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Existence of solutions of Caputo fractional integro-differential equations

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Abstract

In this paper, by using the techniques of measures of non-compactness and the Petryshyn fixed point theorem, we investigate the existence of solutions of a Caputo fractional functional integro-differential equation and obtain some new results. These existence results involve particular results gained from earlier studies under weaker conditions.

Keywords. Measures of non-compactness, Fractional functional integro-differential equation, Fixed point theorem, Existence of solutions.
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1. Introduction

Kuratowski [22] introduced the concept of a measure of noncompactness (MNC). Recently, there have been several successful attempt to apply the concept of MNC in the study of the existence of solutions of nonlinear integral equations and integro-differential equations [5–8, 11, 12, 17, 18, 23–25, 31, 34, 35, 37]. The study integro-differential equations is linked to the wide applications of calculus in mathematical sciences. Therefore, they have received much attention. In recent years, many researchers focus on the development of techniques for discussing the solutions of fractional differential equations and fractional integro-differential equations [3, 4, 9, 10, 14, 19, 26, 36].

Susahab *et al.* [33] investigated the existence of solutions for fractional integro-differential equations of the type

$$\begin{cases} {}^{C}D^{\gamma}(\chi(\vartheta)) = q(\vartheta) + \int_0^{\vartheta} k(\vartheta,\ell,\chi(\ell))\,\mathrm{d}\ell, & \vartheta \in [0,a], \\ \chi^{(i)}(0) = \chi_i, & i = 0,1,\dots,n-1. \end{cases} \tag{1.1}$$

Karthikeyan *et al.* [16], studied the existence existence of solutions for fractional integro-differential equations of the type

$${}^{C}D^{\gamma}(\chi(\tau)) = f(\tau,\chi(\tau)) + \int_0^{\tau} k(\tau,s,\chi(s))ds, \qquad \tau \in [0,a]. \tag{1.2}$$

Recently, in [9], Dadsetadi and et al. established the existence and uniqueness of solution of the following nonlinear fractional Volterra integro-differential equations with the help of Darbo's fixed point theorem,

$$\begin{cases} {}^{C}D^{\gamma}\big(\chi(\vartheta) + \rho(\vartheta,\chi(\vartheta))\big) = h(\vartheta,\chi(\vartheta)) + q\Big(\vartheta,\chi(\vartheta),\int_0^{\vartheta} p(\vartheta,\ell)H(\chi(\ell))\,\mathrm{d}\ell\Big), & \vartheta \in [0,a], \\ \chi^{(i)}(0) = \chi_i, \quad i = 0,1,\dots,n-1. \end{cases} \tag{1.3}$$

In 2021, Samei *et al.* studied the following singular fractional integro-differential equation involving Caputo fractional q-derivative, for $0 < s < 1$,

$${}^{C}D_q^{\sigma}\chi(s) = g\Big(s,\chi(s),\chi'(s),{}^{C}D_q^{\zeta}\chi(s),\int_0^{s}\chi(r)\beta(r)\,\mathrm{d}r\Big), \tag{1.4}$$

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with conditions $\chi(0) = 0$ and $\chi(1) = {}^{C}D_q^{\eta}\chi(\tau)$, where $\chi \in C([0,1])$, $\sigma \in [1,2)$, $\zeta, \eta, \tau \in (0,1)$ [30]. Relevant results have been presented in other studies [1, 2, 15, 20, 28].

In the research, motivated by the above mentioned achievements, we discuss new existence results for Caputo fractional Volterra integro-differential equation (FVIDE) of the form

$$
\begin{aligned}
{}^{C}D^{\sigma}\big(\chi(\vartheta) + g(\vartheta, \chi(\vartheta))\big) &= f(\vartheta, \chi(\alpha(\vartheta))) \\
&+ F\bigg(\vartheta, \chi(\beta(\vartheta)), \frac{1}{\Gamma(\varsigma)} \int_0^{\vartheta} \frac{k(\vartheta, \ell, \chi(\mu(\ell)))}{(\vartheta-\ell)^{1-\varsigma}}\, \mathrm{d}\ell\bigg), \qquad \vartheta \in J_a = [0,a],
\end{aligned} \tag{1.5}
$$

with the initial conditions

$$
\chi^{(i)}(0) = \chi_i, \qquad i = 0, 1, \cdot, n-1, \tag{1.6}
$$

presented and proved. Here ${}^{C}D^{\sigma}$ is the Caputo's fractional derivative and $\chi : J_a \to \mathcal{R}$ is an unknown function and other functions are known. Also $g : J_a \times \mathcal{R} \to \mathcal{R}$, $f : J_a \times \mathcal{R} \to \mathcal{R}$, $k : J_a^2 \times \mathcal{R} \to \mathcal{R}, \alpha, \beta, \mu : J_a \to J_a$ are continuous functions and χ_i, $i = 0, 1, \ldots, n-1$, are constants. The main goal of this article is to investigate the new results on the existence of solution of Eq. (1.5). For this, we use a fixed point theorem due to Petryshyn that has been presented as a generalization of Darbo's fixed point theorem.

The paper is arranged as follows: In section 2, we recall some auxiliary facts and notations of the idea of MNC. Section 3 is involved to prove the main theorem for Eq. (1.5). In section 4, some examples are given to demonstrate the applicability of our results. Finally, section 5 concludes our work.

2. Preliminaries

Definition 2.1 ([21])**.** The Riemann-Liouville fractional integral of order $\sigma > 0$ of a function χ, is defined as

$$
I^{\sigma}\chi(\vartheta) = \frac{1}{\Gamma(\sigma)} \int_0^{\vartheta} (\vartheta-\mu)^{\sigma-1}\chi(\mu)\, \mathrm{d}\mu, \quad \vartheta > 0. \tag{2.1}
$$

Definition 2.2 ([21])**.** The Caputo derivative of fractional order non negative $\sigma \geq 0$ for a function χ, is defined by

$$
\left({}^{C}D^{\sigma}\chi\right)(\vartheta) = \frac{1}{\Gamma(n-\sigma)} \int_0^{\vartheta} (\vartheta-\mu)^{n-\sigma-1}\chi^{(n)}(\mu)\, \mathrm{d}\mu, \qquad n = [\sigma]+1. \tag{2.2}
$$

Lemma 2.3 ([21])**.** *Let $\sigma > 0$ and $n = [\sigma]+1$. If $\chi(\vartheta) \in C^n[0,a]$, then*

(i) $\left(I^{\sigma}\, {}^{C}D^{\sigma}\chi\right)(\tau) = \chi(\tau) - \sum_{i=0}^{n-1} \frac{\chi^{(i)}(0)}{i!}\tau^i$;

(ii) $\left({}^{C}D^{\sigma} I^{\sigma}\chi\right)(\tau) = \chi(\tau)$.

Throughout this paper, assume that $E = C([0,a])$ is a Banach space with the standard norm $\|\cdot\|$. Denote by $\bar{B}_{\delta} = \{z \in E : \| z \| \leq \delta\}$ the closed ball centered at the origin 0 of radius δ. The symbol $\partial\bar{B}_{\delta} = \{z \in E : \|z\| = \delta\}$ represents a sphere in E and around 0 with radius δ.

Definition 2.4 ([22])**.** Let $P \subset E$, so $\alpha(P)$ which is called Kuratowski MNC is

$$
\alpha(P) = \inf\left\{\sigma > 0 : P = \bigcup_{i=1}^{n} P_i,\ \mathrm{diam}(P_i) \leq \sigma,\ i = 1, 2, \ldots, n\right\}. \tag{2.3}
$$

Definition 2.5 ([13])**.** Let $P \subset E$ and

$$
\Delta(P) = \inf\left\{\sigma > 0 :\ \text{P has a finite } \sigma\text{-net in } E\right\}. \tag{2.4}
$$

This quantity is called the Hausdorff MNC.

Theorem 2.6 ([29])**.** *Let $P, Q \subset E$, then*

(i) $\Delta(P \cup Q) = \max\{\Delta(P), \Delta(Q)\}$;
(ii) $\Delta(P+Q) \leq \Delta(P) + \Delta(Q)$;
(iii) $\Delta(tP) =\mid t \mid \Delta(P)$, $t \in \mathcal{R}$;
(iv) $\Delta(P) \leq \Delta(Q)$ *for* $P \subseteq Q$;
(v) $\Delta(\bar{co}P) = \Delta(P)$;
(vi) $\Delta(P)$ *is zero if and only if* P *be precompact.*

Theorem 2.7 ([13])**.** *For all bounded sets* P *from space* $C(J_a)$

$$\Delta(P) = \lim_{\sigma \to 0} \sup_{y \in P} \omega^{J_a}(y, \sigma), \tag{2.5}$$

where $\omega^{J_a}(y,\sigma)$ is modulus of continuity of function y on J_a.

Definition 2.8 ([27])**.** Let $\mho : E \to E$ be a continuous mapping so that $\forall P \subset E$ with P bounded, $\mho(P)$ is bounded and $\Delta(\mho P) \leq \lambda \Delta(P), \lambda \in (0,1)$. If

$$\Delta(\mho P) < \Delta(P), \qquad \forall\, \Delta(P) > 0, \tag{2.6}$$

then $\mho$ is called condensing mapping.

Theorem 2.9 ([29], see also [32])**.** *Let* $\mho : B_\delta \to E$ *be a condensing mapping so that:*

(P) *if* $\mho(x) = \lambda x$, *for some* x *in* ∂B_δ *then* $\lambda \leq 1$.

Then $\mho$ *has at least one fixed point in* B_δ.

3. Main results

In this section, fixed point theorems are used to check the existence of the solution to FVIDE (1.5). Applying Lemma 2.3, we get

$$\begin{aligned}\chi(\vartheta) = &\sum_{i=0}^{n-1} \frac{\chi^{(i)}(0) + g^{(i)}(0,\chi_0)}{i!}\vartheta^i - g\big(\vartheta, \chi(\vartheta)\big)\\ &+ \frac{1}{\Gamma(\sigma)}\int_0^\vartheta \frac{f(\ell, \chi(\alpha(\ell)))}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell + \frac{1}{\Gamma(\sigma)}\int_0^\vartheta \frac{F\big(\ell, \chi(\beta(\ell)), (H\chi)(\ell)\big)}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell,\end{aligned} \tag{3.1}$$

where

$$(H\chi)(\vartheta) = \frac{1}{\Gamma(\varsigma)}\int_0^\vartheta \frac{k(\vartheta, \ell, \chi(\mu(\ell)))}{(\vartheta-\ell)^{1-\varsigma}}\,\mathrm{d}\ell. \tag{3.2}$$

The Eq. (1.5) is equivalent to the above fractional integral equation. So, every solution of (3.1) is a solution of (1.5) and vice versa. In what follows, we consider Eq. (3.1) under the following conditions:

(L1) $g \in C(J_a \times \mathcal{R}, \mathcal{R}), f \in C(J_a \times \mathcal{R}, \mathcal{R}), F \in C(J_a \times \mathcal{R}^2, \mathcal{R}), k \in C(J_a^2 \times \mathcal{R}, \mathcal{R})$, and $\alpha, \beta, \mu : J_a \to J_a$ are continuous;

(L2) There exist non negative constants k_1, k_2, c_1, c_2, and c_3 so that $k_1 < 1$ and

$$\begin{aligned}&|g(\vartheta, \omega_1) - g(\vartheta, \varpi_1)| \leq k_1|\omega_1 - \varpi_1|,\\ &|F(\vartheta, \omega_1, \omega_2) - F(\vartheta, \varpi_1, \varpi_2)| \leq c_1|\omega_1 - \varpi_1| + c_2|\omega_2 - \varpi_2|;\end{aligned} \tag{3.3}$$

(L3) $\exists\, \delta_0 \geq 0$ such that

$$\sup\left\{L + A + \frac{M_1 a^\sigma}{\Gamma(1+\sigma)} + \frac{M_2 a^\sigma}{\Gamma(1+\sigma)}\right\} \leq \delta_0, \tag{3.4}$$

where,

$$L = \sup\left\{\Big|\sum_{i=0}^{n-1}\frac{\chi^{(i)}(0)+g^{(i)}(0,\chi_0)}{i!}\vartheta^i\Big| \,:\, \forall\,\vartheta\in J_a\right\},$$
$$A = \sup\left\{|g(\vartheta,\omega_1)| \,:\forall\vartheta\in J_a,\ \omega_1\in[-\delta_0,\delta_0]\right\},$$
$$M_1 = \sup\left\{|f(\vartheta,\omega_1)| \,:\, \forall\vartheta\in J_a,\ \omega_1\in[-\delta_0,\delta_0]\right\},$$
$$M_2 = \sup\left\{|F(\vartheta,\omega_1,\omega_2)| \,:\, \forall\,\vartheta\in J_a,\ \omega_1,|\omega_2|\le\frac{\vartheta^{\varsigma}B}{\Gamma(1+\varsigma)}\right\},$$
$$B = \sup\left\{|k(\vartheta,\ell,\omega_1)| \,:\, \forall\vartheta,\ell\in J_a,\ \omega_1\in[-\delta_0,\delta_0]\right\}. \tag{3.5}$$

Theorem 3.1. *With the conditions* (L1)-(L3)*, Eq.* (1.5) *with the initial conditions* (1.6)*, has at least one solution in* $E = C(J_a)$*.*

Proof. First, $\mho : B_{\delta_0}\to E$ is define as follows

$$(\mho\chi)(\vartheta) = \sum_{i=0}^{n-1}\frac{\chi^{(i)}(0)+g^{(i)}(0,\chi_0)}{i!}\vartheta^i - g(\vartheta,\chi(\vartheta)) + \frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell + \frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{F\big(\ell,\chi(\beta(\ell)),(H\chi)(\ell)\big)}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell, \tag{3.6}$$

then it is checked that $\mho$ is continuous on B_{δ_0}. Considering $\varepsilon>0$ and for arbitrary values $\chi,\eta\in B_{\delta_0}$ such that $\|\chi-\eta\|\le\varepsilon$, when $\vartheta\in J_a$ we will have

$$\begin{aligned}
\big|(\mho\chi)(\vartheta)-(\mho\eta)(\vartheta)| &\le |g(\vartheta,\chi(\vartheta))-g(\vartheta,\eta(\vartheta))\big| \\
&\quad+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{|f(\ell,\chi(\alpha(\ell)))-f(\ell,\eta(\alpha(\ell)))|}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell \\
&\quad+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{|F(\ell,\chi(\beta(\ell)),(H\chi)(\ell))-F(\ell,\eta(\beta(\ell)),(H\eta)(\ell))|}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell \\
&\le k_1\,\|\chi-\eta\| + \frac{\vartheta^{\sigma}}{\Gamma(1+\sigma)}\omega(f,\omega(\alpha,\varepsilon)) \\
&\quad+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{|F(\ell,\chi(\beta(\ell)),(H\chi)(\ell))-F(\ell,\eta(\beta(\ell)),(H\chi)(\ell))|}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell \\
&\quad+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{|F(\ell,\eta(\beta(\ell)),(H\chi)(\ell))-F(\ell,\eta(\beta(\ell)),(H\eta)(\ell))|}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell \\
&\le k_1\,\|\chi-\eta\| + \frac{\vartheta^{\sigma}}{\Gamma(1+\sigma)}\omega(f,\omega(\alpha,\varepsilon)) + \frac{c_1\vartheta^{\sigma}}{\Gamma(1+\sigma)}\,\|\chi-\eta\| \\
&\quad+\frac{c_2\vartheta^{\sigma}s^{\varsigma}}{\Gamma(1+\sigma)\Gamma(1+\varsigma)}\omega(k,\varepsilon),
\end{aligned}$$

where

$$\omega(f,\omega(\alpha,\varepsilon)) = \sup\left\{|f(\ell,\chi)-f(\ell,\eta)| \,:\, \ell\in J_a,\chi,\eta\in[-\delta_0,\delta_0],\|\chi-\eta\|\le\omega(\alpha,\varepsilon)\right\},$$
$$\omega(\alpha,\varepsilon) = \sup\left\{|\alpha(\ell_1)-\alpha(\ell_1)| \,:\, \ell_1,\ell_2\in J_a,\|\ell_1-\ell_1\|\le\varepsilon\right\},$$
$$\omega(k,\varepsilon) = \sup\left\{|k(\vartheta,\ell,\chi)-k(\vartheta,\ell,\eta)| \,:\, \vartheta,\ell\in J_a,\chi,\eta\in[-\delta_0,\delta_0],\|\chi-\eta\|\le\varepsilon\right\}. \tag{3.7}$$

Now, because the functions $f=f(\ell,\chi)$ and $k=k(\vartheta,\ell,\chi)$ are uniformly continuous on $J_a\times\mathcal{R}$ and $J_a^2\times\mathcal{R}$, respectively, we conclude $\omega(f,\omega(\alpha,\varepsilon))\to 0$ and $\omega(k,\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence, the continuity of $\mho$ on B_{δ_0} results. Now, it is shown $\mho$ satisfies densifying condition. Let ε be an arbitrary positive constant. For $\chi\in P\subset E$ let $\vartheta_1,\vartheta_2\in J_a$ while $\vartheta_1\le\vartheta_2$ and $\vartheta_2-\vartheta_1\le\varepsilon$. Therefore we obtain

$$\begin{aligned}
|(\mho\chi)(\vartheta_2)-(\mho\chi)(\vartheta_1)| &= \Big|\sum_{i=0}^{n-1}\frac{\chi^{(i)}(0)+g^{(i)}(0,\chi_0)}{i!}\vartheta_2^i-g(\vartheta_2,\chi(\vartheta_2))\\
&\quad+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_2}\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta_2-\ell)^{1-\sigma}}\,\mathrm{d}\ell+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_2}\frac{F(\ell,\chi(\beta(\ell)),(H\chi)(\ell))}{(\vartheta_2-\ell)^{1-\sigma}}\,\mathrm{d}\ell\\
&\quad-\sum_{i=0}^{n-1}\frac{\chi^{(i)}(0)+g^{(i)}(0,\chi_0)}{i!}\vartheta_1^i+g(\vartheta_1,\chi(\vartheta_1))\\
&\quad-\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta_1-\ell)^{1-\sigma}}\,\mathrm{d}\ell-\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\frac{F(\ell,\chi(\beta(\ell)),(H\chi)(\ell))}{(\vartheta_1-\ell)^{1-\sigma}}\,\mathrm{d}\ell\Big|\\
&\le\Big|\sum_{i=0}^{n-1}\frac{\chi^{(i)}(0)+g^{(i)}(0,\chi_0)}{i!}(\vartheta_2^i-\vartheta_1^i)\Big|\\
&\quad+|g(\vartheta_1,\chi(\vartheta_1))-g(\vartheta_1,\chi(\vartheta_2))|+|g(\vartheta_1,\chi(\vartheta_2))-g(\vartheta_2,\chi(\vartheta_2))|\\
&\quad+\frac{1}{\Gamma(\sigma)}\Big|\int_0^{\vartheta_1}\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta_2-\ell)^{1-\sigma}}\,\mathrm{d}\ell+\int_{\vartheta_1}^{\vartheta_2}\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta_2-\ell)^{1-\sigma}}\,\mathrm{d}\ell-\int_0^{\vartheta_1}\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta_1-\ell)^{1-\sigma}}\,\mathrm{d}\ell\Big|\\
&\quad+\frac{1}{\Gamma(\sigma)}\Big|\int_0^{\vartheta_1}\frac{F(\ell,\chi(\beta(\ell)),(H\chi)(\ell))}{(\vartheta_2-\ell)^{1-\sigma}}\,\mathrm{d}\ell+\int_{\vartheta_1}^{\vartheta_2}\frac{F(\ell,\chi(\beta(\ell)),(H\chi)(\ell))}{(\vartheta_2-\ell)^{1-\sigma}}\,\mathrm{d}\ell\\
&\quad-\int_0^{\vartheta_1}\frac{F(\ell,\chi(\beta(\ell)),(H\chi)(s))}{(\vartheta_1-\ell)^{1-\sigma}}\,\mathrm{d}\ell\Big|\\
&\le k_1|\chi(\vartheta_1)-\chi(\vartheta_2)|+\omega_g(J_a,\varepsilon)\\
&\quad+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\Big|\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta_2-\ell)^{1-\sigma}}-\frac{f(\ell,\chi(\alpha(\ell))}{(\vartheta_1-\ell)^{1-\sigma}}\Big|\,\mathrm{d}\ell+\frac{1}{\Gamma(\sigma)}\int_{\vartheta_1}^{\vartheta_2}\Big|\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta_2-\ell)^{1-\sigma}}\Big|\,\mathrm{d}\ell\\
&\quad+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\Big|\frac{F(\ell,\chi(\beta(\ell)),(H\chi)(\ell))}{(\vartheta_2-\ell)^{1-\sigma}}-\frac{F(\ell,\chi(\beta(\ell)),(H\chi)(\ell)))}{(\vartheta_1-\ell)^{1-\sigma}}\Big|\,\mathrm{d}\ell\\
&\quad+\frac{1}{\Gamma(\sigma)}\int_{\vartheta_1}^{\vartheta_2}\Big|\frac{F(\ell,\chi(\beta(\ell)),\chi(\varsigma(\ell)),(H\chi)(\ell))}{(\vartheta_2-\ell)^{1-\sigma}}\Big|\,\mathrm{d}\ell.
\end{aligned}$$

For simplicity we use the following notation:

$$\omega_g(J_a,\varepsilon)=\sup\Big\{|g(\vartheta,\omega_1)-g(\bar{\vartheta},\omega_1)|\;:\;|\vartheta-\bar{\vartheta}|\le\varepsilon,\;\vartheta\in J_a,\;\omega_1\in[-\delta_0,\delta_0]\Big\}. \tag{3.8}$$

Then we have

$$\begin{aligned}
|(\mho\chi)(\vartheta_2)-(\mho\chi)(\vartheta_1)| &\le k_1\omega(\chi,\varepsilon)+\omega_g(J_a,\varepsilon)\\
&\quad+\frac{M_1}{\Gamma(1+\sigma)}\{\vartheta_1^\sigma-\vartheta_2^\sigma+(\vartheta_2-\vartheta_1)^\sigma\}+\frac{M_1}{\Gamma(1+\sigma)}(\vartheta_2-\vartheta_1)^\sigma\\
&\quad+\frac{M_2}{\Gamma(1+\sigma)}\{\vartheta_1^\sigma-\vartheta_2^\sigma+(\vartheta_2-\vartheta_1)^\sigma\}+\frac{M_2}{\Gamma(1+\sigma)}(\vartheta_2-\vartheta_1)^\sigma\\
&\le k_1\omega(\chi,\varepsilon)+\omega_g(J_a,\varepsilon)+\frac{3\varepsilon^\sigma M_1}{\Gamma(1+\sigma)}+\frac{3\varepsilon^\sigma M_2}{\Gamma(1+\sigma)}.
\end{aligned}\tag{3.9}$$

This yields the following estimate

$$\omega(\mho\chi,\varepsilon)\leq k_1\omega(\chi,\varepsilon),\qquad \chi\in P. \tag{3.10}$$

Thus, taking the supremum in P, then the limit as $\varepsilon\to 0$, we obtain $\Delta(\mho P)\leq k_1\Delta(P)$. Hence $\mho$ is a condensing map. It remains to verify condition (P) of Theorem 2.9. Let $\chi\in\partial\bar{B}_{\delta_0}$. If $\mho\chi=\lambda\chi$ then we have $\lambda\delta_0=\lambda\|\chi\|=\|\mho\chi\|$, and with the condition (L3), we get

$$\begin{aligned}|\mho\chi(\vartheta)|=\Bigg|&\sum_{i=0}^{n-1}\frac{\chi^{(i)}(0)+g^{(i)}(0,\chi_0)}{i!}\vartheta^i-g(\vartheta,\chi(t))\\&+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{f(\ell,\chi(\alpha(\ell)))}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{F(\ell,\chi(\beta(\ell)),(H\chi)(\ell))}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell\Bigg|\leq\delta_0,\quad \vartheta\in J_a,\end{aligned} \tag{3.11}$$

and hence $\|\mho\chi\|\leq\delta_0$, this means $\lambda\leq 1$. □

The following equation which is the main results of Dadsetadi and *et al.* [9], would be obtained from Theorem 3.1.

$${}^{C}D^{\gamma}\big(x(\vartheta)+g(\vartheta,x(\vartheta))\big)=f(\vartheta,x(\vartheta))+F\Big(\vartheta,x(\vartheta),\int_0^{\vartheta}q(\vartheta,\ell)H\big(x(\ell)\big)\,\mathrm{d}\ell\Big),\qquad \vartheta\in J_a=[0,a], \tag{3.12}$$

with the initial conditions

$$x^{(i)}(0)=x_i,\qquad i=0,1,\cdots,n-1, \tag{3.13}$$

Remark 3.2. By employing Riemann-Liouville fractional integrating and Lemma 2.3, Eq. (3.12) changes into

$$\begin{aligned}x(t)=&\sum_{i=0}^{n-1}\frac{x^{(i)}(0)+g^{(i)}(0,x_0)}{i!}\vartheta^i-g(\vartheta,x(\vartheta))\\&+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{f(\ell,x(\ell))}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{F(\ell,x(\ell),(\psi x)(\ell))}{(\vartheta-\ell)^{1-\sigma}}\,\mathrm{d}\ell.\end{aligned} \tag{3.14}$$

where $(\psi x)(\vartheta)=\int_0^{\vartheta}q(\vartheta,\ell)H(x(\ell))d\ell$.

Corollary 3.3. *Consider Eq. (3.14) under the following conditions*

(M1) *There exist non negative constants $k_1,k_2,c_1,c_2,$ and c_3 so that $k_1<1$ and*

$$\begin{aligned}&|g(\vartheta,\omega_1)-g(\vartheta,\varpi_1)|\leq k_1|\omega_1-\varpi_1|\\&|F(\vartheta,\omega_1,\omega_2)-F(\vartheta,\varpi_1,\varpi_2)|\leq c_1|\omega_1-\varpi_1|+c_2|\omega_2-\varpi_2|;\end{aligned} \tag{3.15}$$

(M1) $\exists\ \delta_0\geq 0$ *such that*

$$\sup\left\{L+A+\frac{M_1a^{\sigma}}{\Gamma(1+\sigma)}+\frac{M_2a^{\sigma}}{\Gamma(1+\sigma)}\right\}\leq\delta_0, \tag{3.16}$$

where

$$\begin{aligned}L&=\sup\left\{\left|\sum_{i=0}^{n-1}\frac{\chi^{(i)}(0)+g^{(i)}(0,\chi_0)}{i!}\vartheta^i\right|\,:\,\forall\,\vartheta\in J_a\right\},\\A&=\sup\Big\{|g(\vartheta,\omega_1)|\,:\,\forall\vartheta\in J_a,\,\omega_1\in[-\delta_0,\delta_0]\Big\},\\M_1&=\sup\Big\{|f(\vartheta,\omega_1)|\,:\,\forall\vartheta\in J_a\,\omega_1\in[-\delta_0,\delta_0]\Big\},\\M_2&=\sup\Big\{|F(\vartheta,\omega_1,\omega_2)|\,:\,\forall\vartheta\in J_a,\,\omega_1\in[-\delta_0,\delta_0],\,|\omega_2|\leq aB\Big\},\\B&=\sup\Big\{|q(\vartheta,\ell)H(x(\ell))|\,:\,\forall\vartheta,\ell\in J_a,\,x\in[-\delta_0,\delta_0]\Big\}.\end{aligned} \tag{3.17}$$

Then Eq. (3.12) has at least a solution in J_a.

Proof. It is clear that Eq. (3.12) is a particular case of Eq. (1.5). Here $\alpha(\vartheta) = \beta(\vartheta) = \mu(\vartheta) = \vartheta, \varsigma = 1$ and $k(\vartheta, \ell, x(\mu(\ell))) = q(\vartheta, \ell)H(x(\ell))$. The proof is done similarly to Theorem 3.1. □

4. Examples

Example 4.1. Consider

$$ {}^{C}D^{5/3}\left(\chi(\vartheta) + \frac{1}{3}e^{-\vartheta}\sin(\chi(\vartheta))\right) = \frac{\vartheta^2}{3(1+\vartheta^2)} + \frac{(1+2\vartheta)\ln\left(1+|\chi(\sqrt{\vartheta})|\right)}{7}e^{-\vartheta} + \frac{\vartheta\sin(\chi(1-\vartheta))}{3} + \frac{\sqrt{\vartheta}}{4+4\sqrt{\vartheta}}\frac{1}{\Gamma(1/2)}\int_0^{\vartheta}\frac{\vartheta e^{-\ell}\left(\frac{1}{3} + \int_0^{\ell}(\sin(\chi(\zeta^2)) + \frac{1}{2}\arctan(\frac{\chi(\zeta^2)}{1+\xi(\zeta^2)}))\,\mathrm{d}\zeta\right)}{(\vartheta-\ell)^{1/2}}\,\mathrm{d}\ell, \tag{4.1} $$

for $\vartheta \in J_a$, with

$$ \chi^{(i)}(0) = \chi_i, \quad i = 0, 1. \tag{4.2} $$

Here, Eq. (4.1) is a particular case of Eq. (1.5) with $\sigma = \frac{5}{3}$, $n = 2$, $a = 1$, and

$$ g(\vartheta, \omega_1) = \frac{1}{3}e^{-\vartheta}\sin(\omega_1), $$
$$ f(\vartheta, \chi(\alpha(\vartheta))) = \frac{\vartheta^2}{3(1+\vartheta^2)} + \frac{(1+2\vartheta)\ln(1+|\chi(\sqrt{\vartheta})|)}{7}e^{-\vartheta}, $$
$$ F(\vartheta, \omega_1, \omega_2) = \frac{\vartheta\sin(\omega_1)}{3} + \frac{\sqrt{\vartheta}}{4+4\sqrt{\vartheta}}\omega_2, $$
$$ \omega_2 = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\vartheta}\frac{k(\vartheta, \ell, \chi(\mu(\ell)))}{(\vartheta-\ell)^{\frac{1}{2}}}\,\mathrm{d}\ell, $$
$$ k(\vartheta, \ell, \chi(\mu(\ell))) = \vartheta e^{-\ell}\left(\frac{1}{3} + \chi(\mu(\ell))\right), $$
$$ \chi(\mu(\ell)) = \int_0^{\ell}\left(\sin\left(\chi(\zeta^2)\right) + \frac{1}{2}\arctan\left(\frac{\chi(\zeta^2)}{1+\xi(\zeta^2)}\right)\right)\mathrm{d}\zeta. $$

It is clear that (L1) holds. Also, conditions (L2) and (L3) are satisfied. we have

$$ |g(\vartheta, \omega_1) - g(\vartheta, \varpi_1)| \leq \frac{1}{3}|\omega_1 - \varpi_1|, \tag{4.3} $$

and

$$ |F(\vartheta, \omega_1, \omega_2) - F(\vartheta, \varpi_1, \varpi_2)| \leq \frac{1}{3}|\omega_1 - \varpi_1| + \frac{1}{4}|\omega_2 - \varpi_2|. \tag{4.4} $$

Here $k_1 = \frac{1}{3} < 1, c_1 = \frac{1}{3}, c_2 = \frac{1}{4}$. Also, suppose that $\|\chi\| \leq \delta_0, \delta_0 > 0$ and $\chi_0 = 0, \chi_1 = 1$, then we have

$$ |\chi(\vartheta)| = \left|\sum_{i=0}^{1}\frac{\chi^{(i)}(0) + g^{(i)}(0, \chi_0, 0)}{i!}\vartheta^i - \left(\frac{1}{3}e^{-\vartheta}\sin(\chi(\vartheta))\right) + \frac{1}{\Gamma(5/3)}\int_0^{\vartheta}\frac{f\left(\ell, \chi(\alpha(\ell))\right)}{(\vartheta-\ell)^{2/3}}\,\mathrm{d}\ell + \frac{1}{\Gamma(5/3)}\int_0^{\vartheta}\frac{F\left(\ell, \chi(\beta(\ell)), \chi(\varsigma(\ell)), (H\chi)(\ell)\right)}{(\vartheta-\ell)^{2/3}}\,\mathrm{d}\ell\right| $$
$$ \leq \frac{5}{3} + \frac{1}{\Gamma(5/3+1)}\left(\frac{1}{3} + \frac{3}{7}\delta_0\right) + \frac{1}{\Gamma(5/3+1)}\left(\frac{1}{3} + \frac{1}{4\Gamma(1.5)}\left(\frac{4}{3} + \frac{\delta_0}{2}\right)\right), \quad \forall\vartheta \in J_a. $$

So, condition (L3) holds if

$$ \frac{5}{3} + \frac{1}{\Gamma(5/3+1)}\left(\frac{1}{3} + \frac{3}{7}\delta_0\right) + \frac{1}{\Gamma(5/3+1)}\left(\frac{1}{3} + \frac{1}{4\Gamma(1.5)}\left(\frac{4}{3} + \frac{\delta_0}{2}\right)\right) \leq \delta_0. \tag{4.5} $$

This shows that $\delta_0 = 3.7974$ is a solution of the above inequality. In view of Theorem 3.1, every problems (4.1)-(4.2) has at least one solution defined on $[0, 1]$.

Example 4.2. Consider

$$ {}^{C}D^{0.5}\left(\chi(\vartheta) + \frac{1+\sqrt{1+\vartheta}}{4} + \frac{\vartheta\ln(1+\chi(\vartheta))}{2}\right) = \frac{1}{2}\vartheta e^{-\vartheta^2} + \frac{\sin(\vartheta)\chi(\vartheta\sqrt[3]{\vartheta})}{6(1+\vartheta)} + \frac{|\chi(\sqrt{\vartheta})|}{6(1+|\chi(\sqrt{\vartheta})|} + \frac{1}{7(e^{\vartheta}+\vartheta)}e^{-\vartheta}\frac{1}{\Gamma(3/4)}\int_0^{\vartheta}\frac{(e^{-\vartheta}+\ell\vartheta\cos(\vartheta))\big(\int_0^{\ell}\frac{\vartheta\sin(\chi(\vartheta))}{4}+\vartheta(1+\frac{1}{9}\chi(\zeta^3))d\zeta\big)}{(\vartheta-\ell)^{1/4}}\,\mathrm{d}\ell, \tag{4.6} $$

for $\vartheta \in [0, 1]$, with

$$ \chi(0) = \chi_0 = 0. \tag{4.7} $$

In view of Eq. (1.5), we have $\sigma = 0.5$, $n = a = 1$, and

$$ \begin{aligned} g(\vartheta, \chi(\vartheta)) &= \frac{1+\sqrt{1+\vartheta}}{4} + \frac{\vartheta\ln(1+\chi(\vartheta))}{2}, \\ f(\vartheta, \chi(\alpha(\vartheta))) &= \frac{1}{2}\vartheta e^{-\vartheta^2} + \frac{\sin(\vartheta)\chi(\vartheta\sqrt[3]{\vartheta})}{6(1+\vartheta)}, \\ F(t, \omega_1, \omega_2) &= \frac{|\omega_1|}{6(1+|\omega_1|)} + \frac{1}{7(e^{\vartheta}+\vartheta)}e^{-\vartheta}\omega_2, \\ \omega_2 &= \frac{1}{\Gamma(3/4)}\int_0^{\vartheta}\frac{k(\vartheta,\ell,\chi(\mu(\ell)))}{(\vartheta-\ell)^{1/4}}\,\mathrm{d}\ell, \\ k(\vartheta,\ell,\chi(\mu(\ell))) &= (e^{-\vartheta}+\ell\vartheta\cos(\vartheta))\chi(\mu(\ell)), \\ \chi(\mu(\ell)) &= \int_0^{\ell}\frac{\vartheta\sin(\chi(\vartheta))}{4} + \vartheta\left(1+\frac{1}{9}\chi(\zeta^3)\right)\mathrm{d}\zeta. \end{aligned} $$

Observe that (L1) holds. We show that conditions (L2) and (L3) are satisfied. So,

$$ |g(\vartheta, \omega_1) - g(\vartheta, \varpi_1)| \le \frac{1}{2}|\omega_1 - \varpi_1|, \tag{4.8} $$

and

$$ |F(\vartheta, \omega_1, \omega_2) - F(\vartheta, \varpi_1, \varpi_2)| \le \frac{1}{6}|\omega_1 - \varpi_1| + \frac{1}{7}|\omega_2 - \varpi_2|. \tag{4.9} $$

Here $k_1 = \frac{1}{2} < 1$, $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{7}$. Also, suppose that $\|\chi\| \le \delta_0$, $\delta_0 > 0$ and $\chi_0 = 0$, then we have

$$ \begin{aligned} |\chi(\vartheta)| &= \Bigg|\chi(0) + g(0,\chi_0) - g(\vartheta,\chi(\vartheta)) + \frac{1}{\Gamma(1/2)}\int_0^{\vartheta}\frac{f\big(\ell,\chi(\alpha(\ell))\big)}{(\vartheta-\ell)^{1/2}}\,\mathrm{d}\ell \\ &\quad + \frac{1}{\Gamma(1/2)}\int_0^{\vartheta}\frac{F\big(\ell,\chi(\beta(\ell)),\chi(\varsigma(\ell)),(H\chi)(\ell)\big)}{(\vartheta-\ell)^{1/2}}\,\mathrm{d}\ell\Bigg| \\ &\le \left(\frac{5}{4}+\frac{1}{2}\delta_0\right) + \frac{1}{\Gamma(1.5)}\left(\frac{1}{2}+\frac{1}{6}\delta_0\right) + \frac{1}{\Gamma(1.5)}\left(\frac{1}{6}+\frac{2}{7\Gamma(1.75)}\left(\frac{5}{4}+\frac{1}{6}\delta_0\right)\right) < \delta_0, \end{aligned} \tag{4.10} $$

for $\vartheta \in J_a$. This shows $\delta_0 = 8.2528$ is a solution of the above inequality. In view of Theorem 3.1, every problems (4.6)-(4.7) has at least one solution defined on $[0, 1]$.

5. Conclusion

This article handles the existence of solutions of a Caputo fractional functional integro-differential Equation (1.5) in the Banach space. With the assistance of appropriate of measure of noncompactness and the Petryshyn fixed point theorem hypothesis, we demonstrated our current results of the studied problem in Banach algebra. We presented some examples to illustrate the efficiency of our results.

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