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Numerical solution of two-dimensional nonlinear schrödinger equation using an alternating direction implicit method

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Abstract

In this paper, an alternating direction implicit (ADI) finite difference scheme for solving two-dimensional time-dependent nonlinear Schrödinger equation is presented. In the scheme, the nonlinear term in the equation is linearized by using the values of the wave function at previous time level at each iteration step. The block tridiagonal system of algebraic equations resulted from the discretization is solved using Gauss-Seidel method via sparse matrix computation. The stability of scheme is studied using matrix analysis and found to be conditionally stable. Numerical examples are provided to demonstrate the efficiency, stability and accuracy of the numerical scheme. The obtained numerical results are in a good agreement with exact solutions.

Keywords. Nonlinear Schr ödinger equation, Time-dependent, Two-dimensional, ADI method, Block tridiagonal system, Sparse matrix, Gauss-Seidel method.

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1. Introduction

The nonlinear Schrödinger equation is widely used to describe several physical phenomena in various fields of science and engineering including quantum mechanics, plasma physics, nonlinear optics, water waves, bimolecular dynamics and electromagnetic propagation [5, 6, 22]. The nonlinear Schrödinger equation in two-dimension can be written in the following form [4]:

$$i\frac{\partial u(x,y,t)}{\partial t} + \alpha\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \beta\left|u(x,y,t)\right|^2 u(x,y,t) + p(x,y)u(x,y,t) = 0, \tag{1.1}$$

$(x, y) \in \Omega$, and $t \in (0, T)$,
with initial condition

$$u(x, y, 0) = f(x, y), (x, y) \in \Omega,$$

and boundary condition
$u(x, y, t) = g(x, y, t), (x, y) \in \partial\Omega$ and $t \in (0, T)$,
where $u(x, y, t)$ is the complex value wave function,$i = \sqrt{-1}$, $\Omega = [a, b] \times [c, d] \subset \mathbb{R}^2$, $\partial\Omega$ is the boundary of Ω, a and b are real constants, f and g are sufficiently smooth functions and $p(x, y)$ is a potential function which is real valued and bounded on Ω.

Due to the importance of nonlinear Schrödinger equation for describing several physical phenomena, finding a solution of the equation is essential. Analytic solutions of nonlinear Schrödinger equation are complicated to obtain [7, 8, 18, 19] and thus numerical techniques are widely used. Several numerical methods have been used by authors to

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solve the nonlinear Schrödinger equation. Xu and Zhang [24] presented four ADI schemes for solving two-dimensional nonlinear Schrödinger equation. The authors confirmed the stability of the numerical schemes and compared their accuracy and CPU time by conducting numerical experiments. Bratsos [2] presented a linearized finite difference method to obtain solution of nonlinear Schrödinger equation. The author replaced the nonlinear term by a parametric linearized expression based on Taylors expansion. Lin et al. [16] performed numerical simulation nonlinear Schrödinger equation using implicit-Euler scheme and approximating the unknown function by Gaussian radial basis function. They verified the efficiency and stability of the numerical scheme through numerical experiments and quantified the error magnitude in solving 3D nonlinear Schrödinger problems. Eskar et al. [6] presented a high order compact finite difference method for solving nonlinear Schrödinger equation. The authors demonstrated that these schemes maintain conservation laws and offer precise and stable solutions for both linear and nonlinear 3D Schrödinger equations. Cavalcanti et al.[3] applied a finite difference scheme to solve higher order nonlinear Schrödinger equation. This scheme is designed to uphold the numerical L2 norm and regulate energy based on chosen parameters of the equation.

Shivanian and Jafarabadi [21] used spectral meshless radial point interpolation technique for solving two-dimensional nonlinear Schrödinger equation. The authors applied a predictor-corrector method to eliminate nonlinearity. They demonstrated the stability and convergent of the numerical method and validated accuracy by taking numerical examples. Pathak et al. [20] introduced a simple, stable, efficient, and accurate numerical technique (the Kansa method with polyharmonic radial basis function) for solving generalized 2-D nonlinear Schrödinger equations, supported by stability analysis. Jiwari et al. [14] used meshfree approach to solve nonlinear Schrödinger equation. The authors employed local radial function-based differential quadrature method to reduce the problem of ill-conditioned. Karaba et al. [15] uses meshless method with radial basis functions based on Frechet derivative to solve nonlinear Schrödinger equation.

Iqbal et al. [12] applied cubic B-spline Galerkin method to solve Schrödinger equation. The efficiency and accuracy of the method was evaluated using three different cases: a single solitary wave, the collision of two solitary waves, and the collision of three solitary waves. Arora et al. [1] used trigonometric cubic B-spline basis function with differential quadrature method to simulate nonlinear Schrödinger equations. This method transforms the nonlinear equation into a collection of ordinary differential equations, which can then be solved using the Runge-Kutta method. The obtained numerical results were found to closely match the exact solution. He and Lin et al. [9] used Lattice Boltzmann method for analysis and simulation of coupled nonlinear Schrödinger equation. The numerical results obtained using this method were compared with those from the finite difference method and analytical method to validate its efficiency. Ismail [13], Hu [10], Iqbal et al. [11] and Wang and Li [23] used different approach of finite element method to solve nonlinear Schrödinger equation. Dehghan et al. [4] used time-space pseudo-spectral method to find the solution of nonlinear Schrödinger equation. The authors verified that this method offers a satisfactory approximation even when using a relatively small number of points. Liu et al. [17] applied Harr wavelets multi-resolution collocation procedures to solve nonlinear Schrödinger equations. The stability analysis of the proposed methods was conducted, indicating their accuracy and efficiency in time compared to other methods.

Several authors have used different techniques to develop linearized numerical schemes to solve nonlinear Schrödinger equation. As per the authors knowledge, some of the linearization techniques required long process for the formulation of the numerical schemes. The aim of this work is to develop an alternating direction implicit scheme by replacing the nonlinear term by values of the unknown variable from previous time level and investigate its practicality for solving nonlinear Schrödinger equation. The numerical scheme has been tested with by solving different nonlinear Schrödinger equations.

2. Numerical Scheme

In this study an alternating direction implicit scheme is used to solve (1.1). This scheme involves two stages of solving block tridiagonal systems of equations along the lines parallel to the x- and y-axis. To solve Eq. (1.1) with the scheme, we divide the interval $[a,b]$ into N_x subintervals with step size Δx, the interval $[c,d]$ into N_y subintervals with step size Δy and time interval $[0,T]$ into N_t subintervals with step size Δx . The grid points of the subdivisions are

$$x_1, x_2, ..., x_{Nx}+1, x_1 = a, x_{Nx}+1 = b, x_j = x_1 + (j-1)\Delta x, j = 2, 3, ..., Nx,$$

$$y_1, y_2, ..., y_{Ny}+1, y_1 = c, y_{Ny}+1 = d, y_k = y_1 + (k-1)\Delta y, k = 2, 3, ..., Ny,$$

$$t_1, t_2, ..., t_{Nt}+1, t_1 = 0, t_{Nt}+1 = T, t_n = t_1 + (n-1)\Delta t, n = 2, 3, ..., Nt.$$

The value of $u(x,y,t)$ at (x_j, y_k, t_n) is approximated as $U^n_{j,k}$ in the numerical approximation. The two stages of the numerical scheme are discussed as follows in the discretization of Eq. (1.1). In the first stage, the derivatives $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ at $(j,k,n+\frac{1}{2})$, $(j,k,n+1)$ and (j,k,n), respectively are approximated by central differences. The nonlinear term and the last term in the left side of Eq. (1.1) are approximated by values of the functions at (j,k,n). From these, we get

$$i\left(\frac{U^{n+1}_{j,k}-U^n_{j,k}}{\Delta t}\right)+\alpha\left(\frac{U^{n+1}_{j+1,k}-2U^{n+1}_{j,k}+U^{n+1}_{j-1,k}}{\Delta x^2}+\frac{U^n_{j,k+1}-2U^n_{j,k}+U^n_{j,k-1}}{\Delta y^2}\right)+\beta\left|U^n_{j,k}\right|^2U^n_{j,k}+p_{i,j}U^n_{j,k}=0,$$

or

$$r_xU^{n+1}_{j-1,k}+(i-2r_x)U^{n+1}_{j,k}+r_xU^{n+1}_{j+1,k}=-r_yU^n_{j,k-1}+(i+2r_y-\beta\Delta t\left|U^n_{j,k}\right|^2-\Delta tp_{i,j})U^n_{j,k}-r_yU^n_{j,k+1}, \qquad (2.1)$$

where $r_x = \frac{\alpha\Delta t}{\Delta x^2}$ and $r_y = \frac{\alpha\Delta t}{\Delta y^2}$.

In the second stage we advance from $(n+1)^{th}$ to $(n+2)^{th}$ time level to approximate $\frac{\partial^2 u}{\partial y^2}$ at $(j,k,n+2)$ to obtain the discretization of Eq. (1.1) as

$$i\left(\frac{U^{n+2}_{j,k}-U^{n+1}_{j,k}}{\Delta t}\right)+\alpha\left(\frac{U^{n+1}_{j+1,k}-2U^{n+1}_{j,k}+U^{n+1}_{j-1,k}}{\Delta x^2}+\frac{U^{n+2}_{j,k+1}-2U^{n+2}_{j,k}+U^{n+2}_{j,k-1}}{\Delta y^2}\right)+\beta\left|U^{n+1}_{j,k}\right|^2U^{n+1}_{j,k}+p_{i,j}U^{n+1}_{j,k}=0,$$

or

$$r_yU^{n+2}_{j-1,k}+(i-2r_y)U^{n+2}_{j,k}+r_yU^{n+2}_{j+1,k}=-r_xU^{n+1}_{j,k-1}+\left(i+2r_x-\beta\Delta t\left|U^{n+1}_{j,k}\right|^2-\Delta tp_{i,j}\right)U^{n+1}_{j,k}-r_xU^{n+1}_{j,k+1}. \qquad (2.2)$$

By finding truncation errors of the discretizations (2.1) and (2.2) it can be shown that the scheme is first order accurate in time and second order accurate in space.

To see the basic form of the matrix equations resulted from (2.1) and (2.2), let us take $N_x = N_y = 4$ and $r_x = r_y = r$. The iterative schemes (2.1) and (2.2) yields matrix equations

$$A_{1x}U^{n+1}_x = A_{2x}U^n_x + b_{1x}, \qquad (2.3)$$

where

$$A_{1x}=\begin{bmatrix} i-2r & r & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ r & i-2r & r & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & r & i-2r & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & i-2r & r & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & r & i-2r & r & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & r & i-2r & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & i-2r & r & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & r & i-2r & r\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & i-2r \end{bmatrix},$$

$$A_{2x}=\begin{bmatrix} i+2r-B_{2,2} & 0 & 0 & -r & 0 & 0 & 0 & 0 & 0\\ 0 & i+2r-B_{3,2} & 0 & 0 & -r & 0 & 0 & 0 & 0\\ 0 & 0 & i+2r-B_{4,2} & 0 & 0 & -r & 0 & 0 & 0\\ -r & 0 & 0 & i+2r-B_{2,3} & 0 & 0 & -r & 0 & 0\\ 0 & -r & 0 & 0 & i+2r-B_{3,3} & 0 & 0 & -r & 0\\ 0 & 0 & -r & 0 & 0 & i+2r-B_{4,3} & 0 & 0 & -r\\ 0 & 0 & 0 & -r & 0 & 0 & i+2r-B_{2,4} & 0 & 0\\ 0 & 0 & 0 & 0 & -r & 0 & 0 & i+2r-B_{3,4} & 0\\ 0 & 0 & 0 & 0 & 0 & -r & 0 & 0 & i+2r-B_{4,4} \end{bmatrix},$$

$B_{j,k}=\beta\Delta t\left|U_{j,k}^{n}\right|^2+\Delta t p_{i,j},$

$$U_x^{n+1}=\begin{bmatrix} U_{2,2}^{n+1}\\ U_{3,2}^{n+1}\\ U_{4,2}^{n+1}\\ U_{2,3}^{n+1}\\ U_{3,3}^{n+1}\\ U_{4,3}^{n+1}\\ U_{2,4}^{n+1}\\ U_{3,4}^{n+1}\\ U_{4,4}^{n+1} \end{bmatrix}, \quad U_x^{n}=\begin{bmatrix} U_{2,2}^{n}\\ U_{3,2}^{n}\\ U_{4,2}^{n}\\ U_{2,3}^{n}\\ U_{3,3}^{n}\\ U_{4,3}^{n}\\ U_{2,4}^{n}\\ U_{3,4}^{n}\\ U_{4,4}^{n} \end{bmatrix}, \quad b_{1x}=\begin{bmatrix} -rU_{1,2}^{n+1}\\ 0\\ -rU_{5,2}^{n+1}\\ -rU_{1,3}^{n+1}\\ 0\\ -rU_{5,3}^{n+1}\\ -rU_{1,4}^{n+1}\\ 0\\ -rU_{5,4}^{n+1} \end{bmatrix}, + \begin{bmatrix} rU_{2,1}^{n}\\ -rU_{3,1}^{n}\\ -rU_{4,1}^{n}\\ 0\\ 0\\ 0\\ -rU_{2,5}^{n}\\ -rU_{3,5}^{n}\\ -rU_{4,5}^{n} \end{bmatrix},$$

$$A_{1y}U_y^{n+2}=A_{2y}U_y^{n+1}+b_{1y}, \tag{2.4}$$

where

$A_{1y}=A_{1x},$

$$A_{2y}=\begin{bmatrix} i+2r-C_{2,2} & 0 & 0 & -r & 0 & 0 & 0 & 0 & 0\\ 0 & i+2r-C_{2,3} & 0 & 0 & -r & 0 & 0 & 0 & 0\\ 0 & 0 & i+2r-C_{2,4} & 0 & 0 & -r & 0 & 0 & 0\\ -r & 0 & 0 & i+2r-C_{3,2} & 0 & 0 & -r & 0 & 0\\ 0 & -r & 0 & 0 & i+2r-C_{3,3} & 0 & 0 & -r & 0\\ 0 & 0 & -r & 0 & 0 & i+2r-C_{3,4} & 0 & 0 & -r\\ 0 & 0 & 0 & -r & 0 & 0 & i+2r-C_{4,2} & 0 & 0\\ 0 & 0 & 0 & 0 & -r & 0 & 0 & i+2r-C_{4,3} & 0\\ 0 & 0 & 0 & 0 & 0 & -r & 0 & 0 & i+2r-C_{4,4} \end{bmatrix},$$

$C_{j,k}=\beta\Delta t\left|U_{j,k}^{n+1}\right|^2+\Delta t p_{i,j},$

$$U_y^{n+2}=\begin{bmatrix} U_{2,2}^{n+2}\\ U_{2,3}^{n+2}\\ U_{2,4}^{n+2}\\ U_{3,2}^{n+2}\\ U_{3,3}^{n+2}\\ U_{3,4}^{n+2}\\ U_{4,2}^{n+2}\\ U_{4,3}^{n+2}\\ U_{4,4}^{n+1} \end{bmatrix}, \quad U_y^{n+1}=\begin{bmatrix} U_{2,2}^{n+1}\\ U_{2,3}^{n+1}\\ U_{2,4}^{n+1}\\ U_{3,2}^{n+1}\\ U_{3,3}^{n+1}\\ U_{3,4}^{n+1}\\ U_{4,2}^{n+1}\\ U_{4,3}^{n+1}\\ U_{4,4}^{n+1} \end{bmatrix}, \quad b_{1y}=\begin{bmatrix} -rU_{2,1}^{n+2}\\ 0\\ -rU_{2,5}^{n+2}\\ -rU_{3,1}^{n+2}\\ 0\\ -rU_{3,5}^{n+2}\\ -rU_{4,1}^{n+2}\\ 0\\ -rU_{4,5}^{n+2} \end{bmatrix} + \begin{bmatrix} -rU_{1,2}^{n+1}\\ -rU_{1,3}^{n+1}\\ -rU_{1,4}^{n+1}\\ 0\\ 0\\ 0\\ -rU_{5,2}^{n+1}\\ -rU_{5,3}^{n+1}\\ -rU_{5,4}^{n+1} \end{bmatrix},$$

The system described in (2.3) and (2.4) can be easily generalized for any mesh size. As it is observed in the above discussion, the scheme requires solving block tridiagonal system of equations. Gauss-Seidel method with sparse matrix computation is applied to solve the system at each stage of the scheme.

Table 1. Spectral radii of $A_{1x}^{-1}A_{2x}$ for solving Schrdinger equation for different mesh sizes.

$\Delta x \downarrow \Delta t \rightarrow$	0.01	0.005	0.001	0.0005	0.0001
0.1	2.002261	1.479790	1.036043	1.009378	1.000380
0.05	3.981984	2.829080	1.370533	1.130283	1.006262
0.025	7.974991	5.650443	2.543146	1.829717	1.089373

Table 2. Spectral radii of $A_{1x}^{-1}A_{2x}$ for solving heat Equation (3.1) for different mesh sizes.

$\Delta x \downarrow \Delta t \rightarrow$	0.01	0.005	0.001	0.0005	0.0001
0.1	1.625842	0.952198	0.990259	0.995118	0.999022
0.05	3.683124	2.573814	0.990199	0.995087	0.999016
0.025	7.566080	5.431009	2.310299	1.477936	0.999014

3. Stability Analysis

Here we discuss the stability of the scheme using matrix analysis. Consider the matrices and vectors in (2.3) and (2.4) for any mesh size with $N_x = N_y$. The vectors b_{1x} and b_{1y} contain values of the wave function at the boundaries and there is no error at the boundaries. Thus, the scheme is stable if the modulus of each eigenvalue of the matrices $A_{1x}^{-1}A_{2x}$ and $A_{1y}^{-1}A_{2y}$ is less than or equal 1. Let us consider the case when $\beta = 0$ and $p(x, y)$ which gives $A_{1x}^{-1}A_{2x} = A_{1y}^{-1}A_{2y}$. The maximum of the modulus of eigenvalues of $A_{1x}^{-1}A_{2x}$ (spectral radii) taking $\Omega = [0, 1] \times [0, 1]$ and $\alpha = 1$ at different spatial and time steps is presented in Table 1. The corresponding spectral radii for solving the two-dimensional heat equation

$$\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial^2 T(x,y,t)}{\partial x^2} + \frac{\partial^2 T(x,y,t)}{\partial y^2}, \tag{3.1}$$

using the scheme at the same domain and step sizes is shown in Table 2.

From Table 1 and Table 2, it is observed that the scheme is conditionally stable for solving two-dimensional Schrödinger and heat equations.

4. Numerical Results and Discussions

In this section, numerical examples are provided to describe the efficiency and accuracy of the numerical scheme. Comparisons of numerical and exact solutions are presented in graphically. In the examples, the same grid points on the x-axis and y-axis,$N_x = N_y = N$, is taken. $R(x, y, t)$ and $I(x, y, t)$ represent the real and imaginary part of u, respectively. The accuracy of the numerical scheme is tested using the absolute maximum error.

$$E = \max_{1 \le j,k \le N} \left| u(x_j, y_k, t_n) - U_{j,k}^n \right|, \tag{4.1}$$

where $u(x_j, y_k, t_n)$ and $U_{j,k}^n$ are the exact and numerical solutions of u , respectively. The computations are carried out using MATLAB codes in PC with Windows 10 OS (64-bit), Intel(R) CORE i7-7500U, CPU@ 2.9 GHz and 8GB RAM memory.

Example 4.1. Consider a two-dimensional nonlinear Schrödinger Equation, (1.1) , with $\alpha = 1$, $\beta = 2\pi^2 - 1$ and potential function $p(x, y) = (2\pi^2 - 1)(1 - cos^2\pi x cos^2\pi y)$ [21]. The initial condition and boundary conditions can be obtained from the exact solution $u(x, y, t) = \cos \pi x \cos \pi y e^{-it}$.

The equation is solved on $\Omega = [0, 1] \times [0, 1]$ for $t > 0$. Figure 1 shows the surface plot of the numerical and exact solutions of Example 1 at $t == 1$ using $N = 40$, $T = 1$ and $\Delta t = 0.005$. The maximum absolute errors for real part and imaginary part are $2.4933e-04$ and $1.811e-04$ respectively.From the computational results, the solutions at

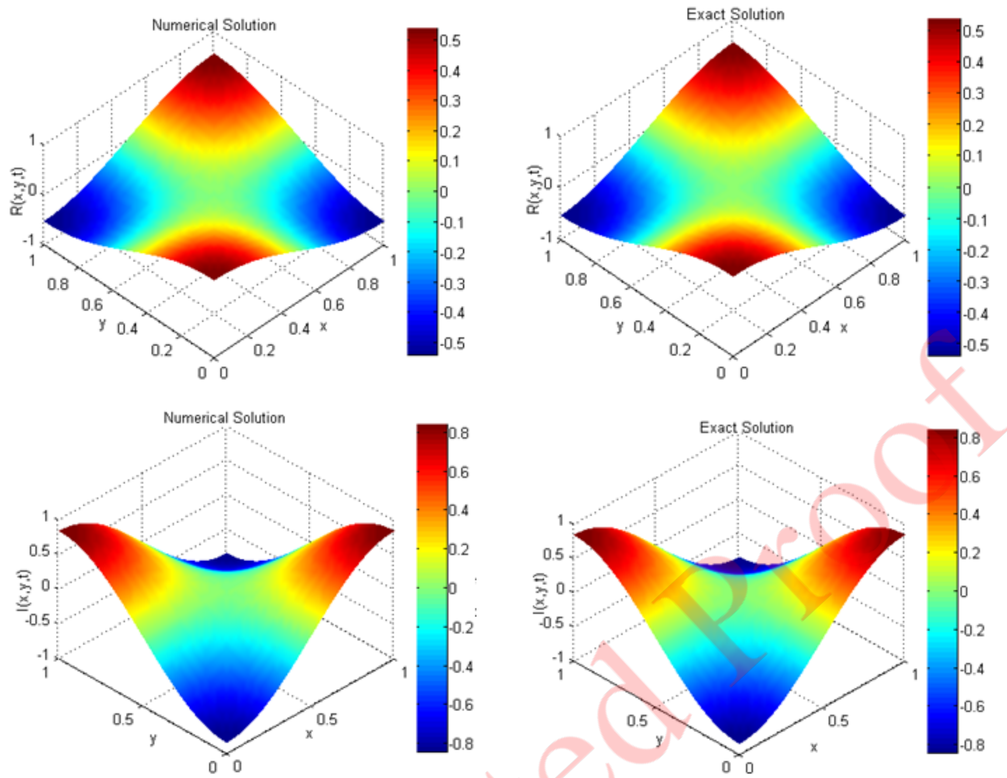

FIGURE 1. Surface plot of numerical and exact solutions for Example 4.1 at $t = 1$ with$N = 40$, $T = 1$ and $\Delta t = 0.005$.

TABLE 3. Maximum absolute errors and CPU times with $\Delta t = 0.001$, $T = 1$ and different mesh sizes for Example 4.1.

	R	I	
N	E	E	CPU time(s)
10	1.933832 e-03	1.065042 e-03	22.415811
20	6.296480e-04	6.811864e-04	148.841399
40	2.385183e-04	1.441837e-04	3035.401684
80	1.717893e-04	6.176003e-06	38047.729067

$y = 0.2$ is displayed in Figure 2. From the figures, we observe that the numerical solution in a good agreement with the exact solution and analogous to [21]. In Table 3, the maximum absolute error and CPU time of the scheme are presented for $N = 10$, 20, 40, 80 and $\Delta t = 0.001$. The absolute error decreases as number of mesh decreases. Table 4 displays the maximum absolute error and CPU time by taking $T = 1$ and $N = 40$ for different time step sizes. These tables show the accuracy and convergence of the numerical scheme.

Example 4.2. Consider (1.1) with $\alpha = \frac{1}{2}$, $\beta = -1$ and potential function $p(x, y) = -1 + \sin^2 x \cos^2 y$ on $\Omega = [0, 2\pi] \times [0, 2\pi]$ for $t > 0$ [24]. The exact solution is $u(x, y, t) = (\sin x \cos x)e^{-2it}$ and the initial condition and boundary conditions are obtained from this solution. Numerical solution of Example 4.2 is obtained at $t = 1$ with $N = 60$, $T = 1$ and $\Delta t = 0.005$. In the computation, the maximum absolute errors of the scheme for real and imaginary part are $3.5435e - 3$ and $2.9207e - 3$, respectively. Figure 3 shows surface plot of numerical and exact solutions for real

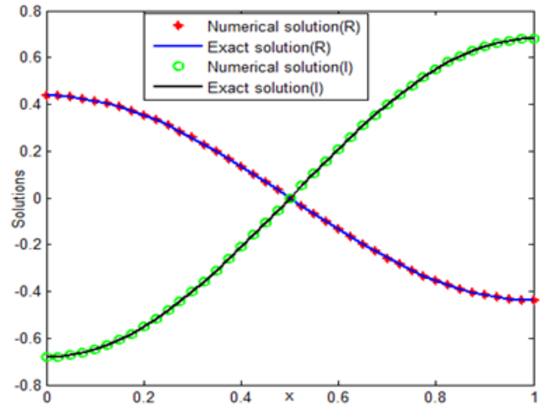

FIGURE 2. Graph of numerical and exact solutions for Example 4.1 at $y = 0.2$ and $t = 1$ with $N = 40$, $T = 1$ and $\Delta t = 0.005$.

TABLE 4. Maximum absolute errors and CPU times with$N = 40$ and $T = 1$ and different time step sizes for Example 4.1.

	R	I	
N_t	E	E	CPU time(s)
10	9.715348 e-03	5.730540 e-03	29.520303
50	2.223621 e-03	7.149175e-04	162.029855
100	1.029718e-03	4.836240e-04	223.20477
1000	2.385183e-04	1.441837e-04	3035.401684

and imaginary part of u. The numerical and numerical solutions at the diagonal, joining the points $(0, 2\pi)$ and $(2\pi, 0)$ of the domain, is displayed in Figure 4. As it is observed from the figures, the numerical solutions coincide with the exact solution showing the accuracy of the numerical scheme.

Example 4.3. Consider Eq. (1.1) with $\alpha = \frac{1}{2}$, $\beta = 1$ and potential function $p(x, y) = 1 - \sinh x \sinh y - sinh^2 x sinh^2 y$ on $\Omega = [0, 1] \times [0, 1]$ for $t > 0$.The exact solution is $u(x, y, t) = (i \sin x \sinh y)e^{it}$. As the previous examples, the initial condition and boundary conditions can be computed from the exact solution.

For computational work, $N = 50$, $T = 1$, $\Delta t = 0.005$ and $t = 1$ are taken. Surface plots of numerical and exact solutions of the real and imaginary part are presented in Figure 5. For more visualization, the graphs of numerical and exact solutions at the diagonal, joining the points $(0, 0)$ and $(1, 1)$, are displayed in Figure 6. The figures show that the numerical results are in a good agreement with the exact solutions.

5. CONCLUSION

In this work, an alternating direction implicit numerical scheme is presented for solving two-dimensional nonlinear Schrödinger equation. Gauss-Seidel method is used to solve the system of algebraic equation resulted from the discretization. The stability of the numerical scheme is analysed and concluded to be conditionally stable. The efficiency and accuracy of the scheme is demonstrated using three test examples. The obtained numerical results are compared with exact solutions and it is observed that all results are analogous to the exact solutions. **Conflict of Interests**

The author declares no conflict of interest.

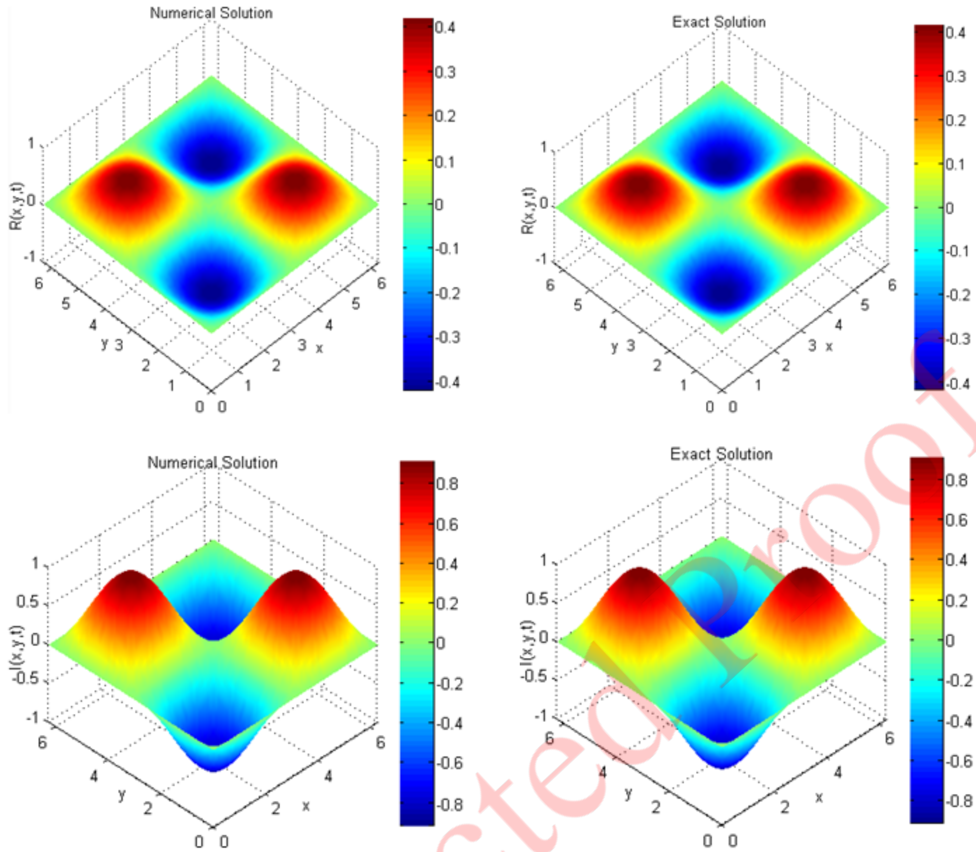

FIGURE 3. Surface plot of numerical and exact solutions for Example 4.2 at $t = 1$ with $N = 60$ with $N = 60$, $T = 1$ and $\Delta t = 0.005$.

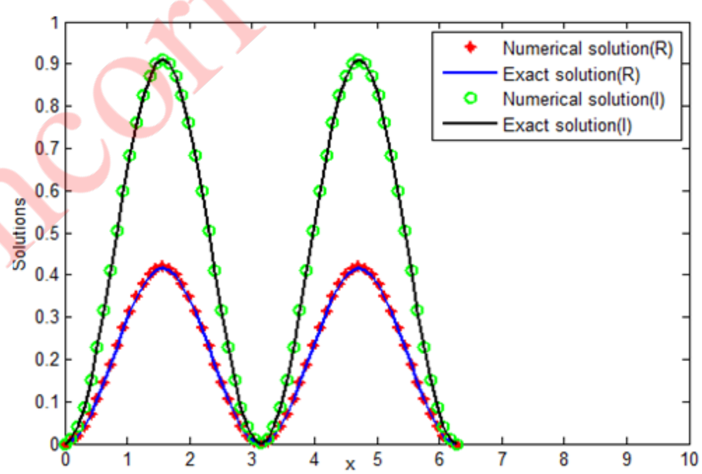

FIGURE 4. Graph of numerical and exact solutions for Example 4.2 at diagonal of the domain (joining the points $(0, 2\pi)$ and $(2\pi, 0)$) and $t = 1$ with $N = 60$,$T = 1$ and $\Delta t = 0.005$.

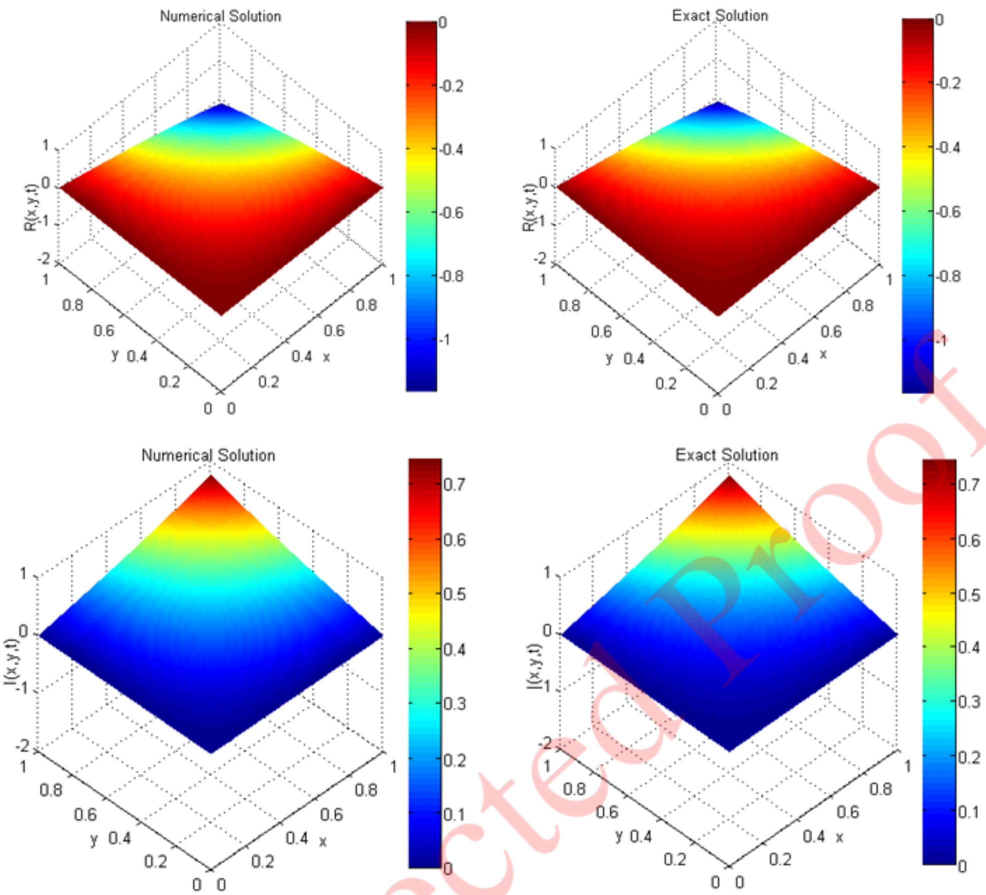

FIGURE 5. Surface plot of numerical and exact solutions for Example 4.3 at $t = 1$ with $N = 50$,$T = 1$ and $\Delta t = 0.005$.

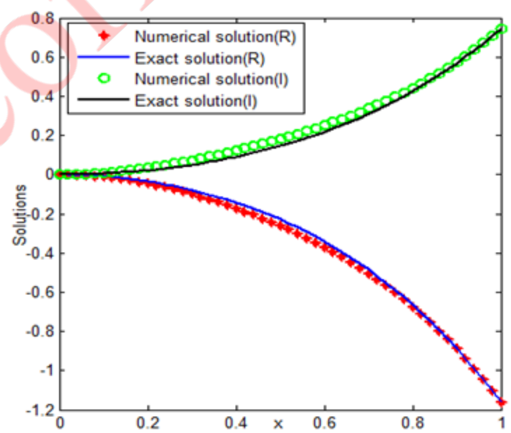

FIGURE 6. Graph of numerical and exat solutions for Example 4.3 at the diagonal (from $(0, 0)$ to $(1, 1)$) and $t = 1$ with $N = 50$, $T = 1$ and $\Delta t = 0.005$.

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