Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-16
DOI:10.22034/cmde.2024.60344.2577

A higher order orthogonal collocation technique for discontinuous two dimensional problems with Neumann boundary conditions

Santosh Kumar Bhal[1], Ashish Kumar Nandi[2,*], and Abedallah Rababah[3]

[1]Department of Mathematics, School of Advanced Sciences and Languages, Vellore Institute of Technology, Bhopal, India.

[2]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai, Tamil Nadu-600127, India.

[3]Department of Mathematical Sciences, United Arab Emirates University, United Arab Emirates.

Abstract

In this paper, the orthogonal spline collocation method (OSCM) is employed to address the solution of the Helmholtz equation in two-dimensional problems. It is characterized by discontinuous coefficients with certain wave numbers. The solution is approximated by employing distinct basis functions namely, monomial along the x-direction and Hermite along the y-direction. Additionally, to solve the two-dimensional problems efficiently in the sense of computational cost with less operation counts, matrix decomposition algorithm (MDA) is used to convert it into a set of one-dimensional problems. As a consequence, the resulting reduced matrix becomes non-singular in discrete cases. To assess the performance of the proposed numerical scheme, a grid refinement analysis is conducted to incorporate various wave coefficients of the Helmholtz equation. The illustrations and examples demonstrate a higher order of convergence compared to existing methods.

Keywords. Helmholtz equation, Orthogonal spline collocation methods, Matrix decomposition algorithm.

2010 Mathematics Subject Classification. 65L10, 65L60, 65L70.

1. Introduction

Helmholtz interface problems are encountered in numerous scientific and physical phenomena such as the study of sound waves traveling from one medium to another [26, 29], radiation from the source of electromagnetic field [30], fluid-solid interaction [12] and many more. There are various methods available in the literature to deal with Helmholtz equation, namely finite difference method [21] and finite element method [24]. In computing the solution, these methods require sufficiently small step size which results in a large number of algebraic equations and expensive computational costs. These equations form sparse matrices that require n^3 operations to reach the solution. To refrain from indulging in numerous computations involving large matrices, orthogonal spline collocation (OSC) method [9] can be used as a powerful and efficient method that reduces the number of operations to n. The well-established scheme exhibits several advantages when compared to existing iterative methods such as finite difference and boundary element methods. Some of these advantages include the following.

- The scheme easily manages the discontinuities that emerge in interface problems by using monomial basis functions [32].
- It can be applied to various types of boundary conditions on nonuniform meshes without sacrificing precision.
- The incorporation of high-order polynomial approximation results in a superior order of convergence, contributing to increased accuracy in the solution.

In the past decade, scientists and researchers have preferred OSC method due to its significant applicability with respect to its ease of implementation. De Boor and Swartz [13] first proposed the OSC method to linear ODEs with boundary value problems, and this inspired Fairweather and Meade [17] to further develop the solver to generate

Received: 30 January 2024 ; Accepted: 31 August 2024.
* Corresponding author. Email: ashish.nandi123@gmail.com.

a sequence of approximate solutions that made a large impact on solving PDEs with different types of boundary conditions. In connection to this, the following authors in the articles [6–8, 15, 27, 28] implemented OSC method for elliptic and parabolic PDEs with mixed type boundary conditions. More specifically, as this method generates high optimal accuracy with super-convergence results, it is worth mentioning that the authors of these articles [2], [4], and [3] have significantly worked on OSC method to solve Helmholtz equation over compact finite difference schemes [18, 20, 21]. One can refer to the article [5], where the authors have utilized the OSC method to derive approximate solutions for nonlinear boundary valued problems. The practical applicability of OSC method is that the discontinuity that occurs at any interior point of the domain can be tackled by considering monomial cubic basis for each divided sub-interval of the domain. In this regard, some of the applications of OSC method can also be seen in the recent articles [1, 19, 33]. Due to its stability for high order computation, the orthogonal spline collocation method effectively calculates the estimated solution of the Helmholtz equation. Motivated by the developments in OSCM, we have considered the two-dimensional Helmholtz equation,

$$u_{xx} + u_{yy} + \gamma_0^2 p(x) u = f(x,y), \quad (x,y) \in \Omega, \tag{1.1}$$

$$u(0,y) = 0, \quad u(1,y) = 0, \text{ on } \partial\Omega, \tag{1.2}$$

$$u_y(x,0) = 0, \quad u_y(x,1) = 0, \text{ on } \partial\Omega, \tag{1.3}$$

where $\Omega = (a,b) \times (c,d)$ is a rectangular domain, γ_0^2 is a wave number or material coefficient, and $p(x)$ is piecewise continuous function having finite jumps across a straight line $\Gamma_i = \{(x,y) : x = x_i\}$ in Ω.

The present focus of this article is to apply orthogonal spline collocation method to the Helmholtz equation with Neumann boundary condition that extends the work of Feng [18]. Neumann boundary conditions, which signify the derivative of the solution at the boundary points, have several challenges while applying conventional numerical schemes like finite difference and finite element approaches to deal Helmholtz equation. The OSC method effectively handles these boundary conditions by incorporating them directly into the collocation process, leading to better accuracy and stability. Consideration of higher degree polynomials generates high order convergence of solutions. By choosing suitable basis functions, the OSC method can significantly reduce the computational cost associated with solving sparse linear systems of equations, which arises from the Helmholtz equation with Neumann boundary conditions. In particular, the MDA algorithm [10] and almost block diagonal (ABD) solver [14] are very useful processes in obtaining the solutions set.

For the notational convenience, we have assumed that $w^2 = \gamma_0^2 p(x)$. Further, the solution $u(x,y)$ is assumed to satisfy the following natural jump conditions across the interface Γ_i:

$$[u] = \alpha, \ [u_x] = \beta, \ [u_y] = 0. \tag{1.4}$$

This might occur for the second or higher order partial derivatives of $u(x,y)$ and the source function $f(x,y)$ with respect to x. To this end, the article is organized as follows: section 2 describes basic notations and formulation of OSCM to solve the Equations (1.1)-(1.3). In section 3, the matrix decomposition algorithm is applied to the linear system of collection equations which gives the almost block diagonal matrix. Finally, the performance of this method to Helmholtz equation for different values of wave number with respect to the Neumann boundary conditions is carried out in the numerical section 4.

2. Orthogonal Spline Collocation Method

In this section, OSCM with monomial cubic basis is applied in x-direction and Hermite cubic basis functions in y-direction to obtain the collocation solution of Eqs. (1.1)-(1.3). In this procedure, the following assumptions have been considered.

Suppose $\pi_x = \{x_i\}_{i=0}^N$ and $\pi_y = \{y_j\}_{j=0}^N$ denotes the partitions of $[a,b]$ and $[c,d]$ such that $x_i = ih_x, \ \ i = 0, \dots, N$, and $y_j = jh_y, \ \ j = 0, \dots, N$, where h_x and h_y are the step sizes. Then we set $I_i = [x_{i-1}, x_i]$ and $I_j = [y_{j-1}, y_j]$, where $i, j = 1, \dots, N$. Further, we define

$$\mathcal{M}_{-1}^3(\pi_x) = \{\Phi : \Phi \in PC[0,1], \ \Phi|_{I_i} \in P_3, \ i = 1, 2, \dots, N\},$$

where $PC[0,1]$ represents the space of continuous piecewise functions on $[0,1]$ and P_3 is a polynomial of degree ≤ 3. Similarly,

$$\begin{aligned}
\mathcal{M}_1^3(\pi_y) &= \left\{\chi : \chi \in C^1[0,1],\ \chi|_{I_j} \in P_3,\ j = 1,2,\ldots,N\right\}, \\
\mathcal{M}_{-1}^{3,0}(\pi_x) &= \mathcal{M}_{-1}^3(\pi_x) \cap \{\chi : \chi(0) = \chi(1) = 0\}, \\
\mathcal{M}_1^{3,0}(\pi_y) &= \mathcal{M}_1^3(\pi_y) \cap \{\chi : \chi(0) = \chi(1) = 0\}.
\end{aligned}$$

Next, the collocation points considered as $\{\xi_i\}_{i=1}^{2N}$ on $[0,1]$ are actually two-point Gauss-Legendre quadrature points are defined by

$$\xi_{2i-1} = x_{i-1} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right) h_i, \quad \text{and} \quad \xi_{2i} = x_{i-1} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) h_i, \quad i = 1,2,\cdots,N.$$

If $x = \xi_{2i-1}$, then $x - x_{i-1} = \frac{1}{2}(1+\rho_1)\,h$, where $\rho_1 = -\frac{1}{\sqrt{3}}$. Likewise, if $x = \xi_{2i}$, then we have $x - x_{i-1} = \frac{1}{2}(1+\rho_2)\,h$, where $\rho_2 = -\rho_1$.

To arrive at an approximate solution to the problems in Eqs. (1.1) and (1.3) through OSCM estimation, we define $U \in \mathcal{M}_{-1}^{3,0} \otimes \mathcal{M}_1^{3,0}$ such that

$$U_{xx}(\xi_i,\xi_j) + U_{yy}(\xi_i,\xi_j) + w^2 U(\xi_i,\xi_j) = f(\xi_i,\xi_j), \ \ i,j = 1,2,3,\cdots,2N. \tag{2.1}$$

Let the monomial basis functions for $\mathcal{M}_{-1}^{3,0}$ be as follows:

$$\{\Phi_m\}_{m=1}^{4N+2} = \{\Phi_1, \Phi_2, \ldots, \Phi_m, \Phi_{m+1}, \ldots, \Phi_{4N}, \Phi_{4N+1}, \Phi_{4N+2}\},$$

$$\{\Phi_{4(k-1)+j}\}_{k=1}^{N} = (x - x_{k-1})^{j-1}, \ \ j = 1,2,3,4 \ \text{ and } \ \Phi_{4N+1} = 1, \ \ \Phi_{4N+2} = (x - x_N),$$

and the Hermite basis function for $\mathcal{M}_1^{3,0}$ be

$$\{\Psi_n\}_{n=1}^{2N} = \{v_0, v_1, \cdots, v_{N-1}, v_N, s_1, \cdots, s_{N-1}\},$$

where $v_i,\ i = 0,1,2,\cdots,N$ are value functions and $s_j,\ j = 1,2,\cdots,N-1$ are slope functions for Hermite cubic basis functions. In this context, see [31] for the expressions of v_i and s_j.

With the help of the above defined basis functions, the collocation approximation is expressed in the form

$$U(x,y) = \sum_{m=1}^{4N+2} \sum_{n=1}^{2N} U_{m,n}\, \Phi_m(x)\, \Psi_n(y). \tag{2.2}$$

Substituting Eq. (2.2) into Eq. (2.1) leads to the following structure of the collocation equations:

$$(A_1 \otimes B_2 + B_1 \otimes A_2)\ u = F, \tag{2.3}$$

where

$$\begin{aligned}
A_1 &= \left[a_{i,j}^{(1)}\right]_{i,j=1}^{4N+2}, \quad a_{i,j}^{(1)} = \Phi_j''(\xi_i) + w^2\, \Phi_j(\xi_i), \\
A_2 &= \left[a_{i,j}^{(2)}\right]_{i,j=1}^{2N}, \quad a_{i,j}^{(2)} = \Psi_j''(\xi_i), \\
B_1 &= \left[b_{i,j}^{(1)}\right]_{i,j=1}^{4N+2}, \quad b_{i,j}^{(1)} = \Phi_j(\xi_i), \\
B_2 &= \left[b_{i,j}^{(2)}\right]_{i,j=1}^{2N}, \quad b_{i,j}^{(2)} = \Psi_j(\xi_i).
\end{aligned}$$

To solve (2.3), A_2 and B_2 are formulated into the generalized eigenvalues and corresponding eigenfunctions as follows.

$$A_2\, \varphi = \lambda B_2\, \varphi. \tag{2.4}$$

As the matrices A_2 and B_2 contains the second derivative of the Hermite basis Ψ, therefore the following classical eigenvalue problem is considered,

$$u''(y) = \lambda\, u(y), \quad 0 < y < 1, \tag{2.5}$$

$$u'(0) = u'(1) = 0. \tag{2.6}$$

We note that the generalized eigenvalues and corresponding eigenfunctions of (2.4) give the Hermite cubic collocation approximations to the eigenvalues and corresponding eigenfunctions of (2.5)-(2.6). It is worth mentioning that these values can be obtained by using the Eqs. (3.2.1 − 3.2.4), (3.2.13 − 3.2.16), and Appendix of [11].

Therefore, the $2N$ generalized eigenvalues of the equations (2.5)-(2.6) are given by

$$\lambda_0^- = 0, \qquad \lambda_N^- = -\frac{9}{h^2},$$

$$\lambda_j^\pm = -\frac{12}{h^2}\left(\frac{8+\eta_j \pm \mu_j}{7-\eta_j}\right), \quad j = 1, 2, \cdots, N-1,$$

where

$$\eta_j = \cos\left(\frac{j\pi}{N}\right), \qquad \mu_j = \sqrt{43 + 40\,\eta_j - 2\,\eta_j^2}.$$

As the system (2.3) is a two-dimensional problem and to reduce it to the set of independent one-dimensional problems, two real non-singular matrices $\mathcal{Z}$ and Λ are required.

Hence from Appendix [11], we now define

$$\mathcal{Z} = 3\sqrt{3}\left[\begin{array}{c|c} C\,\Lambda_\alpha^- & \tilde{C}\,\Lambda_\alpha^+ \\ \hline \mathbf{0} \mid -S\,\Lambda_\beta^- & \mathbf{0} \mid -S\,\Lambda_\beta^+ \end{array}\right], \tag{2.7}$$

and

$$\Lambda = \text{ diag }\left(\lambda_0^-, \lambda_1^-, \cdots, \lambda_{N-1}^-, \lambda_N^-, \lambda_1^+, \cdots, \lambda_{N-1}^+\right), \tag{2.8}$$

where

$$\mathcal{S} = \sqrt{\frac{2}{N}}\sin\left(\frac{mn\pi}{N}\right)_{m,n=1}^{N-1}, \quad \tilde{C} = \sqrt{\frac{2}{N}}\cos\left(\frac{mn\pi}{N}\right)_{m=0,n=1}^{N,N-1}, \quad C = \sqrt{\frac{2}{N}}\cos\left(\frac{mn\pi}{N}\right)_{m=0,n=0}^{N},$$

$$\Lambda_\alpha^\pm = \text{diag}(\alpha_1^\pm, \alpha_2^\pm, \ldots, \alpha_{N-1}^\pm), \quad \Lambda_\beta^- = \text{diag}\Big(\beta_1^-, \beta_2^-, \ldots, \beta_{N-1}^-\Big), \quad \Lambda_\beta^+ = \text{diag}\Big(1, \beta_1^+, \beta_2^+, \ldots, \beta_{N-1}^+, \frac{1}{\sqrt{3}}\Big),$$

$$\alpha_j^\pm = (5 + 4\eta_j \mp \mu_j)\nu_j^\pm, \quad \beta_j^\pm = 18\sin\left(\frac{j\pi}{N}\right)\nu_j^\pm,$$

$$\nu_j^\pm = \Big[27(1+\eta_j)(8+\eta_j \mp \mu_j)^2 + (1-\eta_j)(11 + 7\eta_j \mp 4\mu_j)^2\Big]^{-\frac{1}{2}}.$$

Next, we have discussed the method of finding the solution of (2.3) using the matrices $\mathcal{Z}$ and Λ.

3. Matrix Decomposition Algorithm

In this section, a matrix decomposition algorithm (MDA) is implemented to convert the two dimensional problem to the set of one dimensional problems. The detailed application of this algorithm for two dimensional elliptic boundary value problems can be found in the following articles [6, 7, 9, 10, 16, 25]. Henceforth, the matrices $\mathcal{Z}$ and Λ of the previous section are used to decompose the matrix system (2.3) as follows.

Pre-multiply the matrix $\left(I \otimes \mathcal{Z}^T\, B_2^T\right)$ to the Equation (2.3), we get

$$\left(I \otimes \mathcal{Z}^T\, B_2^T\right)\,(A_1 \otimes B_2 + B_1 \otimes A_2)\; u = \left(I \otimes \mathcal{Z}^T\, B_2^T\right)\; F. \tag{3.1}$$

Further, the expression (3.1) is rewritten as

$$\left(I \otimes \mathcal{Z}^T\, B_2^T\right)\,(A_1 \otimes B_2 + B_1 \otimes A_2)\,(I \otimes \mathcal{Z})\,\left(I \otimes \mathcal{Z}^{-1}\right)\, u = \left(I \otimes \mathcal{Z}^T\, B_2^T\right)\, F. \tag{3.2}$$

Now the left hand side of the expression (3.2) can be simplified as

$$(A_1 \otimes I + B_1 \otimes \Lambda)\,\left(I \otimes \mathcal{Z}^{-1}\right)\,u = \left(I \otimes \mathcal{Z}^T\,B_2^T\right)\,F. \tag{3.3}$$

Note that in (3.3), the following facts are used:

$$\mathcal{Z}^T\,B_2^T\,A_2\,\mathcal{Z} = \Lambda, \quad \text{and} \quad \mathcal{Z}^T\,B_2^T\,B_2\,\mathcal{Z} = I.$$

Assuming that $\left(I \otimes \mathcal{Z}^{-1}\right)\,u = v$ and $\left(I \otimes \mathcal{Z}^T\,B_2^T\right)\,F = g$, then the expression (3.3) becomes

$$(A_1 \otimes I + B_1 \otimes \Lambda)\,v = g. \tag{3.4}$$

Now the matrix system (3.4) can be solved by following the steps mentioned below.

Step 1. Compute $g = \left(I \otimes \mathcal{Z}^T\,B_2^T\right)\,F$.
Step 2. Solve $(A_1 \otimes I + B_1 \otimes \Lambda)\,v = g$.
Step 3. Compute the approximate solution $u = (I \otimes \mathcal{Z})\,v$.

By definition of Kronecker product, the system defined in (3.4) can be simplified in the form

$$(A_1 + \lambda_i B_1)u_{ij} = g_{ij}, \quad i = 1, 2, \ldots, 2N, \quad j = 1, 2, \ldots, 4N + 2. \tag{3.5}$$

Suppose $E_i = A_1 + \lambda_i B_1$, for $i = 1, 2, \ldots, 2N$. Now the system (3.5) becomes

$$E_i u_{ij} = g_{ij}, \quad i = 1, 2, \ldots, 2N, \quad j = 1, 2, \ldots, 4N + 2. \tag{3.6}$$

The matrix form of the system (3.6) is

$$\begin{bmatrix} L_b &&&&&&&& \\ S_1 & T_1 &&&&&&& \\ -C_1 & -D_1 & I_2 &&&&&& \\ && \ddots &&&&&& \\ && S_i & T_i &&&&& \\ && -C_i & -D_i & I_2 &&&& \\ &&&& \ddots &&&& \\ &&&& S_N & T_N && \\ &&&& -C_N & -D_N & I_2 \\ &&&&&& R_b \end{bmatrix} \begin{bmatrix} u_{0,i} \\ v_{0,i} \\ u_{1,i} \\ \vdots \\ u_{j-1,i} \\ v_{j-1,i} \\ \vdots \\ u_{N-1,i} \\ v_{N-1,i} \\ u_{N,i} \end{bmatrix} = \begin{bmatrix} g_0 \\ f_1 \\ \mathbf{0} \\ \vdots \\ f_i \\ \mathbf{d} \\ \vdots \\ f_N \\ \mathbf{0} \\ g_1 \end{bmatrix}, \tag{3.7}$$

where $L_b = [1 \;\; 0]$ and $R_b = [1 \;\; 0]$, arises from the left and right boundary conditions of (1.2), respectively. Furthermore, $u_{ji} = [u_{j_1,i} \;\; u_{j_2,i}]^T$, $v_{ji} = [v_{j_1,i} \;\; v_{j_2,i}]^T$ for $i = 1, 2, \ldots, 2N$, and I_2 is the identity matrix of order 2. In (3.7), $\mathbf{d} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ which is obtained from (1.4). In connection to (3.7), the entries of the matrices S_i and T_i on the intervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{i-1}, x_i]$ are

$$S_i = \begin{bmatrix} \omega_-^2 + \lambda_i & \frac{1}{2}(\lambda_i + 1)\theta_1 h \\ \omega_-^2 + \lambda_i & \frac{1}{2}(\lambda_i + 1)\theta_2 h \end{bmatrix},$$

$$T_i = \begin{bmatrix} 2 + (\omega_-^2 + \lambda_i)(\frac{1}{2}\theta_1 h)^2 & 3\theta_1 h + (\omega_-^2 + \lambda_i)(\frac{1}{2}\theta_1 h)^3 \\ 2 + (\omega_-^2 + \lambda_i)(\frac{1}{2}\theta_2 h)^2 & 3\theta_2 h + (\omega_-^2 + \lambda_i)(\frac{1}{2}\theta_2 h)^3 \end{bmatrix},$$

and on the intervals $[x_i, x_{i+1}]$, $[x_{i+1}, x_{i+2}]$, $\cdots$, $[x_{N-1}, x_N]$

$$S_i = \begin{bmatrix} (\omega_+^2 + \lambda_i) & \frac{1}{2}\theta_1 h \\ (\omega_+^2 + \lambda_i) & \frac{1}{2}\theta_2 h \end{bmatrix},$$

$$T_i = \begin{bmatrix} 2 + (\omega_+^2 + \lambda_i)(\frac{1}{2}\theta_1 h)^2 & 3\theta_1 h + (\omega_+^2 + \lambda_i)(\frac{1}{2}\theta_1 h)^3 \\ 2 + (\omega_+^2 + \lambda_i)(\frac{1}{2}\theta_2 h)^2 & 3\theta_2 h + (\omega_+^2 + \lambda_i)(\frac{1}{2}\theta_2 h)^3 \end{bmatrix},$$

where $\theta_1 = 1 + \rho_1$ and $\theta_2 = 1 + \rho_2$.

It can be observed that for every value of $h = \frac{1}{N}$ with different θ_1 and θ_2, the columns of matrices $T_i,\ i = 1, 2, \ldots, N$ are linearly independent. Therefore, the matrices T_i are non-singular. In the next theorem, it is proved that the system (3.7) has one and only one solution.

Theorem 3.1. *Let $T_i,\ i = 1, 2, \ldots, N$ be non-singular matrices. Then the almost block diagonal linear system of order $4N + 2$ defined in (3.7) has a unique solution.*

Proof. Given that the matrices T_i are non-singular for $i = 1, 2, \ldots, N$, to prove that the coefficient matrix defined in (3.7) is non-singular, it is sufficient to show that there exists a trivial solution to the homogeneous system (3.7). Hence from (3.7), we obtain

$$L_b \ u_{0,i} = 0, \tag{3.8}$$

$$S_1 \, u_{0,i} + T_1 \, v_{0,i} = 0, \tag{3.9}$$

$$-C_1 \, u_{0,i} - D_1 \, v_{0,i} + I_2 \, u_{1,i} = 0. \tag{3.10}$$

From (3.8), we have $u_{0,i} = 0$. Further, substituting $u_{0,i} = 0$ in (3.9), we get

$$T_1 \, v_{0,i} = 0, \tag{3.11}$$

and since T_1 is non-singular implies that $v_{0,i} = 0$. Again substituting $u_{0,i}$ and $v_{0,i}$ in (3.10) gives $u_{1,i} = 0$. Since each T_i are non-singular, solve the i^{th} row equation of (3.7) as follows

$$S_i \, u_{j-1,i} + T_i \, v_{j-1,i} = 0, \tag{3.12}$$

$$-C_i \, u_{j-1,i} - D_i \, v_{j-1,i} + I_2 \, u_{j,i} = 0. \tag{3.13}$$

Similar to the above analysis, it is conclusive that $u_{j-1,i} = v_{j-1,i} = u_{j,i} = 0$ for $i = 1, 2, \ldots, N$. Hence the only solution that exists is trivial. Therefore, the coefficient matrix in (3.7) is invertible and has a unique solution. □

4. Numerical Results

In this section, the numerical implementation of the proposed method is demonstrated to show the accuracy and order of convergence of Helmholtz interface problems that involve different forms of Neumann boundary conditions. Furthermore, we have shown that the order of convergence of the proposed scheme is better (for Neumann boundary conditions) than the order of convergence of discontinuous problems obtained using finite difference scheme mentioned in [18, 20, 22, 23]. The performance of this method is computed using L^∞, L^2 and H^1 norms, where L^∞ error is estimated by determining the maximum absolute error at 10×10 equally spaced points in each sub-interval $I_j \times I_j,\ j = 1, \ldots, N$. To estimate the L^2 and H^1 error, the composite three-point Gauss quadrature is used. It is assumed that u_h is the approximate solution with step size h corresponding to the exact solution $u(x, y)$. Furthermore, the maximum absolute error and the first derivatives at the nodes are computed using the ℓ^∞ norm. It is noted that the input N generates the matrices of size $(4N + 2) \times 2N$ of the required order. In the following examples, the consideration of degree r piecewise polynomials in OSCM ensures the spatial accuracy of $r + 1$ in the L^∞ and L^2 norms, and r in H^1 norm. We also expect $2r - 2$ order superconvergence in the ℓ^∞ norm of the approximation and its first spatial derivative.

In the following example, Helmholtz interface problem is considered with piecewise constant wave numbers and hence computed the error bounds along with order of convergence.

Example 4.1. The two-dimensional interface problem under consideration is

$$\begin{aligned} u_{xx}(x, y) + u_{yy}(x, y) + w^2 u(x, y) &= f(x, y),\ (x, y) \in [0, 1] \times [0, 1], \\ u_x(0, y) = 0,\ u_x(1, y) &= 1 \ \text{ on } \partial\Omega, \\ u_y(x, 0) = 0,\ u_y(x, 1) &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$

Table 1. Order of convergence and error using ℓ^∞ norm.

	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
N	error $\|u-u_h\|_{\ell^\infty}$	order	error $\|u-u_h\|_{\ell^\infty}$	order
4	7.6926×10^{-4}		1.4989×10^{-3}	
8	4.7442×10^{-5}	4.0192	9.4696×10^{-5}	3.9845
12	9.3467×10^{-6}	4.0065	1.8757×10^{-5}	3.9932
16	2.9546×10^{-6}	4.0032	5.9411×10^{-6}	3.9963
20	1.2097×10^{-6}	4.0019	2.4347×10^{-6}	3.9977
24	5.8323×10^{-7}	4.0013	1.1745×10^{-6}	3.9984

Table 2. Order of convergence and error using L^2 norm.

	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
N	error $\|u-u_h\|_{L^2}$	order	error $\|u-u_h\|_{L^2}$	order
4	1.0134×10^{-3}		1.1293×10^{-3}	
8	6.2114×10^{-5}	4.0281	6.9111×10^{-5}	4.0303
12	1.2224×10^{-5}	4.0091	1.3638×10^{-5}	4.0024
16	3.8628×10^{-6}	4.0045	4.3147×10^{-6}	4.0005
20	1.5813×10^{-6}	4.0027	1.7673×10^{-6}	4.0001
24	7.6232×10^{-7}	4.0018	8.5227×10^{-7}	4.0000

where

$$w^2(x) = \begin{cases} w_-^2 \ , & x \in [0, 0.5] \times [0, 1], \\ w_+^2 \ , & x \in (0.5, 1] \times [0, 1]. \end{cases}$$

The exact solution is given by,

$$u(x, y) = \begin{cases} \cos(\pi x)\ \cos(\pi y), & (x, y) \in \Omega^-, \\ x + \cos(\pi x)\ \cos(\pi y), & (x, y) \in \Omega^+. \end{cases}$$

and the interface conditions at $x = \frac{1}{2}$ are

$$[u] = -\frac{1}{2}, \quad [u_x] = 1.$$

The source function $f(x, y)$ is computed with the help of the exact solution, i.e.,

$$f(x, y) = \begin{cases} \left(-2\,\pi^2 + w_-^2\right)\ \cos(\pi x)\ \cos(\pi y), \ (x, y) \in \Omega^-, \\ \left(-2\,\pi^2 + w_+^2\right)\ \cos(\pi x)\ \cos(\pi y) + w_+^2 x, \ \ (x, y) \in \Omega^+. \end{cases}$$

We have computed the approximate solutions by considering the following wave numbers: $w_-^2 = 1, w_+^2 = 9$ and $w_-^2 = 25, w_+^2 = 100$. The size of the matrices versus the order of convergence and the error **concerning** ℓ^∞, L^2, L^∞ and H^1 norms are presented in the Tables: 1-4. It is observed that when the matrices size increases this method confers the expected order of convergence, which is close to 4.

TABLE 3. Order of convergence and error using L^∞ norm.

N	$w_-^2 = 1$ error $\|u - u_h\|_{L^\infty}$	$w_+^2 = 9$ order	$w_-^2 = 25$ error $\|u - u_h\|_{L^\infty}$	$w_+^2 = 100$ order
4	1.3441×10^{-3}		1.2855×10^{-3}	
8	9.2106×10^{-5}	3.8672	8.1217×10^{-5}	3.9844
12	1.8500×10^{-5}	3.9589	1.6088×10^{-5}	3.9930
16	5.8875×10^{-6}	3.9798	5.0960×10^{-6}	3.9961
20	2.4180×10^{-6}	3.9880	2.0885×10^{-5}	3.9976
24	1.1678×10^{-6}	3.9920	1.0075×10^{-6}	3.9984

TABLE 4. Order of convergence and error using H^1 norm.

N	$w_-^2 = 1$ error $\|u - u_h\|_{H^1}$	$w_+^2 = 9$ order	$w_-^2 = 25$ error $\|u - u_h\|_{H^1}$	$w_+^2 = 100$ order
4	1.2322×10^{-2}		1.2702×10^{-2}	
8	1.5131×10^{-3}	3.0256	1.5171×10^{-3}	3.0657
12	4.4684×10^{-4}	3.0082	4.4723×10^{-4}	3.0125
16	1.8829×10^{-4}	3.0041	1.8837×10^{-4}	3.0056
20	9.6353×10^{-5}	3.0024	9.6378×10^{-5}	3.0032
24	5.5744×10^{-5}	3.0016	5.5753×10^{-5}	3.0021

TABLE 5. Order of convergence and error of u_x using ℓ^∞ norm.

N	$w_-^2 = 1$ error $\|u_x - u_{hx}\|_{\ell^\infty}$	$w_+^2 = 9$ order	$w_-^2 = 25$ error $\|u_x - u_{hx}\|_{\ell^\infty}$	$w_+^2 = 100$ order
4	7.2882×10^{-4}		3.4461×10^{-3}	
8	4.7485×10^{-5}	3.9400	2.2542×10^{-4}	3.9343
12	9.3863×10^{-6}	3.9983	3.9770×10^{-5}	3.9882
16	2.9418×10^{-6}	4.0331	1.4269×10^{-5}	3.9882
20	1.2130×10^{-6}	3.9703	5.8541×10^{-6}	3.9928
24	5.8436×10^{-7}	4.0056	2.8256×10^{-6}	3.9952

For the super-convergence result, we have calculated the partial derivatives of the solution with respect to x, y and xy and thereafter its error is elaborated in the Tables: 5-7. Generally at the nodal points, the order of convergence of solution derivatives reduces the actual order of convergence by 1 while OSC method for this problem gives the actual order of convergence that appears in the Tables: 5-7.

In the next example, the wave numbers are defined at two interface conditions and the order of convergence, and error with regard to different norms are evaluated.

TABLE 6. Order of convergence and error of u_y using ℓ^∞ norm.

N	$w_-^2 = 1$ error $\|u_y - u_{hy}\|_{\ell^\infty}$	$w_+^2 = 9$ order	$w_-^2 = 25$ error $\|u_y - u_{hy}\|_{\ell^\infty}$	$w_+^2 = 100$ order
4	1.3472×10^{-3}		3.4403×10^{-3}	
8	8.0430×10^{-5}	4.0661	2.2595×10^{-4}	3.9285
12	1.5748×10^{-5}	4.0218	4.5055×10^{-5}	3.9767
16	4.9673×10^{-6}	4.0108	1.4304×10^{-5}	3.9882
20	2.0316×10^{-6}	4.0065	5.8685×10^{-6}	3.9929
24	9.7900×10^{-7}	4.0043	2.8325×10^{-6}	3.9952

TABLE 7. Order of convergence and error of u_{xy} using ℓ^∞ norm.

N	$w_-^2 = 1$ error $\|u_{xy} - u_{hxy}\|_{\ell^\infty}$	$w_+^2 = 9$ order	$w_-^2 = 25$ error $\|u_{xy} - u_{hxy}\|_{\ell^\infty}$	$w_+^2 = 100$ order
4	1.1690×10^{-3}		7.9871×10^{-3}	
8	7.8204×10^{-5}	3.9018	4.6827×10^{-4}	4.0922
12	1.5633×10^{-5}	3.9706	9.6344×10^{-5}	3.8995
16	4.9733×10^{-6}	3.9811	3.0906×10^{-5}	3.9522
20	2.0460×10^{-6}	3.9803	1.2739×10^{-5}	3.9718
24	9.8862×10^{-7}	3.9894	6.1643×10^{-6}	3.9814

Example 4.2. We consider the following two-dimensional interface problem

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) + w^2 u(x,y) &= f(x,y), \ (x,y) \in [0,1] \times [0,1], \\ u(0,y) = 0, \ \ u(1,y) &= 0 \ \text{ on } \partial\Omega, \\ u_y(x,0) = 0, \ \ u_y(x,1) &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$

where

$$w^2(x) = \begin{cases} w_1^2 \ , & x \in \Omega_1 = [0, 0.25] \times [0,1], \\ w_2^2 \ , & x \in \Omega_2 = (0.25, 0.75] \times [0,1], \\ w_3^2 \ , & x \in \Omega_3 = (0.75, 1] \times [0,1]. \end{cases}$$

The exact solution is

$$u(x,y) = \begin{cases} \sin(\pi x)\ \cos(\pi y), & (x,y) \in \Omega_1, \\ x + \sin(\pi x)\ \cos(\pi y), & (x,y) \in \Omega_2, \\ 1 - x + \sin(\pi x)\ \cos(\pi y), & (x,y) \in \Omega_3, \end{cases}$$

and the interface conditions at $x = \frac{1}{2}$ and at $x = \frac{3}{4}$ are

$$[u] = \frac{1}{4}, \quad [u_x] = 1,$$

$$[u] = -\frac{1}{2}, \quad [u_x] = -2,$$

TABLE 8. Order of convergence and error using ℓ^∞ norm.

	$w_1^2=1 \quad w_2^2=2$	$w_3^2=3$	$w_1^2=1 \quad w_2^2=10$	$w_+^2=100$
N	error $\|u-u_h\|_{\ell^\infty}$	order	error $\|u-u_h\|_{\ell^\infty}$	order
4	6.9359×10^{-4}		7.8240×10^{-4}	
8	4.2944×10^{-5}	4.0135	5.2043×10^{-5}	3.9101
12	8.4669×10^{-6}	4.0047	1.0363×10^{-5}	3.9801
16	2.6772×10^{-6}	4.0023	3.2878×10^{-6}	3.9907
20	1.0962×10^{-6}	4.0014	1.3499×10^{-6}	3.9893
24	5.2857×10^{-7}	4.0009	6.5076×10^{-7}	4.0020

TABLE 9. Order of convergence and error using L^2 norm.

	$w_1^2=1 \quad w_2^2=2$	$w_3^2=3$	$w_1^2=1 \quad w_2^2=10$	$w_+^2=100$
N	error $\|u-u_h\|_{L^2}$	order	error $\|u-u_h\|_{L^2}$	order
4	9.0102×10^{-4}		8.8622×10^{-4}	
8	5.5270×10^{-5}	4.0270	5.4318×10^{-5}	4.0282
12	1.0878×10^{-5}	4.0090	1.0709×10^{-5}	4.0047
16	3.4373×10^{-6}	4.0054	3.3866×10^{-6}	4.0019
20	1.4071×10^{-6}	4.0027	1.3869×10^{-6}	4.0010
24	6.7833×10^{-7}	4.0018	6.6873×10^{-7}	4.0006

respectively. The source function $f(x,y)$ is calculated with the help of the exact solution as

$$f(x,y)=\begin{cases}\left(-2\,\pi^2+w_1^2\right)\,\sin(\pi x)\,\cos(\pi y), & (x,y)\in\Omega_1,\\ \left(-2\,\pi^2+w_2^2\right)\,\cos(\pi x)\,\cos(\pi y)+w_2^2x, & (x,y)\in\Omega_2,\\ \left(-2\,\pi^2+w_3^2\right)\,\cos(\pi x)\,\cos(\pi y)+w_3^2(1-x), & (x,y)\in\Omega_3.\end{cases}$$

In the following Tables: 8-11, the error and order of convergence are computed for the wave numbers $\{w_-^2=1, w_+^2=2, w_3^2=3\}$ and $\{w_1^2=1, w_2^2=10, w^2+3=100\}$. It is also noticed that the fourth order of convergence is obtained even if the size of the matrix is 144. Thereafter, we have presented the error and super-convergence result of the partial derivatives of the solution corresponding to x, y, and xy that are shown in the Tables: 12-14. Additionally, we have plotted the approximate and exact solution graph in Figures 1 and 2.

The following figures are the approximate solution and exact solution plot of Example 4.2 for $N=24$ which gives the matrix of order 4704. The values of (x_i,y_i) vary at the nodal points and the figure plotted corresponding to its $u_h(x_i,y_i)$ and $u(x_i,y_i)$. These graphs signify that although there are discontinuities in the interface problem still the approximate solution is almost equal to the exact solution.

In the last example, the Helmholtz interface problem is solved with piecewise continuous wave function and the order of convergence followed by the errors are presented.

Example 4.3. Let us assume the two-dimensional interface problem

$$\begin{aligned} u_{xx}(x,y)+u_{yy}(x,y)+w^2u(x,y) &= f(x,y),\ (x,y)\in[0,1]\times[0,1],\\ u_x(0,y)=0,\ \ u_x(1,y) &= 1 \ \text{ on } \partial\Omega,\\ u_y(x,0)=0,\ \ u_y(x,1) &= 0 \ \text{ on } \partial\Omega,\end{aligned}$$

Table 10. Order of convergence and error using L^∞ norm.

N	$w_1^2=1$ $w_2^2=2$ error $\|u-u_h\|_{L^\infty}$	$w_3^2=3$ order	$w_1^2=1$ $w_2^2=10$ error $\|u-u_h\|_{L^\infty}$	$w_+^2=100$ order
4	1.1068×10^{-3}		1.0655×10^{-3}	
8	7.6518×10^{-5}	3.8544	7.0895×10^{-5}	3.9097
12	1.5390×10^{-5}	3.9555	1.4092×10^{-5}	3.9845
16	4.9000×10^{-6}	3.9783	4.4610×10^{-6}	3.9984
20	2.0218×10^{-6}	3.9871	1.8262×10^{-6}	4.0025
24	9.7221×10^{-7}	3.9914	8.8007×10^{-7}	4.0039

Table 11. Order of convergence and error using H^1 norm.

N	$w_1^2=1$ $w_2^2=2$ error $\|u-u_h\|_{H^1}$	$w_3^2=3$ order	$w_1^2=1$ $w_2^2=10$ error $\|u-u_h\|_{H^1}$	$w_+^2=100$ order
4	1.2229×10^{-2}		1.2379×10^{-2}	
8	1.5104×10^{-3}	3.0173	1.5123×10^{-3}	3.0331
12	4.4650×10^{-4}	3.0058	4.4668×10^{-4}	3.0077
16	1.8821×10^{-4}	3.0029	1.8825×10^{-4}	3.0036
20	9.6327×10^{-5}	3.0017	9.6339×10^{-5}	3.0021
24	5.5733×10^{-5}	3.0011	5.5738×10^{-5}	3.0014

Table 12. Order of convergence and error of u_x using ℓ^∞ norm.

N	$w_1^2=1$ $w_2^2=2$ error $\|u_x-u_{hx}\|_{\ell^\infty}$	$w_3^2=3$ order	$w_1^2=1$ $w_2^2=10$ error $\|u_x-u_{hx}\|_{\ell^\infty}$	$w_+^2=100$ order
4	1.1170×10^{-3}		1.8858×10^{-3}	
8	6.6909×10^{-5}	4.0613	2.0993×10^{-4}	3.1672
12	1.3109×10^{-5}	4.0202	4.4329×10^{-5}	3.8354
16	4.1357×10^{-6}	4.0100	1.4335×10^{-5}	3.9242
20	1.6917×10^{-6}	4.0060	5.9297×10^{-6}	3.9559
24	8.1525×10^{-7}	4.0040	2.8748×10^{-6}	3.9710

where

$$w^2(x)=\begin{cases} w_-^2=x+1\ , & x\in[0,0.5]\times[0,1],\\ w_+^2=x+2\ , & x\in(0.5,1]\times[0,1].\end{cases}$$

The exact solution is given by

$$u(x,y)=\begin{cases}\cos(\pi x)\,\cos(\pi y), & (x,y)\in\Omega^-\\ x+\cos(\pi x)\,\cos(\pi y), & (x,y)\in\Omega^+.\end{cases}$$

TABLE 13. Order of convergence and error of u_y using ℓ^∞ norm.

	$w_1^2 = 1 \quad w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1 \quad w_2^2 = 10$	$w_+^2 = 100$
N	error $\|u_y - u_{hy}\|_{\ell^\infty}$	order	error $\|u_y - u_{hy}\|_{\ell^\infty}$	order
4	1.1017×10^{-3}		1.3105×10^{-3}	
8	6.6180×10^{-5}	4.0572	1.0064×10^{-4}	3.7028
12	1.2973×10^{-5}	4.0188	2.0654×10^{-5}	3.9058
16	4.0938×10^{-6}	4.0093	6.6217×10^{-6}	3.9542
20	1.6747×10^{-6}	4.0056	2.7288×10^{-6}	3.9728
24	8.0710×10^{-7}	4.0037	1.3203×10^{-6}	3.9819

TABLE 14. Order of convergence and error of u_{xy} using ℓ^∞ norm.

	$w_1^2 = 1 \quad w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1 \quad w_2^2 = 10$	$w_+^2 = 100$
N	error $\|u_{xy} - u_{hxy}\|_{\ell^\infty}$	order	error $\|u_{xy} - u_{hxy}\|_{\ell^\infty}$	order
4	1.4785×10^{-4}		1.7331×10^{-3}	
8	1.0731×10^{-5}	3.7843	4.0482×10^{-4}	2.0980
12	2.5180×10^{-6}	3.5754	9.3134×10^{-5}	3.6240
16	8.4065×10^{-7}	3.8134	3.0888×10^{-5}	3.8364
20	3.5265×10^{-7}	3.8930	1.2918×10^{-5}	3.9066
24	1.7225×10^{-7}	3.9302	6.2992×10^{-6}	3.9392

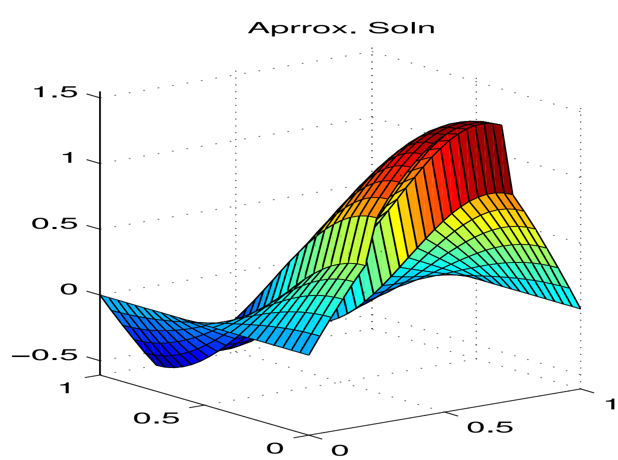

FIGURE 1. The plot of approximate solution for $N = 24$.

and the interface conditions at $x = \frac{1}{2}$ are

$$[u] = -\frac{1}{2}, \quad [u_x] = 1.$$

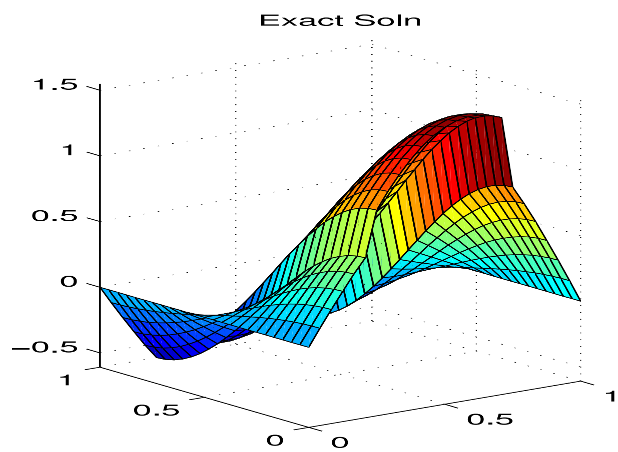

FIGURE 2. The plot of the exact solution for $N = 24$.

TABLE 15. Order of convergence and error using ℓ^∞ and L^∞ norm.

N	$w_-^2 = x+1$ error $\|u-u_h\|_{\ell^\infty}$	$w_+^2 = x+2$ order	$w_-^2 = x+1$ error $\|u-u_h\|_{L^\infty}$	$w_+^2 = x+2$ order
4	7.0745×10^{-4}		1.1208×10^{-3}	
8	4.3823×10^{-5}	4.0129	7.7459×10^{-5}	3.8550
12	8.6405×10^{-6}	4.0045	1.5579×10^{-5}	3.9555
16	2.7381×10^{-6}	4.0023	4.9602×10^{-6}	3.9782
20	1.1187×10^{-6}	4.0014	2.0376×10^{-6}	3.9870
24	5.3942×10^{-7}	4.0009	9.8412×10^{-7}	3.9914

The source function $f(x, y)$ is determined by the help of the exact solution as

$$f(x,y) = \begin{cases} \left(-2\,\pi^2 + w_-^2\right)\,\cos(\pi x)\,\cos(\pi y), & (x,y) \in \Omega^-, \\ \left(-2\,\pi^2 + w_+^2\right)\,\cos(\pi x)\,\cos(\pi y) + w_+^2 x, & (x,y) \in \Omega^+. \end{cases}$$

We have calculated the estimate solutions by assuming the wave functions as $w_-^2 = x+1, w_+^2 = x+2$ and $w_-^2 = x+1, w_+^2 = x+2$. Further, the computation of order of convergence and error concerning various norms is presented in the Tables: 15-17.

5. Conclusion

This article focuses on implementing the MDA algorithm for solving the two-dimensional Helmholtz interface problem with Neumann boundary conditions. The MDA algorithm effectively resolves the problem, and we computed the approximate solution for three different types of wave functions. This method remains effective even in scenarios involving large wave numbers. The associated errors were presented using various norms. Moreover, we observed the desired order of convergence and achieved superconvergence results at nodal points, even with a small matrix size. In future the OSC method for solving Helmholtz interface problems can be extended to the annulus region and in a disc. A theoretical convergence analysis for Helmholtz problems with interfaces and the extension of OSC method to Robbin boundary conditions are topics of future research.

TABLE 16. Order of convergence and error using L^2 and H^1 norm.

	$w_-^2 = x+1$	$w_+^2 = x+2$	$w_-^2 = x+1$	$w_+^2 = x+2$
N	error $\|u - u_h\|_{L^2}$	order	error $\|u - u_h\|_{H^1}$	order
4	9.0184×10^{-4}		1.2230×10^{-2}	
8	5.5321×10^{-5}	4.0270	1.5105×10^{-3}	3.0173
12	1.0888×10^{-5}	4.0090	4.4650×10^{-4}	3.0058
16	3.4404×10^{-6}	4.0045	1.8821×10^{-4}	3.0029
20	1.4084×10^{-6}	4.0027	9.6327×10^{-5}	3.0017
24	6.7896×10^{-7}	4.0018	5.5733×10^{-5}	3.0011

TABLE 17. Order of convergence and error of u_x, u_y and u_{xy} using ℓ^∞ norm.

	$w_-^2 = x+1$			$w_+^2 = 2+x$		
N	$\|u_x - u_{hx}\|_{\ell^\infty}$	order	$\|u_y - u_{hy}\|_{\ell^\infty}$	order	$\|u_{xy} - u_{hxy}\|_{\ell^\infty}$	order
4	1.0987×10^{-3}		1.1527×10^{-3}		1.0878×10^{-4}	
8	6.5998×10^{-5}	4.0572	6.9056×10^{-5}	4.0611	9.8381×10^{-6}	3.4669
12	1.2938×10^{-5}	4.0188	1.3529×10^{-5}	4.0203	2.3422×10^{-6}	3.5396
16	4.0826×10^{-6}	4.0093	4.2683×10^{-6}	4.0101	7.8312×10^{-7}	3.8083
20	1.6701×10^{-6}	4.0056	1.7459×10^{-6}	4.0060	3.3128×10^{-7}	3.8555
24	8.0488×10^{-7}	4.0037	8.4136×10^{-7}	4.0040	1.6201×10^{-7}	3.9232

ACKNOWLEDGEMENTS

We would like to express our sincere thanks and gratitude to the anonymous referees for their valuable comments and suggestions in the improvement of the manuscript.

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