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On analytical solutions of the ZK equation and related equations by using the generalized $(\frac{G'}{G})$-expansion method

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Abstract

The generalized $(\frac{G'}{G})$-expansion method with the aid of Maple is proposed to seek exact solutions of nonlinear evolution equations. For finding exact solutions are expressed three types of solutions that include hyperbolic function solution, trigonometric function solution, and rational solution. The article studies the Zakharov–Kuznetsov (ZK) equation, the generalized ZK (gZK) equation, and the generalized forms of these equations. Exact solutions with traveling wave solutions of nonlinear evolution equations are obtained. It is shown that the proposed method is direct, effective and can be used for many other nonlinear evolution equations.

Keywords. Generalized $(\frac{G'}{G})$-expansion method, Zakharov–Kuznetsov equation, Generalized Zakharov-Kuznetsov equation, Traveling wave solution.

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1. Introduction

The investigation of the traveling wave solutions plays an important role in nonlinear sciences. A variety of powerful methods have been presented, such as the inverse scattering transform [1], Hirota's bilinear method [21], sine-cosine method [41], homotopy perturbation method [6, 7, 15], homotopy analysis method [8, 9], variational iteration method [3, 10, 11, 19], tanh-function method [14], Bäcklund transformation [33], Exp-function method [12, 13, 20, 24, 27] and so on. Here, we use an effective method for constructing a range of exact solutions for the following nonlinear partial differential equations that first proposed by Wang [39]. A new method called the $(\frac{G'}{G})$-expansion method is presented to look for traveling wave solutions of nonlinear evolution equations (NLEEs). Zhang et al. [46] have examined the generalized $(\frac{G'}{G})$-expansion method and its applications. Authors of [45] have used to mKdV equation with variable coefficients using the $(\frac{G'}{G})$-expansion method. Besides, Bekir [5] has investigated an application of the $(\frac{G'}{G})$-expansion method for nonlinear evolution equations. Some applications of the $(\frac{G'}{G})$-expansion method to nonlinear partial differential equations have been applied by Zaved and Gepreel [43]. In [17] the $(\frac{G'}{G})$-expansion method has been applied to solve higher-order nonlinear equations by Gao and Zhao. The explicit exact solutions for the nonlinear Klein-Gordon equation by using a $(\frac{G'}{G})$-expansion method have been studied by Sousaraie and Bagheri [37]. We consider family of generalizations of the Korteweg-de Vries (KdV) equation which take the forms of a Zakharov-Kuznetsov (ZK),

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generalized Zakharov-Kuznetsov (gZK) and generalized form of modified Zakharov-Kuznetsov (gmZK) equations, as follow [13, 42]

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0, \tag{1.1}$$

$$u_t + au^n u_x + b(u_{xx} + u_{yy})_x = 0, \quad n > 1, \tag{1.2}$$

$$u_t + au^{\frac{n}{2}} u_x + b(u_{xx} + u_{yy})_x = 0, \quad n \geq 1. \tag{1.3}$$

The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. The ZK equation, which is a more isotropic two-dimensional, was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions [44]. Several authors studied a Zakharov-Kuznetsov (ZK), generalized Zakharov-Kuznetsov (gZK), generalized form of modified Zakharov-Kuznetsov (gmZK) equations, the reader is advised to read [4, 22, 23, 35, 36, 38, 40, 48].

Authors of [32] explained the generalized fifth-order KdV like equation with prime number $p = 3$ via a generalized bilinear differential operator. N-lump was investigated to the variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation [31]. Applications of $tan(\phi/2)$-expansion method for the Biswas-Milovic equation [28], the Gerdjikov-Ivanov model [30], the Kundu-Eckhaus equation [29] and the fifth-order integrable equations [26] were studied. Lump solutions were analyzed to the fractional generalized CBS-BK equation [47] and the (3+1)-D Burger system [18]. The approximations of one-dimensional hyperbolic equation with non-local integral conditions were constructed by reduced differential transform method [34]. The generalized Hirota bilinear strategy by the number prime was used to the (2+1)-dimensional generalized fifth-order KdV like equation [32]. The traveling wave solutions and analytical treatment of the simplified MCH equation and the combined KdV-mKdV equations were studied [2].

Our purpose of this paper is to obtain analytical solutions of nonlinear a ZK, gZK, and gmZK equations, and to determine the accuracy of the $(\frac{G'}{G})$-expansion method in solving these kinds of problems of this paper. The paper is organized as follows: In section 2, we describe the basic idea of the $(\frac{G'}{G})$-expansion method. In section 3, the application of this method to the ZK equation will be introduced briefly. Section 4 contains the generalized ZK equation. Section 5 contains the generalized form of modified ZK equation. Also, a conclusion is given in section 6. Finally, some references are given at the end of this paper.

2. Basic idea of $(\frac{G'}{G})$-expansion method

We give the detailed description of method which first presented by Wang [39].

Step 1. For a given nonlinear partial differential equations (NLPDEs) with independent variables $X = (x, y, z, t)$ and dependent variable u:

$$\mathcal{P}(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{tt}, u_{tx}, u_{ty}, u_{tz}...) = 0, \tag{2.1}$$

can be converted to an ODE

$$\mathcal{M}(u, -cu', u', u', u', u'', ...) = 0, \tag{2.2}$$

which transformation $\xi = x + y - c$ is wave variable. Also, c is constant to be determined later.

Step 2. We seek its solutions in the more general polynomial form as follows

$$u(\xi) = a_0 + \sum_{k=1}^{m} a_k \left(\frac{G'(\xi)}{G(\xi)}\right)^k, \tag{2.3}$$

where $G(\xi)$ satisfies the second order linear ordinary differential equation (LODE) in the form

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.4}$$

where $a_0, a_k(k = 1, 2, ..., m), \lambda$ and μ are constants to be determined later, $a_m = 0$, but the degree of which is generally equal to or less than $m-1$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (2.2).

Step 3. Substituting (2.3) and Eq. (2.4) into Eq. (2.2) with the value of m obtained in Step 1. Collecting the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^k$, $k = 0, \pm1, \pm2, ...$, then setting each coefficient to zero, we can get a set of over-determined partial differential equations for $a_0, a_i(i = 1, 2, ..., n), \lambda, c$ and μ with the aid of symbolic computation Maple 12.

Step 4. Solving the algebraic equations in Step 3, then substituting $a_i, ..., a_m, c$ and general solutions of Eq. (2.4) into (2.3) we can obtain a series of fundamental solutions of Eq. (2.1) depending of the solution $G(\xi)$ of Eq. (2.4).

3. Zakharov-Kuznetsov Equation

In this section we would like to use the $(\frac{G'}{G})$-expansion method to the ZK equation

$$u_t + 2uu_x + (u_{xx} + u_{yy})_x = 0, \tag{3.1}$$

and by using the wave variable $\xi = x + y - ct$ reduces it to an ODE

$$-cu' + 2uu' + 2u''' = 0. \tag{3.2}$$

Then by integrating Eq. (3.2) and neglecting the constant of integration we obtain

$$-cu + u^2 + 2u'' = 0. \tag{3.3}$$

By using Eq. (2.3) it is easily derived that

$$\begin{aligned} u^2(\xi) &= a_m^2\left(\frac{G'(\xi)}{G(\xi)}\right)^{2m} + ..., \\ u_\xi(\xi) &= -ma_m\left(\frac{G'(\xi)}{G(\xi)}\right)^{m+1} + ..., \\ u_{\xi\xi}(\xi) &= m(m+1)a_m\left(\frac{G'(\xi)}{G(\xi)}\right)^{m+2} + ... \end{aligned} \tag{3.4}$$

u" with u^2 in Eq. (3.3), we required that $m + 2 = 2m \Rightarrow m = 2$. We can suppose that the solutions of Eq. (3.3) is of the form

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad a_2 \neq 0, \tag{3.5}$$

and therefore

$$u^2(\xi) = a_2^2\left(\frac{G'(\xi)}{G(\xi)}\right)^4 + 2a_1a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^3 + (a_1^2 + 2a_0a_2)\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_0a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_0^2, \tag{3.6}$$

$$\begin{aligned} u_{\xi\xi}(\xi) &= 6a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^4 + (2a_1 + 10a_2\lambda)\left(\frac{G'(\xi)}{G(\xi)}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2)\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \\ &+ (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)\left(\frac{G'(\xi)}{G(\xi)}\right) + 2a_2\mu^2 + a_1\lambda\mu. \end{aligned} \tag{3.7}$$

Substituting (3.5)–(3.7) in to Eq. (3.3) and by using Eq. (2.4) in Maple software, we obtain the system of the following results

$$\begin{aligned} &a_0 = -4\lambda^2 - 8\mu + 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \qquad a_1 = -12\lambda, \qquad a_2 = -12, \\ &c = 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} &a_0 = -4\lambda^2 - 8\mu - 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \qquad a_1 = -12\lambda, \qquad a_2 = -12, \\ &c = -4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \end{aligned} \tag{3.9}$$

where λ and μ are arbitrary constants. Substituting Eqs. (3.8) and (3.9) into expression Eq. (3.5), can be written as

$$u(\xi) = -4\lambda^2 - 8\mu + 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu} - 12\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) - 12\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \tag{3.10}$$

where $\xi = x + y - 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}t$, or

$$u(\xi) = -4\lambda^2 - 8\mu - 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu} - 12\lambda\left(\frac{G'(\xi)}{G(\xi)}\right) - 12\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \tag{3.11}$$

where $\xi = x + y + 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}t$. Substituting the general solutions of Eq. (2.4) into (3.10) and (3.11) we have three types of exact solutions of Eq. (3.1) as follows:

I. When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solution

$$\begin{aligned} u_1(\xi) &= -3(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 \\ &+ 3\lambda^2 - 4\lambda^2 - 8\mu + 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \end{aligned} \tag{3.12}$$

where $\xi = x + y - 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}t$, and

$$\begin{aligned} u_2(\xi) &= -3(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 + 3\lambda^2 \\ &- 4\lambda^2 - 8\mu - 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \end{aligned} \tag{3.13}$$

where $\xi = x + y + 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}t$.

II. When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$\begin{aligned} u_3(\xi) &= -3(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 + 3\lambda^2 \\ &- 4\lambda^2 - 8\mu + 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \end{aligned} \tag{3.14}$$

where $\xi = x + y - 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}t$ and

$$\begin{aligned} u_4(\xi) &= -3(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2 + 3\lambda^2 \\ &- 4\lambda^2 - 8\mu - 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \end{aligned} \tag{3.15}$$

where $\xi = x + y + 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}t$.

III. When $\lambda^2 - 4\mu = 0$, we get rational solution

$$u_5(\xi) = \frac{-12C_2^2}{(C_1 + C_2\xi)^2} + 3\lambda^2 - 4\lambda^2 - 8\mu + 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \tag{3.16}$$

where $\xi = x + y - 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}t$ and

$$u_6(\xi) = \frac{-12C_2^2}{(C_1 + C_2\xi)^2} + 3\lambda^2 - 4\lambda^2 - 8\mu - 2\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}, \tag{3.17}$$

where $\xi = x + y + 4\sqrt{4\lambda^4 + 16\lambda^2\mu + 4\mu^2 - 6\lambda\mu}t$. If $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then solution (3.12) and (3.13), respectively, give:

$$u_1(x,y,t) = 3\lambda^2 sech^2\left[\frac{\lambda}{2}\xi\left(x + y - 8\lambda^2 t\right)\right], \tag{3.18}$$

$$u_2(x,y,t) = -\lambda^2\left(3tanh^2\left[\frac{\lambda}{2}(x + y + 8\lambda^2 t)\right] + 5\right). \tag{3.19}$$

In particular, if $\lambda = 0$, then we obtain sets of non-trivial solutions:

$$a_0 = -4\mu, \qquad a_1 = 0, \qquad a_2 = -12, \qquad c = 8\mu, \tag{3.20}$$

or

$$a_0 = -12\mu, \qquad a_1 = 0, \qquad a_2 = -12, \qquad c = -8\mu. \tag{3.21}$$

By using (3.20) and (3.21), expression Eq. (3.5) can be written as

$$u(\xi) = -4\mu - 12\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x + y - 8\mu t, \tag{3.22}$$

$$u(\xi) = -12\mu - 12\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x + y + 8\mu t. \tag{3.23}$$

Again, we have three types of exact solutions of Eq. (3.1) as follows

Case 1: $\mu < 0$.

$$u_7(\xi) = 12\mu\left(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi}\right)^2 - 4\mu, \qquad \xi = x + y - 8\mu t, \tag{3.24}$$

$$u_8(\xi) = 12\mu\left(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi}\right)^2 - 12\mu, \qquad \xi = x + y + 8\mu t. \tag{3.25}$$

Case 2: $\mu > 0$.

$$u_9(\xi) = -12\mu\left(\frac{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^2 - 4\mu, \qquad \xi = x + y - 8\mu t, \tag{3.26}$$

$$u_{10}(\xi) = -12\mu\left(\frac{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^2 - 12\mu, \qquad \xi = x + y + 8\mu t. \tag{3.27}$$

However, if $C_1 \neq 0, C_2 = 0$, then u_7, u_8, u_9 and u_{10}, respectively, give:

$$u_7(x,y,t) = 4[3tanh^2(\sqrt{-\mu}(x + y - 8\mu t)) - 1]\mu, \tag{3.28}$$

$$u_8(x,y,t) = -12sech^2(\sqrt{-\mu}(x + y + 8\mu t))\mu,$$

$$u_9(x,y,t) = -4[tan^2(\sqrt{\mu}(x + y - 8\mu t)) + 1]\mu,$$

$$u_{10}(x,y,t) = -12sec^2(\sqrt{\mu}(x + y + 8\mu t))\mu.$$

The results obtained in the above are exact solutions of the ZK equation.

4. The generalized ZK equation

In this section we study the generalized ZK equation [13, 42]

$$u_t + au^n u_x + b(u_{xx} + u_{yy})_x = 0, \qquad n > 1, \tag{4.1}$$

where a and b are constants. Using the wave variable $\xi = x + y - ct$ reduce it to an ODE

$$-cu + \frac{a}{n+1}u^{n+1} + 2bu'' = 0, \qquad n > 1, \tag{4.2}$$

where is obtained by integrating and neglecting the constant of integration. To get a closed form solution, we use the transformation

$$u(\eta) = v(\eta)^{\frac{1}{n}} \tag{4.3}$$

that will carry Eq. (4.2) into an ODE

$$-cn^2(n+1)v^2 + an^2v^3 + 2bn(n+1)vv'' - 2b(n^2-1)(v')^2 = 0, \tag{4.4}$$

we set

$$v(\xi) = a_0 + \sum_{k=1}^{m} a_k \left(\frac{G'(\xi)}{G(\xi)}\right)^k. \tag{4.5}$$

By the same manipulation as illustrated in the previous section, we can determine value of m by balancing v^3 and (vv'') or $(v')^2$ in Eq. (4.4), we can seen that $3m = 2m + 2$, then conclude $m = 2$. With the aid Eq. (4.5) it is derived that

$$v(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right), \qquad a_2 \neq 0, \tag{4.6}$$

$$v^3(\xi) = \left(a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)\right)^3, \tag{4.7}$$

and

$$\begin{aligned} v_{\xi\xi}(\xi) &= 6a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^4 + (2a_1 + 10a_2\lambda)\left(\frac{G'(\xi)}{G(\xi)}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2)\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \\ &+ (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)\left(\frac{G'(\xi)}{G(\xi)}\right) + 2a_2\mu^2 + a_1\lambda\mu. \end{aligned} \tag{4.8}$$

Substituting (4.6)–(4.8) in to Eq. (4.4), we obtain the system of following results:
If $\lambda = 0$, then we obtain sets of non-trivial solutions:

$$a_0 = a_2\mu, \qquad a_1 = 0, \qquad a_2 = \frac{2b(n^2+3n+2)}{an^2}, \qquad c = -\frac{8b\mu}{n^2}, \tag{4.9}$$

or

$$a_0 = 0, \qquad a_1 = 0, \qquad a_2 = -\frac{8b(n+1)}{an}, \qquad c = \frac{8b\mu}{n}. \tag{4.10}$$

By using (4.9) and (4.10), expression Eq. (4.6) can be written as

$$v(\xi) = \frac{2b(n^2+3n+2)}{an^2}\mu - \frac{2b(n^2+3n+2)}{an^2}\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x + y + \frac{8b\mu}{n^2}t, \tag{4.11}$$

$$v(\xi) = -\frac{8b(n+1)}{an}\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x + y - \frac{8b\mu}{n}t. \tag{4.12}$$

When $\mu < 0$, we get

$$v_1(\xi) = \frac{2b(n^2+3n+2)\mu}{an^2}\left[\left(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi}\right)^2 + 1\right], \quad \xi = x + y + \frac{8b\mu}{n^2}t, \tag{4.13}$$

$$v_2(\xi) = \frac{8b(n+1)}{an}\mu\left(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi}\right)^2, \qquad \xi = x + y - \frac{8b\mu}{n}t. \tag{4.14}$$

When $\mu > 0$, we have

$$v_3(\xi) = \frac{2b(n^2+3n+2)}{an^2}\mu\left[1 - \left(\frac{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^2\right], \quad \xi = x + y + \frac{8b\mu}{n^2}t, \tag{4.15}$$

$$v_4(\xi) = -\frac{8b(n+1)}{an}\mu\left(\frac{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^2, \qquad \xi = x + y - \frac{8b\mu}{n}t. \tag{4.16}$$

If $C_1 \neq 0, C_2 = 0, \mu < 0$, then solution (4.13) and (4.14), respectively, become:

$$v_1(x,y,t) = \frac{2b(n^2+3n+2)\mu}{an^2}\left\{1 + \tanh^2\left(\sqrt{-\mu}\left[x + y + \frac{8b\mu}{n^2}t\right]\right)\right\}, \tag{4.17}$$

$$v_2(x,y,t) = \frac{8b(n+1)\mu}{an}\tanh^2\left(\sqrt{-\mu}\left[x + y - \frac{8b\mu}{n}t\right]\right). \tag{4.18}$$

However, if $C_1 \neq 0, C_2 = 0, \mu > 0$, then solution (4.15) and (4.16), respectively, give:

$$v_3(x,y,t) = \frac{2b(n^2+3n+2)\mu}{an^2}\left\{1 - \tan^2\left(\sqrt{\mu}\left[x + y + \frac{8b\mu}{n^2}t\right]\right)\right\}, \tag{4.19}$$

$$v_4(x,y,t) = -\frac{8b(n+1)\mu}{an}\tan^2\left(\sqrt{\mu}\left[x + y - \frac{8b\mu}{n}t\right]\right).$$

Then by using Eq. (4.3) we obtain

Case 1: $\mu < 0$.

$$u_1(\xi) = \sqrt[n]{\frac{2b(n^2+3n+2)\mu}{an^2}}\left[\left(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi}\right)^2 + 1\right]^{\frac{1}{n}}, \quad \xi = x + y + \frac{8b\mu}{n^2}t, \tag{4.20}$$

$$u_2(\xi) = \sqrt[n]{\frac{8b(n+1)\mu}{an}}\left(\frac{C_1\sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi}\right)^{\frac{2}{n}}, \qquad \xi = x + y - \frac{8b\mu}{n}t. \tag{4.21}$$

Case 2: $\mu > 0$.

$$u_3(\xi) = \sqrt[n]{\frac{2b(n^2+3n+2)\mu}{an^2}}\left[1 - \left(\frac{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^2\right]^{\frac{1}{n}}, \quad \xi = x + y + \frac{8b\mu}{n^2}t, \tag{4.22}$$

$$u_4(\xi) = \sqrt[n]{-\frac{8b(n+1)\mu}{an}}\left(\frac{C_1\sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^{\frac{2}{n}}, \qquad \xi = x + y - \frac{8b\mu}{n}t. \tag{4.23}$$

If particular, $C_1 \neq 0, C_2 = 0, \mu < 0, \mu > 0$ then solution (4.20)-(4.23), respectively, give:

$$u_1(x,y,t) = \sqrt[n]{\frac{2b(n^2+3n+2)\mu}{an^2}}\left\{1 + \tanh^2\left(\sqrt{-\mu}\left[x + y + \frac{8b\mu}{n^2}t\right]\right)\right\}^{\frac{1}{n}}, \tag{4.24}$$

$$u_2(x,y,t) = \sqrt[n]{\frac{8b(n+1)\mu}{an}} \left\{ \tanh^2\left(\sqrt{-\mu}\left[x+y-\frac{8b\mu}{n}t \right] \right) \right\}^{\frac{1}{n}}, \tag{4.25}$$

$$u_3(x,y,t) = \sqrt[n]{\frac{2b(n^2+3n+2)\mu}{an^2}} \left\{ 1-\tan^2\left(\sqrt{\mu}\left[x+y+\frac{8b\mu}{n^2}t \right] \right) \right\}^{\frac{1}{n}}, \tag{4.26}$$

$$u_4(x,y,t) = \sqrt[n]{-\frac{8b(n+1)\mu}{an}} \tan^{\frac{2}{n}}\left(\sqrt{\mu}\left[x+y-\frac{8b\mu}{n}t \right] \right).$$

The results obtained in the above are exact solutions of the generalized ZK equation.

5. Generalized form of mZK equation

Now we close this work by studying a generalized form of mZK equation [13, 42] given by

$$u_t + au^{\frac{n}{2}}u_x + b(u_{xx}+u_{yy})_x = 0, \qquad n \geq 1, \tag{5.1}$$

where a and b are constants. Using the wave variable $\xi = x+y-ct$ reduce it to an ODE

$$-cu + \frac{2a}{n+1}u^{\frac{n+2}{2}} + 2bu'' = 0. \tag{5.2}$$

where is obtained by integrating and neglecting the constant of integration. To get a closed form solution, we use the transformation

$$u(\eta) = v(\eta)^{\frac{2}{n}}, \tag{5.3}$$

that will carry Eq. (5.2) into an ODE

$$-cn^2(n+2)v^2 + 2an^2v^3 + 4bn(n+2)vv'' - 4b(n^2-4)(v')^2 = 0. \tag{5.4}$$

By the same manipulation and by using (4.5)–(4.8), solving the system of algebraic equations gives the following sets of nontrivial solutions:
If $\lambda = 0$, then we obtain sets of non-trivial solutions:

$$a_0 = a_2\mu, \qquad a_1 = 0, \qquad a_2 = -\frac{4b(n^2+6n+8)}{an^2}, \qquad c = -\frac{32b\mu}{n^2}, \tag{5.5}$$

or

$$a_0 = 0, \qquad a_1 = 0, \qquad a_2 = -\frac{8b(n+2)}{an}, \qquad c = \frac{16b\mu}{n}, \tag{5.6}$$

or

$$a_0 = \frac{1}{3}a_2\mu, \qquad a_1 = 0, \qquad a_2 = -\frac{12b(n+2)}{an}, \qquad c = \frac{16b\mu}{n}. \tag{5.7}$$

By using (5.5)–(5.7), expression Eq. (4.6) can be written as

$$v(\xi) = -\frac{4b(n^2+6n+8)}{an^2}\mu - \frac{4b(n^2+6n+8)}{an^2}\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x+y+\frac{32b\mu}{n^2}t, \tag{5.8}$$

$$v(\xi) = -\frac{8b(n+2)}{an}\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x+y-\frac{16b\mu}{n}t, \tag{5.9}$$

$$v(\xi) = -\frac{4b(n+2)}{an}\mu - \frac{12b(n+2)}{an}\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \qquad \xi = x+y-\frac{16b\mu}{n}t. \tag{5.10}$$

When $\mu < 0$, we get

$$v_1(\xi) = \frac{4b(n^2+6n+8)\mu}{an^2}\left[\left(\frac{C_1 \sinh\sqrt{-\mu}\xi + C_2 \cosh\sqrt{-\mu}\xi}{C_1 \cosh\sqrt{-\mu}\xi + C_2 \sinh\sqrt{-\mu}\xi}\right)^2 - 1\right], \quad \xi = x + y + \frac{32b\mu}{n^2}t, \tag{5.11}$$

$$v_2(\xi) = \frac{8b(n+2)\mu}{an}\left(\frac{C_1 \sinh\sqrt{-\mu}\xi + C_2 \cosh\sqrt{-\mu}\xi}{C_1 \cosh\sqrt{-\mu}\xi + C_2 \sinh\sqrt{-\mu}\xi}\right)^2, \qquad \xi = x + y - \frac{16b\mu}{n}t, \tag{5.12}$$

$$v_3(\xi) = \frac{4b(n+2)\mu}{an}\left[3\left(\frac{C_1 \sinh\sqrt{-\mu}\xi + C_2 \cosh\sqrt{-\mu}\xi}{C_1 \cosh\sqrt{-\mu}\xi + C_2 \sinh\sqrt{-\mu}\xi}\right)^2 - 1\right], \quad \xi = x + y - \frac{16b\mu}{n}t. \tag{5.13}$$

When $\mu > 0$, we have

$$v_4(\xi) = -\frac{4b(n^2+6n+8)}{an^2}\mu\left[1 + \left(\frac{C_1 \sin\sqrt{\mu}\xi + C_2 \cos\sqrt{\mu}\xi}{C_1 \cos\sqrt{\mu}\xi + C_2 \sin\sqrt{\mu}\xi}\right)^2\right], \quad \xi = x + y + \frac{32b\mu}{n^2}t, \tag{5.14}$$

$$v_5(\xi) = -\frac{8b(n+2)}{an}\mu\left(\frac{C_1 \sin\sqrt{\mu}\xi + C_2 \cos\sqrt{\mu}\xi}{C_1 \cos\sqrt{\mu}\xi + C_2 \sin\sqrt{\mu}\xi}\right)^2, \qquad \xi = x + y - \frac{16b\mu}{n}t, \tag{5.15}$$

$$v_6(\xi) = -\frac{4b(n+2)\mu}{an}\left[1 + 3\left(\frac{C_1 \sin\sqrt{\mu}\xi + C_2 \cos\sqrt{\mu}\xi}{C_1 \cos\sqrt{\mu}\xi + C_2 \sin\sqrt{\mu}\xi}\right)^2\right], \qquad \xi = x + y - \frac{16b\mu}{n}t. \tag{5.16}$$

If $C_1 \neq 0, C_2 = 0, \mu < 0$, then solution (5.11)–(5.13), respectively, become:

$$v_1(x,y,t) = -\frac{4b(n^2+6n+8)\mu}{an^2} sech^2\left(\sqrt{-\mu}\left[x + y + \frac{32b\mu}{n^2}t\right]\right), \tag{5.17}$$

$$v_2(x,y,t) = \frac{8b(n+2)\mu}{an}\tanh^2\left(\sqrt{-\mu}\left[x + y - \frac{16b\mu}{n}t\right]\right), \tag{5.18}$$

$$v_3(x,y,t) = \frac{4b(n+2)\mu}{an}\left\{3\tanh^2\left(\sqrt{-\mu}\left[x + y - \frac{16b\mu}{n}t\right]\right) - 1\right\}. \tag{5.19}$$

However, if $C_1 \neq 0, C_2 = 0, \mu > 0$, then solution (5.14)–(5.16), respectively, give:

$$v_4(x,y,t) = \frac{4b(n^2+6n+8)\mu}{an^2} sec^2\left(\sqrt{\mu}\left[x + y + \frac{32b\mu}{n^2}t\right]\right), \tag{5.20}$$

$$v_5(x,y,t) = \frac{8b(n+2)\mu}{an}\tan^2\left(\sqrt{\mu}\left[x + y - \frac{16b\mu}{n}t\right]\right), \tag{5.21}$$

$$v_6(x,y,t) = -\frac{4b(n+2)\mu}{an}\left\{3\tan^2\left(\sqrt{\mu}\left[x + y - \frac{16b\mu}{n}t\right]\right) + 1\right\}. \tag{5.22}$$

Then by using Eq. (5.3) we obtain

Case 1: $\mu < 0$.

$$u_1(\xi) = \sqrt[n]{\frac{16b^2(n^2+6n+8)^2\mu^2}{a^2n^4}}\left[\left(\frac{C_1 \sinh\sqrt{-\mu}\xi + C_2 \cosh\sqrt{-\mu}\xi}{C_1 \cosh\sqrt{-\mu}\xi + C_2 \sinh\sqrt{-\mu}\xi}\right)^2 - 1\right]^{\frac{2}{n}}, \tag{5.23}$$

where $\xi = x + y + \frac{32b\mu}{n^2}t$,

$$u_2(\xi) = \sqrt[n]{\frac{64b^2(n+2)^2\mu^2}{a^2n^2}} \left(\frac{C_1 \sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi}\right)^{\frac{4}{n}}, \tag{5.24}$$

where $\xi = x + y - \frac{16b\mu}{n}t$ and

$$u_3(\xi) = \sqrt[n]{\frac{16b^2(n+2)^2\mu^2}{a^2n^2}} \left[3\left(\frac{C_1 \sinh\sqrt{-\mu}\xi + C_2\cosh\sqrt{-\mu}\xi}{C_1\cosh\sqrt{-\mu}\xi + C_2\sinh\sqrt{-\mu}\xi}\right)^2 - 1\right]^{\frac{2}{n}}, \tag{5.25}$$

where $\xi = x + y - \frac{16b\mu}{n}t$.

Case 2: $\mu > 0$.

$$u_4(\xi) = \sqrt[n]{\frac{16b^2(n^2+6n+8)^2\mu^2}{a^2n^4}} \left[1 + \left(\frac{C_1 \sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^2\right]^{\frac{2}{n}}, \tag{5.26}$$

where $\xi = x + y + \frac{32b\mu}{n^2}t$ and

$$u_5(\xi) = \sqrt[n]{\frac{64b^2(n+2)^2\mu^2}{a^2n^2}} \left(\frac{C_1 \sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^{\frac{4}{n}}, \qquad \xi = x + y - \frac{16b\mu}{n}t, \tag{5.27}$$

$$u_6(\xi) = \sqrt[n]{\frac{16b^2(n+2)^2\mu^2}{a^2n^2}} \left[1 + 3\left(\frac{C_1 \sin\sqrt{\mu}\xi + C_2\cos\sqrt{\mu}\xi}{C_1\cos\sqrt{\mu}\xi + C_2\sin\sqrt{\mu}\xi}\right)^2\right]^{\frac{2}{n}}, \tag{5.28}$$

where $\xi = x + y - \frac{16b\mu}{n}t$. If $C_1 \neq 0, C_2 = 0, \mu < 0, \mu > 0$, then solution (5.23)–(5.28), respectively, give:

$$u_1(x,y,t) = \sqrt[n]{\frac{16b^2(n^2+6n+8)^2\mu^2}{a^2n^4}} sech^{\frac{4}{n}}\left(\sqrt{-\mu}\left[x + y + \frac{32b\mu}{n^2}t\right]\right), \tag{5.29}$$

$$u_2(x,y,t) = \sqrt[n]{\frac{64b^2(n+2)^2\mu^2}{a^2n^2}} \tanh^{\frac{4}{n}}\left(\sqrt{-\mu}\left[x + y - \frac{16b\mu}{n}t\right]\right), \tag{5.30}$$

$$u_3(x,y,t) = \sqrt[n]{\frac{16b^2(n+2)^2\mu^2}{a^2n^2}} \left\{3\tanh^2\left(\sqrt{-\mu}\left[x + y - \frac{16b\mu}{n^2}t\right]\right) - 1\right\}^{\frac{2}{n}}, \tag{5.31}$$

$$u_4(x,y,t) = \sqrt[n]{\frac{16b^2(n^2+6n+8)^2\mu^2}{a^2n^4}} sec^{\frac{4}{n}}\left(\sqrt{\mu}\left[x + y + \frac{32b\mu}{n^2}t\right]\right), \tag{5.32}$$

$$u_5(x,y,t) = \sqrt[n]{\frac{64b^2(n+2)^2\mu^2}{a^2n^2}} \tan^{\frac{4}{n}}\left(\sqrt{\mu}\left[x + y - \frac{16b\mu}{n^2}t\right]\right), \tag{5.33}$$

$$u_6(x,y,t) = \sqrt[n]{\frac{16b^2(n+2)^2\mu^2}{a^2n^2}} \left\{1 + 3\tan^2\left(\sqrt{\mu}\left[x + y - \frac{16b\mu}{n^2}t\right]\right)\right\}^{\frac{2}{n}}. \tag{5.34}$$

The results obtained in the above are exact solutions of the mgZK equation.

6. Conclusion

In this article, we investigated the Zakharov-Kuznetsov (ZK) equation, the generalized Zakharov-Kuznetsov (gZK) equation and the generalized form of modified Zakharov-Kuznetsov (gmZK) equation. The generalized $(\frac{G'}{G})$-expansion method is a useful method for finding traveling wave solutions of nonlinear evolution equations. This method has been successfully applied to obtain some new ZK equation, the gZK equation, and the mgZK equation. These exact solutions include three types hyperbolic function solution, trigonometric function solution, and rational solution. The generalized $(\frac{G'}{G})$-expansion method is more powerful in searching for exact solutions of NLPDEs. This method has its own advantages: direct, concise, and elementary that the general solutions of the second order LODE have been well known for the researchers and effective that it can be used for many other nonlinear evolution equations. Also, package Maple has been successfully applied to obtain some new generalized solitonary solutions to the ZK equation, the gZK equation, and the mgZK equation. It can be concluded that this method is a very powerful and efficient technique in finding exact solutions for wide classes of problems.

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