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LQR technique based SMC design for a class of uncertain time-delay Conic nonlinear systems

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Abstract

In this paper, the finite-time sliding mode controller design problem of a class of conic-type nonlinear systems with time-delays, mismatched external disturbance and uncertain coefficients is investigated. The time-delay conic nonlinearities are considered to lie in a known hypersphere with an uncertain center. Conditions have been obtained to design a linear quadratic regulator based on sliding mode control. For this purpose, by applying Lyapunov- Krasovskii stability theory and linear matrix inequality approach, sufficient conditions are derived to ensure the finite-time boundedness of the closed-loop systems over the finite-time interval. Thereafter, an appropriate control strategy is constructed to drive the state trajectories onto the specified sliding surface in a finite time. Finally, an example related to the time-delayed Chua's circuit is given to demonstrate the effectiveness of the suggested method. Also, the efficiency of the suggested method is compared with other methods by using an another numerical example.

Keywords. Finite-time bounded, Integral sliding mode, Time delay system, Linear quadratic regulator, Conic nonlinear systems.

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1. Introduction

In recent years, the issue of stability and control of delayed systems has garnered significant attention from researchers. Due to its relevance in practical and industrial systems like heating systems, chemical reactors, and biological systems [2, 27], numerous studies have been conducted in the field of delayed systems [4, 19, 26]. While most existing research has focused on Lyapunovs asymptotic stability over an infinite time interval, in practical applications, studying a systems behavior over a finite time interval is often crucial. Asymptotic stability may not be suitable in such cases due to potential adverse transient effects caused by large state values. Therefore, finite time stability, which concentrates on the systems transient behavior over a specified time interval, becomes essential [14, 15, 27, 39].
Moreover, the presence of uncertain terms in practical systems is inevitable and can significantly impact controller accuracy. Neglecting uncertain terms in system modeling can lead to suboptimal controller performance [9]. Research has delved into the stability and finite time control of delayed systems with uncertainty [8, 9, 14, 29, 32, 47]. For instance, in [14], time finite stabilization for switching linear systems with soft constrained uncertainty and time-varying latency using the mean residence time method has been explored.
Recently, Khaledi et al. [17] introduced a finite-time sliding mode control (SMC) for a class of nonlinear systems, considering latency in state variables and uncertain matrix coefficients in the dynamic system. They proposed a new SMC theory, specifically the time-based SMC finite time bounded, which is effective for systems with uncertainties and external disturbances. SMC has become a widely used control design method for robust control in linear and nonlinear systems [10, 13, 44]. The SMC technique is employed in the presence of uncertainties and disturbances to achieve asymptotic stability conditions [41].

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Various research has been conducted on sliding mode control in different areas of control theory [16, 18, 20, 21, 25, 33–37, 45], yielding promising results on different aspects of the method [1–3, 5, 30, 38]. Linear matrix inequalities (LMI) play a crucial role in determining stability conditions, often formulated in the form of these inequalities. Concepts like finite-time stability (FTS) and finite-time bounded (FTB) have been extensively studied by researchers [10, 11, 27]. The conic nonlinear system, characterized by conic-type nonlinearities, is found in engineering systems modeling, such as locally sinusoidal nonlinearities, dead zone nonlinearities in diodes and amplifiers, piecewise linear functions, and Lipschitz nonlinearities [12].

This paper focuses on obtaining LMI-type conditions using the sliding mode control method to ensure the FTB condition for the analyzed system [12]. Specifically, by considering the dynamic system described in [17], novel conditions are derived by designing a linear quadratic regulator-based sliding mode control with an integral switching surface. Sufficient conditions for the nonlinear system are obtained using Lyapunov-Krasovskii stability theory and linear matrix inequality. The methods efficiency is demonstrated through a numerical example, inspired by the topic and proof technique in [40].

The subsequent sections of this paper are structured as follows: Section 2 provides initial definitions and describes the system under study. Section 3 defines a suitable integral-type sliding surface and derives the controller and equivalent dynamic system using this surface. Section 4 is divided into two subsections. The first subsection presents a theorem offering sufficient conditions for establishing the FTB condition, discussed with the aid of an LQR-based cost functional. The second subsection designs a controller to place the dynamic systems state variables on the defined sliding surface within a finite time. Section 5 evaluates the suggested methods efficiency and compares it with other methods using a numerical example. Finally, section 6 concludes the paper.

The following notations are used throughout the paper:

$A>(\geq)0$ means a symmetric positive-definite (positive-semidefinite) matrix. $A<(\leq)0$ has a corresponding meaning. $\lambda_{max}(.)$ means maximum eigenvalues of the corresponding matrix and $PRT(X) = X^T.X$, where X is an arbitrary vector. $\parallel . \parallel$ shows the Euclidean norm for vectors or the spectral norm for matrices. As mentioned before, FTB, FTS, LMI, LQR and SMC stand for finite time bounded, finite time stability, linear matrix inequality, linear quadratic regulator, and sliding mode control, respectively. The symbol "$*$" represents the transposition of a symmetric member relative to the original diameter in a symmetric block matrix.

2. Introducing the studied system and necessary definitions

Consider the nonlinear system with time-delay θ as follows:

$$\Sigma : \begin{cases} \dot{z}(s) = h(z(s), z(s-\theta), \lambda(s)) + Bu(s), \\ \dot{\lambda}(s) = \phi\lambda(s), \end{cases} \tag{2.1}$$

where $z(s), z(s-\theta) \in \mathbb{R}^n$ are the state vectors; $u(s) \in \mathbb{R}^m$ is the input vector, $\lambda(s) \in \mathbb{R}^r$ is the external disturbance vector, and $\phi \in \mathbb{R}^{r\times r}$. $h(z(s), z(s-\theta), \lambda(s))$ is an unknown $(z, z(s-\theta), \lambda)$-dependent nonlinear function with the following descriptions:

$$\begin{aligned} &\parallel h(z(s), z(s-\theta), \lambda(s)) - [\bar{A}z(s) + \bar{A}_\theta z(s-\theta) + \bar{F}\lambda(s)] \parallel \\ &\leq \parallel A_r z(s) + A_{r\theta} z(s-\theta) + F_r\lambda(s) \parallel, \end{aligned} \tag{2.2}$$

where $\bar{A} = A + A_\Delta$, $\bar{A}_\theta = A_\theta + A_{\Delta\theta}$ and $\bar{F} = F + F_\Delta$. The matrices A, A_θ, F, B, A_r, $A_{r\theta}$, and F_r in the system under study Σ are considered known. The unknown matrices A_Δ , $A_{\Delta\theta}$, and F_Δ satisfy the following conditions:

$$\begin{cases} A_\Delta = M_0\delta_0(s)N_0, \\ A_{\Delta\theta} = M_1\delta_1(s)N_1, \\ F_\Delta = M_2\delta_2(s)N_2, \end{cases} \tag{2.3}$$

where M_i and N_i $(i = 0, 1, 2)$ are known matrices and $\delta_i(s)$ $(i = 0, 1, 2)$ are assumed to satisfy $\parallel \delta_i(s) \parallel \leq 1$ $(i = 0, 1, 2)$ for all times s.

System $\acute{\Sigma}$ will be obtained from Σ using condition (2.2) as follows:

$$\acute{\Sigma}: \begin{cases} \dot{z}(s) = \bar{A}z(s) + \bar{A}_\theta z(s-\theta) + g(s) + Bu(s) + \bar{F}\lambda(s), \\ \dot{\lambda}(s) = \phi\lambda(s), \end{cases} \tag{2.4}$$

where $g(s) = h(z(s), z(s-\theta), \lambda(s)) - [\bar{A}z(s) + \bar{A}_\theta z(s-\theta) + \bar{F}\lambda(s)]$.
The following inequality will be easily obtained by applying (2.2):

$$\begin{aligned} \| g(s) \|^2 &= \| h(z(s), z(s-\theta), \lambda(s)) - [\bar{A}z(s) + \bar{A}_\theta z(s-\theta) + \bar{F}\lambda(s)] \|^2 \\ &\leq \| A_r z(s) + A_{r\theta} z(s-\theta) + F_r\lambda(s) \|^2 . \end{aligned} \tag{2.5}$$

Definition 2.1. [40] Considering positive constants $c_1, c_2(> c_1)$, σ, a positive symmetric matrix U and a fixed time interval $[0, T_f]$, the system Σ is called to be FTB with respect to $(c_1, c_2, T_f, U, \delta)$ whenever

$$\begin{cases} z^T(0)Uz(0) \leq c_1, \\ \int_0^{T_f} \lambda^T(s)\lambda(s)ds \leq \delta, \end{cases} \qquad \Rightarrow z^T(s)Uz(s) < c_2, s \in [0, T_f]. \tag{2.6}$$

Lemma 2.2. *[22] Assume that A and B are real matrices with appropriate dimensions. Let $\varepsilon > 0$ and $c, d \in \mathbb{R}^n$ then:*

$$2c^T ABd \leq \varepsilon^{-1} c^T A^T Ac + \varepsilon d^T B^T Bd.$$

Proof. See proof in [22]. □

Lemma 2.3. *[28] Assume that A and B be real matrices of appropriate dimensions. Let $\mu > 0$ and $H(s)$ be a matrix such that $H^T(s)H(s) \leq I$, then:*

$$AH(s)B + [AH(s)B]^T \leq \mu^{-1}AA^T + \mu B^T B.$$

Proof. See proof in [28]. □

3. Sliding mode control design

Consider the following switching surface:

$$\varsigma(s) = Lz(s) - \int_0^s L[(A + BK)z(r) + A_\theta z(r-\theta) + F\lambda(r)]dr, \tag{3.1}$$

where $L \in \mathbb{R}^{m\times n}$ and $K \in \mathbb{R}^{m\times n}$ are real matrices that must be determined. We must consider the matrix L such that LB becomes nonsingular. The answer $z(s)$ from system $\acute{\Sigma}$ will be as follows:

$$z(s) = z(0) + \int_0^s [\bar{A}z(r) + \bar{A}_\theta z(r-\theta) + g(r) + Bu(r) + \bar{F}\lambda(r)]dr. \tag{3.2}$$

Combining (3.1) and (3.2), implies that:

$$\varsigma(s) = Lz(0) + L\int_0^s [\bar{A}z(r) + \bar{A}_\theta z(r-\theta) + g(r) + Bu(r) + \bar{F}\lambda(r)]dr \tag{3.3}$$

$$- \int_0^s L[(A + BK)z(r) + A_\theta z(r-\theta) + F\lambda(r)]dr. \tag{3.4}$$

$$\tag{3.5}$$

Hence

$$\varsigma(s) = Lz(0) + \int_0^s L[(A_\Delta - BK)z(r) + A_{\Delta\theta} z(r-\theta) + g(r) + Bu(r) + F_\Delta\lambda(r)]dr.$$

When the state variables reach the desired sliding surface i.e. $\varsigma(s) = c$ and $\dot{\varsigma}(s) = 0$, solving the equation $\dot{\varsigma}(s) = 0$, u_{eq} will be obtained as follows:

$$u_{eq} = -(LB)^{-1}L[(A_\Delta - BK)z(s) + A_{\Delta\theta} z(s-\theta) + g(s) + F_\Delta\lambda(s)]. \tag{3.6}$$

It should be noted that considering the above explanations, the switching surface converges to c, and also when the state variables are placed on the switching surface, they reach c. By substituting u_{eq} into $\acute{\Sigma}$, the following system will be obtained:

$$\tilde{\Sigma}:\begin{cases}\dot{z}(s)=\tilde{A}z(s)+\tilde{A}_\theta z(s-\theta)+\tilde{L}g(s)+\tilde{F}\lambda(s),\\ \dot{\lambda}(s)=\phi\lambda(s),\end{cases}\tag{3.7}$$

where $\tilde{A}=A+\tilde{L}A_\Delta+BK, \tilde{A}_\theta=A_\theta+\tilde{L}A_{\Delta\theta}, \tilde{F}=F+\tilde{L}F_\Delta$ and $\tilde{L}=I-B(LB)^{-1}L$.

4. Main results

A. FTB Analysis

Our goal in this section is to find the matrices K and L in (3.1) in such a way that the system (3.7) satisfies the FTB condition.

To achieve this goal, by applying the following theorem, sufficient conditions will be obtained to ensure that the FTB condition is met. The results are discussed with the help of an LQR based cost functional given by:

$$\int_0^{T_f}(z^T(s)Qz(s)+u^T(s)Ru(s))ds\leq J^*.$$

Theorem 4.1. *Consider the system (2.1).Then the system (3.7) is FTB wrt (c_1,c_2,T_f,U,δ) if there exist six positive scalars ϵ_1, ϵ_2, ϵ_3, ϵ_4, ν_1, and ν_2, a positive-defined symmetric matrices P and X and a real matrix Y, so that the following LMIs are established:*

$$\begin{bmatrix}
\Sigma_{1,1} & A_\theta X & F & 0 & \Sigma_{1,5} & XA_r^T & \epsilon_1\tilde{L} & XN_0^T & \epsilon_2\tilde{L}M_0 & 0 & \epsilon_3\tilde{L}M_1 & 0 & \epsilon_4\tilde{L}M_2 & X^TQ\\
* & -X & 0 & 0 & 0 & XA_{r\theta}^T & 0 & 0 & 0 & XN_1^T & 0 & 0 & 0 & 0\\
* & * & -\alpha\|\phi\|^2I & 0 & 0 & F_r^T & 0 & 0 & 0 & 0 & 0 & N_2^T & 0 & 0\\
* & * & * & 0 & \hat{L}^TR & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -RI & 0 & 0 & 0 & \Sigma_{5,9} & 0 & \Sigma_{5,11} & 0 & \Sigma_{5,13} & 0\\
* & * & * & * & * & -\epsilon_1I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -\epsilon_1I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & -\epsilon_2I & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -\epsilon_2I & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -\epsilon_3I & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & -\epsilon_3I & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & -\epsilon_4I & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & -\epsilon_4I & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & * & -Q
\end{bmatrix}<0,\tag{4.1}$$

$$U\nu_1\leq P\leq\nu_2U,\tag{4.2}$$

$$\nu_2c_1(1+\theta)+\delta\parallel\phi\parallel^2(1-e^{-\alpha T_f})<e^{-\alpha T_f}\nu_1c_2,\tag{4.3}$$

where $\Sigma_{1,1}=X^TA^T+AX+Y^TB^T+BY+X$, $\Sigma_{1,5}=-Y^TB^T\hat{L}^TR$, $\Sigma_{5,9}=\epsilon_2R^T\hat{L}M_0$, $\Sigma_{5,11}=\epsilon_3R^T\hat{L}M_1$, $\Sigma_{5,13}=\epsilon_4R^T\hat{L}M_2$ and $\hat{L}=-(LB)^{-1}L$. Therefore the matrix K, can be obtained by $K=YX^{-1}$.

Proof. Let $z^T(0)Uz(0)\leq c_1$, $\int_0^{T_f}\lambda^T(s)\lambda(s)ds\leq\delta$. Our goal is to find conditions such that $z^T(s)Uz(s)\leq c_2$ $\forall s\in[0,T_f]$. Define a Lyapunov function as

$$V(s)=z^T(s)Pz(s)+\int_{s-\theta}^{s}z^T(\tau)Pz(\tau)d\tau.\tag{4.4}$$

By considering (3.7) and using the time derivative of $V(s)$, the following relation will be obtained:

$$\dot{V}(s)=\dot{z}^T(s)Pz(s)+z^T(s)P\dot{z}(s)+z^T(s)Pz(s)-z^T(s-\theta)Pz(s-\theta)\tag{4.5}$$

$$=(\tilde{A}z(s)+\tilde{A}_\theta z(s-\theta)+\tilde{L}g(s)+\tilde{F}\lambda(s))^TPz(s)\tag{4.6}$$

$$+z^T(s)P(\tilde{A}z(s)+\tilde{A}_\theta z(s-\theta)+\tilde{L}g(s)+\tilde{F}\lambda(s))\tag{4.7}$$

$$+z^T(s)Pz(s)-z^T(s-\theta)Pz(s-\theta)\tag{4.8}$$

$$=z^T(s)\tilde{A}^TPz(s)+z^T(s-\theta)\tilde{A}_\theta^TPz(s)+g^T(s)\tilde{L}^TPz(s)\tag{4.9}$$

$$+\lambda^T(s)\tilde{F}^T Pz(s) + z^T(s)P\tilde{A}z(s) + z^T(s)P\tilde{A}_\theta z(s-\theta) + z^T(s)P\tilde{L}g(s) \tag{4.10}$$
$$+ z^T(s)P\tilde{F}\lambda(s) + z^T(s)Pz(s) - z^T(s-\theta)Pz(s-\theta) \tag{4.11}$$
$$= z^T(s)(\tilde{A}^T P + P\tilde{A} + P)z(s) + z^T(s-\theta)\tilde{A}_\theta^T Pz(s) + \lambda^T(s)\tilde{F}^T Pz(s) \tag{4.12}$$
$$+ z^T(s)P\tilde{F}\lambda(s) + z^T(s)P\tilde{A}_\theta z(s-\theta) + 2g^T(s)\tilde{L}^T Pz(s) \tag{4.13}$$
$$- z^T(s-\theta)Pz(s-\theta). \tag{4.14}$$

With the help of (2.5) and Lemma 2.2, we will have the following inequality:

$$2g^T(s)\tilde{L}^T Pz(s) \le \epsilon_1 z^T(s)P\tilde{L}\tilde{L}^T Pz(s) + \epsilon_1^{-1} \parallel g(s) \parallel^2 \tag{4.15}$$
$$\le \epsilon_1 z^T(s)P\tilde{L}\tilde{L}^T Pz(s) \tag{4.16}$$
$$+ \epsilon_1^{-1} PRT(A_r z(s) + A_{r\theta} z(s-\theta) + F_r\lambda(s)) \tag{4.17}$$
$$= z^T(s)(\epsilon_1 P\tilde{L}\tilde{L}^T P + \epsilon_1^{-1} A_r^T A_r)z(s) \tag{4.18}$$
$$+ \epsilon_1^{-1} z^T(s)A_r^T A_{r\theta} z(s-\theta) + \epsilon_1^{-1} z^T(s)A_r^T F_r\lambda(s) \tag{4.19}$$
$$+ \epsilon_1^{-1} z^T(s-\theta)A_{r\theta}^T A_r z(s) + \epsilon_1^{-1} z^T(s-\theta)A_{r\theta}^T A_{r\theta} z(s-\theta) \tag{4.20}$$
$$+ \epsilon_1^{-1} z^T(s-\theta)A_{r\theta}^T F_r\lambda(s) + \epsilon_1^{-1}\lambda^T(s)F_r^T A_r z(s) \tag{4.21}$$
$$+ \epsilon_1^{-1}\lambda^T(s)F_r^T A_{r\theta} z(s-\theta) + \epsilon_1^{-1}\lambda^T(s)F_r^T F_r\lambda(s). \tag{4.22}$$

By defining an auxiliary function as:

$$J = \dot{V}(s) - \alpha\dot{\lambda}^T(s)\dot{\lambda}(s) = \dot{V}(s) - \alpha \parallel \phi \parallel^2 \lambda^T(s)\lambda(s),$$

we will have:

$$J = z^T(s)(\tilde{A}^T P + P\tilde{A} + P)z(s) + z^T(s-\theta)\tilde{A}_\theta^T Pz(s)$$
$$+ \lambda^T(s)\tilde{F}^T Pz(s) + z^T(s)P\tilde{A}_\theta z(s-\theta) + z^T(s)P\tilde{F}\lambda(s)$$
$$- z^T(s-\theta)Pz(s-\theta) + 2g^T(s)\tilde{L}^T Pz(s) - \alpha \parallel \phi \parallel^2 \lambda^T(s)\lambda(s).$$

From (4.15), we will find that:

$$J \le z^T(s)(\tilde{A}^T P + P\tilde{A} + P + \epsilon_1 P\tilde{L}\tilde{L}^T P + \epsilon_1^{-1} A_r^T A_r)z(s) \tag{4.23}$$
$$+ z^T(s-\theta)(\tilde{A}_\theta^T P + \epsilon_1^{-1} A_{r\theta}^T A_r)z(s) + \lambda^T(s)(\tilde{F}^T P + \epsilon_1^{-1} F_r^T A_r)z(s) \tag{4.24}$$
$$+ z^T(s)(P\tilde{F} + \epsilon_1^{-1} A_r^T F_r)\lambda(s) + z^T(s)(P\tilde{A}_\theta + \epsilon_1^{-1} A_r^T A_{r\theta})z(s-\theta) \tag{4.25}$$
$$+ \lambda^T(s)(\epsilon_1^{-1} F_r^T F_r - \alpha \parallel \phi \parallel^2 I)\lambda(s) + z^T(s-\theta)(\epsilon_1^{-1} A_{r\theta}^T A_{r\theta} - P)z(s-\theta) \tag{4.26}$$
$$+ z^T(s-\theta)(\epsilon_1^{-1} A_{r\theta}^T F_r)\lambda(s) + \lambda^T(s)(\epsilon_1^{-1} F_r^T A_{r\theta})z(s-\theta). \tag{4.27}$$

We write the above inequality in the following matrix form:

$$J \le \Xi^T \Psi \Xi,$$

where

$$\Xi = \begin{bmatrix} z(s) \\ z(s-\theta) \\ \lambda(s) \\ g(s) \end{bmatrix},$$

and

$$\Psi = \begin{bmatrix} \tilde{A}^T P + P\tilde{A} + P + \epsilon_1 P\tilde{L}\tilde{L}^T P + \epsilon_1^{-1} A_r^T A_r & P\tilde{A}_\theta + \epsilon_1^{-1} A_r^T A_{r\theta} & P\tilde{F} + \epsilon_1^{-1} A_r^T F_r & 0 \\ * & \epsilon_1^{-1} A_{r\theta}^T A_{r\theta} - P & \epsilon_1^{-1} A_{r\theta}^T F_r & 0 \\ * & * & \epsilon_1^{-1} F_r^T F_r - \alpha \|\phi\|^2 I & 0 \\ * & * & * & 0 \end{bmatrix}.$$

Next, by using the LQR based cost functional and the control input (3.6), we have:

$$\begin{aligned} J_c &= \dot{V}(s) + z^T(s)Qz(s) + u^T(s)Ru(s) - \alpha\dot{\lambda}^T(s)\dot{\lambda}(s) \\ &\leq \Xi^T \Psi \Xi + z^T(s)Qz(s) + u^T(s)Ru(s). \end{aligned}$$

Considering

$$\hat{L} = -(LB)^{-1}L, \; \bar{Q} = diag(Q, 0, 0, 0),$$

and

$$\Upsilon = \begin{bmatrix} \hat{L}(A_\Delta - BK) & \hat{L}A_{\Delta\theta} & \hat{L}F_\Delta & \hat{L} \end{bmatrix},$$

one can find that:

$$z^T(s)Qz(s) = \Xi^T \bar{Q} \Xi, \text{ and } u^T(s)Ru(s) = \Xi^T \Upsilon^T R \Upsilon \Xi.$$

Hence,

$$J_c = \Xi^T(\Psi + \bar{Q} + \Upsilon^T R \Upsilon)\Xi. \tag{4.28}$$

If $J_c<0$, holds, then using Schur complement, the R.H.S of (4.28) is equivalent to:

$$\Omega = \begin{bmatrix} \Pi_{11} + \epsilon_1 P\tilde{L}\tilde{L}^T P + \epsilon_1^{-1} A_r^T A_r & P\tilde{A}_\theta + \epsilon_1^{-1} A_r^T A_{r\theta} & P\tilde{F} + \epsilon_1^{-1} A_r^T F_r & 0 & (A_\Delta - BK)^T \hat{L}^T R \\ * & \epsilon_1^{-1} A_{r\theta}^T A_{r\theta} - P & \epsilon_1^{-1} A_{r\theta}^T F_r & 0 & A_{\Delta\theta}^T \hat{L}^T R \\ * & * & \epsilon_1^{-1} F_r^T F_r - \alpha \|\phi\|^2 I & 0 & F_\Delta^T \hat{L}^T R \\ * & * & * & 0 & \hat{L}^T R \\ * & * & * & * & -RI \end{bmatrix} <0, \tag{4.29}$$

where, $\Pi_{11} = \tilde{A}^T P + P\tilde{A} + P + Q$.
By decomposition of the last inequality, we will have:

$$\begin{aligned} \Omega = &\begin{bmatrix} \Pi_{11} & P\tilde{A}_\theta & P\tilde{F} & 0 & (A_\Delta - BK)^T \hat{L}^T R \\ * & -P & 0 & 0 & A_{\Delta\theta}^T \hat{L}^T R \\ * & * & -\alpha \parallel \phi \parallel^2 I & 0 & F_\Delta^T \hat{L}^T R \\ * & * & * & 0 & \hat{L}^T R \\ * & * & * & * & -RI \end{bmatrix} \\ &+ \begin{bmatrix} \epsilon_1 P\tilde{L}\tilde{L}^T P + \epsilon_1^{-1} A_r^T A_r & \epsilon_1^{-1} A_r^T A_{r\theta} & \epsilon_1^{-1} A_r^T F_r & 0 & 0 \\ * & \epsilon_1^{-1} A_{r\theta}^T A_{r\theta} & \epsilon_1^{-1} A_{r\theta}^T F_r & 0 & 0 \\ * & * & \epsilon_1^{-1} F_r^T F_r & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} <0. \end{aligned} \tag{4.30}$$

We can rewrite the second matrix in Eq. (4.30) as follows:

$$\epsilon_1 \begin{bmatrix} P\tilde{L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P\tilde{L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \epsilon_1^{-1} \begin{bmatrix} A_r^T \\ A_{r\theta}^T \\ F_r^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_r^T \\ A_{r\theta}^T \\ F_r^T \\ 0 \\ 0 \end{bmatrix}^T.$$

Hence,

$$\Omega = \begin{bmatrix} \Pi_{11} & P\tilde{A}_\theta & P\tilde{F} & 0 & (A_\Delta - BK)^T \hat{L}^T R \\ * & -P & 0 & 0 & A_{\Delta\theta}^T \hat{L}^T R \\ * & * & -\alpha \parallel \phi \parallel^2 I & 0 & F_\Delta^T \hat{L}^T R \\ * & * & * & 0 & \hat{L}^T R \\ * & * & * & * & -RI \end{bmatrix}$$

$$+\epsilon_1 \begin{bmatrix} P\tilde{L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P\tilde{L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$+\epsilon_1^{-1} \begin{bmatrix} A_r^T \\ A_{r\theta}^T \\ F_r^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_r^T \\ A_{r\theta}^T \\ F_r^T \\ 0 \\ 0 \end{bmatrix}^T < 0. \tag{4.31}$$

The following result will be obtained by using Schur's complement to (4.31):

$$\Omega_1 = \begin{bmatrix} \Pi_{11} & P\tilde{A}_\theta & P\tilde{F} & 0 & (A_\Delta - BK)^T \hat{L}^T R & A_r^T & \epsilon_1 P\tilde{L} \\ * & -P & 0 & 0 & A_{\Delta\theta}^T \hat{L}^T R & A_{r\theta}^T & 0 \\ * & * & -\alpha \parallel \phi \parallel^2 I & 0 & F_\Delta^T \hat{L}^T R & F_r^T & 0 \\ * & * & * & 0 & \hat{L}^T R & 0 & 0 \\ * & * & * & * & -RI & 0 & 0 \\ * & * & * & * & * & -\epsilon_1 I & 0 \\ * & * & * & * & * & * & -\epsilon_1 I \end{bmatrix} < 0. \tag{4.32}$$

To remove the unspecified parameters in (4.32), we rewrite the above relation as $\Omega_1 = \Omega_{1D} + \Omega_{1\Delta}$, where

$$\Omega_{1D} = \begin{bmatrix} \acute{\Pi}_{11} & PA_\theta & PF & 0 & -K^T B^T \hat{L}^T R & A_r^T & \epsilon_1 P\tilde{L} \\ * & -P & 0 & 0 & 0 & A_{r\theta}^T & 0 \\ * & * & -\alpha \parallel \phi \parallel^2 I & 0 & 0 & F_r^T & 0 \\ * & * & * & 0 & \hat{L}^T R & 0 & 0 \\ * & * & * & * & -RI & 0 & 0 \\ * & * & * & * & * & -\epsilon_1 I & 0 \\ * & * & * & * & * & * & -\epsilon_1 I \end{bmatrix},$$

and

$$\Omega_{1\Delta} = \begin{bmatrix} P\tilde{L}A_\Delta + A_\Delta^T \tilde{L}^T P & P\tilde{L}A_{\Delta\theta} & P\tilde{L}F_\Delta & 0 & A_\Delta^T \hat{L}^T R & 0 & 0 \\ * & 0 & 0 & 0 & A_{\Delta\theta}^T \hat{L}^T R & 0 & 0 \\ * & * & 0 & 0 & F_\Delta^T \hat{L}^T R & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix},$$

where $\acute{\Pi}_{11} = (A + BK)^T P + P(A + BK) + P + Q$. By substituting the values defined in (2.3) for A_Δ, $A_{\Delta\theta}$ and F_Δ into the $\Omega_{1\Delta}$ and using Lemma 2.3 to show that there exist positive scalars ϵ_2, ϵ_3 and ϵ_4, we have:

$$
\begin{aligned}
\Omega_{1\Delta} = & \begin{bmatrix} P\tilde{L}M_0 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_0 \\ 0 \\ 0 \end{bmatrix} \delta_0(s) \begin{bmatrix} N_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \{ \begin{bmatrix} P\tilde{L}M_0 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_0 \\ 0 \\ 0 \end{bmatrix} \delta_0(s) \begin{bmatrix} N_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \}^T \\
& + \begin{bmatrix} P\tilde{L}M_1 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_1 \\ 0 \\ 0 \end{bmatrix} \delta_1(s) \begin{bmatrix} 0 & N_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \{ \begin{bmatrix} P\tilde{L}M_1 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_1 \\ 0 \\ 0 \end{bmatrix} \delta_1(s) \begin{bmatrix} 0 & N_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \}^T \\
& + \begin{bmatrix} P\tilde{L}M_2 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_2 \\ 0 \\ 0 \end{bmatrix} \delta_2(s) \begin{bmatrix} 0 & 0 & N_2 & 0 & 0 & 0 & 0 \end{bmatrix} + \{ \begin{bmatrix} P\tilde{L}M_2 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_2 \\ 0 \\ 0 \end{bmatrix} \delta_2(s) \begin{bmatrix} 0 & 0 & N_2 & 0 & 0 & 0 & 0 \end{bmatrix} \}^T.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\Omega_{1\Delta} \leq \epsilon_2 & \begin{bmatrix} P\tilde{L}M_0 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P\tilde{L}M_0 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_0 \\ 0 \\ 0 \end{bmatrix}^T + \epsilon_2^{-1} \begin{bmatrix} N_0^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} N_0^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
+\epsilon_3 & \begin{bmatrix} P\tilde{L}M_1 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P\tilde{L}M_1 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_1 \\ 0 \\ 0 \end{bmatrix}^T + \epsilon_3^{-1} \begin{bmatrix} 0 \\ N_1^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ N_1^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
+\epsilon_4 & \begin{bmatrix} P\tilde{L}M_2 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P\tilde{L}M_2 \\ 0 \\ 0 \\ 0 \\ R^T\hat{L}M_2 \\ 0 \\ 0 \end{bmatrix}^T + \epsilon_4^{-1} \begin{bmatrix} 0 \\ 0 \\ N_2^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ N_2^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T.
\end{aligned}
$$

The following matrix inequality will be obtained with the help of Schur's complement:

$$\begin{bmatrix} \acute{\Pi}_{11} & PA_\theta & PF & 0 & -K^TB^T\hat{L}^TR & A_r^T & \epsilon_1 P\tilde{L} & N_0^T & \epsilon_2 P\tilde{L}M_0 & 0 & \epsilon_3 P\tilde{L}M_1 & 0 & \epsilon_4 P\tilde{L}M_2 \\ * & -P & 0 & 0 & 0 & A_{r\theta}^T & 0 & 0 & 0 & N_1^T & 0 & 0 & 0 \\ * & * & -\alpha\|\phi\|^2 I & 0 & 0 & F_r^T & 0 & 0 & 0 & 0 & 0 & N_2^T & 0 \\ * & * & * & 0 & \hat{L}^TR & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -RI & 0 & 0 & 0 & \epsilon_2 R^T\hat{L}M_0 & 0 & \epsilon_3 R^T\hat{L}M_1 & 0 & \epsilon_4 R^T\hat{L}M_2 \\ * & * & * & * & * & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon_2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\epsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\epsilon_3 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\epsilon_3 I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\epsilon_4 I & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & -\epsilon_4 I \end{bmatrix} < 0. \tag{4.33}$$

Now introducing $X = P^{-1}$ and $Y = KX$. Multiplying the sides by $diag\{X, X, I, I, I, I, I, I, I, I, I, I\}$, and using Schure complement we arrive at condition (4.1). It is easy to get

$$\dot{V}(\tau) + z^T(\tau)Qz(\tau) + u^T(\tau)Ru(\tau) - \alpha\dot{\lambda}^T(\tau)\dot{\lambda}(\tau) < 0. \tag{4.34}$$

Multiplying by $e^{-\alpha\tau}$ and integrating from 0 to s, we get:

$$\begin{aligned} \int_0^s e^{-\alpha\tau}(z^T(\tau)Qz(\tau) + u^T(\tau)Ru(\tau))d\tau + \int_0^s e^{-\alpha\tau}\dot{V}(\tau)d\tau \\ - \int_0^s \alpha \parallel \phi \parallel^2 e^{-\alpha\tau}\lambda^T(\tau)\lambda(\tau)d\tau < 0, \end{aligned} \tag{4.35}$$

$$\begin{aligned} \frac{1 - e^{-\alpha s}}{\alpha} \int_0^s z^T(\tau)Qz(\tau) + u^T(\tau)Ru(\tau)d\tau \leq \\ - (e^{-\alpha s}V(s) - V(0)) \\ + \alpha \parallel \phi \parallel^2 \int_0^s e^{-\alpha\tau}\lambda^T(\tau)\lambda(\tau)d\tau. \end{aligned}$$

Since $\int_0^s e^{-\alpha\tau}(z^T(\tau)Qz(\tau) + u^T(\tau)Ru(\tau))d\tau > 0$, it is obvious that

$$V(s) \leq e^{\alpha s}V(0) + e^{\alpha s}\alpha \parallel \phi \parallel^2 \int_0^s e^{-\alpha\tau}\lambda^T(\tau)\lambda(\tau)d\tau. \tag{4.36}$$

From (4.4), it is easy to drive that:

$$V(0) = z^T(0)Pz(0) + \int_{-\theta}^0 z^T(\tau)Pz(\tau)d\tau.$$

Letting $\tilde{P} = U^{\frac{-1}{2}}PU^{\frac{-1}{2}}$, we have

$$\begin{aligned} V(0) &= z^T(0)U^{\frac{1}{2}}\tilde{P}U^{\frac{1}{2}}z(0) + \int_{-\theta}^0 z^T(\tau)U^{\frac{1}{2}}\tilde{P}U^{\frac{1}{2}}z(\tau)d\tau \\ &\leq \nu_2 z^T(0)Uz(0) + \nu_2 \int_{-\theta}^0 z^T(\tau)Uz(\tau)d\tau \\ &\leq \nu_2 z^T(0)Uz(0) + \nu_2\theta z^T(0)Uz(0) \\ &\leq \nu_2 c_1(1 + \theta), \end{aligned}$$

where $\nu_2 = \lambda_{max}(\tilde{P})$. Using the above in (4.36) and from Definition 2.1, we get

$$V(s) \leq e^{\alpha T_f} \nu_2 c_1 (1+\theta) + e^{\alpha T_f} \delta \parallel \phi \parallel^2 (1 - e^{-\alpha T_f}). \tag{4.37}$$

On the other hand $V(s) \geq z^T(s) P z(s) \geq z^T(s) U^{\frac{1}{2}} \tilde{P} U^{\frac{1}{2}} z(s) \geq \nu_1 z^T(s) U z(s)$, where $\nu_1 = \lambda_{min}(\tilde{P})$. Therefore (4.37) becomes

$$\nu_1 z^T(s) U z(s) \leq e^{\alpha T_f} \nu_2 c_1 (1+\theta) + e^{\alpha T_f} \delta \parallel \phi \parallel^2 (1 - e^{-\alpha T_f}).$$

Hence, from the inequality (4.3), one can get $z^T(s) U z(s) \leq c_2$. According to Definition 2.1, system (3.7) with the sliding mode dynamics (3.1) is FTB. To complete the proof, we have to find the value of J^*, for $s \in [0, T_f]$ and $V(s) > 0$, from (4.35), it follows that:

$$\begin{aligned}
&\frac{1 - e^{-\alpha T_f}}{\alpha} \int_0^{T_f} (z^T(\tau) Q z(\tau) + u^T(\tau) R u(\tau)) d\tau \\
&\leq V(0) + \alpha \parallel \phi \parallel^2 \int_0^{T_f} e^{-\alpha\tau} \lambda^T(\tau) \lambda(\tau) d\tau \\
&\leq \nu_2 c_1 (1+\theta) + \alpha \parallel \phi \parallel^2 \int_0^{T_f} e^{-\alpha\tau} \lambda^T(\tau) \lambda(\tau) d\tau \\
&\leq \nu_2 c_1 (1+\theta) + (1 - e^{-\alpha T_f}) \delta \parallel \phi \parallel^2 .
\end{aligned}$$

Therefore,

$$\int_0^{T_f} (z^T(\tau) Q z(\tau) + u^T(\tau) R u(\tau)) d\tau \leq \frac{\alpha \nu_2 c_1 (1+\theta)}{1 - e^{-\alpha T_f}} + \alpha\delta \parallel \phi \parallel^2 = J^*.$$

This completes the proof. □

B. Proper controller design

The main purpose of this section is to design a suitable controller in such a way that it puts the state variables of $\tilde{\Sigma}$ on the sliding surface $\varsigma(s) = \varsigma(z(s)) = c$ in a finite time S^*.

Theorem 4.2. *[17] Consider the system (Σ), the sliding surface of type (3.1), and K resulting from Theorem 4.1. Also, assume that $\tilde{L}$ in (3.7) is chosen such that LB is nonsingular. In this case, by applying the following control law, the state variables of the system $(\tilde{\Sigma})$ are placed on a predefined sliding surface $\varsigma(s) = \varsigma(z(s)) = c$, at a finite time:*

$$u(s) = Kz(s) - \eta(s) sign(\varsigma(s)), \tag{4.38}$$

where $\eta(s)$ is as follows:

$$\begin{aligned}
\eta(s) = \sigma + &\parallel (B^T X B)^{-1} B^T X M_0 \parallel\parallel N_0 z(s) \parallel \\
&+ \parallel (B^T X B)^{-1} X M_1 \parallel\parallel N_1 z(s-\theta) \parallel \\
&+ \parallel (B^T X B)^{-1} B^T X \parallel (\parallel A_r z(s) \parallel \\
&+ \parallel A_{r\theta} z(s-\theta) \parallel + \parallel F_r \lambda(s) \parallel) \\
&+ \parallel (B^T X B)^{-1} B^T X M_2 \parallel\parallel N_2 \lambda(s) \parallel,
\end{aligned} \tag{4.39}$$

where $\sigma > 0$, is a partial constant.

Proof. See proof in [17]. □

5. Numerical example

In this section, a numerical example is presented to study the method numerically and test its efficiency. Chuas circuit is one of the physical systems for which the presence of chaos (in the sense of Shilnikov) has been established experimentally, confirmed numerically, and proven mathematically. In recent years, Chuas circuit has become a standard model for studying chaos in systems described by finite-dimensional ordinary differential equations [23, 24, 43].

Example 5.1. Consider a time-delayed Chua's circuit as follows:

$$\begin{cases} \dot{z}_1(s) = -\alpha_c(1+b)z_1(s) + \alpha_c z_2(s) + 0.01z_1(s-\theta) + 0.01z_2(s-\theta) \\ \qquad +0.001z_3(s-\theta) + g_1(z_1(s)), \\ \dot{z}_2(s) = z_1(s) - z_2(s) + z_3(s) + 0.01z_1(s-\theta) + 0.001z_2(s-\theta) \\ \qquad +0.001z_3(s-\theta) + g_1(z_1(s-\theta)), \\ \dot{z}_3(s) = -\beta_c z_2(s) - \mu z_3(s) + 0.001z_1(s-\theta) + 0.001z_2(s-\theta) \\ \qquad +0.001z_3(s-\theta), \end{cases} \tag{5.1}$$

where $g_1(z(s)) = 0.5\alpha_c(a-b)(|z_1(s)+1| - |z_1(s)-1|)$,
and
$g_1(z(s-\theta)) = 0.5\alpha_c(a-b)(|z_1(s-\theta)+1| - |z_1(s-\theta)-1|)$.
And the parameters are $\alpha_c = 9 * 10^{-12}$, $\beta_c = 16.5811$, $\mu = 0.138083$, $a = -1.39386$, $b = -0.7559$, $\alpha = .1$, $R = .01$, $\theta = 3$, and $Q = diag\{0.1, 0.1, 0.1\}$. Rewriting the system (5.1) as (2.4), we have:

$$A = \begin{bmatrix} -\alpha_c(1+b) & \alpha_c & 0 \\ 1 & -1 & 1 \\ 0 & -\beta_c & -\mu \end{bmatrix}, \tag{5.2}$$

$$B = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}, \tag{5.3}$$

$$F = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{5.4}$$

$$A_\theta = \begin{bmatrix} 0.01 & 0.01 & 0.001 \\ 0.01 & 0.001 & 0.001 \\ 0.001 & 0.001 & 0.001 \end{bmatrix}. \tag{5.5}$$

We define $g(s)$ as follows:

$$g(s) = \begin{bmatrix} 0.5\alpha_c(a-b)(|z_1(s)+1| - |z_1(s)-1|) \\ 0.5\alpha_c(a-b)(|z_1(s-\theta)+1| - |z_1(s-\theta)-1|) \\ 0 \end{bmatrix}.$$

Due to $|x+1| - |x-1| \leq 2|x| \quad \forall x \in R$, the following inequality will be obtained:

$$\begin{aligned} \| g(s) \|^2 \leq &(0.5\alpha_c(a-b)(|z_1(s)+1| - |z_1(s)-1|))^2 \\ &+ (0.5\alpha_c(a-b)(|z_1(s-\theta)+1| - |z_1(s-\theta)-1|))^2. \end{aligned}$$

To write $g(s)$ in the form of (2.5), consider the matrices A_r, $A_{r\theta}$, and F_r as follows:

$$A_r = \begin{bmatrix} \alpha_c(a-b) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A_{r\theta} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha_c(a-b) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$F_r = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Suppose that $\phi = -100$, in (2.4) and the initial condition $z^T(0) =$

$$\begin{bmatrix} -0.155 & 1 & 0.155 \end{bmatrix}. \tag{5.6}$$

Consider the parameters of Definition 2.1 as follows:

$$c_1 = 1, c_2 = 8.8^2, T_f = 1, U = I, \delta = 37.432.$$

Suppose $L = B^T X$, with $X = I$, and the values of M_0, N_0, M_1, N_1, M_2, and N_2 are zero. With the help of Yalmip software in Matlab and SeDuMi solver, the following answers are obtained by solving LMIs (4.1), (4.2), and (4.3):

$$X = 10^{-4} * \begin{bmatrix} 0.2945 & 0 & 0 \\ 0 & 0.2945 & 0 \\ 0 & 0 & 0.2945 \end{bmatrix},$$

$$P = 10^{4} * \begin{bmatrix} 3.3952 & 0 & 0 \\ 0 & 3.3952 & 0 \\ 0 & 0 & 3.3952 \end{bmatrix},$$

$$Y = 10^{-5} * \begin{bmatrix} -0.1057 \\ -0.2642 \\ -0.1057 \end{bmatrix}^T,$$

$$\epsilon_1 = 1.237 * 10^{-11}, \epsilon_2 = 1.001 * 10^3, \epsilon_3 = 1.001 * 10^3$$

,

$$\epsilon_4 = 1.001 * 10^3, \nu_1 = 4.7457 * 10^3, \nu_2 = 6.5323 * 10^4.$$

Therefore, the gain K will be obtained as follows:

$$K = \begin{bmatrix} -0.0359 & -0.0897 & -0.0359 \end{bmatrix}.$$

Considering (3.1), the sliding surface will be as follows:

$$\varsigma(s) = \begin{bmatrix} 2 & 5 & 2 \end{bmatrix} z(s) - \int_0^s \begin{bmatrix} 3.8159 & -41.1225 & 3.5397 \end{bmatrix} z(r)dr + \int_0^s (\begin{bmatrix} 0.0720 & 0.0270 & 0.0090 \end{bmatrix} z(r-\theta) + 9\lambda(r))dr.$$

Suppose σ is equal to 0.03, in which case, given equation (4.38), $u(s)$ will be obtained as follows:

$$u(s) = \begin{bmatrix} -0.0359 & -0.0897 & -0.0359 \end{bmatrix} z(s) - \eta(s) sign(\varsigma(s)),$$

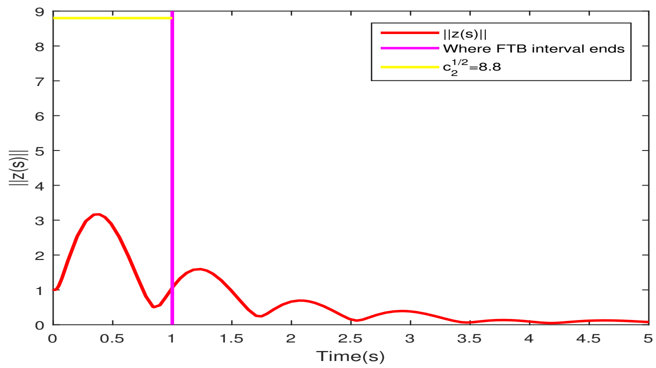

FIGURE 1. Evolution of $\parallel z(s) \parallel$ over time.

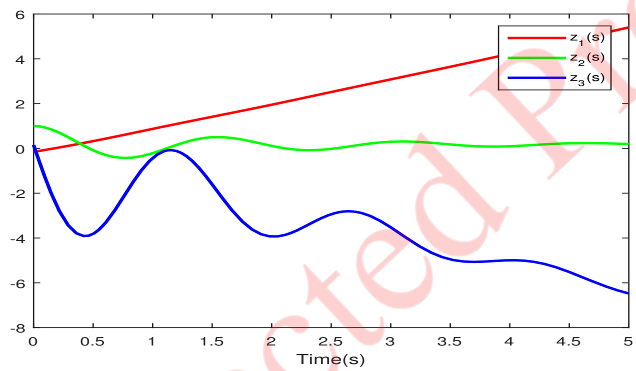

FIGURE 2. Response of the state variables $z_1(s)$, $z_2(s)$ and $z_3(s)$ for the open-loop.

where

$$\eta(s) = 0.03 + 0.1741[\parallel A_r z(s) \parallel + \parallel A_{r\theta} z(s-\theta) \parallel].$$

To eliminate the effect of chattering caused by the input signals, we replace $sign(\varsigma(s))$, with $\varsigma(s)\backslash(0.015+ \parallel \varsigma(s) \parallel)$. According to Figure 1 and Definition 2.1, we observe that based on the defined parameters, the F.T.B condition is established on the time-finite interval $[0, 1s]$. The diagrams of state variables for open-loop and closed-loop systems are shown in Figure 2 and Figure 3, respectively. We observe that in the absence of the control law, the system becomes unstable. Also, by applying the control law, the state variables z_1, z_2, and z_3 reach a predetermined sliding surface in a finite time. Finally, Figure 4 shows the control input $u(s)$, and Figure 5 shows the switching surface function $\varsigma(s)$, and moreover, the minimum upper bound of the cost function is $J^* = 3.7432 * 10^4$.

Then, the efficiency of the studied method is compared with the previous methods. For this purpose, we solve the example presented in [23] with a new method, therefore we ignore the delay and consider other parameters according to [23] as follows:

$$\phi = -100, \quad \alpha_c = 9.1, \quad c_1 = 1, \quad c_2 = 8.8^2, \quad \delta = 1, \quad \alpha = 1.82, \quad z^T(0) = \begin{bmatrix} -8 & 0 & 0 \end{bmatrix},$$

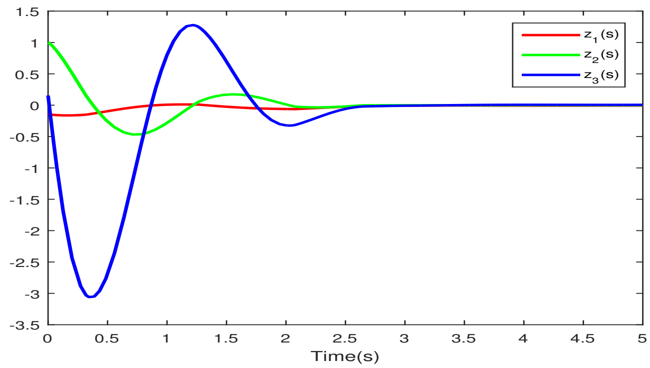

FIGURE 3. Response of the state variables $z_1(s)$, $z_2(s)$ and $z_3(s)$.

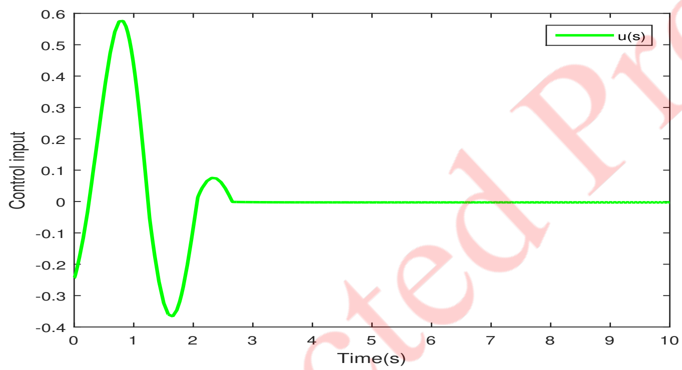

FIGURE 4. Control input $u(s)$.

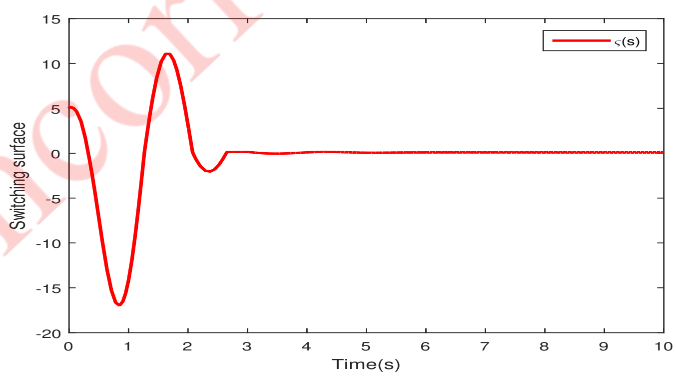

FIGURE 5. Switching surface function $\varsigma(s)$.

and

$$g(z(s), z(s-\theta), \lambda(s)) = \begin{bmatrix} 0.5\alpha_c(a-b)(|z_1(s)+1| - |z_1(s)-1|) \\ 0 \\ 0 \end{bmatrix},$$

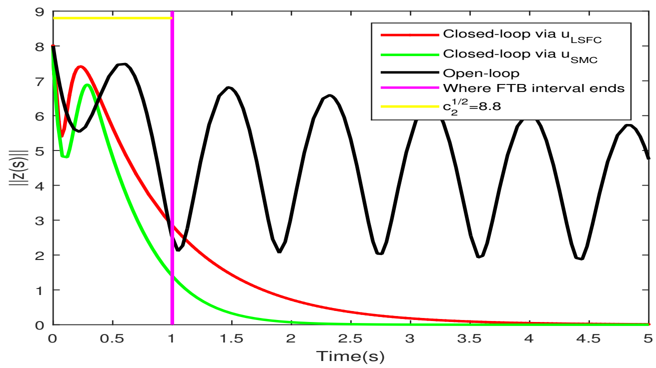

FIGURE 6. Evolution of $\| z(s) \|$ over time for the open-loop and closed-loop via $u_{LSFC}(s)$ and $u_{SMC}(s)$.

other parameters are the same as before.
Using Theorem 2 of [23], the linear state-feedback controller gain will be obtained as follows:

$$K_{LSFC} = \begin{bmatrix} -12.0891 & -10.0109 & 0.3746 \end{bmatrix}.$$

Therefore, the feedback control input will be as follows:

$$u_{LSFC}(s) = K_{LSFC}.z(s).$$

With the help of Yalmip software in Matlab and SeDuMi solver, the following answers are obtained by solving the linear matrix inequalities (4.1), (4.2) and (4.3) in Theorem 4.1:

$$X = 10^{-3} * \begin{bmatrix} 0.1173 & 0 & 0 \\ 0 & 0.1173 & 0 \\ 0 & 0 & 0.1173 \end{bmatrix},$$

$$Y = 10^{-5} * \begin{bmatrix} -0.0863 & -0.2158 & -0.0863 \end{bmatrix},$$

$$K_{SMC} = \begin{bmatrix} -0.0074 & -0.0184 & -0.0074 \end{bmatrix}.$$

Therefore the sliding surface from (3.1) will be as follows:

$$\begin{aligned} \varsigma_{SMC}(s) &= \begin{bmatrix} 2 & 5 & 2 \end{bmatrix} z(s) \\ &\quad - \int_0^s (\begin{bmatrix} 0.3145 & -20.5694 & 4.4810 \end{bmatrix} z(r) + 9\lambda(r))dr. \end{aligned}$$

Hence, we can find the following u_{SMC} controller:

$$u_{SMC}(s) = \begin{bmatrix} -0.0074 & -0.0184 & -0.0074 \end{bmatrix} z(s) - \eta(s) sign(\varsigma_{SMC}(s)),$$

where

$$\eta(s) = 0.03 + 0.1741 \parallel A_r z(s) \parallel .$$

Figure 6 includes the $z^T(s)Uz(s)$ response using the controller defined in [23], the SMC-based controller in this paper, as well as the open-loop system. We observe that, using u_{SMC}, chart $z^T(s)Uz(s)$, is always below chart $z^T(s)Uz(s)$, using u_{LSFC}. Also, chart $z^T(s)Uz(s)$, sits down after about 2 seconds using u_{SMC}, while it sits down after about 4 seconds using u_{LSFC}.

Example 5.2. Consider the following parameters for the system (2.4):

$$A = \begin{bmatrix} -am_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 1 \end{bmatrix}, \quad A_\theta = \begin{bmatrix} -c & 0 & 0 \\ -c & 0 & 0 \\ 2c & 0 & -c \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2.5 \\ 2 \end{bmatrix}, \quad F = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.7 \end{bmatrix},$$
$$A_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{r\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_r = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$
$$M_0 = M_1 = M_2 = \begin{bmatrix} 0.2 & 0.3 & 0.7 \end{bmatrix}^T,$$
$$N_0 = \begin{bmatrix} 0.4 & 0.2 & 0.3 \end{bmatrix},$$
$$N_1 = \begin{bmatrix} 0.3 & 0.5 & 0.3 \end{bmatrix},$$

$N_2 = 0.1,\ \theta = 3,$ where $a = 9, b = 14.28, c = 0.1, m_1 = (2/7), \alpha = 0.1, c_1 = 0.4, c_2 = 5, T_f = 5, U = I, \delta = 0.4, R = 3, Q = diag\{0.1, 0.1, 0.01\}, \phi = 12$ and $L = B^T X$ with $X = I$. Consider the initial condition as $z(0) = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix}^T$.

We also define the conic-type nonlinearity as follows:

$$g(s) = \begin{bmatrix} 0.5(|z_1(s)+1| - |z_1(s)-1|) \\ 0.05(|z_1(s-\theta(s))+1| - |z_1(s-\theta(s))-1|) \\ 0 \end{bmatrix}.$$

Due to the inequality $|x+1| - |x-1| \leq 2|x|$ that holds for any arbitrary real number x, one can find that:

$$\| g(s) \|^2 \leq (0.5(|z_1(s)+1| - |z_1(s)-1|))^2 + (0.05(|z_1(s-\theta)+1| - |z_1(s-\theta)-1|))^2.$$

Therefore, considering the matrices A_r,$A_{r\theta}$ and F_r, $g(s)$ is written in the form of (2.5).
By solving LMIs (4.1), (4.2) and (4.3), we will have the following solutions:

$$X = 10^{-12} \begin{bmatrix} 0.3052 & -0.0571 & -0.2146 \\ -0.0571 & 0.0609 & 0.1476 \\ -0.2146 & 0.1476 & 0.6636 \end{bmatrix}, Y = 10^{-12} \begin{bmatrix} 0.0521 \\ -0.1321 \\ -0.2008 \end{bmatrix}^T.$$

Hence, the gain K can be calculated as:

$$K = \begin{bmatrix} -0.1808 & -3.1738 & 0.3449 \end{bmatrix}.$$

Therefore, by using (3.1), the sliding surface is obtained as follows:

$$\begin{aligned} \varsigma(s) &= \begin{bmatrix} 1 & 2.5 & 2 \end{bmatrix} z(s) \\ &\quad - \int_0^s (\begin{bmatrix} -2.1050 & -57.7651 & 8.3803 \end{bmatrix} z(r) + \begin{bmatrix} 0.5 & 0 & -0.2 \end{bmatrix} z(r-\theta(r)) + 2.55\lambda(r))dr. \end{aligned}$$

By choosing the parameter σ as 0.0150 then the designed $u(s)$ in (4.38) is found to be:

$$u(s) = \begin{bmatrix} -0.1808 & -3.1738 & 0.3449 \end{bmatrix} z(s) - \eta(s) sign(\varsigma(s)).$$

Where

$$\eta(s) = 0.0150 + 0.4245 \| z(s) \| + 0.1837 \| z(s-\theta(s)) \| + 0.0235 \| \lambda(s) \|$$

and moreover the minimum upper bound of the cost function is $J^* = 1.1275$.
To reduce the chattering related to control input signals, we replace $sign(\varsigma(s))$ with $\varsigma(s) \backslash (0.015 + \| \varsigma(s) \|)$. From

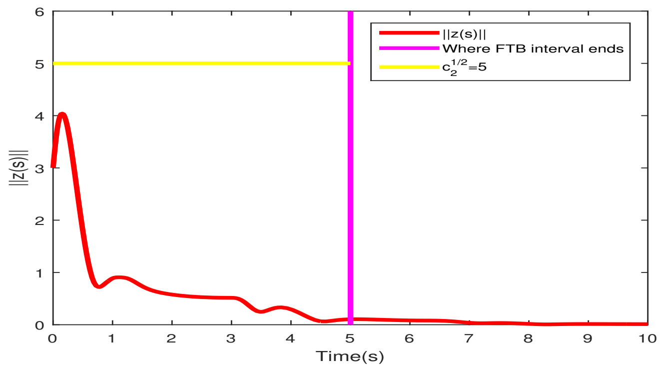

FIGURE 7. Evolution of $\parallel z(s) \parallel$ over time.

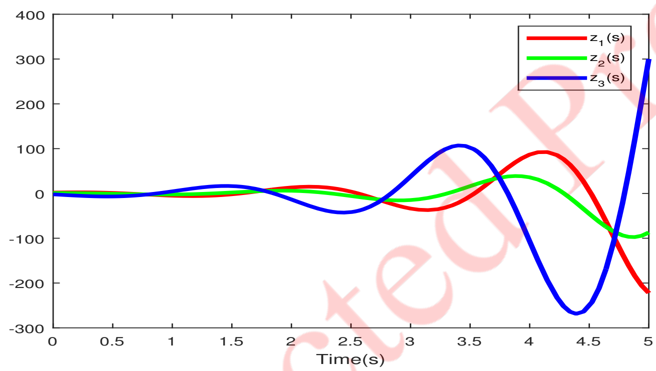

FIGURE 8. Response of the state variables $z_1(s)$,$z_2(s)$ and $z_3(s)$ for the open-loop.

Figure 7, we see that with the defined parameters, the F.T.B condition is established based on Definition (2.1) over the finite-time interval $[0, 5s]$. In Figures 8-10, the state representations of the considered nonlinear system are represented. Figure 8 and Figure 9 show the state trajectories of the system for the open-loop and closed-loop systems. We can observe that the designed SMC law gives the better F.T.B performance and the absence of the controller in the system leads to the unboundedness of the state trajectories. Figure 9 shows the diagrams of the state variables z_1,z_2 and z_3. We observe that these variables have reached the sliding surface in a finite time. Finally Figure 10 shows the response of the designed control input $u(s)$ and Figure 11 shows the finite-time reachability of the switching surface function $\varsigma(s)$ this illustrate the effectiveness of the proposed method.

6. Conclusion

The problem of robust FTB of nonlinear time delay systems using LQR based SMC law has been investigated. Using the Lyapunov-Krasovskii stability theory and LMI technique, sufficient conditions are obtained to ensure the required results for the system. Furthermore, a proposed SMC scheme shown that by synthesizing an SMC law, the system trajectories can be driven onto the predefined switching surface in a finite time. Finally, an example is presented to demonstrate the effectiveness of the proposed results.

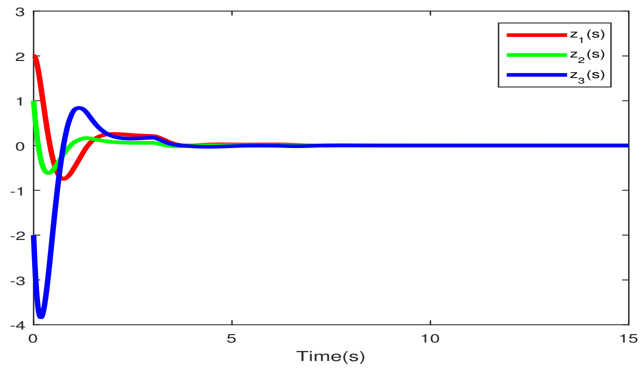

FIGURE 9. Response of the state variables $z_1(s)$,$z_2(s)$ and $z_3(s)$.

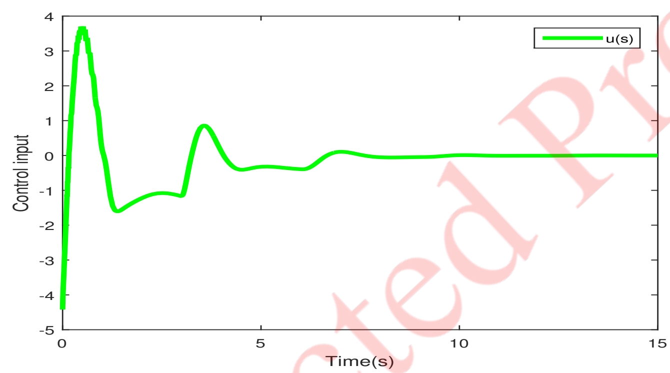

FIGURE 10. Control input $u(s)$.

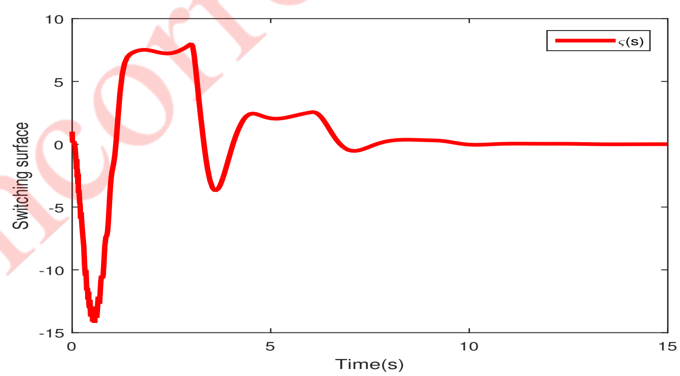

FIGURE 11. Switching surface function $\varsigma(s)$.

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