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Two-dimensional temporal fractional advection-diffusion problem resolved through the Sinc-Galerkin method

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Abstract

The Sinc-Galerkin method, even for issues spanning infinite and semi-infinite intervals, is known as exponentially fading mistakes and, in certain circumstances, as the optimum convergence rate. Additionally, this approach does not suffer from the normal instability issues that often arise in other methods. Therefore, a numerical technique based on the Sinc-Galerkin method is devised in this study to solve the two-dimensional time fractional advection-diffusion problem. To be precise, the spatial and temporal discretizations of the Sinc-Galerkin and finite difference methods are coupled to provide the suggested approach. Additionally, the suggested method's convergence is looked at. Two numerical examples are provided in depth in the conclusion to demonstrate the effectiveness and precision of the suggested approach.

Keywords. Time fractional advection-diffusion equation, Sinc-Galerkin method, Caputo's fractional derivative, Convergence analysis.

2010 Mathematics Subject Classification. 65M12.

1. Introduction

The traditional time derivative is insufficient to describe the historical dependency process. Examples include resonant radiation transport in plasma, Anomalous (scattering) transport in disordered semiconductors, non-Debb relaxation in solid dielectrics, and light beam penetration across turbulent media [9]. Fractional time derivative, or more precisely, the integral term in the definition of fractional time derivative, is the mathematical answer for modeling memory processes. As a result, fractional time derivative issues have grown in prominence for modeling a variety of phenomena, such as anomalous diffusion, which is both an important scientific topic and a real-world issue that often arises in engineering. Often, the classical diffusion law cannot properly represent the anomalous diffusion behavior characterized by gradual dissipation or rapid diffusion over time and spatial correlation over a long time. Such physical and mechanical processes may be precisely described by fractional differential equations. References for the theory and use of fractional differential operators include [9, 25].

In this paper, we concentrate on solving the time-fractional advection-diffusion problem numerically. One of the important study models for anomalous solute diffusion and migration in complex environments is the fractional advection-diffusion equation [5]. When the medium structure of certain complex media changes during the diffusion process, the diffusion behavior changes accordingly. Therefore, the process cannot be described by differential equations of integer order. The fractional equations should then be taken into consideration once the diffusion equation has to be adjusted for time or space. Variable order (or variable structure) operators were first proposed by Lorenzo and Hartley [10]. In the paper [15], n order to model the motion of a spherical particle sedimenting in a quiescent viscous liquid, the authors created a variable-order differential equation of motion. The authors in [22] discusses variable-order differential operators in anomalous diffusion modeling, and in [24] compares constant-order and variable-order fractional models for describing system memory properties.

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The main goal of this research is to use the Sinc-Galerkin method to solve the two-dimensional time fractional advection-diffusion equation. We consider the following fractional partial differential equation with the boundary and initial conditions

$$
\begin{aligned}
&{}^{C}D_t^{\alpha}u(x,y,t) - \Delta u(x,y,t) + p_1(x,y)\frac{\partial u(x,y,t)}{\partial x} + p_2(x,y)\frac{\partial u(x,y,t)}{\partial y} + p_3(x,y)u(x,y,t) = f(x,y,t)\\
&\qquad u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \qquad 0 < t < T, \quad (x,y) \in \partial\mathcal{D},\\
&\qquad u(x,y,0) = h(x,y), \qquad (x,y) \in \mathcal{D},
\end{aligned} \tag{1.1}
$$

where $\mathcal{D} = \{(x,y)|\ x,y \in [0,1]\}$ is a domain that is bounded in $\mathbb{R}^2$ and has the boundary $\partial\mathcal{D}$, $p_1(x,y)$ and $p_2(x,y)$ are the advection coefficients and $h(x,y)$, $p_3(x,y)$ and $f(x,y,t)$ are given functions. In addition, ${}^{C}D_t^{\alpha}u(x,y,t)$ is the left Caputo fractional derivative for $(x,y,t) \in Q_T = (0,T) \times \mathcal{D}$ and $\alpha \in (0,1]$ by

$$
{}^{C}D_t^{\alpha}u(x,y,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_s(s,x,y)}{(t-s)^{\alpha}}ds.
$$

Sinc numerical methods, such as [1, 2, 4, 11, 16, 19–21, 27, 28], have become more and more popular for solving partial differential equations. These techniques inspired writers to utilize them for a variety of reasons. The Sinc methods' excellent accuracy in the vicinity of singularities, which is related to their uncomplicated and simple implementation, might be considered their most significant advantage. There are very useful papers for solving numerical partial differential equations such as [8, 12–14, 17, 18, 23]. Also, for problems in semi-infinite and infinite intervals, they exhibit exponentially decreasing errors and, under some circumstances, an ideal convergence rate. Last but not least, these approaches do not experience the normal instability issues that often arise in other methods, such as [11, 19], etc. This is because of their quick convergence rate. We achieve this by using the finite difference method (FDM) for the temporal fractional derivative and the Sinc-Galerkin method for the spatial dimension. The structure of this paper is as follows: A computational approach is provided in section 2 to find a rough solution to the time fractional advection-diffusion equation (1.1) based on finite difference and Sinc-Galerkin methods. Additionally, we provide a convergence study of the suggested method for section 3. Two numerical examples are offered in the last section to demonstrate how well a novel method works.

2. The proposed method

2.1. **Finite difference method.** Suppose the time step is $t_k = k\tau$ for $k = 0, 1, \cdots, n$ with $\tau = T/n$. The function ${}^{C}D_t^{\alpha}u(x,y,t)$ is the time fractional derivative at t_n within the scope of the general time discretization scheme ([6, 7]):

$$
{}^{C}D_t^{\alpha}u(x,y,t_n) \simeq \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k \frac{u^{n-k} - u^{n-k-1}}{\tau}, \tag{2.1}
$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ and $u^k = u(x,y,t_k)$ for $k = 0, 1 \ldots, n$. Substituting (2.1) into the problem (1.1), we get:

$$
\begin{aligned}
{}^{C}D_t^{\alpha}u(x,y,t_n) &= \frac{1}{\Gamma(2-\alpha)\tau^{\alpha}}\sum_{k=0}^{n-1} b_k\left(u^{n-k} - u^{n-k-1}\right) + \mathcal{R}\\
&= \frac{1}{\lambda}u^n - \frac{1}{\lambda}\sum_{k=0}^{n-1}\omega_k u^{n-k} - \frac{b_{n-1}}{\lambda}u^0 + \mathcal{R},
\end{aligned}
$$

where $|\mathcal{R}| \le C\tau^{2-\alpha}$ for some constant $C > 0$ [7], $U^n = u(x,y,t_n)$, $\lambda = \Gamma(2-\alpha)\tau^{\alpha}$ and $\omega_k = b_{k-1} - b_k$. The discrete form of (1.1) is provided by omitting the short term $\mathcal{R}$ and using the following form:

$$
\frac{1}{\lambda}U^n - \frac{1}{\lambda}\sum_{k=0}^{n-1}\omega_k U^{n-k} - \frac{b_{n-1}}{\lambda}u^0 - \Delta U^n + p_1(x,y)\frac{\partial U^n}{\partial x} + p_2(x,y)\frac{\partial U^n}{\partial y} + p_3(x,y)U^n = f(x,y,t_n).
$$

Simplifying we obtain:

$$
\begin{aligned}
&-\Delta U^n + p_1(x,y)\frac{\partial U^n}{\partial x} + p_2(x,y)\frac{\partial U^n}{\partial y} + \left(p_3(x,y) + \frac{1}{\lambda}\right)U^n = f(x,y,t_n) + \frac{1}{\lambda}\sum_{k=0}^{n-1}\omega_k U^{n-k} + \frac{b_{n-1}}{\lambda}u^0 \\
&U^n(x,0) = U^n(x,1) = U^n(0,y) = U^n(1,y) = 0, \qquad (x,y) \in \partial\mathcal{D}.
\end{aligned}
\tag{2.2}
$$

2.2. **Sinc-Galerkin method.** The Sinc function is defined on the interval $x \in (-\infty, \infty)$ by

$$
\mathrm{Sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}
$$

Sinc functions for time and space variables with uniformly spaced nodes are presented as:

$$
S(k,h_x)(z) = \mathrm{Sinc}\left(\frac{z-kh_x}{h_x}\right), \quad S^*(k,h_y)(z) = \mathrm{Sinc}\left(\frac{z-kh_y}{h_y}\right),
$$

for h_x, $h_y > 0$, and $k = 0, \pm 1, \pm 2, \ldots$. In order to approximate on the interval $(0,1)$, which is required in this paper, the eye-shaped domain in the z-plane, $D_E = \left\{z = x + iy : \left|\arg\left(\frac{z}{1-z}\right)\right| < d \leq \frac{\pi}{2}\right\}$ is mapped onto the infinite strip according to $D_S = \left\{v = \rho + i\sigma : \ |\sigma| < d \leq \frac{\pi}{2}\right\}$ with

$$
v = \phi(z) = \ln\left(\frac{z}{1-z}\right).
$$

So, the following compositions define the basis elements on the interval $(0,1)$:

$$
\begin{aligned}
S_j(x) &= S(j,h_x)\mathrm{o}\phi(x) = \mathrm{Sinc}\left(\frac{\phi(x) - jh_x}{h_x}\right), \\
S_j^*(y) &= S^*(j,h_y)\mathrm{o}\phi(y) = \mathrm{Sinc}\left(\frac{\phi(y) - jh_y}{h_y}\right),
\end{aligned}
$$

where h_x and h_y are the mesh sizes in D_S for the identical grids kh_x and kh_y with $k = 0, \pm 1, \pm 2, \ldots$. The inverse of the expression $v = \phi(z)$ is

$$
z = \phi^{-1}(v) = \psi(v) = \frac{\exp(v)}{\exp(v) + 1},
$$

It enables us to acquire the following inverse images of the grids with equal spacing:

$$
x_k = \psi(kh_x), \quad y_k = \psi(kh_y), \qquad k = 0, \pm 1, \pm 2, \ldots.
$$

Theorem 2.1. *[19] Let the function ϕ be a conformal one-to-one map with the simple connected domain D_E onto D_S, then*

$$
\begin{aligned}
\delta_{i,k}^{(0)} &= [S(i,h)\mathrm{o}\phi(x)]\,|_{x=x_k} = \begin{cases} 0, & k \neq i, \\ 1, & k = i, \end{cases} \\
\delta_{i,k}^{(1)} &= h\frac{d}{d\phi}\,[S(i,h)\mathrm{o}\phi(x)]\,|_{x=x_k} = \begin{cases} \dfrac{(-1)^{(k-i)}}{(k-i)}, & k \neq i, \\ 0, & k = i, \end{cases} \\
\delta_{i,k}^{(2)} &= h^2\frac{d^2}{d\phi^2}\,[S(i,h)\mathrm{o}\phi(x)]\,|_{x=x_k} = \begin{cases} \dfrac{-2(-1)^{(k-i)}}{(k-i)^2}, & k \neq i, \\ \dfrac{-\pi^2}{3}, & k = i. \end{cases}
\end{aligned}
$$

The following theorem outlines the circumstances in which the Sinc-Galerkin method yields exponential convergence.

Theorem 2.2. *[19, 28] Let $G(x,y) \in B(D_E)$ for $h_x, h_y > 0$ and each fixed y. Also, suppose that ϕ is one-to-one conformal map of domain D_E onto D_S and $x_i = \phi^{-1}(ih_x)$, $y_j = \phi^{-1}(jh_y)$ and $\Gamma_x = \phi^{-1}(\mathbb{R})$, $\Gamma_y = \phi^{-1}(\mathbb{R})$. Suppose there are positive constants α_x, β_x, and $C_x(y)$ such that*

$$\left|\frac{G(x,y)}{\phi'(x)}\right| \leq C_x(y) \begin{cases} \exp(-\alpha_x\,|\phi(x)|), & x \in \Gamma_a^{(x)}, \\ \exp(-\beta_x\,|\phi(x)|), & x \in \Gamma_b^{(x)}, \end{cases}$$

where $\Gamma_a^{(x)} \equiv \{x \in \Gamma_x : \ \phi(x) = u \in (-\infty, 0)\}$, $\Gamma_b^{(x)} \equiv \{x \in \Gamma_x : \ \phi(x) = u \in [0,\infty)\}$. Also for each fixed x, let $G(x,y) \in B(D_E)$ and assume there are positive constants α_y, β_y, and $C_y(x)$ such that

$$\left|\frac{G(x,y)}{\phi'(y)}\right| \leq C_y(x) \begin{cases} \exp(-\alpha_y\,|\phi(y)|), & y \in \Gamma_a^{(y)}, \\ \exp(-\beta_y\,|\phi(y)|), & y \in \Gamma_b^{(y)}, \end{cases}$$

where $\Gamma_a^{(y)} \equiv \{y \in \Gamma_y : \ \phi(y) = u \in (-\infty, 0)\}$, $\Gamma_b^{(y)} \equiv \{y \in \Gamma_y : \ \phi(y) = u \in [0,\infty)\}$. Then the Sinc *trapezoidal quadrature rule is*

$$\begin{aligned} \int_{\Gamma_y}\int_{\Gamma_x} G(x,y)dxdy &= h_x h_y \sum_{i=-M_x}^{N_x}\sum_{j=-M_y}^{N_y} \frac{G(x_i,y_j)}{\phi'(x_i)\phi'(y_j)} \\ &+\mathcal{O}\left(\exp\left(-\alpha_x M_x h_x\right)\right) + \mathcal{O}\left(\exp\left(-\beta_x N_x h_x\right)\right) + \mathcal{O}\left(\exp\left(-\frac{2\pi d}{h_x}\right)\right) \\ &+\mathcal{O}\left(\exp\left(-\alpha_y M_y h_y\right)\right) + \mathcal{O}\left(\exp\left(-\beta_y N_y h_y\right)\right) + \mathcal{O}\left(\exp\left(-\frac{2\pi d}{h_y}\right)\right). \end{aligned}$$

Using the selections

$$N_x = \left[\left|\tfrac{\alpha_x}{\beta_x} M_x + 1\right|\right], \quad M_y = \left[\left|\tfrac{\alpha_x}{\alpha_y} M_x + 1\right|\right], \quad N_y = \left[\left|\tfrac{\alpha_x}{\beta_y} M_x + 1\right|\right]$$

and $h \equiv h_x = h_y$, with

$$h = \sqrt{\frac{2\pi d}{\alpha_x M_x}}.$$

The Sinc *trapezoidal quadrature rule will have an exponential order of $\mathcal{O}(\exp(-\sqrt{2\pi d \alpha_x M_x}))$.*

Note that, for $S_j^*(y) = S^*(j, h_y) o \phi(y)$ a same theorem as Theorem 2.1 is established. Now, consider the boundary value problem (2.2) and for simplicity, assume that $U^0 = h(x,y) = 0$. Therefore, an approximate solution for $U^n(x,y) = U(t,x,y_n)$ is presented by:

$$\bar{U}^n(x,y) = \sum_{i=-M_x}^{M_x}\sum_{j=-M_y}^{M_y} c_{i,j}^n S_i(x) S_j^*(y). \tag{2.3}$$

According to the functions $S_{i,j} := S_i(x)S_j^*(y)$, the residual is orthogonal, that is, to obtain the coefficients $c_{i,j}^n$ in (2.3), we get

$$\begin{aligned} &-\left\langle \bar{U}_{xx}^n, S_{i,j}\right\rangle - \left\langle \bar{U}_{yy}^n, S_{i,j}\right\rangle + \left\langle p_1\, \bar{U}_x^n, S_{i,j}\right\rangle + \left\langle p_2\, \bar{U}_y^n, S_{i,j}\right\rangle + \left\langle \left(p_3 + \frac{1}{\lambda}\right)\, \bar{U}^n, S_{i,j}\right\rangle \\ &\qquad = \left\langle f(x,y,t_n), S_{i,j}\right\rangle + \frac{1}{\lambda}\sum_{k=0}^{n-1} \omega_k \left\langle \bar{U}^{n-k}, S_{i,j}\right\rangle \end{aligned} \tag{2.4}$$

where $-M_x \leq i \leq M_x$ and $-M_y \leq j \leq M_y$. The weighted inner product here is defined by

$$\langle f, g\rangle = \int_0^1\int_0^1 f(x,y)g(x,y)w(x)w^*(y)dxdy,$$

where $w(x)w^*(y)$ is a product weight function. Approximations for the integrals in (2.4) are then derived by using the Sinc quadrature procedure described in Theorem 2.2. So, we get

$$\begin{aligned}
-\left\langle \bar{U}^n_{xx}, S_{i,j}\right\rangle &= -\int_0^1\int_0^1 \bar{U}^n_{xx} S_i(x)w(x)S^*_j(y)w^*(y)dxdy \\
&= -\int_0^1\int_0^1 \bar{U}^n(x,y)\frac{\partial^2}{\partial x^2}\left(S_i(x)w(x)\right)S^*_j(y)w^*(y)dxdy \\
&\approx -h_x h_y \sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y} \frac{c^n_{p,q}S^*_j(y_q)w^*(y_q)\frac{\partial^2}{\partial x^2}\left(S_i(x)w(x)\right)\big|_{x=x_p}}{\phi'(x_p)\phi'(y_q)} \\
&\approx -h_x h_y \sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y} \frac{c^n_{p,q}S^*_j(y_q)w^*(y_q)}{\phi'(x_p)\phi'(y_q)}\sum_{s=0}^{2}\frac{\delta^{(s)}_{i,p}}{h^s_x}g_{2,s}(x_p),
\end{aligned}$$

where
$x_p=\phi^{-1}(ph_x)$, $y_q=\phi^{-1}(qh_y)$, $g_{2,2}(x)=w(x)\left(\phi'(x)\right)^2$, $g_{2,1}(x)=w(x)\phi''(x)+2w'(x)\phi'(x)$, $g_{2,0}(x)=w''(x)$ and

$$w(x)=\frac{1}{\sqrt{\phi'(x)}},\qquad w^*(y)=\frac{1}{\sqrt{\phi'(y)}}.$$

Also, we have

$$\begin{aligned}
-\left\langle \bar{U}^n_{yy}, S_{i,j}\right\rangle &= -\int_0^1\int_0^1 \bar{U}^n_{yy} S_i(x)w(x)S^*_j(y)w^*(y)dxdy \\
&= -\int_0^1\int_0^1 \bar{U}^n(x,y)\frac{\partial^2}{\partial y^2}\left[S^*_j(y)w^*(y)\right]S_i(x)w(x)dxdy \\
&\approx -h_x h_y \sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y} \frac{c^n_{p,q}S_i(x_p)w(x_p)\frac{\partial^2}{\partial y^2}\left[S^*_j(y)w^*(y)\right]\big|_{y=y_q}}{\phi'(x_p)\phi'(y_q)} \\
&\approx -h_x h_y \sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y} \frac{c^n_{p,q}S_i(x_p)w(x_p)}{\phi'(x_p)\phi'(y_q)}\sum_{s=0}^{2}\frac{\delta^{(s)}_{j,q}}{h^s_y}g_{2,s}(y_q),
\end{aligned}$$

and

$$\begin{aligned}
\left\langle p_1\,\bar{U}^n_x, S_{i,j}\right\rangle &= \int_0^1\int_0^1 p_1(x,y)\bar{U}^n_x S_i(x)w(x)S^*_j(y)w^*(y)dxdy \\
&= -\int_0^1\int_0^1 \bar{U}^n(x,y)\frac{\partial}{\partial x}\left(p_1(x,y)S_i(x)w(x)\right)S^*_j(y)w^*(y)dxdy \\
&= -\int_0^1\int_0^1 \bar{U}^n(x,y)\frac{\partial}{\partial x}\left(p_1(x,y)\right)S_i(x)w(x)S^*_j(y)w^*(y)dxdy \\
&\quad -\int_0^1\int_0^1 \bar{U}^n(x,y)\frac{\partial}{\partial x}\left(S_i(x)w(x)\right)p_1(x,y)S^*_j(y)w^*(y)dxdy \\
&\approx -h_x h_y \sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y} \frac{c^n_{p,q}S_i(x_p)w(x_p)S^*_j(y_q)w^*(y_q)\frac{\partial}{\partial x}\left(p_1(x,y)\right)\big|_{x=x_p}}{\phi'(x_p)\phi'(y_q)} \\
&\quad -h_x h_y \sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y} \frac{c^n_{p,q}p_1(x_p,y_q)S^*_j(y_q)w^*(y_q)}{\phi'(x_p)\phi'(y_q)}\sum_{s=0}^{1}\frac{\delta^{(s)}_{i,p}}{h^s_x}g^*_{1,s}(x_p),
\end{aligned}$$

where $g^*_{1,0}(x) = w'(x)$ and $g^*_{1,1}(x) = w(x)\phi'(x)$. In addition, we can get:

$$
\begin{aligned}
\langle p_2\, \bar{U}^n_y, S_{i,j} \rangle &= \int_0^1 \int_0^1 p_2(x,y)\bar{U}^n_y S_i(x)w(x)S^*_j(y)w^*(y)dxdy \\
&= -\int_0^1 \int_0^1 \bar{U}^n(x,y)S_i(x)w(x)\frac{\partial}{\partial y}\left(p_2(x,y)S^*_j(y)w^*(y)\right)dxdy \\
&= -\int_0^1 \int_0^1 \bar{U}^n(x,y)S_i(x)w(x)\frac{\partial}{\partial y}\left(p_2(x,y)\right)S^*_j(y)w^*(y)dxdy \\
&\quad -\int_0^1 \int_0^1 \bar{U}^n(x,y)S_i(x)w(x)p_2(x,y)\frac{\partial}{\partial y}\left(S^*_j(y)w^*(y)\right)dxdy \\
&\approx -h_x h_y \sum_{p=-M_x}^{M_x} \sum_{q=-M_y}^{M_y} \frac{c^n_{p,q} S_i(x_p)w(x_p)S^*_j(y_q)w^*(y_q)\frac{\partial}{\partial y}\left(p_2(x,y)\right)\big|_{y=y_q}}{\phi'(x_p)\phi'(y_q)} \\
&\quad -h_x h_y \sum_{p=-M_x}^{M_x} \sum_{q=-M_y}^{M_y} \frac{c^n_{p,q} p_2(x_p,y_q)S_i(x_p)w(x_p)}{\phi'(x_p)\phi'(y_q)} \sum_{s=0}^{1} \frac{\delta^{(s)}_{j,q}}{h^s_y} g^*_{1,s}(y_q),
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle \left(p_3 + \frac{1}{\lambda}\right)\, \bar{U}^n, S_{i,j} \right\rangle &= \int_0^1 \int_0^1 \left(p_3(x,y) + \frac{1}{\lambda}\right)\bar{U}^n(x,y)S_i(x)w(x)S^*_j(y)w^*(y)dxdy \\
&\approx h_x h_y \sum_{p=-M_x}^{M_x} \sum_{q=-M_y}^{M_y} \frac{c^n_{p,q}\left(p_3(x_p,y_q) + \frac{1}{\lambda}\right)S_i(x_p)w(x_p)S^*_j(y_q)w^*(y_q)}{\phi'(x_p)\phi'(y_q)}.
\end{aligned}
$$

Finally for the right side, we obtain:

$$
\begin{aligned}
\langle f(x,y,t_n), S_{i,j} \rangle &= \int_0^1 \int_0^1 f(x,y,t_n)S_i(x)w(x)S^*_j(y)w^*(y)dxdy \\
&\approx h_x h_y \sum_{p=-M_x}^{M_x} \sum_{q=-M_y}^{M_y} \frac{f(x_p,y_q,t_n)S_i(x_p)w(x_p)S^*_j(y_q)w^*(y_q)}{\phi'(x_p)\phi'(y_q)},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\lambda}\sum_{k=0}^{n-1} \omega_k \langle \bar{U}^{n-k}, S_{i,j} \rangle &= \frac{1}{\lambda}\sum_{k=0}^{n-1} \omega_k \int_0^1 \int_0^1 \bar{U}^{n-k}(x,y)S_i(x)w(x)S^*_j(y)w^*(y)dxdy \\
&\approx \frac{h_x h_y}{\lambda}\sum_{k=0}^{n-1} \omega_k \sum_{p=-M_x}^{M_x} \sum_{q=-M_y}^{M_y} \frac{c^{n-k}_{p,q} S_i(x_p)w(x_p)S^*_j(y_q)w^*(y_q)}{\phi'(x_p)\phi'(y_q)}.
\end{aligned}
$$

Theorem 2.3. *If (2.3) is the approximat solution of problem (1.1), then for determination of the unknown coefficients by using the discrete Sinc-Galerkin system $\{c_{i,j}\}$, we get*

$$\begin{aligned}
&\sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y}\frac{c_{p,q}^n}{\phi'(x_p)\phi'(y_q)}\Bigg\{-S_j^*(y_q)w^*(y_q)\sum_{s=0}^{2}\frac{\delta_{i,p}^{(s)}}{h_x^s}g_{2,s}(x_p)-S_i(x_p)w(x_p)\sum_{s=0}^{2}\frac{\delta_{j,q}^{(s)}}{h_y^s}g_{2,s}(y_q)\\
-&\quad\frac{\partial}{\partial x}\left(p_1(x,y)\right)\big|_{x=x_p}S_i(x_p)w(x_p)S_j^*(y_q)w^*(y_q)-p_1(x_p,y_q)S_j^*(y_q)w^*(y_q)\sum_{s=0}^{1}\frac{\delta_{i,p}^{(s)}}{h_x^s}g_{1,s}^*(x_p)\\
-&\quad\frac{\partial}{\partial y}\left(p_2(x,y)\right)\big|_{y=y_q}S_i(x_p)w(x_p)S_j^*(y_q)w^*(y_q)-p_2(x_p,y_q)S_i(x_p)w(x_p)\sum_{s=0}^{1}\frac{\delta_{j,q}^{(s)}}{h_y^s}g_{1,s}^*(y_q)\\
+&\quad\left(p_3(x_p,y_q)+\frac{1}{\lambda}\right)S_i(x_p)w(x_p)S_j^*(y_q)w^*(y_q)\Bigg\}\\
=&\quad\sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y}\frac{f(x_p,y_q,t_n)S_i(x_p)w(x_p)S_j^*(y_q)w^*(y_q)}{\phi'(x_p)\phi'(y_q)}\\
+&\quad\frac{1}{\lambda}\sum_{k=0}^{n-1}\omega_k\sum_{p=-M_x}^{M_x}\sum_{q=-M_y}^{M_y}\frac{c_{p,q}^{n-k}S_i(x_p)w(x_p)S_j^*(y_q)w^*(y_q)}{\phi'(x_p)\phi'(y_q)}.
\end{aligned}$$

3. Convergence analysis of the proposed method

We now demonstrate the unconditional stability and convergence of the aforementioned time-distance technique.

Lemma 3.1. *Suppose that p_1, $p_2\in C^1(\mathcal{D})$ and $p_3\in L_2(\mathcal{D})$. If there are some positive constants c_1 and c_2 which are the lower and upper bounds of p_1,p_2 and p_3, respectively for $(x,y)\in\mathcal{D}$, then, the time discrete scheme (2.2) is stable in $L_2(\mathcal{D})$.*

Proof. Use the symbols U^n and $\tilde{U}^n$ to denote the precise and approximative solutions of (2.2), respectively, and $e^n=U^n-\tilde{U}^n$ to denote the roundoff error. in order to write

$$-\Delta e^n+p_1(x,y)e_x^n+p_2(x,y)e_y^n+\left(p_3(x,y)+\frac{1}{\lambda}\right)e^n=\frac{1}{\lambda}\sum_{k=0}^{n-1}\omega_k e^{n-k}. \tag{3.1}$$

Multiplying both sides of (3.1) by e^n and using Green's formula over $L_2(\mathcal{D})$, yields

$$\begin{aligned}
&\langle\nabla e^n,\nabla e^n\rangle_{L_2(\mathcal{D})}+\langle p_1e_x^n,e^n\rangle_{L_2(\mathcal{D})}+\langle p_2e_y^n,e^n\rangle_{L_2(\mathcal{D})}+\left\langle\left(p_3+\frac{1}{\lambda}\right)e^n,e^n\right\rangle_{L_2(\mathcal{D})}\\
&\qquad=\frac{1}{\lambda}\sum_{k=0}^{n-1}\omega_k\langle e^{n-k},e^n\rangle_{L_2(\mathcal{D})}.
\end{aligned}$$

We have $\langle e_x^n,e_x^n\rangle_{L_2(\mathcal{D})}\geq 0$ and also

$$\begin{aligned}
\langle p_1e_x^n,e^n\rangle_{L_2(\mathcal{D})}&\geq c_1\langle e_x^n,e^n\rangle_{L_2(\mathcal{D})}\\
&=c_1\int_0^1\int_0^1 e_x^n(x,y)e^n(x,y)dxdy\\
&=c_1\int_0^1\int_0^1\frac{1}{2}\frac{\partial}{\partial x}\left((e^n(x,y))^2\right)dxdy=0.
\end{aligned}$$

Similarly, we can obtain:

$$\langle p_2e_y^n,e^n\rangle_{L_2(\mathcal{D})}\geq 0,\quad\langle p_3e^n,e^n\rangle_{L_2(\mathcal{D})}\geq 0.$$

As a result, we obtain:

$$\langle e^n, e^n \rangle_{L_2(\mathcal{D})} \leq \sum_{k=1}^{n-1} \omega_k \left\langle e^{n-k}, e^n \right\rangle_{L_2(\mathcal{D})}.$$

Applying Cauchy-Schwarz inequality leads to:

$$\|e^n\|^2_{L_2(\mathcal{D})} \leq \sum_{k=1}^{n-1} |\omega_k| \|e^{n-k}\|_{L_2(\mathcal{D})} \|e^n\|_{L_2(\mathcal{D})},$$

and

$$\begin{aligned} \|e^n\|_{L_2(\mathcal{D})} &\leq \sum_{k=1}^{n-1} \omega_k \|e^{n-k}\|_{L_2(\mathcal{D})} \\ &\leq (\omega_1 + \omega_2 + \cdots + \omega_{n-1}) \left\|e^1\right\|_{L_2(\mathcal{D})} \\ &= (b_0 - b_{n-1}) \left\|e^1\right\|_{L_2(\mathcal{D})} \\ &\leq \left\|e^1\right\|_{L_2(\mathcal{D})}, \end{aligned}$$

which is the desired result. □

We now consider the convergence property of the above time discrete scheme.

Lemma 3.2. *Assume that the conditions of Lemma 3.1 is satisfied. Let $u^n = u(x, t_n)$ and $U^n = u(x, t_n)$ be the exact solutions of (1.1) and (2.2), respectively. That is U^n is an approximation solution of (1.1), then the time discrete solution is convergent in $L_2(\mathcal{D})$.*

Proof. Again, we have

$$-\Delta E^n + p_1(x,y) E^n_x + p_2(x,y) E^n_y + \left(p_3(x,y) + \frac{1}{\lambda}\right) E^n = \frac{1}{\lambda} \sum_{k=0}^{n-1} \omega_k E^{n-k} + \mathcal{R},$$

where $E^n = u^n - U^n$. Multiplying both sides by E^n and using Green's formula over $L_2(\mathcal{D})$, yields

$$\begin{aligned} &\langle \nabla E^n, \nabla E^n \rangle_{L_2(\mathcal{D})} + \langle p_1 E^n_x, E^n \rangle_{L_2(\mathcal{D})} + \left\langle p_2 E^n_y, E^n \right\rangle_{L_2(\mathcal{D})} + \left\langle \left(p_3 + \frac{1}{\lambda}\right) E^n, E^n \right\rangle_{L_2(\mathcal{D})} \\ &\quad = \frac{1}{\lambda} \sum_{k=0}^{n-1} \omega_k \left\langle E^{n-k}, E^n \right\rangle_{L_2(\mathcal{D})} + \langle \mathcal{R}, E^n \rangle_{L_2(\mathcal{D})}. \end{aligned}$$

Similar to Lemma 3.1, one can easily obtain

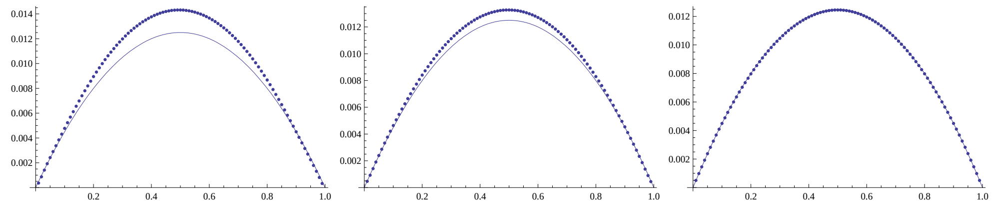
$$\langle E^n, E^n \rangle_{L_2(\mathcal{D})} \leq \sum_{k=1}^{n-1} \omega_k \left\langle E^{n-k}, E^n \right\rangle_{L_2(\mathcal{D})} + \lambda \left\langle \mathcal{R}, E^n \right\rangle_{L_2(\mathcal{D})}.$$

Applying Cauchy-Schwarz inequality leads to:

$$\|E^n\|^2_{L_2(\mathcal{D})} \leq \sum_{k=1}^{n-1} |\omega_k| \|E^{n-k}\|_{L_2(\mathcal{D})} \|E^n\|_{L_2(\mathcal{D})} + \lambda |\mathcal{R}| \|E^n\|_{L_2(\mathcal{D})},$$

and

$$\|E^n\|_{L_2(\mathcal{D})} \leq \sum_{k=1}^{n-1} |\omega_k| \|E^{n-k}\|_{L_2(\mathcal{D})} + \lambda |\mathcal{R}|.$$



(a) $M_x = M_y = 6$. (b) $M_x = M_y = 8$. (c) $M_x = M_y = 10$.

FIGURE 1. The function $\bar{U}^1(x,y)$ with different values of M_x and M_y for Example 4.1.

Since $|\mathcal{R}| \le C\tau^{2-\alpha}$ and $\lambda = \Gamma(2-\alpha)\tau^{\alpha}$, we obtain

$$\|E^n\|_{L_2(\mathcal{D})} \le \sum_{k=1}^{n-1} |\omega_k| \|E^{n-k}\|_{L_2(\mathcal{D})} + C\Gamma(2-\alpha)\tau^2.$$

Such as Lemma 3.1, we get

$$\begin{aligned}
\|E^n\|_{L_2(\mathcal{D})} &\le \sum_{k=1}^{n-1} \omega_k \|E^n\|_{L_2(\mathcal{D})} + C\Gamma(2-\alpha)\tau^2 \\
&\le (\omega_1 + \omega_2 + \cdots + \omega_{n-1}) \left\|E^1\right\|_{L_2(\mathcal{D})} + C\Gamma(2-\alpha)\tau^2 \\
&= (b_0 - b_{n-1}) \left\|E^1\right\|_{L_2(\mathcal{D})} + C\Gamma(2-\alpha)\tau^2 \\
&\le \left\|E^1\right\|_{L_2(\mathcal{D})} + C\Gamma(2-\alpha)\tau^2,
\end{aligned}$$

which completes the proof. □

4. Numerical examples

In this section, we apply the proposed method for three problems. In the examples, the maximum absolute error at points $(0.1i,\, 0.5, t_1)$, $i = 1, 2, \cdots 9$ is taken as

$$E_G = \max_{i \in \{1,2,\cdots 9\}} |u(0.1i,\, 0.5, t_1) - \bar{u}(0.1i,\, 0.5, t_1)|\,.$$

Also, we take $T = 1$, $d = \frac{\pi}{2}$, $\alpha = 0.2, 0.5, 0.8$, $n = 5$ and a sequence of runs for $M_x = M_y = 6, 8$ and 10. Note that all the experiments are performed in Mathematica 12.0 by a system with this specification: Intel(R) Core(TM) i7-7500U CPU @ 2.70GHz 2.90GHz

Example 4.1. Consider the problem

$$\begin{aligned}
&{}^{C}D_t^{0.5} u(x,y,t) - \Delta u(x,y,t) + \frac{\partial u(x,y,t)}{\partial x} + \frac{\partial u(x,y,t)}{\partial y} + u(x,y,t) = f(x,y,t) \\
&u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \quad 0 < t < 1,\ (x,y) \in \partial\mathcal{D} \\
&u(x,y,0) = 0, \qquad (x,y) \in \mathcal{D}.
\end{aligned} \tag{4.1}$$

The exact solution of the problem is $u(x,y,t) = txy(1-x)(1-y)$. The function $\bar{U}^1(x,y)$ is the approximate solution with different values of M_x and M_y are shown in Figures 1, 2, and 3. Therefore, the maximum absolute error E_G for different values of α, M_x and M_y is listed in Table 1. In these figures and the table, we can clearly see that the exact solution is consistent with the approximate solution.

Example 4.2. Again we consider the problem (4.1) by a different exact solution

$$u(x,y,t) = t\left(1 - \sqrt[4]{x}\right)\sin(\pi x)\sin(\pi y).$$

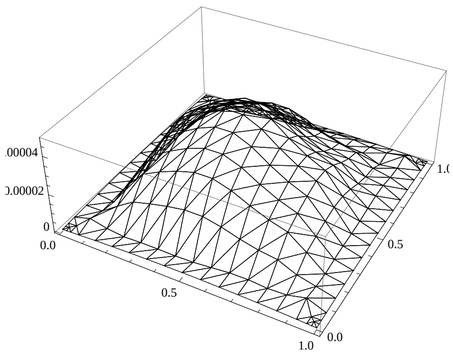


FIGURE 2. The error function $\left|u(x,y,t_1) - \bar{U}^1(x,y)\right|$ for $M_x = M_y = 10$ for Example 4.1.

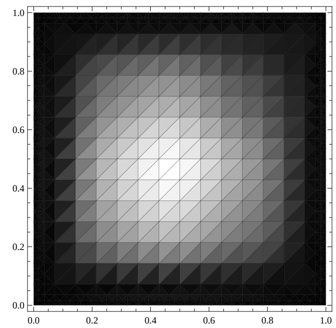
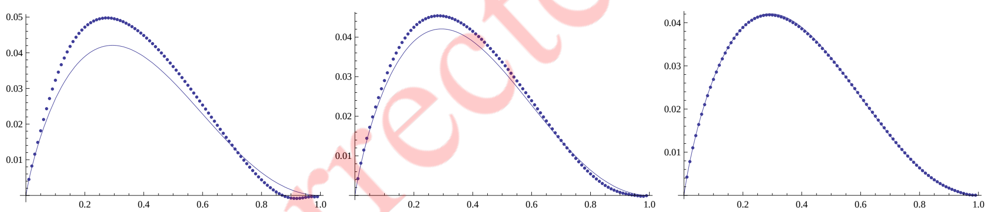


FIGURE 3. The density plot of the error function $\left|u(x,y,t_1) - \bar{U}^1(x,y)\right|$ for $M_x = M_y = 10$ for Example 4.1.

TABLE 1. The maximum absolute errors E_G for different values of α, M_x and $M_y$for Example 4.1.

α_i	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$	CPU Time(s)
Maximum absolute error	E_G	E_G	E_G	
$M_x = M_y = 6$	1.906×10^{-3}	1.818×10^{-3}	1.709×10^{-3}	543
$M_x = M_y = 8$	8.064×10^{-4}	7.717×10^{-4}	7.299×10^{-4}	721
$M_x = M_y = 10$	2.622×10^{-6}	2.013×10^{-6}	1.273×10^{-6}	934



(a) $M_x = M_y = 6$. (b) $M_x = M_y = 8$. (c) $M_x = M_y = 10$.

FIGURE 4. Approximation of the solution $\bar{U}^1(x,y)$ for various values of M_x and M_y for Example 4.2.

In Figures 4, 5, and 6, the approximation of the solution $\bar{U}^1(x,y)$ for various values of M_x and M_y is shown. Additionally, Table 2 lists the maximum absolute errors E_G of the suggested technique for various values of α, M_x and M_y. We can see from these figures and table that the approximate solution also yields the desired result.

TABLE 2. The maximum absolute errors E_G for different values of α, M_x and M_y for Example 4.2.

α_i	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$	CPU Time(s)
Maximum absolute error	E_G	E_G	E_G	
$M_x = M_y = 6$	8.473×10^{-3}	8.228×10^{-3}	7.918×10^{-3}	578
$M_x = M_y = 8$	3.639×10^{-3}	3.547×10^{-3}	3.430×10^{-3}	784
$M_x = M_y = 10$	1.145×10^{-4}	1.122×10^{-4}	1.041×10^{-4}	986

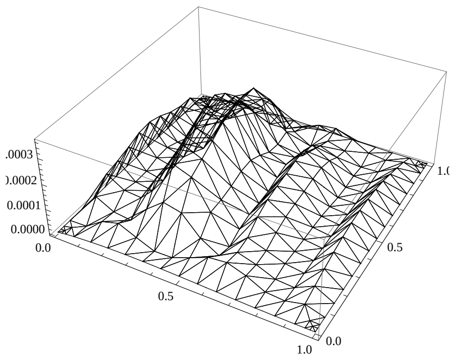


FIGURE 5. The error function $\left|u(x,y,t_1) - \bar{U}^1(x,y)\right|$ for $M_x = M_y = 10$ for Example 4.2.

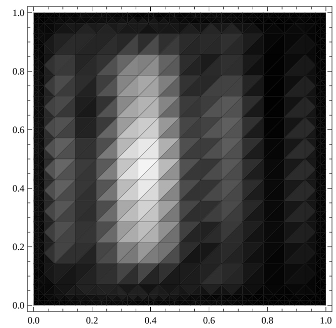


FIGURE 6. The density plot of the error function $\left|u(x,y,t_1) - \bar{U}^1(x,y)\right|$ for $M_x = M_y = 10$ for Example 4.2.

TABLE 3. The maximum absolute errors E_G for different values of α, M_x and M_y for Example 4.3.

α_i	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$	CPU Time(s)
Maximum absolute error	E_G	E_G	E_G	
$M_x = M_y = 6$	7.563×10^{-4}	6.005×10^{-4}	6.624×10^{-4}	604
$M_x = M_y = 8$	3.476×10^{-4}	3.532×10^{-4}	4.939×10^{-4}	703
$M_x = M_y = 10$	3.26×10^{-5}	4.709×10^{-5}	6.923×10^{-5}	1018

Example 4.3. In order to compare the proposed method with other methods, we use [26], where a time-dependent fractional advection diffusion equation is given and solved by moving least squares (MLS) approximation. The numerical results for different value of M_x, M_y and α are given in Table 3. The numerical results show that the proposed method given an effective method compare to the methods eg. MLS method technique.
Consider the following problem

$$
\begin{aligned}
&{}^{C}D_t^{\alpha} u(x,y,t) - \Delta u(x,y,t) + \frac{\partial u(x,y,t)}{\partial x} + \frac{\partial u(x,y,t)}{\partial y} + u(x,y,t) = f(x,y,t) \\
&\quad u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \quad 0 < t < 1,\ (x,y) \in \partial\mathcal{D} \qquad (4.2)\\
&\quad u(x,y,0) = 0, \qquad (x,y) \in \mathcal{D}. \qquad (4.3)
\end{aligned}
$$

The exact solution of the problem is $u(x,y,t) = t^{2+\alpha} e^{x+y} xy(1-x)(1-y)$.

CONCLUSION

In this paper, we used finite difference and Sinc-Galerkin methods. The time fractional advection-diffusion equation has an approximate solution, and a computer approach is developed to find it. We can quickly find appropriate solutions in a rectangular domain thanks to the characteristics of the Sinc-Galerkin method. The suggested method's convergence was also provided. The numerical results based on the suggested method were then presented, demonstrating its numerical accuracy and convergence.

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