopy, nonlinear dynamics [40], plasma physics [8], fiber optic theory [1], theory of shallow water waves [36], telecommunication systems [31], soliton theory [12], space technology [30], quantum mechanics [27], electromagnetic field [16], fluid mechanics [7], nonlinear dispersive water waves [27] and many other contemporary fields of science and engineering [6, 32]. The study of nonlinear dynamics is rapidly rising to the top of the list of important fields for understanding the dynamics of major natural phenomena [33]. The distance across the sea optical fibers can be traveled in femtoseconds by optical solitons, which are essentially localized electromagnetic waves [46]. In the present age of science and technology, the idea of solitons has greatly contributed to the advancement of communication technology [43]. Considering the next generation of high-speed communication technology, this technology is one of the most stimulating areas of research in recent years [5, 13, 15, 24, 38].

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Analytical solutions to nonlinear partial differential equations (NPDEs) is a growing area of research interest in the engineering sciences, mathematical physics, and other technological advanced domains. In the recent past, a variety of analytical techniques are being introduced to deal with such models efficiently, including the extended auxiliary equation mapping method [37], the new extended direct algebraic method [44], the multiple exp-function scheme [47], the He's semi-inverse method [48], the $\exp(-\xi(\psi))$ expansion method [34], the improved $\tan(\phi/2)$-expansion technique [2, 20], the homogeneous balance method [28], the extended Fan-sub equation method [35, 42], the modified Jacobi elliptic expansion method [22], the unified auxiliary equation method [3], the auxiliary equation method [29], the generalized exponential rational function method [42], the Sine-Cosine method [21], the F-expansion technique [19], the Darboux transformation method [39], the Hirota bilinear method [45], the Sine-Gordon expansion method [9], and the inverse scattering transform method [26]; which have been implemented to develop travelling wave solutions of many prominent NPDEs. The development of analytical solutions to the nonlinear evolution equation is a crucial field of study in the analysis of nonlinear physical phenomena. With the help of exact solutions, it is possible to comprehend the mechanisms underlying the complex physical occurrences and dynamical processes that nonlinear evolution equations more efficiently [23, 25].

2. Fordy-Gibbons-Jimbo-Miwa Equation

A NPDE with two spatial and one temporal coordinate that represents the evolution of long, nonlinear, small-amplitude waves with a gradual dependence on the transverse coordinate is the Kadomtsev-Petviashvili equation (KP). The second KP-hierarchy equation [18], which has two types of solitary wave solutions such that the solitary waves of the second type do not interact elastically [10], is shown to be conditionally integrable. From the KP hierarchy, the Fordy-Gibbons-Jimbo-Miwa equation can be derived through equivalent reductions [41]. It is reasonable to assume that these systems' soliton solutions can be found from those of the KP hierarchy through reduction. Nevertheless, this process is not straightforward, and it is still not evident how to derive the soliton solutions to these Jimbo Miwa equations through reduction. Initially, Fordy and Gibbons [11] proposed the the Fordy-Gibbons-Jimbo-Miwa equation in 1991. In this study, some novel travelling wave solutions for the Fordy-Gibbons-Jimbo-Miwa equation with higher order dispersion and nonlinear terms are successfully developed using the extended GREM approach [4]. The Fordy-Gibbons-Jimbo-Miwa equation is expressed as

$$9U_t + (U_{xxxx} - 5U^2U_{xx} - 5U_xU_{xx} - 5UU_{xx}^2 + U^5)_x = 0. \tag{2.1}$$

In 1999, Xing-Biao Hu et al. developed the multi-soliton solutions for the governing model [17] by employing the Hirota bilinear approach [14].

The structure of this article is as follows: In section 2, the Fordy-Gibbons-Jimbo-Miwa equation is introduced, while in section 3, the solution to the governing model are developed, the stability analysis is analysed in section 4, the physical descriptions for the established results are illustrated in section 5. The conclusions were drawn at the end.

3. Mathematical Analysis

Consider the transformation

$$U(x,t) = u(\xi), \;\; \xi = x - t\omega, \tag{3.1}$$

where ω is nonzero constants to be determined later. By using the transformation Eq. (3.2) into Eq. (3.1)

$$u^5 + u^4 - 5u^2u'' - 5u\,(u')^2 - 5u'u'' - 9u\omega = 0. \tag{3.2}$$

The solution to the Eq. (3.2) is of the form

$$u(\xi) = a_0 + a_1\psi(\xi) + \frac{a_{-1}}{\psi(\xi)}, \tag{3.3}$$

and ψ is given as follow

$$\psi'(\xi) = \eth_0 + \eth_1\psi(\xi) + \eth_2\psi(\xi)^2, \tag{3.4}$$

where $\eth^i(\mathrm{i} = 0,\ 1,\ 2)$ are constants. By substituting Eq. (3.3) along with Eq. (3.4) in Eq. (3.2) and gathering the coefficients of ψ^i, several families of analytical solutions to the nonlinear model (2) are discovered using the most recent computational tools like Mathematica or Maple.

Family I:

$$a_0 = -\frac{\eth_1}{2},\ a_{-1} = -\eth_0,\ a_1 = 0,\ \eth_2 = \frac{\eth_1^2 - 12\sqrt{\omega}}{4\eth_0}.$$

when $4\eth_0\eth_2 - \eth_1^2 < 0$ and $\eth_2\eth_1 \neq 0$ or $\eth_0\eth_2 \neq 0$, the solutions of (3.4) are

$$\psi_1(\xi) = \frac{\sqrt{4\eth_0\eth_2 - \eth_1^2}\tan\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1}{2\eth_2},$$

and the solution is

$$U_1 = a_0 + \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\tan\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1\right) + \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\tan\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1}, \tag{3.5}$$

$$\psi_2(\xi) = \frac{\sqrt{4\eth_0\eth_2 - \eth_1^2}\cot\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1}{2\eth_2},$$

and the solution is

$$U_2 = a_0 + \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\cot\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1} + \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\cot\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1\right), \tag{3.6}$$

$$\psi_3(\xi) = \frac{1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tan\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \sec\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right),$$

and the solution is

$$\begin{aligned} U_3 = a_0 &+ \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tan\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \sec\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right) \\ &+ \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tan\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \sec\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1}, \end{aligned} \tag{3.7}$$

$$\psi_4(\xi) = \frac{1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\cot\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \csc\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right),$$

and the solution is

$$\begin{aligned} U_4 = a_0 &+ \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\cot\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \csc\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1} \\ &+ \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\cot\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \csc\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right), \end{aligned} \tag{3.8}$$

$$\psi_5(\xi) = \frac{1}{4\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tan\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \cot\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - 2\eth_1\right),$$

and the solution is

$$U_5 = a_0 + \frac{a_1}{4\eth_2}\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\left(\tan\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\cot\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\right)-2\eth_1\right)$$
$$+\frac{4a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2-\eth_1^2}\left(\tan\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\cot\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\right)-2\eth_1}, \tag{3.9}$$

$$\psi_6(\xi) = \frac{1}{2\eth_2}\left(\frac{-\sqrt{(4\eth_0\eth_2-\eth_1^2)(A^2-B^2)}-A\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{A\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B}-\eth_1\right),$$

and the solution is

$$U_6 = a_0 + \frac{2a_{-1}\eth_2}{\frac{-\sqrt{(4\eth_0\eth_2-\eth_1^2)(A^2-B^2)}-A\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{A\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B}-\eth_1}$$
$$+\frac{a_1}{2\eth_2}\left(\frac{-\sqrt{(4\eth_0\eth_2-\eth_1^2)(A^2-B^2)}-A\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{A\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B}-\eth_1\right), \tag{3.10}$$

$$\psi_7(\xi) = \frac{1}{2\eth_2}\left(\frac{A\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{(4\eth_0\eth_2-\eth_1^2)(A^2-B^2)}}{A\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B}-\eth_1\right),$$

and the solution is

$$U_7 = a_0 + \frac{2a_{-1}\eth_2}{\frac{A\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{(4\eth_0\eth_2-\eth_1^2)(A^2-B^2)}}{A\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B}-\eth_1}$$
$$+\frac{a_1}{2\eth_2}\left(\frac{A\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{(4\eth_0\eth_2-\eth_1^2)(A^2-B^2)}}{A\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B}-\eth_1\right), \tag{3.11}$$

where A and B are two nonzero real constants and satisfy$A^2-B^2>0$.

$$\psi_8(\xi) = -\frac{2\eth_0\cos\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\sin\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\eth_1\cos\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)},$$

and the solution is

$$U_8 = a_0 - \frac{2a_1\eth_0\cos\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\sin\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\eth_1\cos\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}$$
$$-\frac{a_{-1}\sec\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{2\eth_0}\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\sin\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\eth_1\cos\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\right), \tag{3.12}$$

$$\psi_9(\xi) = \frac{2\eth_0\sin\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\eth_1\sin\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)},$$

and the solution is

$$
\begin{aligned}
U_9 = a_0 &+ \frac{2a_1\eth_0 \sin\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\eth_1\sin\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)} \\
&+ \frac{a_{-1}\csc\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{2\eth_0}\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\eth_1\sin\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\right), \qquad (3.13)
\end{aligned}
$$

$$
\psi_{10}(\xi) = -\frac{2\eth_0\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\eth_1\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{4\eth_0\eth_2-\eth_1^2}},
$$

and the solution is

$$
U_{10} = a_0 - \frac{2a_1\eth_0\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\eth_1\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{4\eth_0\eth_2-\eth_1^2}} - \frac{R}{2\eth_0}, \qquad (3.14)
$$

where

$$
R = a_{-1}\sec\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\eth_1\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{4\eth_0\eth_2-\eth_1^2}\right),
$$

$$
\psi_{11}(\xi) = \frac{2\eth_0\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{-\eth_1\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{4\eth_0\eth_2-\eth_1^2}},
$$

and the solution is

$$
U_{11} = a_0 + \frac{2a_1\eth_0\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{-\eth_1\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{4\eth_0\eth_2-\eth_1^2}} + \frac{R}{2\eth_0}, \qquad (3.15)
$$

where,

$$
R = a_{-1}\csc\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\left(-\eth_1\sin\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+\sqrt{4\eth_0\eth_2-\eth_1^2}\cos\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{4\eth_0\eth_2-\eth_1^2}\right),
$$

$$
\psi_{12}(\xi) = \frac{4\eth_0\sin\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\Delta_2-2\eth_1\sin\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)},
$$

and the solution is

$$
U_{12} = a_0 + \frac{4a_1\eth_0\sin\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\Delta_2-2\eth_1\sin\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)} + \frac{R}{4\eth_0}, \qquad (3.16)
$$

where,

$$
R = a_{-1}\csc\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\sec\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\left(\Delta_2-2\eth_1\sin\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\right),
$$

and

$$
\Delta_2 = 2\sqrt{4\eth_0\eth_2-\eth_1^2}\cos^2\left(\frac{1}{4}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)-\sqrt{4\eth_0\eth_2-\eth_1^2},
$$

Family II:

$$a_0 = -\eth_1,\ a_{-1} = 0,\ a_1 = -2\eth_2,\ \eth_0 = \frac{\eth_1^2 + 3\sqrt{\omega}}{4\eth_2}.$$

When $4\eth_0\eth_2 - \eth_1^2 > 0$ and $\eth_2\eth_1 \neq 0$ or $\eth_0\eth_2 \neq 0$, the solutions of (3.4) are

$$\psi_{13}(\xi) = \frac{1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\tanh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1\right),$$

and the solution is

$$U_{13} = a_0 + \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\tanh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1} + \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\tanh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1\right),$$

$$\psi_{14}(\xi) = \frac{1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\coth\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1\right), \tag{3.17}$$

and the solution is

$$U_{14} = a_0 + \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\coth\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1\right) + \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\coth\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1},$$

$$\psi_{15}(\xi) = \frac{1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tanh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \operatorname{sech}\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right), \tag{3.18}$$

and the solution is

$$\begin{aligned} U_{15} = a_0 &+ \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tanh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \operatorname{sech}\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1} \\ &+ \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tanh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \operatorname{sech}\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right), \end{aligned} \tag{3.19}$$

$$\psi_{16}(\xi) = \frac{1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\coth\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \operatorname{csch}\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right),$$

and the solution is

$$\begin{aligned} U_{16} = a_0 &+ \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\coth\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \operatorname{csch}\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right) \\ &+ \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\coth\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \operatorname{csch}\left(\frac{1}{2}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1}, \end{aligned} \tag{3.20}$$

$$\psi_{17}(\xi) = \frac{1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tanh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \coth\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right),$$

and the solution is

$$\begin{aligned} U_{17} = a_0 &+ \frac{2a_{-1}\eth_2}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tanh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \coth\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1} \\ &+ \frac{a_1}{2\eth_2}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\left(\tanh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \coth\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right) - \eth_1\right), \end{aligned} \tag{3.21}$$

$$\psi_{18}(\xi) = \frac{1}{2\eth_2}\left(\frac{-\sqrt{(4\eth_0\eth_2 - \eth_1^2)\left(A^2 + B^2\right)} - A\sqrt{4\eth_0\eth_2 - \eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)}{A\sinh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + B} - \eth_1\right),$$

and the solution is

$$\begin{aligned} U_{18} &= a_0 + \frac{a_1}{2\eth_2}\left(\frac{-\sqrt{(4\eth_0\eth_2-\eth_1^2)\left(A^2+B^2\right)} - A\sqrt{4\eth_0\eth_2-\eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{A\sinh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B} - \eth_1\right) \\ &+ \frac{2a_{-1}\eth_2}{\frac{-\sqrt{(4\eth_0\eth_2-\eth_1^2)(A^2+B^2)} - A\sqrt{4\eth_0\eth_2-\eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{A\sinh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B} - \eth_1}, \end{aligned} \tag{3.22}$$

$$\psi_{19}(\xi) = \frac{1}{2\eth_2}\left(\frac{A\sqrt{4\eth_0\eth_2-\eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \sqrt{(4\eth_0\eth_2-\eth_1^2)\left(A^2+B^2\right)}}{A\sinh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B} - \eth_1 2\eth_2\right),$$

and the solution is

$$\begin{aligned} U_{19} &= a_0 + \frac{a_1}{2\eth_2}\left(\frac{A\sqrt{4\eth_0\eth_2-\eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \sqrt{(4\eth_0\eth_2-\eth_1^2)\left(A^2+B^2\right)}}{A\sinh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B} - \eth_1\right) \\ &+ \frac{2a_{-1}\eth_2}{\frac{A\sqrt{4\eth_0\eth_2-\eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \sqrt{(4\eth_0\eth_2-\eth_1^2)(A^2+B^2)}}{A\sinh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)+B} - \eth_1}, \end{aligned} \tag{3.23}$$

where A and B are two nonzero real constants.

$$\psi_{20}(\xi) = \frac{2\eth_0\cosh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\sinh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \eth_1\cosh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)},$$

and the solution is

$$\begin{aligned} U_{20} &= a_0 + \frac{2a_1\eth_0\cosh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\sinh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \eth_1\cosh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)} + \frac{a_{-1}\operatorname{sech}\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{2\eth_0} \\ &\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\sinh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \eth_1\cosh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\right), \end{aligned} \tag{3.24}$$

$$\psi_{21}(\xi) = -\frac{2\eth_0\sinh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\eth_1\sinh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \sqrt{4\eth_0\eth_2-\eth_1^2}\cosh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)},$$

and the solution is

$$\begin{aligned} U_{21} &= a_0 - \frac{2a_1\eth_0\sinh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\eth_1\sinh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \sqrt{4\eth_0\eth_2-\eth_1^2}\cosh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)} \\ &- \frac{a_{-1}\operatorname{csch}\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{2\eth_0}\left(\eth_1\sinh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \sqrt{4\eth_0\eth_2-\eth_1^2}\cosh\left(\frac{1}{2}\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)\right), \end{aligned} \tag{3.25}$$

$$\psi_{22}(\xi) = \frac{2\eth_0\cosh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2-\eth_1^2}\sinh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - \eth_1\cosh\left(\sqrt{4\eth_0\eth_2-\eth_1^2}\xi\right) - i\sqrt{4\eth_0\eth_2-\eth_1^2}},$$

and the solution is

$$U_{22} = a_0 + \frac{2a_1\eth_0 \cosh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)}{\sqrt{4\eth_0\eth_2 - \eth_1^2}\sinh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1\cosh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - i\sqrt{4\eth_0\eth_2 - \eth_1^2}} + \frac{R}{2\eth_0}, \tag{3.26}$$

where

$$R = a_{-1}\text{sech}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\sinh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \eth_1\cosh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - i\sqrt{4\eth_0\eth_2 - \eth_1^2}\right),$$

$$\psi_{23}(\xi) = \frac{2\eth_0 \sinh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)}{-\eth_1\sinh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \sqrt{4\eth_0\eth_2 - \eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \sqrt{4\eth_0\eth_2 - \eth_1^2}},$$

and the solution is

$$U_{23} = a_0 + \frac{2a_1\eth_0 \sinh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)}{-\eth_1\sinh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \sqrt{4\eth_0\eth_2 - \eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \sqrt{4\eth_0\eth_2 - \eth_1^2}} + \frac{R}{2\eth_0}, \tag{3.27}$$

where

$$R = a_{-1}\text{csch}\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\left(-\eth_1\sinh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \sqrt{4\eth_0\eth_2 - \eth_1^2}\cosh\left(\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) + \sqrt{4\eth_0\eth_2 - \eth_1^2}\right),$$

$$\psi_{24}(\xi) = \frac{4\eth_0 \sinh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\cosh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)}{\Delta_1 - 2\eth_1\sinh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\cosh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)},$$

and the solution is

$$U_{24} = a_0 + \frac{4a_1\eth_0 \sinh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\cosh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)}{\Delta_1 - 2\eth_1\sinh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\cosh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)} + \frac{R}{4\eth_0}, \tag{3.28}$$

where,

$$R = a_{-1}\text{csch}\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\text{sech}\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\left(\Delta_1 - 2\eth_1\sinh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\cosh\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right)\right),$$

and

$$\Delta_1 = 2\sqrt{4\eth_0\eth_2 - \eth_1^2}\cosh^2\left(\frac{1}{4}\sqrt{4\eth_0\eth_2 - \eth_1^2}\xi\right) - \sqrt{4\eth_0\eth_2 - \eth_1^2},$$

Family III:

$$a_{-1} = 0,\ a_0 = \sqrt{3}\sqrt[4]{\omega},\ a_1 = \eth_2,\ \eth_1 = 2\sqrt{3}\sqrt[4]{\omega},$$

When $\eth_2\eth_1 \neq 0$ and $\eth_0 = 0$, the solutions of (3.4) and (3.2) are

$$\psi_{25}(\xi) = -\frac{\eth_1 f}{\eth_2\left(-\sinh(\eth_1\xi) + \cosh(\eth_1\xi) + f\right)},$$

and the solution is

$$U_{25} = a_0 - \frac{a_{-1}\eth_2\left(-\sinh(\eth_1\xi) + \cosh(\eth_1\xi) + f\right)}{\eth_1 f} - \frac{a_1\eth_1 f}{\eth_2\left(-\sinh(\eth_1\xi) + \cosh(\eth_1\xi) + f\right)}, \tag{3.29}$$

$$\psi_{26}(\xi) = -\frac{\eth_1\left(\sinh(\eth_1\xi) + \cosh(\eth_1\xi)\right)}{\eth_2\left(\sinh(\eth_1\xi) + \cosh(\eth_1\xi) + f\right)},$$

and the solution is

$$U_{26} = a_0 - \frac{a_{-1}\eth_2\left(\sinh\left(\eth_1\xi\right)+\cosh\left(\eth_1\xi\right)+f\right)}{\eth_1\left(\sinh\left(\eth_1\xi\right)+\cosh\left(\eth_1\xi\right)\right)} - \frac{a_1\eth_1\left(\sinh\left(\eth_1\xi\right)+\cosh\left(\eth_1\xi\right)\right)}{\eth_2\left(\sinh\left(\eth_1\xi\right)+\cosh\left(\eth_1\xi\right)+f\right)}, \tag{3.30}$$

where f is an arbitrary constant.
Family IV:

$$a_{-1}=0,\ a_0=\sqrt{3}\sqrt[4]{\omega},\ a_1=\eth_2,$$

When $\eth_2 \neq 0$ and $\eth_1 = \eth_0 = 0$, the solutions of (3.4) and (3.2) are

$$\psi_{27}(\xi) = -\frac{1}{\eth_2\xi+p},$$

and the solution is

$$U_{27} = a_0 + a_{-1}\left(-\eth_2\xi - p\right) - \frac{a_1}{\eth_2\xi+p}, \tag{3.31}$$

where p is an arbitrary constant.

4. Stability Analysis

For the stability analysis of given model we define the hamiltonian momentum of Eq. (2.1), which is written as

$$S = \frac{1}{2}\int_{-\infty}^{\infty} U^2\, dx. \tag{4.1}$$

By putting the solitons wave solution which we get from the Eq. (3.5) into Eq. (4.1)as a result we obtain,

$$S = \frac{1}{2}\int_{-1}^{1}\left(-\frac{\eth_1^2-12\sqrt{\omega}}{2\left(2\sqrt{3}\sqrt{-\sqrt{\omega}}\tan\left(\sqrt{3}\sqrt{-\sqrt{\omega}}(x-\omega)\right)-\eth_1\right)} - \frac{\eth_1}{2}\right)^2 dx. \tag{4.2}$$

In above framework S, U are momentum and field potential respectively. Now we define the fundamental element for the soliton stability that is

$$\frac{\partial S}{\partial \omega} > 0, \tag{4.3}$$

the frequency is denoted by ω, which reduces to

$$\begin{aligned}
&\Big[324\omega\Big\{6\sqrt[4]{\omega}(\cosh\left(4\sqrt{3}\omega^{5/4}\right) + (5\omega-1)\cosh\left(2\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)\\
&+\cosh\left(4\sqrt{3}\sqrt[4]{\omega}\right) - (5\omega+1)\cosh\left(2\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right))\\
&-\sqrt{3}(\sinh\left(2\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)+\sinh\left(4\sqrt{3}\sqrt[4]{\omega}\right)-\sinh\left(2\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right))\Big\}\\
&+648\eth_1\omega^{3/4}\Big\{\sqrt{3}\sqrt[4]{\omega}(-2\sinh\left(4\sqrt{3}\omega^{5/4}\right)+(1-5\omega)\sinh\left(2\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)\\
&+(5\omega+1)\sinh\left(2\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right))+\cosh\left(2\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)-\cosh\left(2\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+54\eth_1^2\sqrt{\omega}\Big\{\sqrt{3}\left(-2\sinh\left(2\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)+\sinh\left(4\sqrt{3}\sqrt[4]{\omega}\right)+2\sinh\left(2\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right)\right)\\
&-6\sqrt[4]{\omega}\left(\cosh\left(4\sqrt{3}\sqrt[4]{\omega}\right)-3\cosh\left(4\sqrt{3}\omega^{5/4}\right)\right)\Big\}+54\sqrt{3}\eth_1^3\sqrt{\omega}\Big\{-2\sinh\left(4\sqrt{3}\omega^{5/4}\right)\\
&+(5\omega-1)\sinh\left(2\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)-(5\omega+1)\sinh\left(2\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right)\Big\}\\
&+\frac{9}{4}\eth_1^4\Big\{6\sqrt[4]{\omega}(\cosh\left(4\sqrt{3}\omega^{5/4}\right)+(1-5\omega)\cosh\left(2\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)+\cosh\left(4\sqrt{3}\sqrt[4]{\omega}\right)\\
&+(5\omega+1)\cosh\left(2\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right))-\sqrt{3}(-\sinh\left(2\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)\\
&+\sinh\left(4\sqrt{3}\sqrt[4]{\omega}\right)+\sinh\left(2\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right))\Big\}\Big]\\
&\times\Big[8\omega^{3/4}\left(\sqrt{3}\eth_1\cosh\left(\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)-6\sqrt[4]{\omega}\sinh\left(\sqrt{3}(\omega-1)\sqrt[4]{\omega}\right)\right)^2\\
&\left(\sqrt{3}\eth_1\cosh\left(\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right)-6\sqrt[4]{\omega}\sinh\left(\sqrt{3}\sqrt[4]{\omega}(\omega+1)\right)\right)^2\Big]^{-1}>0,
\end{aligned}
\tag{4.4}
$$

as a concluding remark, the stability criterion is validated from the above inequality. Hence, Eq. (2.1) is a stable nonlinear partial diffential equation.

5. Discussion and Results

The Fordy-Gibbons-Jimbo-Miwa equation is a nonlinear partial differential equation that arises in various areas of physics, including soliton theory, integrable systems, and quantum field theory. Its solutions have physical interpretations in different contexts. For instance, bright solitons represent localized energy packets in nonlinear optics and Bose-Einstein condensates. Dark solitons correspond to regions of lower density in Bose-Einstein condensates and nonlinear optics. Periodic solitons appear in periodic structures such as crystal lattices and periodic optical waveguides. Anti-kink solitons represent a type of topological defect in the field theory. Optical solitons can travel long distances without spreading due to their self-focusing and self-phase modulation properties. The precise understanding of these solutions is vital for furthering our knowledge and application of various fields of physics.

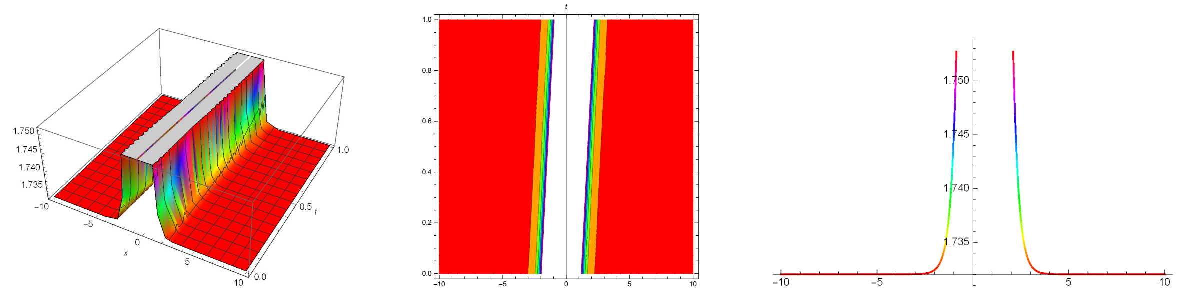
With the aid of the most recent scientific instruments, the graphical visualisation of the Fordy-Gibbons-Jimbo-Miwa model has been demonstrated in this part. The extended GREM approach is used to examine a range of solitary wave solutions for a given set of parameters. In Figures 1-8, the graphical solutions of the Fordy-Gibbons-Miwa model are shown for various values of the constants involving $\omega, \eth_2, \eth_1$. A single bell-shaped soliton solution is shown in Figure 1. Figure 2 shows elliptic solutions and bright-dark soliton waves. In the Figure 3, bright-dark-and-multisoliton is shown. The precise solitary wave solutions are shown in Figure 6. For the constant values indicated below, the anti-kink solitons wave solution is depicted in Figure 7. The contour and 2D charts show the nature of the nonlinear waves produced by the governing model Eq. (2.1).

6. Conclusions

In this paper, we have presented a study on the Fordy-Gibbons-Miwa equation using the extended generalized Riccati equation mapping approach. We have obtained several exact solutions for this equation, including bright, dark, periodic, anti-kink, and optical solitons. The obtained solutions have been demonstrated through 3D, contour, and 2D plots to show their characteristics under different values of the relevant parameters. Our findings indicate that the extended generalized Riccati equation mapping approach is an effective and reliable tool for solving nonlinear partial differential equations. Moreover, the identified solutions for the Fordy-Gibbons-Miwa equation could be useful in understanding the underlying physical phenomena associated with this equation. Future research can be focused on investigating the stability and dynamics of these solutions, as well as extending this approach to other nonlinear partial differential equations.

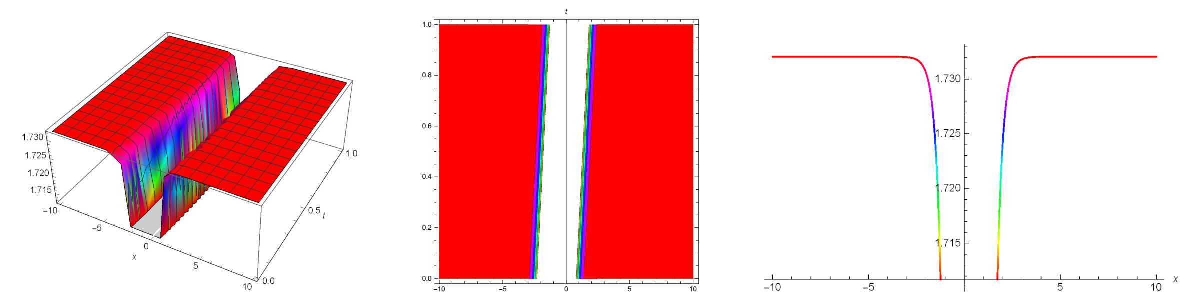
Declarations

Conflict of Interest. The authors declare that they do not have any conflicts of interest to disclose.
Ethical Approval. The authors confirm that no experiments involving animals were carried out for this research.



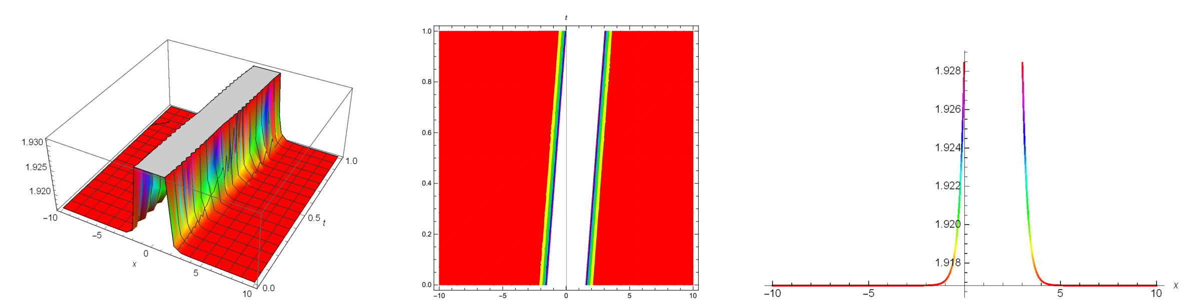
(a) (b) (c)

FIGURE 1. $U_1(\zeta,\tau):\ \omega = 1, \eth_0 = 1, \eth_1 = 2.$



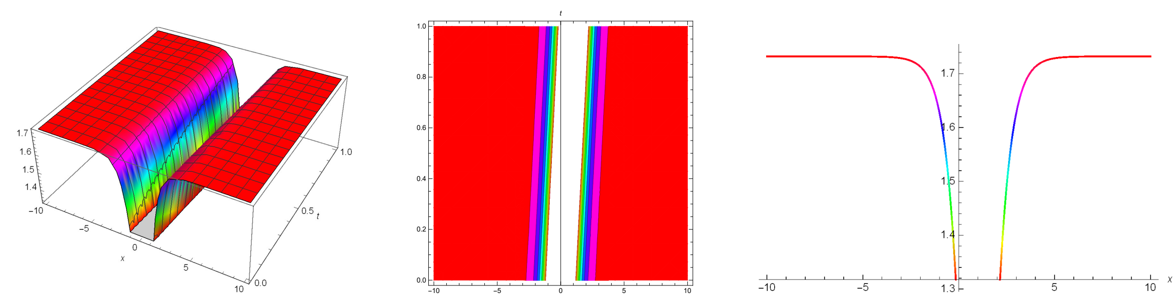
(a) (b) (c)

FIGURE 2. $U_5(\zeta,\tau):\ \omega = 1, \eth_0 = 2, \eth_1 = 3.$



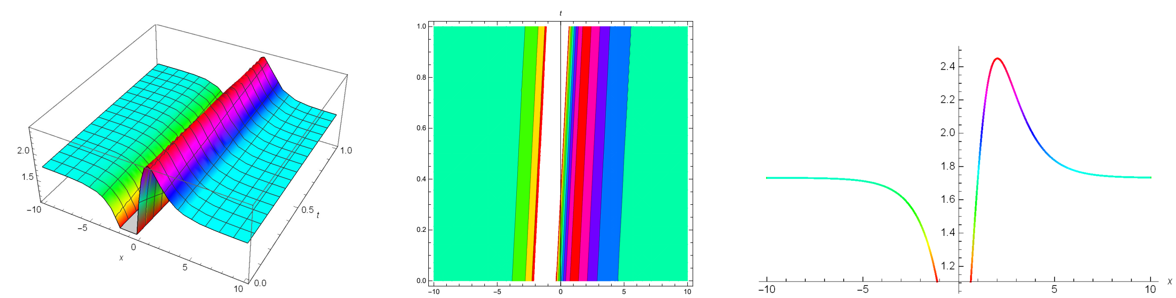
(a) (b) (c)

FIGURE 3. $U_9(\zeta,\tau):\ \omega = 1.5, \eth_0 = 1, \eth_1.$



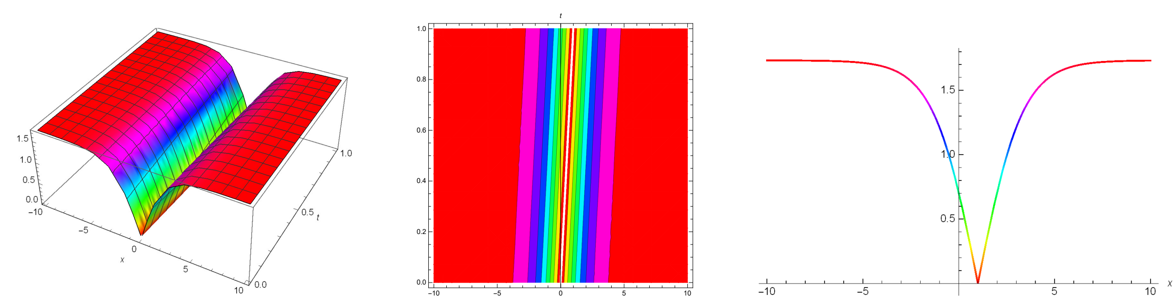
(a) (b) (c)

FIGURE 4. $U_{13}(\zeta,\tau):\ \omega=1,\eth_2=2,\eth_1=3.$



(a) (b) (c)

FIGURE 5. $U_{15}(\zeta,\tau):\ \omega=1,\eth_2=1,\eth_1=3.$



(a) (b) (c)

FIGURE 6. $U_{16}(\zeta,\tau):\ \omega=1,\eth_2=1,\eth_1=3.$

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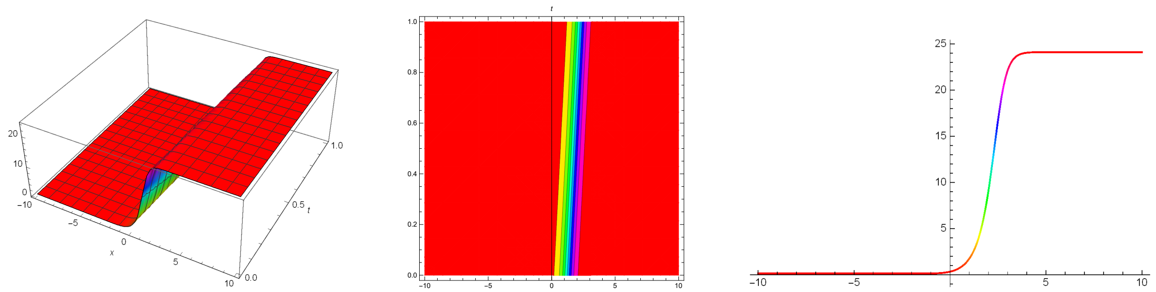
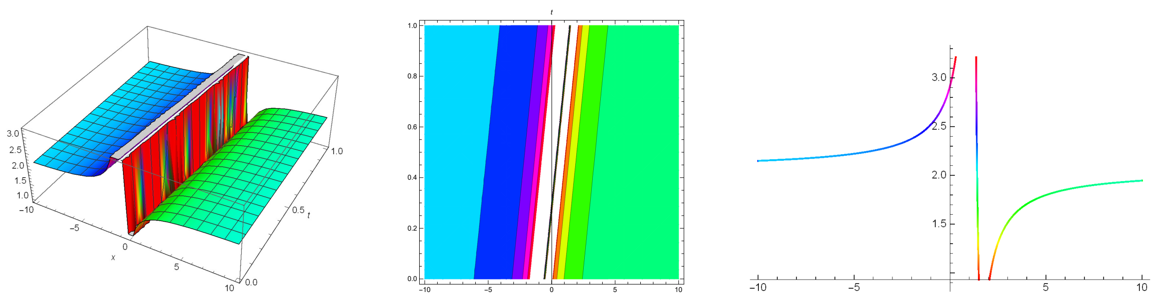
Author Contributions.

(1) Fazal Badshah: conceptualization, funding, identification of the problem.
(2) Kalim U. Tariq: Methodology, supervision, resource.
(3) Hadi Rezazadeh: Software, scientific computation, visualization.
(4) Medhat Ilyas: Validation, writing original draft.



(a) (b) (c)

FIGURE 7. $U_{22}(\zeta,\tau):\ \omega=1, \eth_2=1, \eth_1=1.5.$



(a) (b) (c)

FIGURE 8. $U_{27}(\zeta,\tau):\ \omega=2, \eth_2=3, p=2.5.$

(5) Mir Sajjad Hashemi: Formal analysis and investigation.
(6) Mohammad Ali Hosseinzadeh: Modelling, Investigation, Modifying the concept and computations, editing and reviewing of the work.

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