Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-15
DOI:10.22034/cmde.2024.61045.2621

Spectral collocation algorithm for the fractional Bratu equation via Hexic shifted chebyshev polynomials

Ahmed Gamal Atta[1], **Jomana Farag Soliman**[2], **Elaf Wael Elsaeed**[2], **Mostafa Wael Elsaeed**[2], **and Youssri Hassan Youssri**[3,*]

[1]Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo 11341, Egypt.

[2]Department of Mathematics, Faculty of Science, Galala University, Egypt.

[3]Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt.

Abstract

This paper offers a numerical collocation scheme for solving the fractional nonlinear Bratu differential equation. We obtain a system of nonlinear equations using our spectral collocation method, which we then solve iteratively using Newton's method to obtain an approximate solution. Additionally, numerical comparisons are made between the proposed strategy and several numerical strategies documented in various literatures. The numerical findings verify the accuracy, computational efficiency, and ease of use of the recommended approach.

Keywords. Chebyshev polynomials, Collocation method, Bratu differential equation.

2010 Mathematics Subject Classification. 65M70; 26A33; 34B15; 41A50.

1. Introduction

Numerical spectral methods are among the most widely used numerical approaches developed and adjusted to solve differential equations with particular numerical values. The basic idea underlying these methods is to use a truncated series of basis functions, usually orthogonal polynomials like ultraspherical, Chebyshev, Legendre, or other linearly independent polynomials, to represent the unknown solution of the differential equation. The main benefit of these methods is that they can yield findings with a reasonable degree of accuracy with a comparatively smaller number of degrees of freedom. A significant amount of research addressing all three kinds of spectral techniques has been conducted recently. For example, the author in [7] provided a spectral collocation approach to handle the nonlinear generalized fractional Riccati equation. A novel spectral Tau approach to fractional delay differential equations was developed in [1]. Furthermore, a Galerkin algorithm is used in [16] to solve the time-fractional diffusion equation. For further studies, see [11, 21, 27, 28, 30].

The Bratu differential equation is a second-order nonlinear ordinary differential equation in mathematical physics that arises from the study of boundary value problems. The Bratu equation is highly nonlinear due to its exponential component. It is widely used to describe a wide range of physical processes, including the buckling of elastic plates and the propagation of flames in combustion theory. Due to the difficulty of obtaining its analytical solutions, most solutions are estimated using numerical methods. The Bratu issue can be solved numerically using various techniques, including iterative, shooting, and finite difference approaches.

In mathematical physics, the Bratu differential equation is a second-order nonlinear ordinary differential equation that arises from the study of boundary value problems. Because of its exponential component, the Bratu equation is extremely nonlinear. It is frequently employed to explain a broad variety of physical phenomena, such as the spreading of flames in combustion theory and the buckling of elastic plates. Due to the difficulty of obtaining its analytical solutions, there have been many attempts to solve this problem. For example, the author in [8] presented an efficient method for solving fractional integral and differential equations of Bratu type. In [26], the authors proposed an

Received: 30 March 2024 ; Accepted: 29 September 2024.
∗ Corresponding author. Email: youssri@cu.edu.eg.

approximate solution of the Bratu differential equation using the Falker-type method. In [25], the authors proposed a K-step block hybrid Nyström-type method for the solution of the Bratu problem with impedance boundary conditions. For further work on Bratu's problem, the interested reader can refer to the following works: [10, 13, 14, 18, 22, 24].

Orthogonal polynomials have recently attracted new interest for their use in spectral approaches. The first, second, third, and fourth varieties of Chebyshev polynomials, in particular, have been crucial to the development of spectral approaches for PDEs. The accurate representation of smooth functions provided by the finite Chebyshev expansion is one benefit of utilizing Chebyshev polynomials. Furthermore, as m goes to infinity, the coefficients in the Chebyshev expansion approach zero faster than any inverse power in m. Masjed-Jamei introduced Chebyshev polynomials of the fifth and sixth sorts in his PhD thesis [15] in 2006. Many researchers have given these polynomials a great deal of attention. For example, the authors in [5] solved the fractional Rayleigh–Stokes problem via sixth-kind Chebyshev polynomials. In [4], the authors solved the nonlinear mixed partial integro-differential equations with continuous kernels to solve the reaction-diffusion-convection problem using sixth-kind Chebyshev polynomials. In [3], the authors solved the time-fractional diffusion equation using seventh-kind Chebyshev polynomials.

We point out here that the novelty of our contribution to this paper can be summarized as follows:

- Some specific integer and fractional derivatives of the shifted sixth-kind Chebyshev polynomials are expressed in terms of their original ones.
- Constructing theoretical background concerning these polynomials. More precisely, the fundamental formulas of these formulas, such as their analytics inversion, integer derivatives, and fractional derivatives of the formulas, are established. These formulas will be the backbone of applying various numerical methods to different DEs.
- Introducing new method based on collocation method to solve fractional Bratu equation.
- analyzeing in detail the error bound of this method.

To the best of our knowledge, the contributions using the sixth-kind Chebyshev polynomials are few in numerical analysis. We also refer to the approach followed, which has many advantages since by choosing shifted sixth-kind Chebyshev polynomials as basis functions a few terms of the retained modes, it is possible to produce approximations with excellent precision. Less calculation is required, and the resulting errors are small. We also note here that the presented collocation algorithm for treating the fractional Bratu equation is new, which motivates us to analyze it.

1.1. Properties of the hexic Chebyshev polynomials.

The hexic (sixth)-kind Chebyshev polynomials $Y_j(\xi)$, $j \geq 0$ are a sequence of orthogonal polynomials defined on $[-1, 1]$, that can be easily generated with the aid of the following recurrence relation:

$$Y_i(\xi) = \xi\, Y_{i-1}(\xi) - \alpha_i\, Y_{i-2}(\xi), \; Y_0(\xi) = 1, \quad Y_1(\xi) = \xi, \quad i \geq 2, \tag{1.1}$$

$$\alpha_i = \frac{i\,(i+1) + (-1)^i\,(2\,i+1) + 1}{4\,i\,(i+1)}. \tag{1.2}$$

These polynomials satisfy the orthogonality formula ([2]):

$$\int_{-1}^{1} \sqrt{1-\xi^2}\,\xi^2\, Y_i(\xi)\, Y_j(\xi)\, d\xi = h_i\, \delta_{i,j}, \tag{1.3}$$

where

$$h_i = \frac{\pi}{2^{2\,i+3}} \begin{cases} 1, & \text{if } i \text{ even}, \\ \frac{i+3}{i+1}, & \text{if } i \text{ odd}, \end{cases} \tag{1.4}$$

and

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.5}$$

The shifted orthogonal Chebyshev polynomials of the sixth-kind $C_i(\xi)$ are defined on $[0,1]$ as

$$C_i(\xi) = Y_i\,(2\,\xi - 1)\,. \tag{1.6}$$

$C_i(\xi)$ can be generated with the aid of the following recursive formula:

$$C_i(\xi) = (2\,\xi - 1)\ C_{i-1}(\xi) - \alpha_i\, C_{i-2}(\xi), \quad C_0(\xi) = 1, \quad C_1(\xi) = 2\,\xi - 1, \quad i \geq 2, \tag{1.7}$$

where α_i is defined in (1.2).
The orthogonality relation of $C_i(\xi)$ on $[0,1]$ is given by:

$$\int_0^1 \omega\, C_i(\xi)\, C_j(\xi)\; d\xi = \bar{h}_i\, \delta_{i,j}, \tag{1.8}$$

where

$$\omega = (2\,\xi - 1)^2\,\sqrt{\xi - \xi^2}, \tag{1.9}$$

and

$$\bar{h}_i = \frac{1}{4}\, h_i. \tag{1.10}$$

The analytic formula of $C_j(\xi)$ is [2]

$$C_j(\xi) = \sum_{\ell=0}^{j} B_{\ell,j}\, \xi^{\ell}, \tag{1.11}$$

where

$$B_{\ell,j} = \frac{2^{2\ell-j}}{(2\ell+1)!}\begin{cases} \displaystyle\sum_{k=\lfloor\frac{\ell+1}{2}\rfloor}^{\lfloor\frac{j}{2}\rfloor} \frac{(-1)^{\frac{j}{2}+k+\ell}(2k+\ell+1)!}{(2k-\ell)!}, & \text{if } j \text{ even},\\[2ex] \displaystyle\frac{2}{j+1}\sum_{k=\lfloor\frac{\ell}{2}\rfloor}^{\lfloor\frac{j-1}{2}\rfloor} \frac{(-1)^{\frac{j+1}{2}+k+\ell}(k+1)(2k+\ell+2)!}{(2k-\ell+1)!}, & \text{if } j \text{ odd}. \end{cases} \tag{1.12}$$

The inversion formula $C_j(\xi)$ is [2]

$$\xi^j = \sum_{m=0}^{j} Q_{m,j}\, C_m(\xi), \tag{1.13}$$

where

$$Q_{m,j} = \frac{(2j+1)!2^{m-2j+2}}{(j-m)!(m+j+4)!}\begin{cases} (m+2)\left(m(m+4)+j^2+j+3\right), & \text{if } m \text{ even},\\ (m+1)(m(m+4)+j(j+3)+6), & \text{if } m \text{ odd}. \end{cases} \tag{1.14}$$

Corollary 1.1. *Let $m \geq 2$. The second-order derivative of the polynomials $C_m(\xi)$ can be expressed explicitly as [6]:*

$$D^2\, C_m(\xi) = \sum_{\ell=0}^{m-2} \lambda_{\ell,m}\, C_\ell(\xi), \tag{1.15}$$

where

$$\lambda_{\ell,m} = 2^{2-m+\ell}\begin{cases} (2+\ell)(-8+m(4+m)-\ell(4+\ell)), & \textit{if } m \textit{ even and } \frac{m-\ell-2}{2} \textit{ even},\\ (m-\ell)(2+\ell)(4+m+\ell), & \textit{if } m \textit{ even and } \frac{m-\ell-4}{2} \textit{ even},\\ \dfrac{(1+\ell)(4+m+\ell)\left(-4+2m+m^2-(2+m)\ell\right)}{1+m}, & \textit{if } m \textit{ odd and } \frac{m-\ell-2}{2} \textit{ even},\\ \dfrac{(m-\ell)(1+\ell)(2(2+\ell)+m(6+m+\ell))}{1+m}, & \textit{if } m \textit{ odd and } \frac{m-\ell-4}{2} \textit{ even},\\ 0, & \textit{otherwise}. \end{cases}$$

2. Collocation scheme for the fractional Bratu equation

Consider the following fractional Bratu equation [17, 20]:

$$D^{\alpha}\upsilon + \lambda\, e^{\upsilon} = 0; \quad 1 < \alpha \leq 2, \tag{2.1}$$

subject to the initial conditions:

$$\upsilon(0) = a_1, \quad \upsilon'(0) = a_2, \tag{2.2}$$

or the boundary conditions:

$$\upsilon(0) = a_1, \quad \upsilon(1) = a_3, \tag{2.3}$$

where λ, is a given real parameter, and a_1, a_2 and a_3 are constants. Also, D^{α} is the fractional-order derivative in Caputo sense [19]

$$D^{\alpha}h(\xi) = \frac{1}{\Gamma(m-\alpha)}\int_0^{\xi}(\xi-y)^{m-\alpha-1}h^{(m)}(y)dy, \quad \alpha > 0, \quad \xi > 0, \tag{2.4}$$

where

$$m-1 < \alpha < m, \quad m \in \mathbb{N}.$$

The collection of C_i form an orthogonal basis of $L^2_{\omega(\xi)}(0,1)$. And so, for any given function $\upsilon(\xi) \in L^2_{\omega}(0,1)$, one has

$$\upsilon(\xi) = \sum_{i=0}^{\infty}\hat{\upsilon}_i\, C_i(\xi), \tag{2.5}$$

and approximated as

$$\upsilon(\xi) \approx \upsilon_N(\xi) = \sum_{i=0}^{N}\hat{\upsilon}_i\, C_i(\xi). \tag{2.6}$$

Therefore, we can write the residual $\mathbf{R}(\xi)$ of Eq. (2.1) as

$$\mathbf{R}(\xi) = D^{\alpha}\upsilon_N + \lambda\, e^{\upsilon_N}. \tag{2.7}$$

Now, we will solve the Bratu Equation (2.1) for the cases corresponding to $\alpha = 2$ and $1 < \alpha < 2$.
Case 1: At $\alpha = 2$.
By virtue of Corollary 1.1 along with the expansion (2.6), $\mathbf{R}(\xi)$ may be rewritten as

$$\begin{aligned}\mathbf{R}(\xi) &= D^2\upsilon_N + \lambda\, e^{\upsilon_N}\\ &= \sum_{i=0}^{N}\hat{\upsilon}_i\, D^2\, C_i(\xi) + \lambda\, e^{\sum_{i=0}^{N}\hat{\upsilon}_i\, C_i(\xi)}\\ &= \sum_{i=0}^{N}\sum_{p=0}^{i-2}\hat{\upsilon}_i\,\lambda_{p,i}\, C_p(\xi) + \lambda\, e^{\sum_{i=0}^{N}\hat{\upsilon}_i\, C_i(\xi)},\end{aligned} \tag{2.8}$$

Moreover, we get the following initial conditions

$$\begin{aligned} &\sum_{i=0}^{N} \hat{\upsilon}_i \, C_i(0) = a_1, \\ &\sum_{i=0}^{N} \hat{\upsilon}_i \, \frac{d\, C_i(0)}{d\, \xi} = a_2, \end{aligned} \tag{2.9}$$

or the boundary conditions

$$\begin{aligned} &\sum_{i=0}^{N} \hat{\upsilon}_i \, C_i(0) = a_1, \\ &\sum_{i=0}^{N} \hat{\upsilon}_i \, C_i(1) = a_3, \end{aligned} \tag{2.10}$$

Now, the application of collocation method enables us to write

$$\mathbf{R}(\xi_r) = 0, \quad r = 1, 2, ..., N-1, \tag{2.11}$$

which can be rewritten as

$$\sum_{i=0}^{N} \sum_{p=0}^{i-2} \hat{\upsilon}_i \, \lambda_{p,i} \, C_p(\xi_r) + \lambda \, e^{\sum_{i=0}^{N} \hat{\upsilon}_i \, C_i(\xi_r)} = 0, \quad r = 1, 2, ..., N-1, \tag{2.12}$$

the set $\{\xi_r : r = 1, 2, ..., N-1\}$ represents the first $(N-1)$ distinct roots of $C_i(\xi)$. Consequently, Eq. (2.12), coupled with either the initial conditions (2.9) or the boundary conditions (2.10), forms a system of $(N+1)$ equations. This system can be effectively solved using Newton's iterative method to determine the unknown expansion coefficients $\hat{\upsilon}_i$.
Case 2: At $1 < \alpha < 2$.
Now, in order to derive our proposed method when $1 < \alpha < 2$, it is required to express the fractional derivative of the basis $C_j(\xi)$ in terms of the polynomials $C_k(\xi)$.

Theorem 2.1. *The following formula holds for* $1 < \alpha < 2$

$$D^{\alpha} C_j(\xi) = \sum_{k=0}^{N} \zeta_{k,r,j,\alpha} \, C_k(\xi), \tag{2.13}$$

where

$$\zeta_{k,r,j,\alpha} = \sum_{r=2}^{j} \frac{r! \, B_{r,j} \, \rho_{k,r,\alpha}}{\Gamma(r+1-\alpha)}. \tag{2.14}$$

Proof. From Eq. (1.11), we have

$$C_j(\xi) = \sum_{r=0}^{j} B_{r,j} \, \xi^r. \tag{2.15}$$

Using the following property of fractional Caputo derivative [19]

$$D^{\alpha} \, \xi^r = \begin{cases} 0, & \text{if } r \in \mathbb{N}_0 \quad and \quad r < \lceil \alpha \rceil, \\ \frac{r!}{\Gamma(r-\alpha+1)} \, \xi^{r-\alpha}, & \text{if } r \in \mathbb{N}_0 \quad and \quad r \geq \lceil \alpha \rceil, \end{cases} \tag{2.16}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\lceil \alpha \rceil$ is the ceiling function, yields

$$D^{\alpha} C_j(\xi) = \sum_{r=2}^{j} B_{r,j} \, \frac{r!}{\Gamma(r+1-\alpha)} \, \xi^{r-\alpha}. \tag{2.17}$$

Now, $\xi^{r-\alpha}$ is approximated in terms of $C_k(\xi)$ as

$$\xi^{r-\alpha} \approx \sum_{k=0}^{N} \rho_{k,r,\alpha}\, C_k(\xi). \tag{2.18}$$

To find $\rho_{k,r,\alpha}$. Based on the orthogonality relation of $C_k(\xi)$ in (1.8), we get

$$\begin{aligned}
\rho_{k,r,\alpha} &= \frac{1}{\bar{h}_k} \int_0^1 \xi^{r-\alpha}\, C_k(\xi)\, \omega\, d\xi \\
&= \frac{1}{\bar{h}_k} \sum_{m=0}^{k} B_{m,k} \int_0^1 \xi^{r+m-\alpha}\, (2\,\xi-1)^2 \sqrt{\xi-\xi^2}\, d\xi \\
&= \frac{1}{\bar{h}_k} \sum_{m=0}^{k} B_{m,k} \int_0^1 \left(4\,\xi^{r+m-\alpha+\frac{5}{2}} + \xi^{r+m-\alpha+\frac{1}{2}} - 4\,\xi^{r+m-\alpha+\frac{3}{2}}\right) \sqrt{1-\xi}\, d\xi \\
&= \frac{1}{\bar{h}_k} \sum_{m=0}^{k} B_{m,k} \left[4\,\beta\left((r+m-\alpha+\frac{7}{2},\frac{3}{2}\right) + \beta\left((r+m-\alpha+\frac{3}{2},\frac{3}{2}\right) - 4\,\beta\left((r+m-\alpha+\frac{5}{2},\frac{3}{2}\right)\right],
\end{aligned} \tag{2.19}$$

where $\beta(.,.)$ is the well-known beta function.
Now, inserting Eq. (2.18) into Eq. (2.17), we get the desired result of Theorem 2.1. □

Now, using Theorem 2.1 along with the expansion (2.6), the residual $\mathbf{R}(\xi)$ of Eq. (2.1) is given by

$$\begin{aligned}
\mathbf{R}(\xi) &= D^{\alpha} \upsilon_N + \lambda\, e^{\upsilon_N} \\
&= \sum_{i=0}^{N} \hat{\upsilon}_i\, D^{\alpha}\, C_i(\xi) + \lambda\, e^{\sum_{i=0}^{N} \hat{\upsilon}_i\, C_i(\xi)} \\
&= \sum_{i=0}^{N} \sum_{k=0}^{N} \hat{\upsilon}_i\, \zeta_{k,r,i,\alpha}\, C_k(\xi) + \lambda\, e^{\sum_{i=0}^{N} \hat{\upsilon}_i\, C_i(\xi)},
\end{aligned} \tag{2.20}$$

Now, the application of collocation method enables us to write

$$\mathbf{R}(\xi_r) = 0, \quad r = 1, 2, ..., N-1, \tag{2.21}$$

that can be rewritten as

$$\sum_{i=0}^{N} \sum_{k=0}^{N} \hat{\upsilon}_i\, \zeta_{k,r,i,\alpha}\, C_k(\xi_r) + \lambda\, e^{\sum_{i=0}^{N} \hat{\upsilon}_i\, C_i(\xi_r)} = 0, \quad r = 1, 2, ..., N-1, \tag{2.22}$$

where $\{\xi_r : r = 1, 2, ..., N-1\}$ are the first $(N-1)$ distinct roots of $C_i(\xi)$. And hence, Eq. (2.22) along with the initial conditions (2.9) or the boundary conditions (2.10), consist $(N+1)$ system of equations that can be solved with the aid of the well-known Newton's iterative method to get the unknown expansion coefficients $\hat{\upsilon}_i$.

Remark 2.2. The following algorithm summarizes the main steps for solving the fractional Bratu equation

3. Error bound

Comparing the numerical solution υ_N with an acceptable orthogonal projection $\pi_N \upsilon$ of the analytic solution υ in a suitable Sobolev space with the norm $||.||_S$ and applying the triangle inequality is a standard approach in error analysis.

$$||\upsilon - \upsilon_N||_S \leq ||\upsilon - \pi_N \upsilon||_S + ||\pi_N \upsilon - \upsilon_N||_S. \tag{3.1}$$

Algorithm 1 Coding algorithm for the proposed scheme.

Input α, N, λ, a_1, a_2 and a_3.
Step 1. Assume an approximate solution $\upsilon_N(\xi)$ as in (2.6).
Step 2. Compute $\mathbf{R}(\xi)$ as in (2.8) for the case $\alpha = 2$ or (2.20) for the case $1 < \alpha < 2$.
Step 3. Apply the collocation method to obtain Eq. (2.12) for the case $\alpha = 2$ or Eq. (2.22) for the case $1 < \alpha < 2$ along with the initial conditions (2.9) or the boundary conditions (2.10) .
Step 4. Use *FindRoot* command with initial guess $\{\hat{\upsilon}_i = 10^{-i},\ i : 0, 1, ..., N\}$, to solve the system resulting from Eq. (2.12) or Eq. (2.22) along with the initial conditions (2.9) or the boundary conditions (2.10) to get $\hat{\upsilon}_i$.
Output $\upsilon_N(\xi)$

Hence, the error estimate $||\upsilon - \pi_N \upsilon||_S$ will be the main concern of this section.
Let $\pi_N : L^2_\omega \to P_N$ be the L^2_ω-orthogonal projection operator defined as follows

$$(\upsilon - \pi_N \upsilon, \upsilon_N)_\omega = 0, \qquad \forall \upsilon_N \in P_N, \tag{3.2}$$

where $P_N = \{C_0(t), C_1(t), C_2(t), \dots, C_N(t)\}$. Therefore, we have

$$||\upsilon - \pi_N \upsilon||_\omega = \inf_{\upsilon_N \in P_N} ||\upsilon - u_N||_\omega. \tag{3.3}$$

Assume that the following Chebyshev-weighted Sobolev space

$$\mathbf{H}^m_\omega(I) = \{\upsilon : D^{\alpha+k}\,\upsilon \in L^2_\omega(I),\ 0 \le k \le m\}, \tag{3.4}$$

where $I = (0, 1)$, equipped with the inner product, norm, and semi-norm

$$\begin{aligned} &(\upsilon, u)_{\mathbf{H}^m_\omega} = \sum_{k=0}^{m} (D^{\alpha+k}\,\upsilon, D^{\alpha+k}\,u)_{L^2_\omega}, \\ &||\upsilon||^2_{\mathbf{H}^m_\omega} = (\upsilon, \upsilon)_{\mathbf{H}^m_\omega}, \qquad |\upsilon|_{\mathbf{H}^m_\omega} = ||D^{\alpha+m}\,\upsilon||_{L^2_\omega}, \end{aligned} \tag{3.5}$$

where $0 < \alpha < 1$ and $m \in \mathbb{N}$.

Lemma 3.1. *[29] For $n \ge 1$, $n + r > 1$ and $n + s > 1$, where r, s, are any constants, we have*

$$\frac{\Gamma(n + r)}{\Gamma(n + s)} \le \mathbf{o}^{r,s}_n\, n^{r-s}, \tag{3.6}$$

where

$$\mathbf{o}^{r,s}_n = \exp\left(\frac{r - s}{2\,(n + s - 1)} + \frac{1}{12\,(n + r - 1)} + \frac{(r - s)^2}{n}\right). \tag{3.7}$$

Remark 3.2. $\mathbf{o}^{r,s}_n$ can be expressed as follows for fixed r, s:

$$\mathbf{o}^{r,s}_n = 1 + O(n^{-1}).$$

Theorem 3.3. *Suppose $1 < \alpha < 2$, $\pi_N \upsilon$ is the orthogonal projection approximation of $\upsilon \in \mathbf{H}^m_\omega(I)$. Then for $0 \le k \le m \le N + 1$, where $m \in \mathbb{N}$, we get*

$$||D^{\alpha+k}\,(\upsilon - \pi_N \upsilon)||_{L^2_\omega} \lesssim N^{-\frac{7}{4}\,(m-k)}\,|\upsilon|^2_{\mathbf{H}^m_\omega}, \tag{3.8}$$

where $\mathcal{A} \lesssim \mathcal{B}$ indicates the existence of a constant ν such that $\mathcal{A} \le \nu\,\mathcal{B}$.

Proof. Based on the definitions of $\upsilon = \sum_{i=0}^{\infty} \hat{\upsilon}_i\, C_i(\xi)$ and $\pi_N \upsilon = \sum_{i=0}^{N} \hat{\upsilon}_i\, C_i(\xi)$, we can write

$$\begin{aligned} ||D^{\alpha+k}\,(\upsilon - \pi_N \upsilon)||^2_{L^2_\omega} &= \sum_{n=N+1}^{\infty} |c_n|^2\, ||D^{\alpha+k}\, C_n(\xi)||^2_{L^2_\omega} \\ &= \sum_{n=N+1}^{\infty} |c_n|^2\, \frac{||D^{\alpha+k}\, C_n(\xi)||^2_{L^2_\omega}}{||D^{\alpha+m}\, C_n(\xi)||^2_{L^2_\omega}}\, ||D^{\alpha+m}\, C_n(\xi)||^2_{L^2_\omega} \\ &\leq \frac{||D^{\alpha+k}\, C_{N+1}(\xi)||^2_{L^2_\omega}}{||D^{\alpha+m}\, C_{N+1}(\xi)||^2_{L^2_\omega}}\, |\upsilon|^2_{\mathbf{H}^m_\omega}. \end{aligned} \tag{3.9}$$

To estimate the factor $\frac{||D^{\alpha+k}\, C_{N+1}(\xi)||^2_{L^2_\omega}}{||D^{\alpha+m}\, C_{N+1}(\xi)||^2_{L^2_\omega}}$, we firstly find $||D^{\alpha+k}\, C_{N+1}(\xi)||^2_{L^2_\omega}$.

$$||D^{\alpha+k}\, C_{N+1}(\xi)||^2_{L^2_\omega} = \int_0^1 D^{\alpha+k}\, C_{N+1}(\xi)\, D^{\alpha+k}\, C_{N+1}(\xi)\, \omega\, d\,\xi. \tag{3.10}$$

Eq. (1.11), along with the fractional derivative operator (2.4) allows us to write

$$\begin{aligned} &||D^{\alpha+k}\, C_{N+1}(\xi)||^2_{L^2_\omega} = \sum_{p=k+2}^{N+1} \sum_{i=k+2}^{N+1} B_{p,N+1}\, B_{i,N+1} \\ &\times \left(\frac{p!\, i!}{\Gamma(p-k-\alpha+1)\, \Gamma(i-k-\alpha+1)} \int_0^1 \xi^{p+i-2\,k-2\,\alpha}\, (2\,\xi-1)^2\, \sqrt{\xi-\xi^2}\, d\,\xi \right) \\ &= \sum_{p=k+2}^{N+1} \sum_{i=k+2}^{N+1} B_{p,N+1}\, B_{i,N+1} \\ &\times \left(\frac{\sqrt{\pi}\, i!\, p!\, \left(4\,\alpha^2 - 2\,\alpha(2\,i-4\,k+2\,p+1) + (i-2\,k+p)^2 + i - 2\,k + p + 3\right) \Gamma\left(i+p-2\,k-2\,\alpha+\frac{3}{2}\right)}{2\,\Gamma(i-k-\alpha+1)\Gamma(p-k-\alpha+1)\Gamma(i+p-2\,k+-2\,\alpha+5)} \right). \end{aligned} \tag{3.11}$$

The following inequality can be obtained after applying the Stirling formula [23]

$$\frac{i!\, p!\, \Gamma\left(i+p-2\,k-2\,\alpha+\frac{3}{2}\right)}{2\,\Gamma(i-k-\alpha+1)\Gamma(p-k-\alpha+1)\Gamma(i+p-2\,k+-2\,\alpha+5)} \lesssim p^{k+\alpha}\, i^{k+\alpha}\, (i+p-2\,k-2\,\alpha)^{-\frac{7}{2}}. \tag{3.12}$$

By virtue of Stirling formula and Lemma 3.1, $||D^{\alpha+k}\, C_{N+1}(\xi)||^2_{L^2_\omega}$ can be written as

$$\begin{aligned} ||D^{\alpha+k}\, C_{N+1}(\xi)||^2_{L^2_\omega} &\lesssim {\lambda_1}^*\, {\lambda_2}^*\, (N+1)^{2\,(k+\alpha)}\, (N-k-\alpha+1)^{-\frac{7}{2}} \sum_{p=k+2}^{N+1} \sum_{i=k+2}^{N+1} 1 \\ &= {\lambda_1}^*\, {\lambda_2}^*\, (N+1)^{2\,(k+\alpha)}\, (N-k-\alpha+1)^{-\frac{7}{2}}\, (N-k)^2 \\ &= {\lambda_1}^*\, {\lambda_2}^* \left(\frac{\Gamma(N+2)}{\Gamma(N+1)}\right)^{2\,(k+\alpha)} \left(\frac{\Gamma(N+2-k-\alpha)}{\Gamma(N+1-k-\alpha)}\right)^{-\frac{7}{2}} \left(\frac{\Gamma(N-k+1)}{\Gamma(N-k)}\right)^2 \\ &\lesssim {\lambda_1}^*\, {\lambda_2}^*\, N^{2\,(k+\alpha)}\, (N-k)^{-\frac{3}{2}}, \end{aligned} \tag{3.13}$$

where ${\lambda_1}^* = \max_{0\leq p\leq N+1} \{B_{p,N+1}\}$ and ${\lambda_2}^* = \max_{0\leq i\leq N+1} \{B_{i,N+1}\}$.
Similarly, we have

$$||D^{\alpha+m}\, C_{N+1}(\xi)||^2_{L^2_\omega} \lesssim {\lambda_1}^*\, {\lambda_2}^*\, N^{2\,(m+\alpha)}\, (N-m)^{-\frac{3}{2}} \tag{3.14}$$

Hence,

$$\begin{aligned}\frac{||D^{\alpha+k}\,\phi^{a}_{N+1}(\xi)||^2_{L^2_\omega}}{||D^{\alpha+m}\,\phi^{a}_{N+1}(\xi)||_{L^2_\omega}} &\lesssim N^{2\,(k-m)}\left(\frac{N-k}{N-m}\right)^{-\frac{3}{2}} \\ &\lesssim N^{2\,(k-m)}\left(\frac{\Gamma(N-k+1)}{\Gamma(N-m+1)}\right)^{-\frac{3}{2}} \\ &\lesssim N^{-\frac{7}{2}\,(m-k)}.\end{aligned} \tag{3.15}$$

Inserting Eq. (3.15) into Eq. (3.18), one has

$$||D^{\alpha+k}\,(\upsilon-\pi_N\upsilon)||^2_{L^2_\omega} \lesssim N^{-\frac{7}{2}\,(m-k)}\,|\upsilon|^2_{\mathbf{H}^m_\omega}. \tag{3.16}$$

Therefore, we get the desired result. □

Theorem 3.4. *Assume that $\pi_N\upsilon$ is the orthogonal projection approximation of $\upsilon\in\mathbf{H}^m_\omega(I)$, when $\alpha=0$. Then for $0\le k\le m\le N+1$, where $m\in\mathbb{N}$, we get*

$$||D^{k}\,(\upsilon-\pi_N\upsilon)||_{L^2_\omega} \lesssim N^{-\frac{7}{4}\,(m-k)}\,||D^m\,\upsilon||_{L^2_\omega}, \tag{3.17}$$

Proof. The definitions of υ and $\pi_N\upsilon$ enable us to write

$$\begin{aligned}||D^{k}\,(\upsilon-\pi_N\upsilon)||^2_{L^2_\omega} &= \sum_{n=N+1}^{\infty}|c_n|^2\,||D^k\,C_n(\xi)||^2_{L^2_\omega} \\ &= \sum_{n=N+1}^{\infty}|c_n|^2\,\frac{||D^k\,C_n(\xi)||^2_{L^2_\omega}}{||D^m\,C_n(\xi)||^2_{L^2_\omega}}\,||D^m\,C_n(\xi)||^2_{L^2_\omega} \\ &\le \frac{||D^k\,C_{N+1}(\xi)||^2_{L^2_\omega}}{||D^m\,C_{N+1}(\xi)||^2_{L^2_\omega}}\,||D^m\,\upsilon||_{L^2_\omega}.\end{aligned} \tag{3.18}$$

Now, imitating similar steps as in Theorem 3.3, we get the desired result. □

Theorem 3.5. *Let $\mathbf{R}(\xi)$ be the residual of Eq. (2.1), that defined as*

$$\mathbf{R}(\xi) = D^{\alpha}\pi_N\upsilon + \lambda\,e^{\pi_N\upsilon}, \tag{3.19}$$

then $||\mathbf{R}(\xi)||_{L^2_\omega}\to 0$ as $N\to\infty$.

Proof. Subtracting Eq. (3.19) from Eq. (2.1), we get

$$\begin{aligned}\mathbf{R}(\xi) =& D^{\alpha}(\upsilon-\pi_N\upsilon)+\lambda\,[e^{\upsilon}-e^{\pi_N\upsilon}] \\ &= D^{\alpha}(\upsilon-\pi_N\upsilon)+\lambda\left[(\upsilon-\pi_N\upsilon)+\frac{(\upsilon^2-\pi_N\upsilon^2)}{2}+\frac{(\upsilon^3-\pi_N\upsilon^3)}{6}+.......\right]\end{aligned} \tag{3.20}$$

Now, taking $||.||_{L^2_\omega}$ for the last equation to get

$$\begin{aligned}||\mathbf{R}(\xi)||_{L^2_\omega} &= \left|\left|D^{\alpha}(\upsilon-\pi_N\upsilon)+\lambda\left[(\upsilon-\pi_N\upsilon)+\frac{(\upsilon^2-\pi_N\upsilon^2)}{2}+\frac{(\upsilon^3-\pi_N\xi^3)}{6}+.......\right]\right|\right|_{L^2_\omega} \\ &\le ||D^{\alpha}(\upsilon-\pi_N\upsilon)||_{L^2_\omega}+\lambda\,||(\upsilon-\pi_N\upsilon)||_{L^2_\omega}+\frac{\lambda}{2}\,||(\upsilon-\pi_N\upsilon)||_{L^2_\omega}\;||(\upsilon+\pi_N\upsilon)||_{L^2_\omega}+.......\end{aligned} \tag{3.21}$$

Now, the application of Theorems 3.3 and 3.4 enables us to write the last equation as

$$\begin{aligned}||\mathbf{R}(\xi)||_{L^2_\omega} &\lesssim N^{-\frac{7}{4}\,(m-k)}\,|\upsilon|^2_{\mathbf{H}^m_\omega}+\lambda\,N^{-\frac{7}{4}\,(m-k)}\,||D^m\,\upsilon||_{L^2_\omega} \\ &+\frac{\lambda}{2}\,N^{-\frac{7}{4}\,(m-k)}\,||D^m\,\upsilon||_{L^2_\omega}\;\;||(\upsilon+\pi_N\upsilon)||_{L^2_\omega}+.......\end{aligned} \tag{3.22}$$

Table 1. Maximum absolute errors of Example 4.1.

N	5	10	15	20	25	30
Error	2.13113×10^{-3}	1.06576×10^{-5}	2.38942×10^{-9}	1.3225×10^{-11}	7.03881×10^{-14}	3.53051×10^{-14}

Table 2. Comparison of absolute errors for Example 4.1.

t	Our method	Method in [17]
0	3.80699×10^{-17}	0
0.1	7.28584×10^{-17}	1.4×10^{-10}
0.2	3.46945×10^{-17}	3.499×10^{-8}
0.3	1.11022×10^{-16}	9.1185×10^{-7}
0.4	8.32667×10^{-17}	9.3271×10^{-6}
0.5	1.11022×10^{-16}	5.73698×10^{-5}
0.6	5.55112×10^{-17}	2.567388×10^{-4}
0.7	1.11022×10^{-16}	9.260041×10^{-4}
0.8	1.33227×10^{-15}	2.8639825×10^{-3}
0.9	8.99281×10^{-15}	7.9152872×10^{-3}
1	3.53051×10^{-14}	2.014183×10^{-3}

Or

$$\begin{aligned} ||\mathbf{R}(\xi)||_{L^2_\omega} &\lesssim N^{-\frac{7}{4}\,(m-k)}\,||D^m\,\upsilon||_{L^2_\omega} + \lambda\,N^{-\frac{7}{4}\,(m-k)}\,||D^m\,\upsilon||_{L^2_\omega} \\ &+ \frac{\lambda}{2}\,N^{-\frac{7}{4}\,(m-k)}\,||D^m\,\upsilon||_{L^2_\omega}\;\;||(\upsilon+\pi_N\upsilon)||_{L^2_\omega} + \ldots\ldots \end{aligned} \tag{3.23}$$

Therefore, it is clear that $||\mathbf{R}(\xi)||_{L^2_\omega} \to 0$ as $N \to \infty$. □

4. Illustrative examples

Example 4.1. [17] Consider the following equation

$$D^\alpha\upsilon - 2\,e^\upsilon = 0;\quad 0 \le t \le 1, \tag{4.1}$$

subject to the initial conditions:

$$\upsilon(0) = \upsilon'(0) = 0, \tag{4.2}$$

where the exact solution is $\upsilon(t) = -2\,\ln(\cos(t))$ at $\alpha = 2$.
Table 1 shows the maximum absolute errors at different values of N. Table 2 presents a comparison of absolute errors between our method and method in [17]. Figure 1 illustrates the absolute errors at different values of N. Also, Figure 2 shows that the approximate solutions have smaller variations for values of α near the value $\alpha = 2$ when $N = 10$. These results prove that the approximate solution is quite near to the analytic one.

Example 4.2. [17] Consider the following equation

$$D^\alpha\upsilon - \pi^2\,e^\upsilon = 0;\quad 0 \le t \le 1, \tag{4.3}$$

subject to the initial conditions:

$$\upsilon(0) = \upsilon'(0) = 0, \tag{4.4}$$

where the exact solution is $\upsilon(t) = -\log(1 - \cos(\pi(t + 0.5)))$ at $\alpha = 2$.
Table 3 shows the maximum absolute errors at different values of N. Figure 3 illustrates the absolute errors at different values of N. Also, Figure 4 shows that the approximate solutions have smaller variations for values of α near the value $\alpha = 2$ when $N = 10$. These results prove that the approximate solution is quite near to the analytic one.

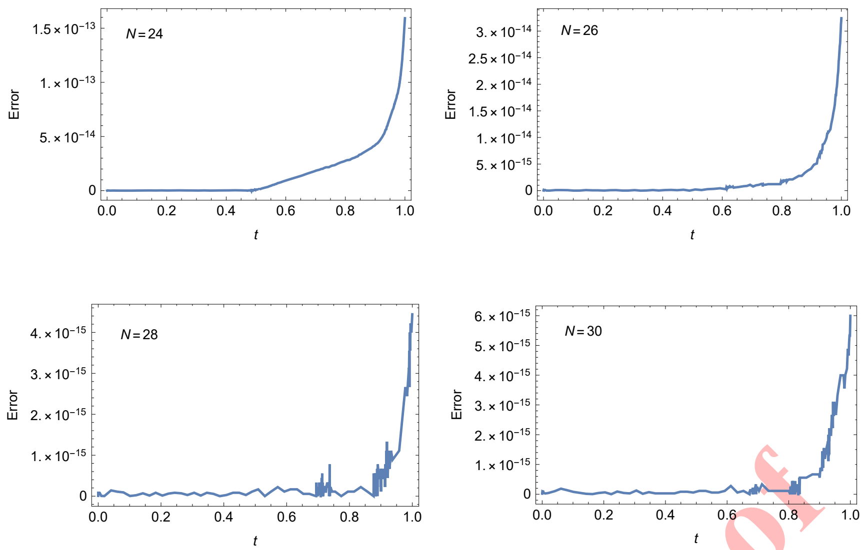

FIGURE 1. The absolute errors of Example 4.1 at different values of N.

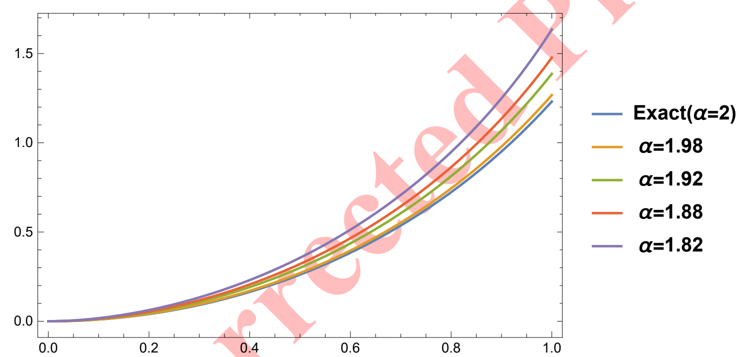

FIGURE 2. Different solutions of Example 4.1.

TABLE 3. Maximum absolute errors of Example 4.2.

N	5	10	15	20	25	30
Error	1.68343×10^{-2}	4.16448×10^{-5}	1.77073×10^{-8}	1.86694×10^{-10}	3.39311×10^{-13}	5.80534×10^{-13}

Example 4.3. [20] Consider the following equation

$$D^{\alpha}\upsilon + e^{\upsilon} = 0; \quad 0 \le t \le 1, \tag{4.5}$$

subject to the boundary conditions:

$$\upsilon(0) = \upsilon(1) = 0, \tag{4.6}$$

where the exact solution is $\upsilon(t) = -2\log\left(\frac{\cosh\left(\frac{\theta}{2}\left(t-\frac{1}{2}\right)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right)$ at $\theta = 2.3576$ when $\alpha = 2$.

Table 4 shows the maximum absolute errors at different values of N. Figure 5 illustrates the absolute errors at different values of N. Also, Figure 6 shows that the approximate solutions have smaller variations for values of α near the value $\alpha = 2$ when $N = 10$. These results prove that the approximate solution is quite near to the analytic one.

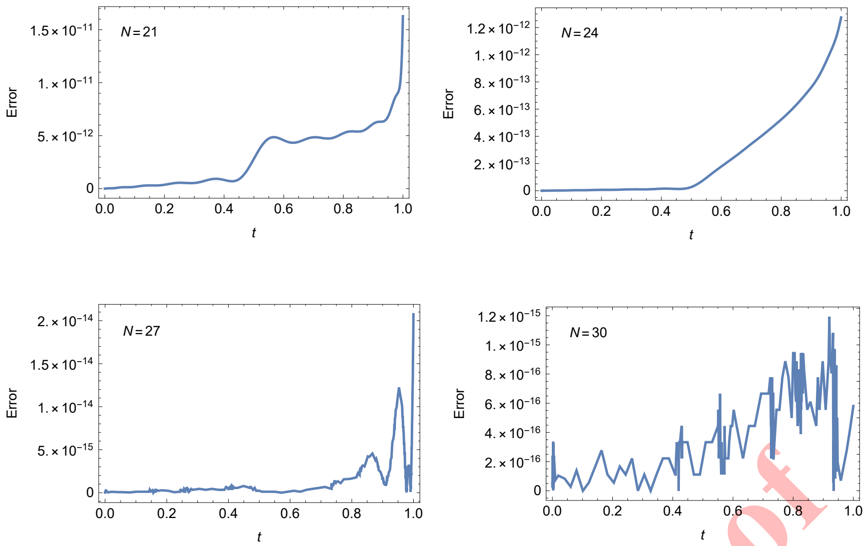

Figure 3. The absolute errors of Example 4.2 at different values of N.

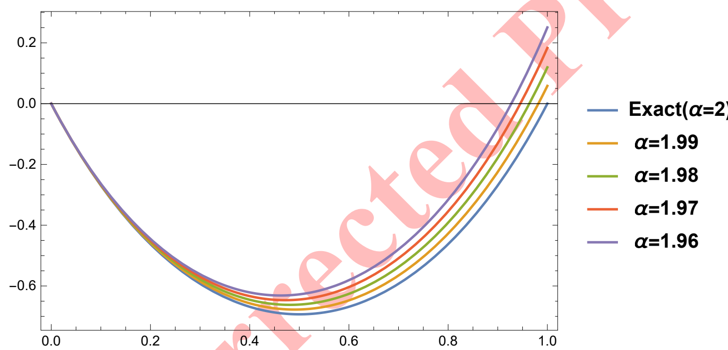

Figure 4. Different solutions of Example 4.2.

Table 4. Maximum absolute errors of Example 4.3.

N	4	9	14	19	24
Error	2.58482×10^{-3}	5.41337×10^{-7}	3.29419×10^{-10}	4.18554×10^{-14}	4.66184×10^{-16}

Table 5. Comparison of absolute errors for Example 4.3.

t	Our method	Method in [20]
0.1	1.24911×10^{-16}	6.2868×10^{-10}
0.3	5.55112×10^{-17}	1.2860×10^{-8}
0.5	5.55112×10^{-17}	3.4980×10^{-10}
0.7	1.11022×10^{-16}	1.7153×10^{-9}
0.9	1.80411×10^{-16}	6.9360×10^{-10}

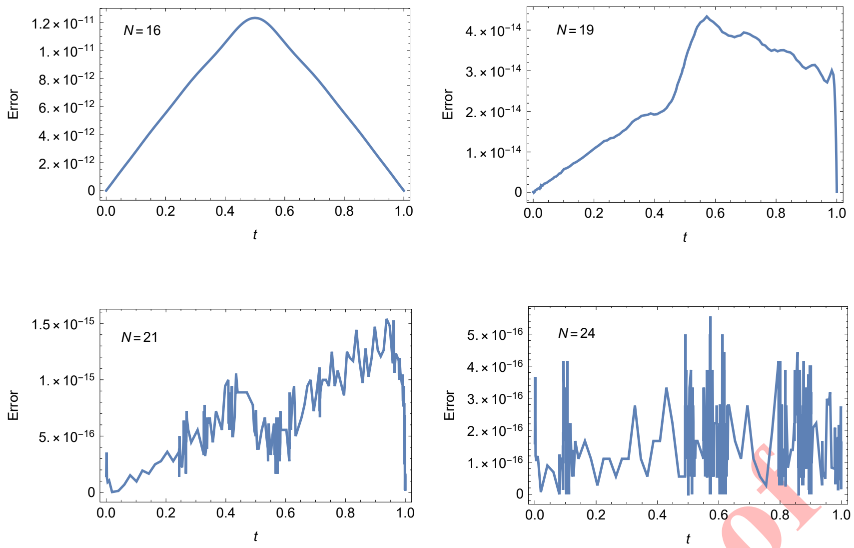

FIGURE 5. The absolute errors of Example 4.3 at different values of N.

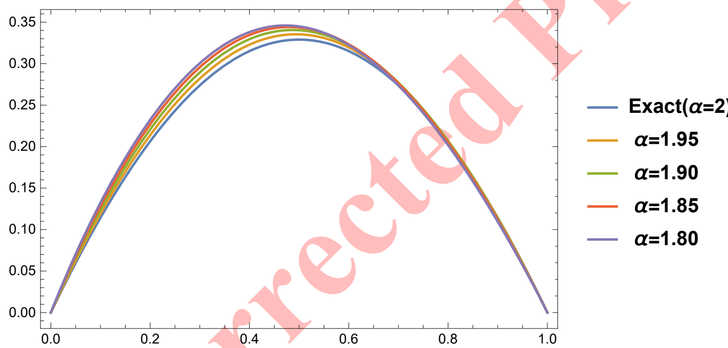

FIGURE 6. Different solutions of Example 4.3.

5. Conclusion

In this study, we have discussed one of the well-known equations in mathematical physics, namely, the fractional Bratu differential equation. This problem was solved numerically by employing the spectral collocation approach and directly estimating the solution using shifted Chebyshev polynomials of the sixth kind as basis functions. The numerical results presented in the preceding section demonstrate the high accuracy of this method. Our aim is to generalize the presented algorithm for more advanced models in applied mathematics and physics. As an expected future work, we aim to employ the developed theoretical results in this paper along with suitable spectral methods to treat some other problems, for instance, [9, 12].

References

[1] W. M. Abd-Elhameed, Y. H. Youssri, and A. G. Atta, *Tau algorithm for fractional delay differential equations utilizing seventh-kind Chebyshev polynomials*, J. Math. Model., (2024), 277–299.

[2] W. M. Abd-Elhameed and Y. H. Youssri, *Sixth-kind Chebyshev spectral approach for solving fractional differential equations*, Int. J. Nonlinear Sci. Numer. Simul., *20*(2), 191–203, 2019.

[3] W. M. Abd-Elhameed, Y. H. Youssri, and A. G. Atta , *Adopted spectral tau approach for the time-fractional diffusion equation via seventh-kind Chebyshev polynomials*, Boundary Value Probl., *2024*(1) (2024), 1-24, .

[4] A. M. Al-Bugami, M. A. Abdou, and A. Mahdy, *Sixth-kind Chebyshev and Bernoulli polynomial numerical methods for solving nonlinear mixed partial integrodifferential equations with continuous kernels*, J. Funct. Spaces, (2023).

[5] A. G. Atta, W. M. Abd-Elhameed, G. M. Moatimid, and Y. H. Youssri, *A fast Galerkin approach for solving the fractional Rayleigh–Stokes problem via sixth-kind Chebyshev polynomials*, Math., *10*(11) (2022), 1843.

[6] A. G. Atta, W. M. Abd-Elhameed, G. M. Moatimid, and Y. H. Youssri, *Advanced shifted sixth-kind Chebyshev tau approach for solving linear one-dimensional hyperbolic telegraph type problem*, Math. Sci., *17*(4) (2023), 415–429.

[7] A. G. Atta, *Spectral collocation approach with shifted Chebyshev third-kind series approximation for nonlinear generalized fractional Riccati equation*, Int. J. Appl. Comput. Math., *10*(2) (2024), 59.

[8] S. Aydınlık and A. Kırış, *An efficient method for solving fractional integral and differential equations of Bratu type*, TWMS J. Appl. Eng. Math., 2024.

[9] M. Bisheh-Niasar, A. Saadatmandi, and M. Akrami-Arani, *A new family of high-order difference schemes for the solution of second order boundary value problems*, Iran. J. Math. Chem., *9*(3) (2018), 187–199.

[10] V. P. Dubey, R. Kumar, and D. Kumar, *Analytical study of fractional Bratu-type equation arising in electro-spun organic nanofibers elaboration*, Physica A: Stat. Mech. Appl., *521* (2019), 762–772.

[11] A. Ebrahimzadeh, R. Khanduzi, and Z. Ebrahimzadeh, *Coupling Chebyshev Collocation with TLBO to Optimal Control Problem of Reservoir Sedimentation: A Case Study on Golestan Dam, Gonbad Kavous City, Iran*, Math. Interdisc. Res., *8*(2) (2023), 141–159.

[12] H. Jafari and H. Tajadodi, *Electro-spunorganic nanofibers elaboration process investigations using BPs operational matrices*, Math. Interdisc. Res., *7*(1) (2016), 19–27.

[13] A. Khalouta and A. Kadem, *Solution of the fractional Bratu-type equation via fractional residual power series method*, Tatra Mt. Math. Publ., *76*(1) (2020), 127–142.

[14] A. Khalouta, *A novel representation of numerical solution for fractional Bratu-type equation*, Adv. Stud. Euro-Tbilisi Math. J., *15*(1) (2022), 93–109.

[15] M. Masjed-Jamei, *Some new classes of orthogonal polynomials and special functions*, PhD thesis, University of Kassel, (2006).

[16] M. Moustafa, Y. H. Youssri, and A. G. Atta, *Explicit Chebyshev Galerkin scheme for the time-fractional diffusion equation*, Int. J. Mod. Phys. C (IJMPC), *35*(01) (2024), 1–15.

[17] O. Odetunde, A. Ajanı, A. Taıwo, and S. Onıtılo, *Numerical solution of Bratu-type initial value problems by Aboodh Adomian decomposition method*, Cankaya Univ. J. Sci. Eng., *20*(2) (2023), 64–75.

[18] P. Pirmohabbati, A. H. Refahi Sheikhani, and A. Abdolahzadeh Ziabari, *Numerical solution of nonlinear fractional Bratu equation with hybrid method*, Int. J. Appl. Comput. Math., *6* (2020), 1–22.

[19] I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional37 differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

[20] M. A. Z. Raja and S. -u.- I. Ahmad, *Numerical treatment for solving one-dimensional Bratu problem using neural networks*, Neural Comput. Appl., *24*(3) (2014), 549–561.

[21] K. Sadri, D. Amilo, K. Hosseini, E. Hinçal, and A. R. Seadawy, *A Tau-Gegenbauer spectral approach for systems of fractional integro-differential equations with the error analysis*, AIMS Math., *9*(2) (2024), 3850–3880.

[22] S. M. Sayed, A. S. Mohamed, E. M. Abo-Eldahab, Y. H. Youssri, *A compact combination of second-kind Chebyshev polynomials for Robin boundary value problems and Bratu-type equations*, J.Umm Al-Qura Univ. Appll. Sci., (2024).

[23] J. Shen, T. Tang, and L. Wang, *Spectral methods: algorithms, analysis and applications*, 41. Springer Science & Business Media, 2011.

[24] H. Singh, A. K. Singh, R. K. Pandey, D. Kumar, and J. Singh, *An efficient computational approach for fractional Bratu's equation arising in electrospinning process*, Math. Methods Appl. Sci., *44*(13) (2021), 10225–10238.
[25] G. O. Theophilus, A. G. Olusegun, and M. Umaru, *K-step block hybrid Nyström-type method for the solution of Bratu problem with impedance boundary condition*, Cent. Asian J. Math. Theory Comput. Sci., *5*(1), 34–46, 2024.
[26] N. S. Yakusak and E. O. Taiwo, *Approximate solution of Bratu differential equation using Falker-type method*, Scholar J., *2*(1) (2024).
[27] Y. H. Youssri and A. G. Atta, *Fejér-quadrature collocation algorithm for solving fractional integro-differential equations via Fibonacci polynomials*, Contemp. Math., (2024), 296–308.
[28] F. Yin and Y. Fu, *Explicit high accuracy energy-preserving Lie-group sine pseudo-spectral methods for the coupled nonlinear Schrödinger equation*, Appl. Numer. Math., *195* (2024), 1–16.
[29] X. Zhao, L.-L. Wang, and Z. Xie, *Sharp error bounds for Jacobi expansions and Gegenbauer–Gauss quadrature of analytic functions*, SIAM J. Numer. Anal., *51*(3) (2013), 1443–1469.
[30] W. Zheng, Y. Chen, and J. Zhou, *A Legendre spectral method for multidimensional partial Volterra integro-differential equations*, J. Comput. Appl. Math., *436* (2024), 115302.