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# Application of $tan(\phi/2)$-expansion method for solving the fractional Biswas-Milovic equation for Kerr law nonlinearity

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**Abstract**

In this paper, the improved tan $(\Phi(\xi)/2)$-expansion method (ITEM) is proposed to obtaining the fractional Biswas-Milovic equation. The exact particular solutions containing four types hyperbolic function solution, trigonometric function solution, exponential solution and rational solution. We obtained the further solutions comparing with other methods as [2]. Recently this method is developed for searching exact travelling wave solutions of nonlinear partial differential equations. These solutions might play important role in nonlinear optic and physics fields. It is shown that this method, with the help of symbolic computation, provide a straightforward and powerful mathematical tool for solving problems in nonlinear optic.

**Keywords.** Improved tan $(\Phi(\xi)/2)$-expansion method, Factional Biswas-Milovic equation, Exact soliton solution.
**2010 Mathematics Subject Classification.** 65L05, 34K06, 34K28.

## 1. Introduction

In the recent decade, fractional nonlinear differential equations have been demonstrated applications in numerous seemingly different fields of engineering sciences, physics, finance, applied mathematics and others [9, 11, 19, 42, 44]. Different researchers worked in nonlinear fractional equations. In this paper, we consider the fractional Biswas-Milovic equation [2, 7] as follows

$$iD_t^{\alpha}q^n + \lambda D_x^{2\beta}q^n + \mu F(|q|^2)q^n = 0, \tag{1.1}$$

where $\lambda\mu > 0$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $n \geq 1$ and $q = q(x,t)$ is a complex valued function. The coefficients $\lambda$ and $\mu$ represent the coefficients of group velocity dispersion and nonlinearity, respectively. In Eq. (1.1), $F$ is a real-valued algebraic function and in order to satisfy the necessary condition of having smoothness of the complex function $F(|q|^2)q$, the function $F(|q|^2)q$ is considered to be $k$ times continuously differentiable [6, 27]. A real understanding the dynamics of optical solitons with a generalized flavor considered by using the BM equation. Also, the governing equation is of special interest in the nonlinear fiber optics community [27]. For further information in about the dynamics of solitons in optical fibers refer to ([5]–[45]). To solve the BM equation with variable physical properties has been proposed different methods by authors ([3, 6, 17, 18, 21, 43]). The nonlinear partial differential equations play a key role in describing key scientific phenomena. In fact, it has been discovered that many models in mathematics and physics are described by nonlinear partial differential equations. With the rapid development of nonlinear sciences based on computer algebraic system, many effective methods have been presented, such as, the homotopy analysis method [13], the variational iteration method [15], the Adomian decomposition method [20], the homotopy perturbation method [12], the tanh-coth method [28], the Exp-function method [14, 23, 29], the $G'/G$-expansion method [32, 33], the homogeneous balance method [49], the formal linearization method [40] and so on. In this paper we have two goals. First, we introduce a general form of the ITEM [1, 24, 25, 30, 31, 34, 35, 46, 48], which is a new method. Next, we obtain

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the exact solutions of the Biswas-Milovic equation for one type of nonlinearity by the aforementioned method. Authors of [10] explained the generalized fifth-order KdV like equation with prime number $p = 3$ via a generalized bilinear differential operator. N-lump was invstigated to the variable-coefficient CaudreyDoddGibbonKoteraSawada equation [36]. Applications of $tan(\phi/2)$-expansion method for the BiswasMilovic equation [37], the GerdjikovIvanov model [38], the KunduEckhaus equation [39] and the fifth-order integrable equations [22] were studied. Lump solutions were analyzed to the fractional generalized CBS-BK equation [47] and the (3+1)-D Burger system [16]. The approximations of one-dimensional hyperbolic equation with non-local integral conditions were constructed by reduced differential transform method [41]. The generalized Hirota bilinear strategy by the number prime was used to the (2+1)-dimensional generalized fifth-order KdV like equation [26]. The traveling wave solutions and analytical treatment of the simplified MCH equation and the combined KdVmKdV equations were studied [4].
The outline of this paper is organized as follows:
In section 2, we describe the ITEM. In section 3, we apply mathematical analysis of Biswas-Milovic equation. In section 4 will be further analyzed Kerr law nonlinearity. Also conclusion is given in section 5.

## 2. Description of the ITEM

The ITEM is well-known analytical method which was improved and developed by Manafian.
**Step 1**. We suppose that given nonlinear partial differential equation for u(x, t) to be in the form

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{2.1}$$

which can be converted to an ODE

$$\mathcal{Q}(u, u', -\mu u', u'', \mu^2 u'', ...) = 0, \tag{2.2}$$

by the transformation $\xi = x - \mu t$ is the wave variable. Also, $\mu$ is constant to be determined later.
**Step 2**. Suppose the traveling wave solution of Eq. (2.2) can be expressed as follows:

$$u(\xi) = S(\phi) = \sum_{k=-m}^{m} A_k \left[p + \tan(\phi/2)\right]^k, \tag{2.3}$$

where $A_k(0 \le k \le m)$ and $A_{-k} = B_k(1 \le k \le m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$ and $\phi = \phi(\xi)$ satisfies the following ordinary differential equation:

$$\phi'(\xi) = a\sin(\phi(\xi)) + b\cos(\phi(\xi)) + c. \tag{2.4}$$

We will consider the following special solutions of Equation (2.4):
**Family 1**: **When** $\Delta = a^2 + b^2 - c^2 < 0$ **and** $b - c \neq 0$**,** then
$\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]$.
**Family 2**: **When** $\Delta = a^2 + b^2 - c^2 > 0$ **and** $b - c \neq 0$**,** then
$\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right]$.
**Family 3**: **When** $\Delta = a^2 + b^2 - c^2 > 0$**,** $b \neq 0$ **and** $c = 0$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b} + \frac{\sqrt{b^2+a^2}}{b}\tanh\left(\frac{\sqrt{b^2+a^2}}{2}\bar{\xi}\right)\right]$.
**Family 4**: **When** $\Delta = a^2 + b^2 - c^2 < 0$**,** $c \neq 0$ **and** $b = 0$, then $\phi(\xi) = 2\tan^{-1}\left[-\frac{a}{c} + \frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}\bar{\xi}\right)\right]$.
**Family 5**: **When** $\Delta = a^2 + b^2 - c^2 > 0$**,** $b - c \neq 0$ **and** $a = 0$, then $\phi(\xi) = 2\tan^{-1}\left[\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\bar{\xi}\right)\right]$.
**Family 6**: **When** $a = 0$ **and** c = 0, then $\phi(\xi) = \tan^{-1}\left[\frac{e^{2b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}}+1}\right]$.
**Family 7**: **When** $b = 0$ **and** $c = 0$, then $\phi(\xi) = \tan^{-1}\left[\frac{2e^{a\bar{\xi}}}{e^{2a\bar{\xi}}+1}, \frac{e^{2a\bar{\xi}}-1}{e^{2a\bar{\xi}}+1}\right]$.
**Family 8**: **When** $a^2 + b^2 = c^2$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{a\bar{\xi}+2}{(b-c)\bar{\xi}}\right]$.
**Family 9**: **When** $a = b = c = ka$, then $\phi(\xi) = 2\tan^{-1}\left[e^{ka\bar{\xi}} - 1\right]$.
**Family 10**: **When** $a = c = ka$ **and** b = −ka, then $\phi(\xi) = -2\tan^{-1}\left[\frac{e^{ka\bar{\xi}}}{-1+e^{ka\bar{\xi}}}\right]$.

**Family 11**: **When** $c = a$**,** then $\phi(\xi) = -2\tan^{-1}\left[\frac{(a+b)e^{b\bar{\xi}}-1}{(a-b)e^{b\bar{\xi}}-1}\right]$.
**Family 12**: **When** $a = c$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{(b+c)e^{b\bar{\xi}}+1}{(b-c)e^{b\bar{\xi}}-1}\right]$.
**Family 13**: **When** $c = -a$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{e^{b\bar{\xi}}+b-a}{e^{b\bar{\xi}}-b-a}\right]$.
**Family 14**: **When** $b = -c$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{ae^{a\bar{\xi}}}{1-ce^{a\bar{\xi}}}\right]$.
**Family 15**: **When** $b = 0$ **and** a = c, then $\phi(\xi) = -2\tan^{-1}\left[\frac{c\bar{\xi}+2}{c\bar{\xi}}\right]$.
**Family 16**: **When** $a = 0$ **and** $b = c$**,** then $\phi(\xi) = 2\tan^{-1}\left[c\bar{\xi}\right]$.
**Family 17**: **When** $a = 0$ **and** $b = -c$**,** then $\phi(\xi) = -2\tan^{-1}\left[\frac{1}{c\bar{\xi}}\right]$.
**Family 18**: **When** $a = 0$ **and** $b = 0$**,** then $\phi(\xi) = c\xi + C$.
**Family 19**: **When** $b = c$ then $\phi(\xi) = 2\tan^{-1}\left[\frac{e^{a\bar{\xi}}-c}{a}\right]$, where $\bar{\xi} = \xi + C, p, A_0, A_k, B_k (k = 1, 2, ..., m), a, b$ and $c$ are constants to be determined later.
**Step 3**. Determine $m$. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest-order nonlinear term(s) in Eq. (2.2). But, the positive integer $m$ can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (2.2).
**Step 4**. Substituting (2.3) into Eq. (2.2) with the value of $m$ obtained in Step 2. Collecting the coefficients of $\tan(\phi/2)^k$, $\cot(\phi/2)^k (k = 0, 1, 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined equations for $A_0, A_k, B_k (k = 1, 2, ..., m)$ $a, b$, $c$ and $p$ with the aid of symbolic computation Maple.
**Step 5**. Solving the algebraic equations in Step 3, then substituting $A_0, A_1, B_1, ..., A_m, B_m, \mu, p$ in (2.3).

## 3. Mathematical analysis of the fractional BM equation

In this section we consider the dimensionless form of the fractional BM equation to be studied in this paper is given in [**?** ]

$$iD_t^{\alpha} q^n + \lambda D_x^{2\beta} q^n + \mu F(|q|^2) q^n = 0, \tag{3.1}$$

where $\lambda\mu > 0$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $n \geq 1$ and $q = q(x, t)$ is a complex valued function. The coefficients $\lambda$ and $\mu$ represent the coefficients of group velocity dispersion and nonlinearity, respectively. Thus if $n = 1$, Eq. (3.1) collapses to NLSE that arises in nonlinear optics, fluid dynamics, plasma physics, mathematical biology and several other areas. In this paper we search for the stationary solution to (3.1). The starting hypothesis is taken to be

$$q(x,t) = u(\xi)\exp(i\theta), \qquad \xi = x - \frac{\eta t^{\alpha}}{\Gamma(1+\alpha)}, \qquad \theta = \frac{st^{\beta}}{\Gamma(1+\beta)} + \frac{rt^{\alpha}}{\Gamma(1+\alpha)}, \tag{3.2}$$

where, $\alpha$ represents the soliton wave number, $\beta$ is the soliton frequency and $\gamma$ is the phase constant. Thus, from (3.1)

$$D_t^{\alpha} q^n = nu^{n-1}(-\eta u' + iru)e^{in\theta}, \tag{3.3}$$
$$D_x^{\beta} q^n = nu^{n-1}(u' + isu)e^{in\theta}, \tag{3.4}$$
$$D_x^{2\beta} q^n = nu^{n-2}[(n-1)(u' + isu)^2 + u(u'' + 2isu' - s^2 u)]e^{in\theta}. \tag{3.5}$$

Inserting (3.3) to (3.5) separating into real and imaginary parts, the results are
Real part:

$$-n(r + \lambda ns^2)u^n + \lambda n(n-1)u^{n-2}(u')^2 + \lambda nu^{n-1}u'' + \mu F(u^2)u^n = 0. \tag{3.6}$$

Imaginary part:

$$iu'u^{n-1}[-\eta n + 2\lambda n^2 s] = 0, \qquad \Rightarrow \eta = 2\lambda ns. \tag{3.7}$$

## 4. Kerr law nonlinearity

This section will be further analyzed the Kerr law nonlinearity via $tan(\phi/2)$-expansion method.

4.1. **The Case n=1 case for the fractional BME.** We start our study by assuming $\mathbf{n = 1}$ in (3.6), therefore we have

$$-(\mathrm{r} + \lambda \mathrm{s}^2)\mathrm{u}(\xi) + \lambda \mathrm{u}''(\xi) + \mu \mathrm{F}(\mathrm{u}^2(\xi))\mathrm{u}(\xi) = 0. \tag{4.1}$$

In the presence of perturbation terms, the fractional BME with Kerr law nonlinearity $(\mathrm{F(w) = w})$ is given by

$$-(\mathrm{r} + \lambda \mathrm{s}^2)\mathrm{u}(\xi) + \lambda \mathrm{u}''(\xi) + \mu \mathrm{u}^3(\xi) = 0. \tag{4.2}$$

The next step is to expand the unknowns $u(\xi)$ in power series in terms of $p + \tan(\phi/2)$,

$$u(\xi) = \sum_{k=-m}^{m} A_k \left[p + \tan(\phi(\xi)/2)\right]^k, \tag{4.3}$$

which $A_{-k} = B_k$. In order to determine value of $m$, we balance the linear term of the highest order $u''$ with the highest order nonlinear term $u^3$ in Eq. (4.2) we get

$$u(\xi) = A_m \left(\tan(\phi(\xi)/2)\right)^m + ..., \tag{4.4}$$

$$u^3(\xi) = A_m^3 \left(\tan(\phi(\xi)/2)\right)^{3m} + ..., \tag{4.5}$$

$$\frac{du(\xi)}{d\xi} = \frac{m(c-b)}{2} A_m \left(\tan(\phi(\xi)/2)\right)^{m+1} + ..., \tag{4.6}$$

$$\frac{d^2u(\xi)}{d\xi^2} = \frac{m(m+1)(c-b)^2}{2} A_m \left(\tan(\phi(\xi)/2)\right)^{m+2} + .... \tag{4.7}$$

By considering the homogeneous balance principle between the highest order derivatives $u''$ and nonlinear terms $u^3$, we obtain $m + 2 = 3m$, then $m = 1$. Suppose that the solutions for Eq. (4.2) can be expressed in the following form

$$u(\xi) = \sum_{k=-1}^{1} A_k (p + \tan(\phi/2))^k, \tag{4.8}$$

Substituting (4.8) and (2.4) into Eq. (4.2) and collecting all terms with the same order of $tan\,(\Phi(\xi)/2)$ together, and setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for $a, b, c, k, w, A_0, A_1$, and $B_1$ as follows:
Coefficients of $Y = \tan(\phi(\xi)/2)$

$$\begin{aligned}
Y^0 \;:\;\; & \lambda(b+c)(B_1 b - B_1 pa + A_1 p^3 a + B_1 c) + \\
& 2(B_1 + A_1 p^2 + pA_0)(\mu A_1^2 p^4 + 2\mu A_0 p^3 A_1 + 2p^2 \mu B_1 A_1 - p^2 \lambda s^2 - p^2 r + \mu A_0^2 p^2 + 2\mu A_0 B_1 p + \mu B_1^2) = 0, \\
Y^1 \;:\;\; & \lambda[p(2a^2 + c^2 - b^2)(A_1 p^2 - B_1) + 3a(A_1 p^2 + B_1)(b+c)] + 4p(B_1 + 2A_1 p^2)(3\mu B_1 A_1 - \lambda s^2 + 3\mu A_0^2 - r) \\
& + 12\mu A_1^3 p^5 + 30\mu A_0 A_1^2 p^4 + 6A_0 p^2(\mu A_0^2 - r + 6\mu B_1 A_1 - \lambda s^2) + 6\mu A_0 B_1^2 = 0, \\
Y^2 \;:\;\; & -3ap\lambda(p^2 A_1 b - p^2 A_1 c - A_1 b + B_1 c - B_1 b - A_1 c) + \lambda(-b^2 + c^2 + 2a^2)(B_1 + 3A_1 p^2) + 30\mu A_1^3 p^4 \\
& + 6A_0 p(10p^2 \mu A_1^2 + \mu A_0^2 - r + 6\mu B_1 A_1 - \lambda s^2) + 2(6A_1 p^2 + B_1)(3\mu B_1 A_1 + 3\mu A_0^2 - \lambda s^2 - r) = 0, \\
Y^3 \;:\;\; & p\lambda[A_1 c^2(p^2+3) + A_1 b^2(p^2-3) - 2p^2 A_1 bc - B_1 b^2 + 2B_1 bc - B_1 c^2 + 6A_1 a^2] - \\
& a\lambda(9A_1 bp^2 - 9A_1 cp^2 - A_1 b - A_1 c - B_1 c + B_1 b) + \\
& 8A_1 p(5\mu A_1^2 p^2 + 3\mu B_1 A_1 + 3\mu A_0^2 + \lambda s^2 - r) + 2A_0(\mu A_0^2 + 30\mu A_1^2 p^2 + 6\mu B_1 A_1 - r - \lambda s^2) = 0, \\
Y^4 \;:\;\; & 3p^2 \lambda A_1 (b-c)^2 + \lambda A_1(-9abp + 9acp + 2a^2 - b^2 + c^2) + 30\mu A_1^3 p^2 + \\
& 2A_1(3\mu B_1 A_1 + 15A_1 \mu A_0 p + 3\mu A_0^2 - r - \lambda s^2) = 0, \\
Y^5 \;:\;\; & -3\lambda A_1(b-c)(-pb + pc + a) + 6\mu A_1^2(2pA_1 + A_0) = 0, \\
Y^6 \;:\;\; & A_1(2\mu A_1^2 + \lambda c^2 + \lambda b^2 - 2\lambda cb) = 0.
\end{aligned} \tag{4.9}$$

Solving the set of algebraic equations using Maple, we get the following results:
**Case I** will be as:

$$a=a,\quad b=b,\quad c=c,\quad p=p,\quad \Delta=a^2+b^2-c^2,\quad A_0=(a+p(b-c))\sqrt{\frac{-\lambda}{2\mu}},\quad A_1=(b-c)\sqrt{\frac{-\lambda}{2\mu}}, \tag{4.10}$$

$$B_1=0,\quad s=s,\quad r=-\frac{\lambda}{2}(2s^2+\Delta),\quad u(\xi)=A_0+A_1\left[p+\tan(\phi(\xi)/2)\right].$$

By using of (4.10) and Families 1, 2, 6, 8, 12 and 15, respectively, can be written as

$$u_1(\xi)=(a+2p(b-c))\sqrt{\frac{-\lambda}{2\mu}}+\sqrt{\frac{-\lambda}{2\mu}}\left[a-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right], \tag{4.11}$$

$$u_2(\xi)=(a+2p(b-c))\sqrt{\frac{-\lambda}{2\mu}}+\sqrt{\frac{-\lambda}{2\mu}}\left[a+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right],$$

$$u_3(\xi)=2pb\sqrt{\frac{-\lambda}{2\mu}}+b\sqrt{\frac{-\lambda}{2\mu}}\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1},\ \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right),$$

$$u_4(\xi)=(a+2p(b-c))\sqrt{\frac{-\lambda}{2\mu}}+\sqrt{\frac{-\lambda}{2\mu}}\left[\frac{a(\xi+C)+2}{(\xi+C)}\right],$$

$$u_5(\xi)=(c+2p(b-c))\sqrt{\frac{-\lambda}{2\mu}}+(b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right],$$

$$u_6(\xi)=(c-2pc)\sqrt{\frac{-\lambda}{2\mu}}-\sqrt{\frac{-\lambda}{2\mu}}\left[\frac{c(\xi+C)+2}{(\xi+C)}\right],$$

where $q(x,t)=u(\xi)e^{i\theta}$, $\xi=x-\frac{2\lambda st^{\alpha}}{\Gamma(1+\alpha)}$ and $\theta=\frac{st^{\beta}}{\Gamma(1+\beta)}-\frac{\lambda(2s^2+\Delta)t^{\alpha}}{2\Gamma(1+\alpha)}$.
**Case II** will be as:

$$a=0,\quad b=b,\quad c=c,\quad p=0,\quad A_0=0,\quad A_1=(b-c)\sqrt{\frac{-\lambda}{2\mu}}, \tag{4.12}$$

$$B_1=0,\quad s=s,\quad r=-\frac{\lambda}{2}(2s^2+b^2-c^2),\quad u(\xi)=A_1\left[p+\tan(\phi(\xi)/2)\right].$$

By using of (4.12) and Families 5, 6, 11 and 17, respectively, can be written as

$$u_7(\xi)=\sqrt{\frac{-\lambda(b^2-c^2)}{2\mu}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right), \tag{4.13}$$

$$u_8(\xi)=b\sqrt{\frac{-\lambda}{2\mu}}\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1},\ \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right),$$

$$u_9(\xi)=b\sqrt{\frac{-\lambda}{2\mu}}\left[\frac{be^{b(\xi+C)}-1}{be^{b(\xi+C)}+1}\right],\qquad u_{10}(\xi)=2\sqrt{\frac{-\lambda}{2\mu}}\frac{1}{\xi+C},$$

where $q(x,t)=u(\xi)e^{i\theta}$, $\xi=x-\frac{2\lambda st^{\alpha}}{\Gamma(1+\alpha)}$ and $\theta=\frac{st^{\beta}}{\Gamma(1+\beta)}-\frac{\lambda(2s^2+b^2-c^2)t^{\alpha}}{2\Gamma(1+\alpha)}$.
**Case III** will be as:

$$a=a,\quad b=b,\quad c=c,\quad p=p,\quad \Delta=a^2+b^2-c^2,\quad A_0=(a+p(b-c))\sqrt{\frac{-\lambda}{2\mu}},\quad A_1=0, \tag{4.14}$$

$$B_1 = -(p^2(b-c)+2ap-b-c)\sqrt{\frac{-\lambda}{2\mu}}, \quad s=s, \quad r=-\frac{\lambda}{2}(2s^2+\Delta), \quad u(\xi)=A_0+B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.14) and Families 1, 2, 6, 8, 12 and 15, respectively, can be written as

$$u_{11}(\xi) = (a+p(b-c))\sqrt{\frac{-\lambda}{2\mu}} - (p^2(b-c)+2ap-b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right]^{-1}, \tag{4.15}$$

$$u_{12}(\xi) = (a+p(b-c))\sqrt{\frac{-\lambda}{2\mu}} - (p^2(b-c)+2ap-b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right]^{-1},$$

$$u_{13}(\xi) = pb\sqrt{\frac{-\lambda}{2\mu}} - b(p^2-1)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}, \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right)\right]^{-1},$$

$$u_{14}(\xi) = (a+p(b-c))\sqrt{\frac{-\lambda}{2\mu}} - (p^2(b-c)+2ap-b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{a(\xi+C)+2}{(b-c)(\xi+C)}\right]^{-1},$$

$$u_{15}(\xi) = (c+p(b-c))\sqrt{\frac{-\lambda}{2\mu}} - (p^2(b-c)+2cp-b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},$$

$$u_{16}(\xi) = c(1-p)\sqrt{\frac{-\lambda}{2\mu}} + c(p-1)^2\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{c(\xi+C)+2}{c(\xi+C)}\right]^{-1},$$

where $q(x,t)=u(\xi)e^{i\theta}$, $\xi = x-\frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta=\frac{st^\beta}{\Gamma(1+\beta)}-\frac{\lambda(2s^2+\Delta)t^\alpha}{2\Gamma(1+\alpha)}$.

**Case IV** will be as:

$$a=a, \quad b=c, \quad c=c, \quad p=p, \quad A_0=a\sqrt{\frac{-\lambda}{2\mu}}, \quad A_1=0, \tag{4.16}$$

$$B_1=(ap-c)\sqrt{\frac{-2\lambda}{\mu}}, \quad s=s, \quad r=-\frac{\lambda}{2}(2s^2+a^2), \quad u(\xi)=A_0+B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.16) and Families 7, 13 and 16, respectively, can be written as

$$u_{17}(\xi) = a\sqrt{\frac{-\lambda}{2\mu}} + (ap-c)\sqrt{\frac{-2\lambda}{\mu}}\left[p+\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)}+1}, \frac{e^{2a(\xi+C)}-1}{e^{2a(\xi+C)}+1}\right]\right)\right]^{-1}, \tag{4.17}$$

$$u_{18}(\xi) = -c\sqrt{\frac{-\lambda}{2\mu}} - c(p+1)\sqrt{\frac{-2\lambda}{\mu}}\left[\frac{e^{b(\xi+C)}+2c}{e^{b(\xi+C)}}\right]^{-1}, \qquad u_{19}(\xi) = -c\sqrt{\frac{-2\lambda}{\mu}}\frac{1}{p+c(\xi+C)},$$

where $q(x,t)=u(\xi)e^{i\theta}$, $\xi = x-\frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta=\frac{st^\beta}{\Gamma(1+\beta)}-\frac{\lambda(2s^2+a^2)t^\alpha}{2\Gamma(1+\alpha)}$.

**Case V** will be as:

$$a=a, \quad b=b, \quad c=c, \quad p=-\frac{a}{b-c}, \quad A_0=0, \quad A_1=0, \quad B_1=[p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}, \tag{4.18}$$

$$\Delta=a^2+b^2-c^2, \quad s=s, \quad r=-\frac{\lambda}{2}(2s^2+\Delta), \quad u(\xi)=B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.14) and Families 1, 2, 6, 8, 12 and **15**, respectively, can be written as

$$u_{19}(\xi) = -[p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}\frac{b-c}{\sqrt{-\Delta}}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right), \tag{4.19}$$

$$u_{20}(\xi) = [p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}\frac{b-c}{\sqrt{\Delta}}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right),$$

$$u_{21}(\xi) = b\sqrt{\frac{-\lambda}{2\mu}}\cot\left(\frac{1}{2}\arctan\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}, \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right),$$

$$u_{22}(\xi) = [p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}\left[-\frac{a}{b-c}+\frac{a(\xi+C)+2}{(b-c)(\xi+C)}\right]^{-1},$$

$$u_{23}(\xi) = [p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}\left[-\frac{c}{b-c}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},$$

$$u_{24}(\xi) = [-p^2c+c]\sqrt{\frac{-\lambda}{2\mu}}\left[-1+\frac{c(\xi+C)+2}{c(\xi+C)}\right]^{-1},$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{\lambda(2s^2+\Delta)t^\alpha}{2\Gamma(1+\alpha)}$.

**Case VI** will be as:

$$a = 0, \quad b = \sqrt{-\frac{\mu}{2\lambda}}B_1, \quad c = \sqrt{-\frac{\mu}{2\lambda}}B_1, \quad p = p, \quad A_0 = 0, \quad A_1 = 0, \tag{4.20}$$

$$B_1 = B_1, \quad s = s, \quad r = -\lambda s^2, \quad u(\xi) = B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.20) and Family 16, we can write

$$u_{25}(\xi) = \frac{B_1}{c(\xi+C)}, \tag{4.21}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{\lambda s^2 t^\alpha}{\Gamma(1+\alpha)}$.

**Case VII** will be as:

$$a = 0, \quad b = b, \quad c = c, \quad p = 0, \quad A_0 = 0, \quad A_1 = (b-c)\sqrt{\frac{-\lambda}{2\mu}}, \quad B_1 = (b+c)\sqrt{\frac{-\lambda}{2\mu}}, \tag{4.22}$$

$$s = s, \quad r = -\frac{\lambda}{2}(2s^2+4(b^2-c^2)), \quad u(\xi) = A_1\left[p+\tan(\phi(\xi)/2)\right] + B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.22) and Families 5, 6, and **11**, respectively, can be written as

$$u_{26}(\xi) = \sqrt{\frac{-\lambda}{2\mu}}\sqrt{b^2-c^2}\left[\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)+\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right], \tag{4.23}$$

$$u_{27}(\xi) = b\sqrt{\frac{-\lambda}{2\mu}}\left[\tan\left(\frac{1}{2}\tan^{-1}\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}, \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right)+\cot\left(\frac{1}{2}\tan^{-1}\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}, \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right)\right],$$

$$u_{28}(\xi) = 2b\sqrt{\frac{-\lambda}{2\mu}}\left[\frac{b^2e^{2b(\xi+C)}+1}{b^2e^{2b(\xi+C)}-1}\right],$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{\lambda(2s^2+4(b^2-c^2))t^\alpha}{2\Gamma(1+\alpha)}$.

**Case VIII** will be as:

$$a = pA_1\sqrt{-\frac{2\mu}{\lambda}}, \quad b = -(A_1p^2-A_1+B_1)\sqrt{-\frac{\mu}{2\lambda}}, \quad c = (A_1p^2+A_1+B_1)\sqrt{-\frac{\mu}{2\lambda}}, \quad p = p, \quad A_0 = 0, \quad A_1 = A_1, \tag{4.24}$$

$$B_1 = B_1, \quad s = s, \quad r = 2\mu A_1 B_1 - \lambda s^2, \quad u(\xi) = A_1\left[p+\tan(\phi(\xi)/2)\right] + B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.24) and Families 1 and **2**, respectively, can be written as

$$u_{29}(\xi) = -\frac{2\lambda}{\mu}\sqrt{A_1B_1}\tan\left(\frac{1}{2}\sqrt{-\frac{2\mu A_1B_1}{\lambda}}(\xi+C)\right) - \frac{\mu}{2\lambda}\sqrt{A_1B_1}\cot\left(\frac{1}{2}\sqrt{-\frac{2\mu A_1B_1}{\lambda}}(\xi+C)\right), \tag{4.25}$$

$$u_{30}(\xi) = \frac{2\lambda}{\mu}\sqrt{-A_1B_1}\tanh\left(\frac{1}{2}\sqrt{\frac{2\mu A_1B_1}{\lambda}}(\xi+C)\right) + \frac{\mu}{2\lambda}\sqrt{-A_1B_1}\cot\left(\frac{1}{2}\sqrt{\frac{2\mu A_1B_1}{\lambda}}(\xi+C)\right),$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} + \frac{(2\mu A_1B_1 - \lambda s^2)t^\alpha}{\Gamma(1+\alpha)}$.

**Case IX** will be as:

$$a = pA_1\sqrt{-\frac{2\mu}{\lambda}}, \quad b = -(A_1p^2 - A_1 - B_1)\sqrt{-\frac{\mu}{2\lambda}}, \quad c = (A_1p^2 + A_1 - B_1)\sqrt{-\frac{\mu}{2\lambda}}, \quad p = p, \quad A_0 = 0, \quad A_1 = A_1, \tag{4.26}$$

$$B_1 = B_1, \quad s = s, \quad r = 4\mu A_1B_1 - \lambda s^2, \quad u(\xi) = A_1\left[p + \tan(\phi(\xi)/2)\right] + B_1\left[p + \tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.26) and Families 1 and **2**, respectively, can be written as

$$u_{31}(\xi) = -\frac{2\lambda}{\mu}\sqrt{-A_1B_1}\tan\left(\frac{1}{2}\sqrt{\frac{2\mu A_1B_1}{\lambda}}(\xi+C)\right) - \frac{\mu}{2\lambda}\sqrt{-A_1B_1}\cot\left(\frac{1}{2}\sqrt{\frac{2\mu A_1B_1}{\lambda}}(\xi+C)\right), \tag{4.27}$$

$$u_{32}(\xi) = \frac{2\lambda}{\mu}\sqrt{A_1B_1}\tanh\left(\frac{1}{2}\sqrt{-\frac{2\mu A_1B_1}{\lambda}}(\xi+C)\right) + \frac{\mu}{2\lambda}\sqrt{A_1B_1}\cot\left(\frac{1}{2}\sqrt{-\frac{2\mu A_1B_1}{\lambda}}(\xi+C)\right),$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} + \frac{(4\mu A_1B_1 - \lambda s^2)t^\alpha}{\Gamma(1+\alpha)}$.

**Case X** will be as:

$$a = -iB_1\sqrt{-\frac{\mu}{2\lambda}}, \quad b = 0, \quad c = B_1\sqrt{-\frac{\mu}{2\lambda}}, \quad p = i, \quad A_0 = 0, \quad A_1 = -\frac{1}{2}B_1, \tag{4.28}$$

$$B_1 = B_1, \quad s = s, \quad r = -2\mu B_1^2 - \lambda s^2, \quad u(\xi) = A_1\left[p + \tan(\phi(\xi)/2)\right] + B_1\left[p + \tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.28) and Family 1, we can write

$$u_{33}(\xi) = -\frac{\sqrt{2}}{2}B_1\tan\left(\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}B_1(\xi+C)\right) + \frac{\sqrt{2}}{2}B_1\cot\left(\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}B_1(\xi+C)\right), \tag{4.29}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} - \frac{(2\mu B_1^2 + \lambda s^2)t^\alpha}{\Gamma(1+\alpha)}$.

**Case XI** will be as:

$$a = -iB_1\sqrt{-\frac{\mu}{2\lambda}}, \quad b = 0, \quad c = B_1\sqrt{-\frac{\mu}{2\lambda}}, \quad p = i, \quad A_0 = 0, \quad A_1 = -\frac{1}{2}B_1, \tag{4.30}$$

$$B_1 = B_1, \quad s = s, \quad r = -2\mu B_1^2 - \lambda s^2, \quad u(\xi) = A_1\left[p + \tan(\phi(\xi)/2)\right] + B_1\left[p + \tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.30) and Family 1, we can write

$$u_{34}(\xi) = \frac{\sqrt{2}}{2}B_1\tan\left(\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}B_1(\xi+C)\right) + \frac{\sqrt{2}}{2}B_1\cot\left(\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}B_1(\xi+C)\right), \tag{4.31}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} + \frac{(\mu B_1^2 - \lambda s^2)t^\alpha}{\Gamma(1+\alpha)}$.

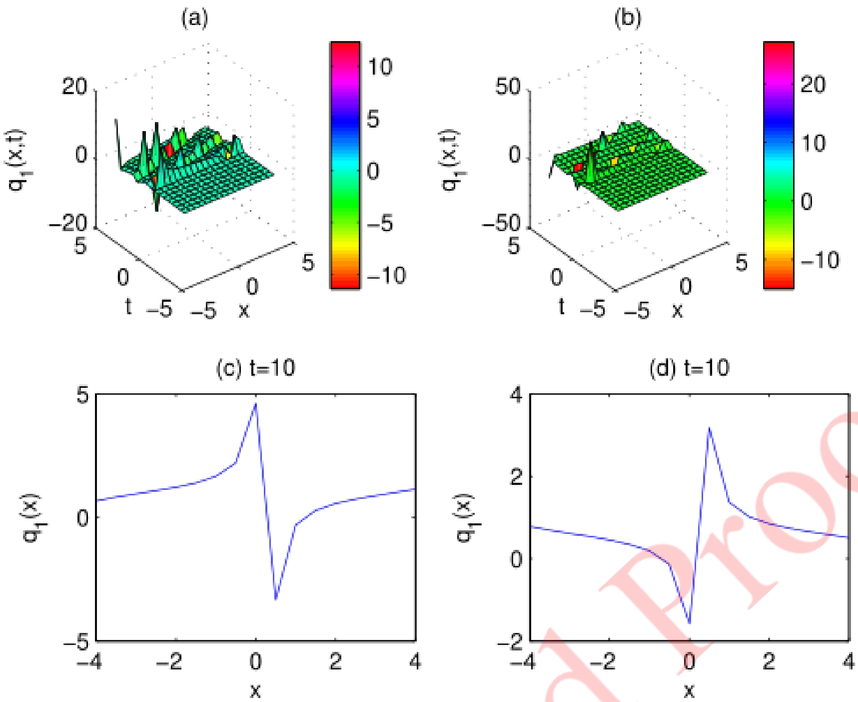


FIGURE 1. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 2, b = 2, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.5, \beta = 0.1$.

**Remark 4.1.** In Figures 1-8, we plot two dimensional and three dimensional graphics of absolute values of (4.11), which denote the dynamics of solutions with appropriate parametric selections. We plot two and three dimensional graphics of Figs 1-4 when $-4 < x < 4, -4 < t < 4$. Moreover, we plot two and three dimensional graphics of Figs 5-8 when $0 < x < 8, 0 < t < 8$. To the best of our knowledge, these optical soliton solutions have not been submitted to literature in advance. Figures 1 and 2 show periodic wave solutions, Figure 3 and 4 present soliton wave solutions. Also, Figures 5 and 6 show rational wave solutions. Moreover, Figures 7 and 8 present exponential wave solutions. We test our results based on different $\alpha$ and $\beta$.

4.2. **The Case n $\geq 2$ case for the fractional BME.** Now we close this work by assuming $n \geq 2$ in (3.6), therefore we have

$$-\mathrm{n}(\mathrm{r} + \mathrm{n}\lambda \mathrm{s}^2)\mathrm{u}^2(\xi) + \mathrm{n}(\mathrm{n}-1)\lambda(\mathrm{u}'(\xi))^2 + \mathrm{n}\lambda \mathrm{u}(\xi)\mathrm{u}''(\xi) + \mu \mathrm{F}(\mathrm{u}^2(\xi))\mathrm{u}^2(\xi) = 0. \tag{4.32}$$

In the presence of perturbation terms, the fractional BME with Kerr law nonlinearity (F(w) = w) is given by

$$-\mathrm{n}(\mathrm{r} + \mathrm{n}\lambda \mathrm{s}^2)\mathrm{u}^2(\xi) + \mathrm{n}(\mathrm{n}-1)\lambda(\mathrm{u}'(\xi))^2 + \mathrm{n}\lambda \mathrm{u}(\xi)\mathrm{u}''(\xi) + \mu \mathrm{u}^4(\xi) = 0. \tag{4.33}$$

In order to determine value of $m$, we balance the linear term of the highest order $uu''$ with the highest order nonlinear term $u^4$ in Eq. (4.33) we get

$$u(\xi) = A_m \left(\tan(\phi(\xi)/2)\right)^m + ..., \tag{4.34}$$

$$u^4(\xi) = A_m^4 \left(\tan(\phi(\xi)/2)\right)^{4m} + ..., \tag{4.35}$$

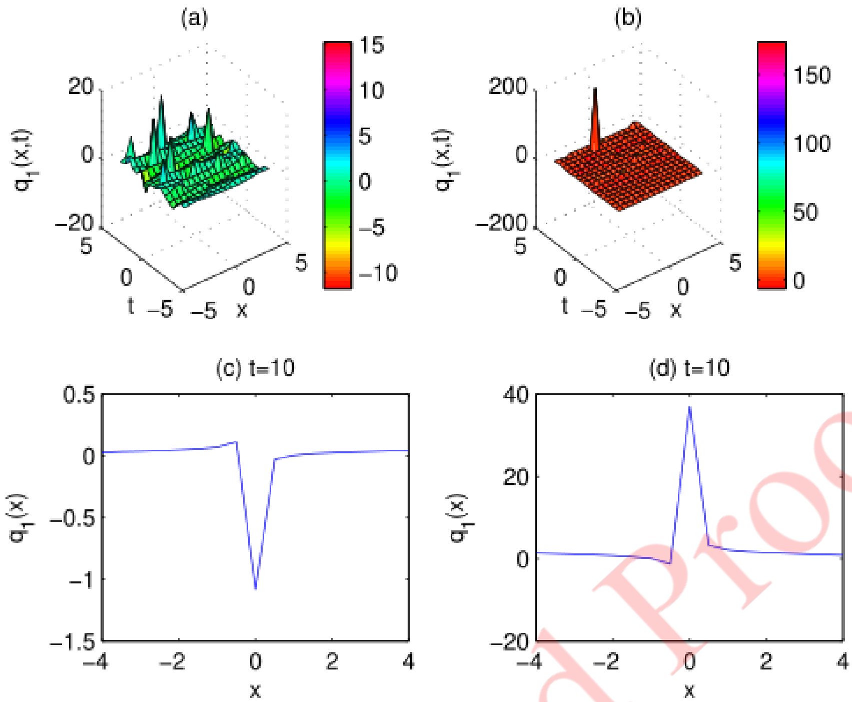


FIGURE 2. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 2, b = 2, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.99, \beta = 0.99$.

$$\frac{du(\xi)}{d\xi} = \frac{m(c-b)}{2} A_m \left(\tan(\phi(\xi)/2)\right)^{m+1} + ..., \tag{4.36}$$

$$\frac{d^2u(\xi)}{d\xi^2} = \frac{m(m+1)(c-b)^2}{2} A_m \left(\tan(\phi(\xi)/2)\right)^{m+2} + ..., \tag{4.37}$$

$$u\frac{d^2u(\xi)}{d\xi^2} = \frac{m(m+1)(c-b)^2}{2} A_m^2 \left(\tan(\phi(\xi)/2)\right)^{2m+2} + .... \tag{4.38}$$

By considering the homogeneous balance principle between the highest order derivatives $uu''$ and nonlinear terms $u^4$, we obtain $2m + 2 = 4m$, then $m = 1$. For simplicity we set $p = 0$ in (2.3). Then the trail solution is

$$u(\xi) = \sum_{k=-1}^{1} A_k(p + \tan^k(\phi/2))^k, \tag{4.39}$$

Substituting (4.39) and (2.4) into Eq. (4.33) and collecting all terms with the same order of $tan\,(\Phi(\xi)/2)$ together, and setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for $a, b, c, k, w, A_0, A_1$, and $B_1$ as follows:

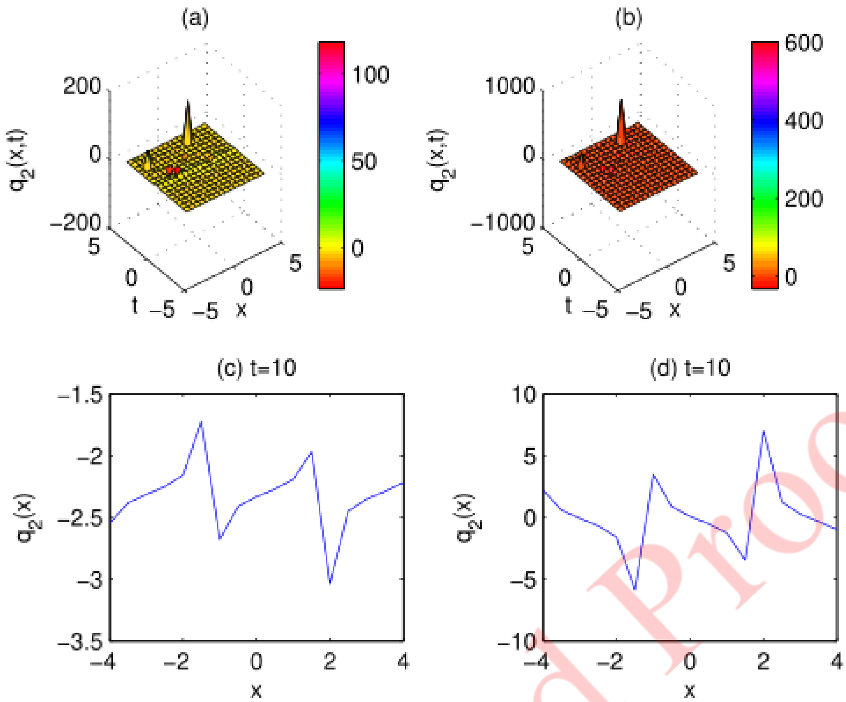


FIGURE 3. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 2, b = 3, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.5, \beta = 0.1$.

Coefficients of $Y = \tan(\phi(\xi)/2)$

$$\left\{\begin{array}{ll}
Y^0: & B_1^2(4\mu B_1^2 + 2n\lambda cb + 2n^2\lambda cb + n^2\lambda b^2 + n\lambda c^2 + n^2\lambda c^2 + n\lambda b^2) = 0,\\
Y^1: & 2B_1(8A_0\mu B_1^2 + 2B_1 n^2\lambda ab + B_1 n\lambda ab + 2B_1 n^2\lambda ac + B_1 n\lambda ac + n\lambda A_0 b^2 + n\lambda A_0 c^2 + 2n\lambda A_0 cb) = 0,\\
Y^2: & 2B_1(8\mu A_1 B_1^2 - B_1 n^2\lambda b^2 + 2B_1 n^2\lambda a^2 + 12B_1\mu A_0^2 + B_1 n^2\lambda c^2 - 2B_1 n^2\lambda s^2 - 2B_1 nr + 4n\lambda cbA_1 +\\
& 3n\lambda A_0 ba - n^2\lambda b^2 A_1 + 2n\lambda c^2 A_1 - n^2\lambda c^2 A_1 + 2n\lambda b^2 A_1 - 2n^2\lambda cbA_1 + 3n\lambda A_0 ca) = 0,\\
Y^3: & 2B_1(-2B_1 n^2\lambda ab - B_1 n\lambda ac + B_1 n\lambda ab + 2B_1 n^2\lambda ac + 24B_1\mu A_0 A_1 + 8\mu A_0^3 - 4n^2\lambda abA_1 - 4nrA_0 +\\
& 2n\lambda A_0 a^2 - n\lambda A_0 b^2 - 4n^2\lambda s^2 A_0 + n\lambda A_0 c^2 + 8n\lambda abA_1 - 4n^2\lambda acA_1 + 8n\lambda caA_1) = 0,\\
Y^4: & -n\lambda(8na^2 A_1 B_1 - 16a^2 A_1 B_1 + 2A_0 baB_1 - 2A_0 baA_1 - 2A_0 caB_1 - 2A_0 caA_1 - nb^2 B_1^2 - 4nb^2 A_1 B_1 +\\
& b^2 B_1^2 + b^2 A_1^2 + 8b^2 A_1 B_1 - nb^2 A_1^2 - 2ncbA_1^2 - 2cbB_1^2 + 2ncbB_1^2 + 2cbA_1^2 - 8c^2 A_1 B_1 + 4nc^2 A_1 B_1 - nc^2 B_1^2 -\\
& nc^2 A_1^2 + c^2 B_1^2 + c^2 A_1^2) - 4n^2\lambda s^2 A_0^2 + 4\mu A_0^4 - 4nrA_0^2 - 8nrA_1 B_1 + 48\mu A_0^2 A_1 B_1 - 8n^2\lambda s^2 A_1 B_1 + 24\mu A_1^2 B_1^2 = 0,\\
Y^5: & 2A_1(2n^2\lambda abA_1 - n\lambda caA_1 - n\lambda abA_1 + 2n^2\lambda acA_1 + 24A_1\mu A_0 B_1 + 8\mu A_0^3 - 4nrA_0 + n\lambda A_0 c^2 +\\
& 2n\lambda A_0 a^2 - 8n\lambda abB_1 - n\lambda A_0 b^2 - 4n^2\lambda s^2 A_0 + 4n^2\lambda abB_1 + 8n\lambda caB_1 - 4n^2\lambda acB_1) = 0,\\
Y^6: & 2A_1(8\mu B_1 A_1^2 - 2nrA_1 + n^2\lambda c^2 A_1 + 2A_1 n^2\lambda a^2 - n^2\lambda b^2 A_1 - 2n^2\lambda s^2 A_1 + 12\mu A_0^2 A_1 + 2n^2\lambda cbB_1 + 3n\lambda A_0 ca -\\
& n^2\lambda B_1 b^2 + 2n\lambda B_1 c^2 + 2n\lambda B_1 b^2 - n^2\lambda B_1 c^2 - 4n\lambda cbB_1 - 3n\lambda A_0 ba) = 0,\\
Y^7: & -2A_1(-8\mu A_0 A_1^2 - 2n^2\lambda acA_1 - n\lambda caA_1 + 2n^2\lambda abA_1 + n\lambda abA_1 - n\lambda A_0 c^2 - n\lambda A_0 b^2 + 2n\lambda A_0 cb) = 0,\\
Y^8: & A_1^2(4\mu A_1^2 + b^2\lambda n + c^2\lambda n + n^2\lambda c^2 + n^2\lambda b^2 - 2n\lambda cb - 2n^2\lambda cb) = 0.
\end{array}\right. \tag{4.40}$$

Solving the set of algebraic equations using Maple, we get the following results:
**Case I** will be as:

$$a = 0, \quad b = \sqrt{-\frac{\mu}{n(n+1)\lambda}}B_1, \quad c = \sqrt{-\frac{\mu}{n(n+1)\lambda}}B_1, \quad p = 0, \quad A_0 = 0, \quad A_1 = 0, \tag{4.41}$$

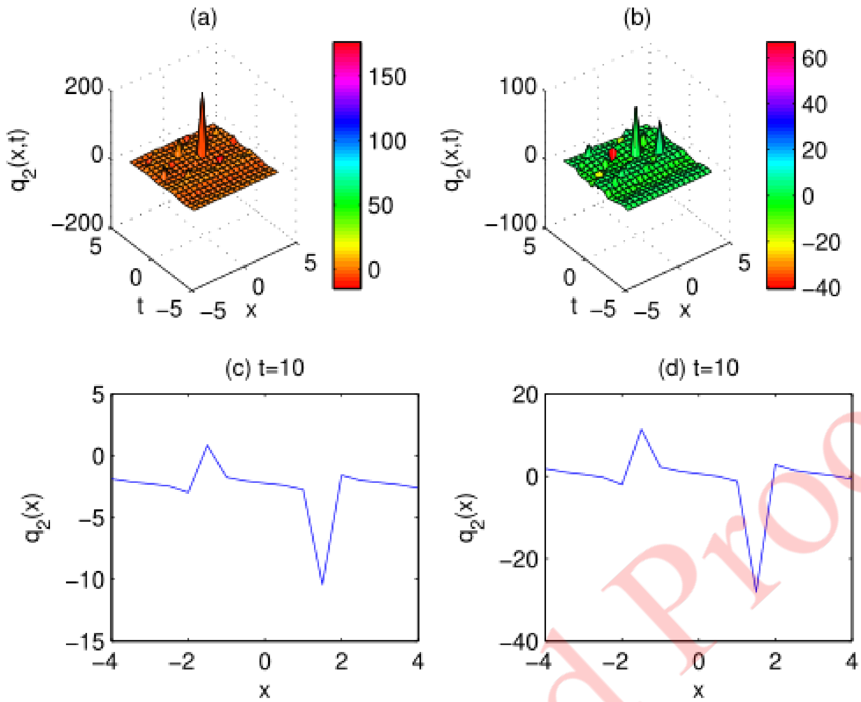


FIGURE 4. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 2, b = 3, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.99, \beta = 0.99$.

$$B_1 = B_1, \quad s = s, \quad r = -n\lambda s^2, \quad u(\xi) = B_1 \cot(\phi(\xi)/2).$$

By using of (4.41) and Family 16, we can write as

$$u_1(\xi) = \frac{B_1}{c(\xi + C)}, \tag{4.42}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda n s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^\alpha}{\Gamma(1+\alpha)}$.

**Case II** will be as:

$$a = 0, \quad b = -\sqrt{-\frac{\mu}{n(n+1)\lambda}}A_1, \quad c = \sqrt{-\frac{\mu}{n(n+1)\lambda}}A_1, \quad p = 0, \quad A_0 = 0, \quad A_1 = A_1, \tag{4.43}$$

$$B_1 = 0, \quad s = s, \quad r = -n\lambda s^2, \quad u(\xi) = A_1 \tan(\phi(\xi)/2).$$

By using of (4.43) and Family 17, we can write as

$$u_2(\xi) = -\frac{A_1}{c(\xi + C)}, \tag{4.44}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda n s t^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^\alpha}{\Gamma(1+\alpha)}$.

**Case III** will be as:

$$a = 0, \quad b = b, \quad c = c, \quad p = 0, \quad A_0 = 0, \quad A_1 = -\sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b - c), \tag{4.45}$$

$$B_1 = \sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b + c), \quad s = s, \quad r = n\lambda(b^2 - c^2 - s^2), \quad u(\xi) = A_1 \tan(\phi(\xi)/2) + B_1 \cot(\phi(\xi)/2).$$

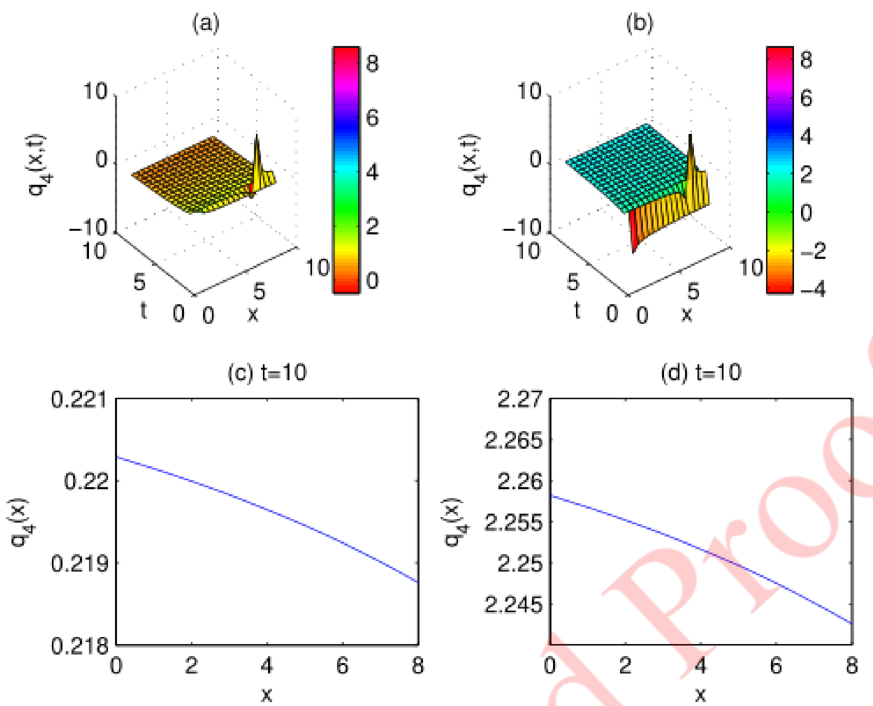


FIGURE 5. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 3, b = 4, c = 5, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.5, \beta = 0.1$.

By using of (4.45) and Families 5, 6, 11, 16, we can write as

$$u_3(\xi) = -\sqrt{-\frac{n(n+1)\lambda}{4\mu}}\sqrt{b^2-c^2}\left\{\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\bar{\xi}\right) - \coth\left(\frac{\sqrt{b^2-c^2}}{2}\bar{\xi}\right)\right\}, \tag{4.46}$$

where $\bar{\xi} = \xi + C$, $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} + \frac{n\lambda(b^2-c^2-s^2)t^\alpha}{\Gamma(1+\alpha)}$.

$$u_4(\xi) = -\sqrt{-\frac{n(n+1)\lambda}{4\mu}}b\left\{\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}}+1}\right]\right) - \cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}}+1}\right]\right)\right\},$$

$$u_5(\xi) = -\sqrt{-\frac{n(n+1)\lambda}{4\mu}}b\left\{\frac{be^{b(\xi+C)}-1}{be^{b(\xi+C)}+1} - \frac{be^{b(\xi+C)}+1}{be^{b(\xi+C)}-1}\right\},$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^\alpha}{\Gamma(1+\alpha)}$ and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} + \frac{n\lambda(b^2-s^2)t^\alpha}{\Gamma(1+\alpha)}$.

$$u_6(\xi) = -\sqrt{-\frac{n(n+1)\lambda}{\mu}}\frac{1}{(\xi+C)}, \qquad q(x,t) = u(\xi)e^{i\theta}, \qquad \xi = x - \frac{2\lambda nst^\alpha}{\Gamma(1+\alpha)} \qquad \theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{n\lambda s^2t^\alpha}{\Gamma(1+\alpha)},$$

**Case IV** will be as:

$$a = -\sqrt{c^2-b^2}, \qquad b = b, \qquad c = c, \qquad p = 0, \qquad A_0 = \sqrt{-\frac{(b^2-c^2)n(n+1)\lambda}{4\mu}}, \qquad A_1 = \sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b-c),$$

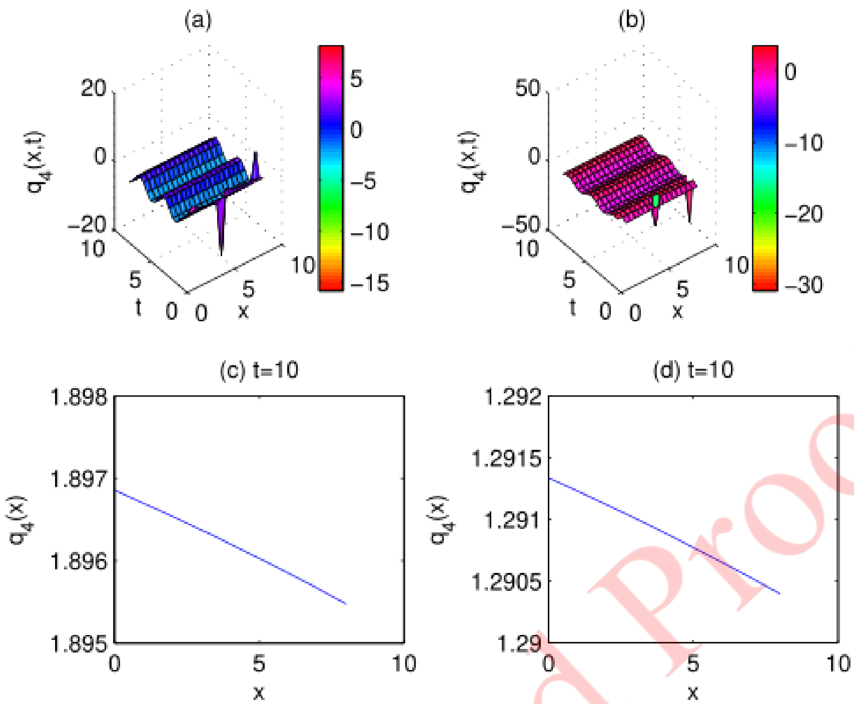


FIGURE 6. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 3, b = 4, c = 5, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.99, \beta = 0.99$.

$$B_1 = 0, \quad s = s, \quad r = -n\lambda s^2, \quad u(\xi) = A_0 + A_1 \tan(\phi(\xi)/2). \tag{4.47}$$

By using of (4.47) and Families 8, we can write as

$$u_7(\xi) = \sqrt{-\frac{(b^2-c^2)n(n+1)\lambda}{4\mu}} + \sqrt{-\frac{n(n+1)\lambda}{4\mu}}\frac{a\bar{\xi}+2}{\bar{\xi}}, \tag{4.48}$$

where $\bar{\xi} = \xi + C$, $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^{\alpha}}{\Gamma(1+\alpha)}$ and $\theta = \frac{st^{\beta}}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^{\alpha}}{\Gamma(1+\alpha)}$.

**Case V** will be as:

$$a = \sqrt{c^2-b^2}, \quad b = b, \quad c = c, \quad p = 0, \quad A_0 = \sqrt{-\frac{(b^2-c^2)n(n+1)\lambda}{4\mu}}, \quad A_1 = 0, \tag{4.49}$$

$$B_1 = \sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b+c), \quad s = s, \quad r = -n\lambda s^2, \quad u(\xi) = A_0 + B_1\cot(\phi(\xi)/2).$$

By using of (4.49) and Families 8, we can write as

$$u_8(\xi) = \sqrt{-\frac{(b^2-c^2)n(n+1)\lambda}{4\mu}} + \sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b^2-c^2)\frac{\bar{\xi}}{a\bar{\xi}+2}, \tag{4.50}$$

where $\bar{\xi} = \xi + C$, $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^{\alpha}}{\Gamma(1+\alpha)}$ and $\theta = \frac{st^{\beta}}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^{\alpha}}{\Gamma(1+\alpha)}$.

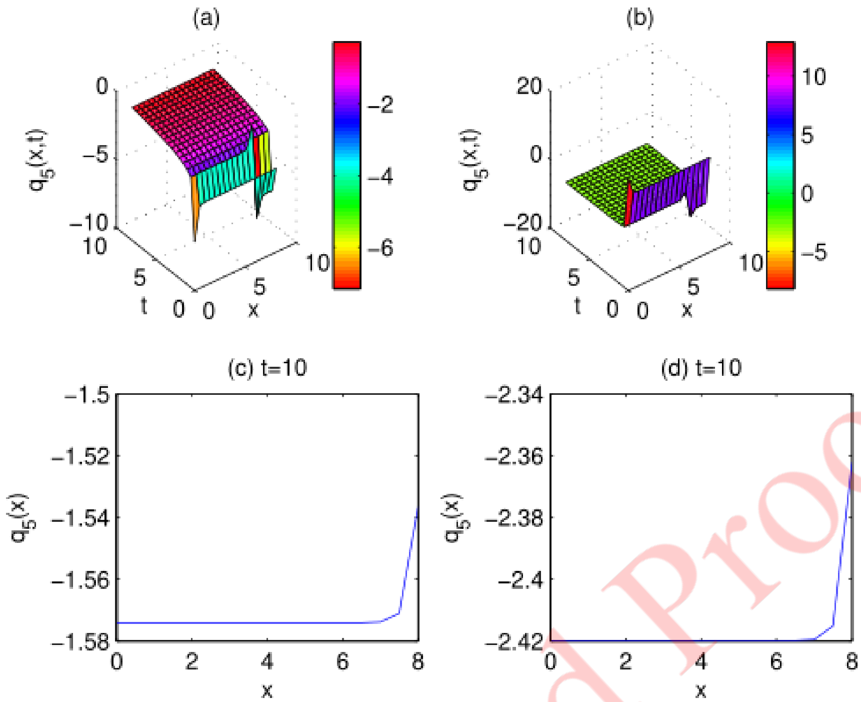


Figure 7. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 3, b = 5, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.5, \beta = 0.1$.

## 5. Conclusion

In this paper, we presented the improved $\tan(\Phi(\xi)/2)$-expansion method for solving the fractional Biswas-Milovic (FBM) equation. We obtained abundant results for FBME. The exact particular solutions containing four types hyperbolic function solution, trigonometric function solution, exponential solution and rational solution. Abundant exact travelling wave solutions including solitons, kink, periodic and rational solutions are attained. It is worth mentioning that some of newly obtained solutions are identical to already published results. It has been shown that the applied method is effective and more wide-ranging than the Exp-function method and sine-cosine method because it gives many new solutions. Therefore, this method can be applied to study many other nonlinear partial differential equations which frequently arise in engineering, mathematical physics and nonlinear optic.

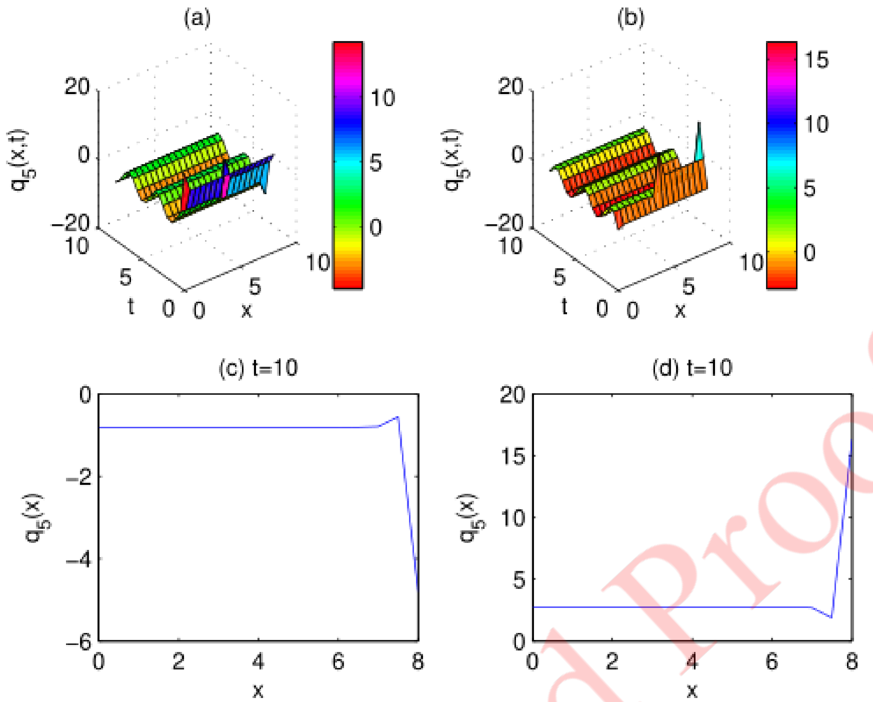


FIGURE 8. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 3, b = 5, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.99, \beta = 0.99$.

## References


[1] M. F. Aghdaei and J. Manafian, *Optical soliton wave solutions to the resonant Davey-Stewartson system*, Opt. Quant. Electron, *48* (2016), 1-33.

[2] S. Ahmadiana and M. T. Darvishi, *A new fractional Biswas-Milovic model with its periodic soliton solutions*, Optik-International Journal for Light and Electron Optics, *38* (2016), 3763-3767.

[3] I. Ahmed, C. Mu, and F. Zhang, *Exact solution of the Biswas-Milovic equation by Adomian decomposition method*, Int. J. Appl. Math. Research, *2* (2011), 418-422.

[4] N. H. Ali, S. A. Mohammed, and J. Manafian, *Study on the simplified MCH equation and the combined KdV-mKdV equations with solitary wave solutions*, Partial Diff. Eq. Appl. Math., *9* (2024), 100599.

[5] A. Biswas, C. Cleary, J. E. Watson, and D. Milovic, *Optical soliton perturbation with time-dependent coefficients in a log law media*, Appl. Math. Comput., *217* (2010), 2891-2894.

[6] A. Biswas and D. Milovic, *Bright and dark solitons of the generalized nonlinear Schrödinger's equation*, Commu. Nonlinear Sci. Num. Simul., *15* (2010), 1473-1484.

[7] A. Biswas and D. Milovic, *Optical solitons with log law nonlinearity*, Commu. Nonlinear Sci. Num. Simul., *15* (2010), 3763-3767.

[8] A. Biswas and S. Konar, *Introduction to Non-Kerr Law Optical Solitons*, CRC Press, Boca Raton, FL, USA, (2006).

[9] M. Caputo, *Elasticita e dissipazione*, Zani-Chelli, Bologna, 1969.

[10] H. Chen, A. Shahi, G. Singh, J. Manafian, B. Baharak Eslami, and N. A. Alkader, *Behavior of analytical schemes with non-paraxial pulse propagation to the cubic-quintic nonlinear Helmholtz equation*, Math. Comput. Simul., *220* (2024), 341-356.

[11] L. Debnath, *Fractional integrals and fractional differential equations in fluid mechanics*, Frac. Calc. Appl. Anal., *6* (2003), 119-155.

[12] M. Dehghan and J. Manafian, *The solution of the variable coefficients fourth–order parabolic partial differential equations by homotopy perturbation method*, Z. Naturforsch, *64a* (2009), 420-430.

[13] M. Dehghan, J. Manafian, and A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Num. Meth. Partial Diff.l Eq. J., *26* (2010), 448-479.

[14] M. Dehghan, J. Manafian, and A. Saadatmandi, *Application of the Exp-function method for solving a partial differential equation arising in biology and population genetics*, Int J. Num. Methods Heat Fluid Flow, *21* (2011), 736-753.

[15] M. Dehghan, J. Manafian, and A. Saadatmandi, *Application of semi–analytic methods for the Fitzhugh–Nagumo equation*, which models the transmission of nerve impulses, Math. Meth. Appl. Sci., *33* (2010), 1384-1398.

[16] Y. Gu, S. Malmir, J. Manafian, O. A. Ilhan, A. A. Alizadeh, and A. J. Othman, *Variety interaction between k-lump and k-kink solutions for the (3+1)-D Burger system by bilinear analysis*, Results Phys., *43* (2022), 106032.

[17] H. Jafari, A. Soorakia, and C. M. Khalique, *Dark solitons of the Biswas-Milovic equation by the first integral method*, Optik, *124* (2013), 3929-3932.

[18] C. M. Khalique, *Stationary solutions for the BiswasMilovic equation*, Appl. Math. Comput., *217* (2011) 7400-7404.

[19] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[20] X. G. Luo, Q. B. Wub, and B. Q. Zhang, *Revisit on partial solutions in the Adomian decomposition method: Solving heat and wave equations*, J. Math. Anal. Appl., *321* (2006), 353-363.

[21] R. Kohla, R. Tinaztepeb, and A. Chowdhury, *Soliton perturbation theory of Biswas-Milovic equation*, Optik, *125* (2014), 1926-1936.

[22] M. Lakestani, J. Manafian, A. R. Najafizadeh, and M. Partohaghighi, *Some new soliton solutions for the nonlinear the fifth-order integrable equations*, Comput. Meth. Diff. Equ., *10*(2) (2022).

[23] J. Manafian, *On the complex structures of the Biswas-Milovic equation for power, parabolic and dual parabolic law nonlinearities*, Eur. Phys. J. Plus, *130* (2015), 1-20.

[24] J. Manafian, *Optical soliton solutions for Schrödinger type nonlinear evolutionequations by the* $\tan(\phi/2)$*-expansion method*, Optik-Int. J. Elec. Opt., *127* (2016), 4222-4245.

[25] J. Manafian, M. F. Aghdaei, and M. Zadahmad, *Analytic study of sixth-order thin-film equation by tan($\phi/2$)-expansion method*, Opt. Quant. Electron, *48* (2016), 1-16.

[26] J. Manafian, L. A. Dawood, and M. Lakestani, *New solutions to a generalized fifth-order KdV like equation with prime number $p = 3$ via a generalized bilinear differential operator*, Partial Diff. Eq. Appl. Math., *9* (2024), 100600.

[27] J. Manafian and M. Lakestani, *Application of tan($\phi/2$)-expansion method for solving the BiswasMilovic equation for Kerr law nonlinearityJalil*, Optik, *127* (2016), 2040-2054.

[28] J. Manafian and M. Lakestani, *A new analytical approach to solve some the fractional-order partial differential equations*, Indian J. Phys., *90* (2016), 1-16.

[29] J. Manafian and M. Lakestani, *Optical solitons with Biswas-Milovic equation for Kerr law nonlinearity*, Eur. Phys. J. Plus, *130* (2015), 1-12.

[30] J. Manafian and M. Lakestani, *Application of tan($\phi/2$)-expansion method for solving the Biswas-Milovic equation for Kerr law nonlinearity*, Optik-Int. J. Elec. Opt., *127* (2016), 2040-2054.

[31] J. Manafian and M. Lakestani, *Dispersive dark optical soliton with Tzitzéica type nonlinear evolution equations arising in nonlinear optics*, Opt. Quant. Electron, *48* (2016), 1-32.

[32] J. Manafian and M. Lakestani, *Solitary wave and periodic wave solutions for Burgers, Fisher, Huxley and combined forms of these equations by the $G'/G$-expansion method*, Pramana- J. Phys., *130* (2015), 31-52.

[33] J. Manafian, M. Lakestani, and A. Bekir, *Comparison between the generalized tanh-coth and the $G'/G$-expansion methods for solving NPDE's and NODE's*, Pramana . J. Phys., *87* (2016), 1-14.

[34] J. Manafian, M. Lakestani, and A. Bekir, *Study of the analytical treatment of the (2+1)-dimensional Zoomeron, the Duffing and the SRLW equations via a new analytical approach*, Int. J. Appl. Comput. Math., *2* (2016), 243-268.

[35] J. Manafian and M. Lakestani, *New improvement of the expansion methods for solving the generalized Fitzhugh-Nagumo equation with time-dependent coefficients*, Int. J. Eng. Math., *2015* (2015), 1-35.

[36] J. Manafian and M. Lakestani, *N-lump and interaction solutions of localized waves to the (2+ 1)- dimensional variable-coefficient CaudreyDoddGibbonKoteraSawada equation*, J. Geom. Phys., *150* (2020), 103598.

[37] J. Manafian and M. Lakestani, *Application of tan($\phi/2$)-expansion method for solving the BiswasMilovic equation for Kerr law nonlinearity*, Optik, *127*(4) (2016), 2040-2054.

[38] J. Manafian and M. Lakestani, *Optical soliton solutions for the GerdjikovIvanov model via tan($\phi/2$)-expansion method*, Optik, *127*(20) (2016), 9603-9620.

[39] J. Manafian and M. Lakestani, *Abundant soliton solutions for the KunduEckhaus equation via tan($\phi(\xi)$)-expansion method*, Optik, *127*(14) (2016), 5543-5551.

[40] M. Mirzazadeh and M. Eslami, *Exact multisoliton solutions of nonlinear Klein-Gordon equation in $1 + 2$ dimensions*, The Eur. Phys. J. Plus, *128* (2015), 1-9.

[41] S. R. Moosavi, N, Taghizadeh, and J. Manafian, *Analytical approximations of one-dimensional hyperbolic equation with non-local integral conditions by reduced differential transform method*, Comput. Meth. Diff. Equ., *8*(3) (2020), 537-552.

[42] K. Oldham, *Fractional differential equations in electrochemistry*, Adv. Eng. Softw., *41* (2010), 9-17.

[43] B. Sturdevant, *Topological 1-soliton solution of the Biswas-Milovic equation with power law nonlinearity*, Nonlinear Analysis; Real World Appl., *11* (2010), 2871-2874.

[44] G. O. Young, *Definition of physical consistent damping laws with fractional derivatives*, Z. Angew. Math. Mech., *75* (1995), 623-635.

[45] Z. Y. Zhang, Z. H. Liu, X. J. Miao, and Y. Z. Chen, *Qualitative analysis and traveling wave solutions for the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity*, Phys. Let. A, *375* (2011), 1275-1280.

[46] S. Zhang, J. Manafian, O. A. Ilhan, A. T. Jalil, Y. Yasin, and M. A. Gatea, *Nonparaxial pulse propagation to the cubic-quintic nonlinear Helmholtz equation*, International Journal of Modern Physics B, *38*(8) (2024), 2450117.

[47] M. Zhang, X. Xie, J. Manafian, O. A. Ilhan, and G. Singh, *Characteristics of the new multiple rogue wave solutions to the fractional generalized CBS-BK equation*, J. Adv. Res., *38* (2022), 131-142.

[48] N. Zhao, J. Manafian, O. A. Ilhan, G. Singh, and R. Zulfugarova, *Abundant interaction between lump and k-kink, periodic and other analytical solutions for the (3+1)-D Burger system by bilinear analysis*, Int. J. Modern Phys.

B, *35*(13) (2021), 2150173.

[49] X. Zhao, L. Wang, and W. Sun, *The repeated homogeneous balance method and its applications to nonlinear partial differential equations*, Chaos Solitons Fract., *28* (2006), 448-453.