



## A nonlinear mathematical model of the delayed predator-prey system that incorporates intraspecific predator competition and fear effect in prey

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### Abstract

In ecological systems, predator-prey contact is seen as something that happens naturally. How does the density of prey populations effect predators ? This is a naturally occurring issue in ecosystems. Even though it plays a little role in population dynamics, predators in most ecological models lower prey numbers by direct killing. Research on vertebrates has shown that predator aversion may impact prey population dynamics and reproductive rates. There has been new research on mathematical models of predator-prey systems that include a range of predator functional responses that include the fear effect. Researchers in these research failed to account for the impact of fear on prey mortality rates. In light of the above, our study focuses on analysing a predator-prey system that incorporates the cost of perceived fear into reproductive processes using a Holling type-IV functional response. The scheme also includes intraspecific competition within the predators and a gestation delay to make the interactions more realistic and natural. The increase of the predator population is constrained for high predator to prey density ratios by this extra intraspecific competition term. These dynamic model's fundamental aspects such as non-negative, boundedness of solutions, and viability of equilibria are investigated, and adequate conditions are discovered. Both the local and global stability of the system are obtained with sufficient conditions on its functionals and parameters. This study makes a major impact in that it creates a novel technique to quantify some important, regulating system resilience parameters, and it studies the presence of Hopf bifurcation when the time lag parameters exceed the critical values by looking at the relating characteristic equation. Furthermore, we addressed how time delay factors reaching thresholds causes the Hopf bifurcation. Numerous numerical examples are used to validate all of these theoretical inferences, and simulations are given to help visualise the examples.

**Keywords.** Prey-predator model, Fear effect, Functional response, Time delay, Hopf bifurcation, Simulations.

**2010 Mathematics Subject Classification.** 93E15, 60H40, 34D20.

### 1. INTRODUCTION

In order to sustain ecological variety and build community structure, analysing the phenomenon of predator-prey involvement is crucial. An extensive amount of papers have been written in the last few decades in order to clarify the complex changing aspects of the predator-prey interaction by undertaking mathematical modelling. While it is undeniable that predators have an impact on prey dynamics directly through killing, numerous recent studies [12, 15] have demonstrated that fear effects also have a significant impact on ecology and evolutionary biology. Despite the fact that various predators have diverse effects on target populations, fear of predation reduces population fitness overall [1, 2].

This could potentially occur without wiping out the entire prey populace. Additionally, fear can alter the behaviour of prey, and terrified prey decrease their ability to reproduce. Furthermore, the presence of predators may cause more severe changes in the physiology and behaviour of prey than actual predation [9, 18]. Prey populations use a number of anti-predator tactics, including habitat use, foraging activity, awareness, and physiological changes [11, 21–24, 28], to escape predator attacks. For instance, mule deer reduce their food intake after a lion assault [3]. Elk's reproductive

Received: 28 May 2024 ; Accepted: 30 July 2024.

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physiology changes in the Greater Yellowstone Ecosystem when wolves assault them [10]. In the end, antipredator measures occasionally may be helpful for survival, but they have long-term negative effects on reproduction, which in turn reduce population numbers [11]. The connection between prey and predator should involve additionally to just direct predation, according to ecologists and evolutionary biologists [24]. They also believe that fear should be taken into account. However, a mathematical model combining the dread impact into predator-prey behaviours was non-existent. Wang et al. [27] created a mathematical model that included fear in 2016, confirming the significance of fear for the development of prey. Additionally, Wang and Zou [33] and Mondal et al. [20] have published works that mimic the fear effect in delayed predator-prey encounters. Intraspecific rivalry among predator species is another component of our concept. There is intraspecific rivalry among predators when there are many more predators than prey, which reduces the fitness of the predator population because food is scarce [16, 17, 25]. Blue crab populations exhibit agonistic activity, which results in damage when available food is limited, and this is indicative of intraspecific competition [8]. Carnivorous (predator) fish eat detritivorous (prey) fish in the Sundarban mangrove habitat. Predators may readily capture prey and display intraspecific rivalry in their search for food since there is no sanctuary for prey [26].

In ecology, the manner in which predators ingest their prey and their interactions with one another have a substantial influence on the changing features of the whole ecosystem. The computational model relies significantly on this interaction, known as the "functional response", which furnishes the pace at which the predator consumes prey per unit of population [13, 14]. By incorporating this rate into the model, its dynamics can be examined in greater detail. Numerous functional responses are occasionally found in ecological literature. A few of these include the Holling types II, III, and IV, the Monod-Haldane functional response, the Beddington-DeAngelis functional response, the Crowley-Martin functional response, and the ratio-dependent functional responses, among others [7, 19, 20, 27, 29–31].

Holling type IV was one of these and was proposed by Andrews [4], it differs from the above types in that it has a collective defence capacity and does not respond monotonously. As the prey biomass reaches a specific level, this group defence exercise demonstrates that the prey increases its defence in an effort to contain the predator's escalating rate. This phenomenon is seen in aquatic ecosystems where aquatic snails (*Nucella lamellosa*) defend themselves by hardening their shells [5] in response to crab predation. Numerous studies on predator-prey interactions have been conducted since the prey's ability to defend itself has a significant impact on its capacity to reproduce.

According to the research efforts mentioned above, fear can reduce both prey reproduction and predator population, and a functional reaction can control predator densities. Therefore, it makes sense to research the predator-prey relationship while allowing for the predator's fear component and functional reaction with gestation delay. By employing an improved Leslie-Gower framework involving Holling type IV operations, the chief objective of this investigation is to explore the effect of dread on a predator-prey system. This study is structured by way of: We construct the mathematical model in section 2. In the same part, topics including positivity, boundedness, equilibrium existence, stability, and the impact of fear on prey and predator equilibrium densities are covered. In Section 3, a standard for global stability for the non-delayed system is developed. In Section 4, the stability analysis of the delay-induced model is investigated with Hopf bifurcation. To demonstrate our theoretical conclusions in section 5, we ran several numerical simulations. Finally, we discuss the biological importance of our findings in section 6.

## 2. MATHEMATICAL FORMULATION

This section entails the development of a predator-prey framework that incorporates a specific obstacle: (i) Holling type IV schemes using an enhanced Leslie-Gower system [6] (ii) Fear effect [32]. Fear has the potential to impede the expansion of prey populations, according to recent field research. Due to this, Wang et al. [32] were inspired to include the fear factor  $F(\delta, v) = \frac{\alpha}{1 + \delta v}$  in the predator-prey model, which was later applied in [18–20, 20]. We suggest the



pair of equations below based on the aforementioned facts:

$$\frac{du}{dt} = ru \left(1 - \frac{u}{k}\right) + \frac{\alpha u}{1 + \delta v} - \frac{\beta uv}{u^2 + a}, \tag{2.1}$$

$$\frac{dv}{dt} = sv \left(1 - \frac{v}{nu}\right), \tag{2.2}$$

Table 1 lists the model (2.1)-(2.2)'s parameters.

The non-monotonic functional response [27]  $g(u) = \frac{\beta u}{u^2 + a}$ , where  $\beta$  and  $a$  are positive constants and  $g(u) >$

TABLE 1. Description of model(2.1)-(2.2) parameters.

Parameters:	Description
$u$	the densities of prey
$v$	the densities of predator
$r$	biotic potential of the population $u$
$k$	carrying capacity of $u$
$\alpha$	maximum cost of fear
$\beta$	the predator maximum consumption rate per capita
$a$	the quantity of prey required to reach half of the maximum rate $\beta$
$s$	intrinsic growth rate of $v$
$n$	a measure of the food's nutritional content, which prey provides to predators for their own reproduction
$\delta$	level of fear

0, for all  $u > 0$  characterises the occurrence of defence group formation in antipredator behaviour (APB)[12, 34]. We also take into account the intraspecific rivalry that is seen in populations of predators when their biomass is disproportionately high. The process of producing the offspring of a predator after consuming prey is mediated in almost every ecosystem by the time latency ( $\tau$ ) required for digestion and gestation. Our final model takes into account intraspecific competition within predator species, gestation delay, and the fear impact in prey species:

$$\frac{du}{dt} = ru \left(1 - \frac{u}{k}\right) + \frac{\alpha u}{1 + \delta v} - \frac{\beta uv}{u^2 + a}, \tag{2.3}$$

$$\frac{dv}{dt} = sv(t - \tau) \left(1 - \frac{v(t - \tau)}{nu(t - \tau)}\right), \tag{2.4}$$

The parameters for model (2.3)-(2.4) are the same as those for models(2.1)-(2.2).

**2.1. Positivity and boundedness of solutions.** In this section, we first look at the non-delayed model (2.1)-(2.2)'s positivity and boundedness. Since they pertain to the biological assurance, these are essential. We start by proving the positivity.

**Lemma 2.1.** *Given any  $t > 0$ , every solution  $(u(t), v(t))$  in system (2.1)-(2.2) having initial values  $(u_0, v_0) \in \mathbb{R}_+^2$  are bound to be positive.*

*Proof.* The following equations can be used to determine whether  $u(t)$  and  $v(t)$  are positive.

$$u(t) = u_0 \exp \left\{ \int_0^t \left[ r - \frac{ru}{k} + \frac{\alpha}{1 + \delta v} - \frac{\beta v}{a} + u^2(s) \right] ds \right\},$$

$$v(t) = v_0 \exp \left\{ \int_0^t \left[ s \left(1 - \frac{v}{nu}\right) \right] ds \right\}.$$



□

with  $u_0, v_0 > 0$ . As  $u_0 > 0$  then  $u(t) > 0, \forall t > 0$ . In a similar manner, we may demonstrate that  $v(t) > 0$ .

**Lemma 2.2.** *All solutions of system (2.1)-(2.2) that represent in  $\mathbb{R}_+^2$  will be included in set*

$$B = \left\{ (u, v) \in \mathbb{R}_+^2 : u \leq \frac{k\alpha}{r}, v \leq \frac{nr}{a} \right\}.$$

*Proof.* We can write the following inequality on using the Equation (2.1),

□

$$\frac{du}{dt} \leq u \left( \alpha - \frac{r}{k}u \right), \quad (2.5)$$

which implies that

$$\limsup_{t \rightarrow \infty} u(t) \leq \frac{k\alpha}{r}.$$

We can obtain the below inequality on using the estimate of  $u(t)$  in the Equation (2.2)

$$\frac{dv}{dt} \leq sv \left( 1 - \frac{vr}{nk\alpha} \right), \quad (2.6)$$

which implies that

$$\limsup_{t \rightarrow \infty} v(t) \leq \frac{nk\alpha}{r} = \mu.$$

Therefore, the scheme (2.1)-(2.2) is bounded.

**2.2. Behaviour of valid steady states.** Equilibrium points are the points of intersection of two curves (2.1)-(2.2) in non-negative quadrant  $\mathbb{R}_+^2 = \{(u(t), v(t)) : u(t) \geq 0, v(t) \geq 0\}$ . System (2.1)-(2.2) has three biologically feasible equilibria in  $\mathbb{R}_+^2$ , specifically:

1. Trivial equilibrium point:  $E_0 = (0, 0)$ ,
2. Auxiliary equilibrium point:  $E_1(u, 0) = \left( k \left( 1 + \frac{\alpha}{r} \right), 0 \right)$ , and
3. Interior equilibrium point:  $\bar{E}(\bar{u}, \bar{v})$ .

Here  $\bar{u}$  and  $\bar{v}$  have to be positive and meet the following equations.

$$r \left( 1 - \frac{u}{k} \right) + \frac{\alpha}{1 + \delta v} - \frac{\beta v}{u^2 + a} = 0, \quad (2.7)$$

$$1 - \frac{v}{nu} = 0, \quad (2.8)$$

from Eq. (2.8), we have  $\bar{v} = n\bar{u}$ , we obtain the following equation on using Eq. (2.8) in Eq. (2.7),

$$d_0\bar{u}^4 + 4d_1\bar{u}^3 + 6d_2\bar{u}^2 + 4d_3\bar{u} + d_4 = 0, \quad (2.9)$$

where  $d_0 = rn\delta$ ,  $d_1 = \frac{1}{4}(r - rkn\delta)$ ,  $d_2 = \frac{1}{6}(ar\delta n + k\beta\delta n^2 - kr - k\alpha)$ ,  $d_3 = \frac{1}{4}(ar + kn\beta - arkn\delta)$ ,  $d_4 = -ka(\alpha + r)$ .

Prior to examine whether Eq. (2.9) roots exist, we need  $I = d_0d_4 - 4d_1d_3 + 3d_2^2$ ,

$$J = \begin{vmatrix} d_0 & d_1 & d_2 \\ d_1 & d_2 & d_3 \\ d_2 & d_3 & d_4 \end{vmatrix},$$

$I^3 - 27J^2 = D$ , which represents discriminant of Eq. (2.9). The criteria for positive conditions are listed below.

**Theorem 2.3.** *Assume that  $\alpha > -r$ . Equation (2.9) accepts*

- (i) if  $D < 0$ , strictly a single positive root of  $\bar{u}$ , and
- (ii) if  $D > 0$ , multiple roots of  $\bar{u}$ .



*Proof.* The Equation (2.9) has more than a single positive root as well as a negative root due to  $d_4 < 0$ . It takes two real and two imaginary roots for  $D < 0$ . As a result, Equation (2.9) has precisely a single positive root in case (i) All of Equation (2.9)'s roots can be either real or imaginary when  $D > 0$ . Since it has already been established that Equation (2.9) allows for more than a single positive root, all of the roots in case (ii) are real. This guarantees that there are multiple roots. Hence the theorem.  $\square$

**Theorem 2.4.** Assume that Theorem (2.3)s condition (i) is true. Assume also that  $\frac{2\beta\bar{v}\bar{u}^2}{(\bar{u}^2 + a)^2} > \left(\frac{r\bar{u}}{k} + \frac{s\bar{v}}{n\bar{u}}\right)$  and  $\bar{v} < \frac{1}{2}$ . Then  $\bar{E}$  becomes stable.

*Proof.* The community matrix of the scheme (2.1)-(2.2) at  $\bar{E}$  is  $\square$

$$J(\bar{E}) = \begin{pmatrix} -\frac{r\bar{u}}{k} + \frac{2\beta\bar{v}\bar{u}^2}{(\bar{u}^2 + a)^2} & -\frac{\alpha\delta\bar{u}}{(1 + \delta\bar{v})^2} - \frac{\beta\bar{u}}{\bar{u}^2 + a} \\ \frac{s\bar{v}^2}{n\bar{u}^2} & -\frac{s\bar{v}}{n\bar{u}} \end{pmatrix}. \tag{2.10}$$

The characteristic equation around  $\bar{E}$  is

$$\lambda^2 + \rho_1\lambda + \rho_2 = 0. \tag{2.11}$$

where  $\rho_1 = -\frac{r\bar{u}}{k} + \frac{2\beta\bar{v}\bar{u}^2}{(\bar{u}^2 + a)^2} - \frac{s\bar{v}}{n\bar{u}}$ ,  $\rho_2 = \frac{rs\bar{v}}{n\bar{u}} + \frac{\alpha s\delta\bar{v}}{n\bar{u}(1 + \delta\bar{v})^2} + \frac{\beta s\bar{v}[\bar{u}^2(1 - 2\bar{v}) + a]}{n(\bar{u}^2 + a)^2}$ .

Now,  $\rho_1 > 0$  and  $\rho_2 > 0$  if  $\frac{2\beta\bar{v}\bar{u}^2}{(\bar{u}^2 + a)^2} > \left(\frac{r\bar{u}}{k} + \frac{s\bar{v}}{n\bar{u}}\right)$  and  $\bar{v} < \frac{1}{2}$ . In that case, Equation (2.11) has two roots with nonpositive real parts. As a result, the Theorem (2.4) assumption causes  $\bar{E}$  to develop into a stable one.

**2.3. Local stability.** For the purpose of analysing the local permanency properties near the equilibrium points  $E_0$  and  $E_1$ , the notion of linear stability studies is employed. We examine the local stability around the coexistence steady state  $\bar{E}$  by looking at the signs of the trace and determinant of the appropriate variational matrix because the components of the point  $\bar{E}$  are not explicitly expressed. With the aid of a graphical method, local stability can be visualised more effectively. The variational matrix,  $J(u, v)$  of system (2.1)-(2.2) at any point  $(u, v)$  is given by

$$J(u, v) = \begin{pmatrix} -\frac{ru}{k} + \frac{2\beta vu^2}{(u^2 + a)^2} & -\frac{\alpha\delta u}{(1 + \delta v)^2} - \frac{\beta u}{u^2 + a} \\ \frac{sv^2}{nu^2} & -\frac{sv}{nu} \end{pmatrix} \tag{2.12}$$

**Theorem 2.5.** At all times, the point of trivial equilibrium  $E_0$  is an instability saddle.

The variational matrix  $J(E_1)$  at  $E_1 = \left(k\left(1 + \frac{\alpha}{r}\right), 0\right)$  has been provided by

$$J(E_1) = \begin{pmatrix} -(r + \alpha) & -k\left(1 + \frac{\alpha}{r}\right) \left[ \alpha\delta + \frac{\beta r^2}{k^2(r + \alpha)^2 + ar^2} \right] \\ 0 & s \end{pmatrix} \tag{2.13}$$

The latent values of  $J(E_1)$  given by  $-(r + \alpha)$  and  $s$ . Hence the predator-free equilibrium point is always saddle since  $r + \alpha < 0$  and  $s > 0$ .

**Theorem 2.6.** The auxiliary equilibrium point  $E_1$  is a saddle point if  $r + \alpha < 0$  and  $s > 0$ .



**2.4. The effect of fear factor.** Next, we will examine how the presence of fear influences the concentrations at a steady state. Initially, we shall illustrate the impact of  $\delta$  on  $\bar{u}$  and  $\bar{v}$ . It was previously mentioned that the coordinates of  $\bar{E}$  must meet the requirements

$$r \left(1 - \frac{\bar{u}}{k}\right) + \frac{\alpha}{1 + \delta\bar{v}} - \frac{\beta\bar{v}}{\bar{u}^2 + a} = 0, \quad (2.14)$$

$$1 - \frac{\bar{v}}{n\bar{u}} = 0, \quad (2.15)$$

From Eq. (2.15),  $\bar{v} = n\bar{u}$ . Differentiating it with respect to  $\delta$ , we have

$$\frac{d\bar{v}}{d\delta} = n \frac{d\bar{u}}{d\delta}. \quad (2.16)$$

Differentiating (2.14) with respect to  $\delta$ , we have

$$-\frac{r}{k} \frac{d\bar{u}}{d\delta} - \frac{\alpha}{(1 + \delta\bar{v})^2} \left\{ \bar{v} + \delta \frac{d\bar{v}}{d\delta} \right\} - \beta \frac{\left\{ (\bar{u}^2 + a) \frac{d\bar{v}}{d\delta} - \bar{v}(2\bar{u}) \frac{d\bar{u}}{d\delta} \right\}}{(\bar{u}^2 + a)^2} = 0. \quad (2.17)$$

Using (2.16) in (2.17)

$$\frac{d\bar{u}}{d\delta} \left\{ \frac{r}{k} + \frac{\alpha\delta n}{(1 + \delta\bar{v})^2} + \frac{\beta n}{(\bar{u}^2 + a)} - \frac{2\beta\bar{u}\bar{v}}{(\bar{u}^2 + a)^2} \right\} = -\frac{\alpha\bar{v}}{(1 + \delta\bar{v})^2}. \quad (2.18)$$

Here we consider two cases.

Case 1: If  $\frac{r}{k} + \frac{\alpha\delta n}{(1 + \delta\bar{v})^2} + \frac{\beta n}{(\bar{u}^2 + a)} > \frac{2\beta\bar{u}\bar{v}}{(\bar{u}^2 + a)^2}$  then  $\frac{d\bar{u}}{d\delta} < 0$  which also denotes that  $\frac{d\bar{v}}{d\delta} < 0$ .

Case 2: If  $\frac{r}{k} + \frac{\alpha\delta n}{(1 + \delta\bar{v})^2} + \frac{\beta n}{(\bar{u}^2 + a)} < \frac{2\beta\bar{u}\bar{v}}{(\bar{u}^2 + a)^2}$  then  $\frac{d\bar{u}}{d\delta} > 0$  which also denotes that  $\frac{d\bar{v}}{d\delta} > 0$ .

The initial instance demonstrates that as fear levels rise, the density of both predator and prey species decreases; conversely, the second instance reveals the opposite to be true. The circumstance of the second instance imply that the stable positive equilibrium point does not exist.

### 3. GLOBAL STABILITY ANALYSIS

Next, we'll use the Lyapunov function to examine the coexistence equilibrium point  $\bar{E}$ 's global stability.

**Theorem 3.1.** *Let the assumption of Theorem (2.5) be satisfied. Further assume that*

$$4 \left[ \frac{r}{k} - \frac{2\beta\mu k\alpha}{r(\bar{u}^2 + a)} \right] \frac{s}{n\bar{u}} > \left[ \frac{\alpha\delta}{(1 + \delta\bar{v})} + \frac{s\mu}{n\bar{u}} + \frac{\beta}{\bar{u}^2 + a} \right]^2.$$

*Then  $\bar{E}$  is globally asymptotically stable.*

*Proof.* Consider a function  $\Lambda : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  defined by

$$\Lambda(u, v) = \left( u - \bar{u} - \bar{u} \ln \frac{u}{\bar{u}} \right) + \left( v - \bar{v} - \bar{v} \ln \frac{v}{\bar{v}} \right). \quad (3.1)$$

Differentiating  $\Lambda$  with respect to time, we have

$$\begin{aligned} \frac{d\Lambda}{dt} &= (u - \bar{u}) \frac{\dot{u}}{u} + (v - \bar{v}) \frac{\dot{v}}{v} \\ &= (u - \bar{u}) \left\{ r \left(1 - \frac{u}{k}\right) + \frac{\alpha}{1 + \delta v} - \frac{\beta v}{u^2 + a} \right\} + (v - \bar{v}) \left( s - \frac{sv}{nu} \right) \\ &\leq -(u - \bar{u})^2 \left[ \frac{r}{k} - \frac{2\beta\mu k\alpha}{r(\bar{u}^2 + a)} \right] + |u - \bar{u}| |v - \bar{v}| \left[ \frac{\alpha\delta}{(1 + \delta\bar{v})} + \frac{s\mu}{n\bar{u}} + \frac{\beta}{\bar{u}^2 + a} \right] - \frac{s}{n\bar{u}} (v - \bar{v})^2, \end{aligned}$$



$$\frac{d\Lambda}{dt} = -X^T B X, \tag{3.2}$$

where  $X = \{|u - \bar{u}|, |v - \bar{v}|\}$  and  $B = [b_{ij}]_{2 \times 2}$ . The components of the matrix  $B$  are

$$b_{11} = \frac{r}{k} - \frac{2\beta\mu k\alpha}{r(\bar{u}^2 + a)}, \quad b_{12} = b_{21} = -\frac{1}{2} \left[ \frac{\alpha\delta}{(1 + \delta\bar{v})} + \frac{s\mu}{n\bar{u}} + \frac{\beta}{\bar{u}^2 + a} \right], \quad b_{22} = \frac{s}{n\bar{u}}.$$

Hence  $B$  is positive definite if  $4 \left[ \frac{r}{k} - \frac{2\beta\mu k\alpha}{r(\bar{u}^2 + a)} \right] \frac{s}{n\bar{u}} > \left[ \frac{\alpha\delta}{(1 + \delta\bar{v})} + \frac{s\mu}{n\bar{u}} + \frac{\beta}{\bar{u}^2 + a} \right]^2$ , which gives  $\frac{d\Lambda}{dt} < 0$ .

Hence  $\bar{E}$  is globally asymptotically stable. □

#### 4. DELAYED MODEL AND HOPF-BIFURCATION

When studying biological processes, our major objective is to identify the specific characteristics that contribute to the durability of the system about the coexistence steady state  $E(\bar{u}, \bar{v})$ . This stability is crucial for maintaining ecological diversity. This section aims to investigate the local stability of the cohabitation steady state in the delay-induced framework (2.3) and (2.4). We already know that at least one co-existence steady state exists if  $r + \alpha > 0$ . The following describes the community matrix for the delayed framework (2.3)-(2.4):

$$J(u, v) = \begin{pmatrix} -\frac{ru}{k} + \frac{2\beta vu^2}{(u^2 + a)^2} & -\frac{\alpha\delta u}{(1 + \delta v)^2} - \frac{\beta u}{u^2 + a} \\ \frac{sv^2}{nu^2} e^{-\lambda\tau} & s \left( 1 - \frac{2v}{nu} \right) e^{-\lambda\tau} \end{pmatrix}, \tag{4.1}$$

The latent equation is given by  $\lambda^2 + T\lambda + D = 0$ .

We write it as

$$\lambda^2 + L_1\lambda + e^{-\lambda\tau} (L_2\lambda + L_3) = 0, \tag{4.2}$$

where  $L_1 = \frac{ru}{k} - \frac{2\beta vu^2}{(u^2 + a)^2}$ ,  $L_2 = -s \left( 1 - \frac{2v}{nu} \right)$  and

$$L_3 = s \left( 1 - \frac{2v}{nu} \right) \left\{ -\frac{ru}{k} + \frac{2\beta vu^2}{(u^2 + a)^2} \right\} + \frac{sv^2}{nu^2} \left\{ \frac{\alpha\delta u}{(1 + \delta v)^2} + \frac{\beta u}{u^2 + a} \right\}.$$

In the absence of any delay ( $\tau = 0$ ), the latent equation associated with the condition is

$$\lambda^2 + (L_1 + L_2)\lambda + L_3 = 0. \tag{4.3}$$

The following requirements must be met in order for cohabitation to be stable.

$$L_1 + L_2 > 0, \text{ and } L_3 > 0. \tag{4.4}$$

In the delay-induced scheme (2.3)-(2.4), the coexistence steady state  $\bar{E}(\bar{u}, \bar{v})$  resilience is determined by the sign of the real components of the roots of the latent Equation (4.1) associated with it. Local asymptotically stable  $\bar{E}$  is denoted by all nonpositive real components; otherwise, it lacks stability. We begin by presuming that  $\bar{E}$  is locally asymptotically stable for unaltered system. Next, we examine the specific conditions under which  $\bar{E}$  remains steady despite any delays. The Equation (4.1) will only have solutions with positive real parts if and only if it possesses entirely fictitious roots, as indicated by the continuity requirement of the delay factor  $\tau$ . Since Equation (4.1) contains roots with a nonpositive real portion and the coexistence equilibrium state  $\bar{E}$  is asymptotically stable, we can infer the parametric circumstances under which these two phenomena occur.

Consider  $\lambda(\tau) = \phi(\tau) + i\gamma(\tau)$  be the root of Equation (4.1). Since  $\bar{E}$  is stable for non delayed system, we have  $\phi(0) < 0$ . By the continuity of  $\tau$ , we can say that  $\phi(\tau < 0)$  any small enough  $\tau > 0$  and  $\bar{E}$  stay steady. We aim to identify a critical value of  $\tau$ , denoted as  $\tau^*$ , at which  $\lambda(\tau)$  turns entirely fictitious, meaning that  $\phi(\tau^*) = 0$  and





$\gamma(\tau^*) \neq 0$ . Under these circumstances, the state of cohabitation experiences a lack of stability. The steady state  $\bar{E}$  will remain constant regardless of any  $\tau$ , though, if such a  $\tau^*$  does not exist for which  $\lambda(\tau^*)$  is totally fictitious. When  $\lambda = \phi + i\gamma$  is replaced Equation (3.2), we get

$$(\phi + i\gamma)^2 + L_1(\phi + i\gamma) + e^{-(\phi+i\gamma)\tau} [L_2(\phi + i\gamma) + L_3] = 0. \quad (4.5)$$

The aforementioned equation becomes  $\phi = 0$  for roots that are entirely imaginary,

$$-\gamma^2 + iL_1\gamma + e^{-i\gamma\tau} (iL_2\gamma + L_3) = 0. \quad (4.6)$$

The following equations can be obtained on equating the real and imaginary portions to zero,

$$L_3 \cos \gamma \tau + L_2 \gamma \sin \gamma \tau = \gamma^2, \quad (4.7)$$

$$L_2 \gamma \cos \gamma \tau - L_3 \sin \gamma \tau = -L_1 \gamma. \quad (4.8)$$

By taking the square of each equation and combining them together, we obtain a bi-quadratic equation for  $\gamma$ :

$$\gamma^4 + (L_1^2 - L_2^2) \gamma^2 - L_3^2 = 0. \quad (4.9)$$

Taking  $\gamma^2 = \xi$ ,  $A_1 = L_1^2 - L_2^2$  and  $A_2 = -L_3^2$ , (4.9) reduces to

$$\xi^2 + A_1 \xi + A_2 = 0. \quad (4.10)$$

Through basic algebraic procedure, we deduce that

$A_1 = L_1^2 - L_2^2 = u^2 \left\{ \frac{r}{k} - \frac{2\beta v u}{(u^2 + a)^2} \right\}^2 - s^2 \left( 1 - \frac{2v}{nu} \right)^2$  is always positive. Now, if  $\tau < 0$ , there is only one real root,

lets assume  $\xi^*$ , in equation (4.10). As a result, the two real roots of equation (4.9) are  $\gamma^* = \pm \sqrt{\xi^*}$ . It suggests that we may determine a threshold value of  $\tau$ , namely  $\tau^*$ , from (4.7) and (4.8) that gives us  $\phi(\tau^*) = 0$  and  $\gamma(\tau^*) = \gamma^*$ , in other words, the solutions to the latent Equation (4.2) have become entirely fictitious ( $\pm i\gamma^*$ ). Equations (4.7) and (4.8) are utilised to determine the threshold quantity  $\tau$  as

$$\tau_j^* = \frac{1}{\gamma^*} \cos^{-1} \left[ \frac{\gamma^{*2} (L_3 - L_2 L_1)}{L_3^2 + L_2 \gamma^*} \right] + \frac{2\pi j}{\gamma^*}, \quad j = 0, 1, 2, \dots \quad (4.11)$$

and  $\tau^* = \min_j \tau_j^*$ . We also inspect to see if the transversality criterion,  $\left[ \frac{d}{d\tau} \text{Re}(\lambda) \right]_{\tau=\tau^*} = \left[ \frac{d\phi}{d\tau} \right]_{\tau=\tau^*} > 0$ , is met or not.

The desired result of differentiating (4.2) with regard to  $\tau$  is

$$\left[ \frac{d\lambda}{d\tau} \right]^{-1} = -\frac{2\lambda + L_1}{\lambda^3 + L_1 \lambda^2} - \frac{\tau}{\lambda} + \frac{L_2}{L_2 \lambda^2 + L_3 \lambda}. \quad (4.12)$$

At  $\tau = \tau^*$ ,

$$\left[ \frac{d\tau}{d\lambda} \right]_{\tau=\tau^*} = \left[ \frac{d\phi}{d\tau} \right]_{\tau=\tau^*} = \text{Re} \left[ -\frac{2i\gamma + L_1}{(i\gamma)^3 + L_1 (i\gamma)^2} - \frac{\tau}{i\gamma} + \frac{L_2}{L_2 (i\gamma)^2 + L_3 i\gamma} \right] = \frac{L_2^2 \gamma^4 + L_3^2 (L_1^2 + 2\gamma^2)}{\gamma^2 (L_1^2 + \gamma^2) (L_2^2 \gamma^2 + L_3^2)} > 0.$$

Since the transversality criterion is met as a result,  $\phi(\tau)$  assumes a positive value for  $\tau > \tau^*$  and is continuous. In this instance, as the latency value  $\tau$  approaches the critical level  $\tau^*$ , the system's cohabitation state of equilibrium (2.3)-(2.4) undergoes a Hopf bifurcation.





### 5. NUMERICAL SIMULATIONS

In this section, we illustrate our results through numerical simulations. The purpose of such investigations is to determine the impact of varying the parameters values and justify our analytical results. The particular coefficients chosen do not necessarily have biological meaning or apply to specific population. Numerical simulations are carried out with the help of MATLAB R2024a software package for a hypothetical set of data. We shall first show the effect of delay and fear factor. For numerical experiment, we fix the parameters as follows:

**Example 5.1.** We take the following parameter values as  $r = 0.098, k = 0.068, \alpha = 0.7, \delta = 0.00999568, s = 0.0658, n = 0.68795, a = 0.069, \beta = 0.598$  with initial conditions  $u_0 = 0.5, v_0 = 0.10$ . We build the Hopf-bifurcation diagrams (Figures 1, 2, and 3) by choosing the proper parameter values, if  $\tau$  is increased continually, a threshold  $\tau_0$  can be established. The system is stable if  $\tau < \tau_0$ , otherwise the system deviates from stability, leading to a Hopf-bifurcation at  $\tau = \tau_0$ . These results demonstrate the critical function that latency performs within the system.

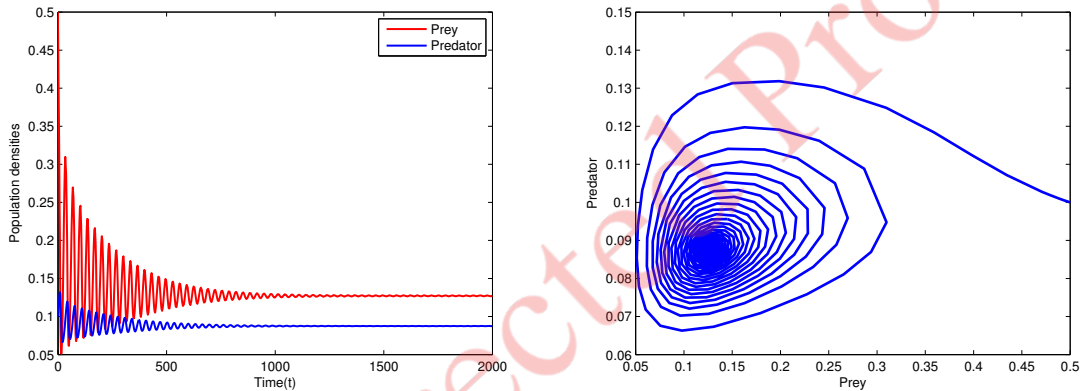


FIGURE 1. The paths and phase diagrams of the framework (2.3)-(2.4) with  $\tau = 0.2 < \tau_0 = 0.32$ .

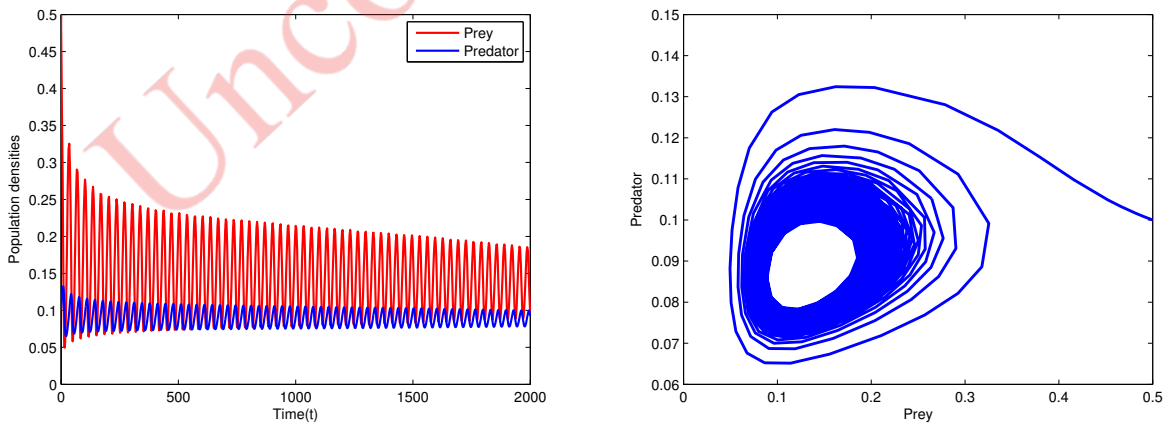


FIGURE 2. The paths and phase diagrams of the framework (2.3)-(2.4) with  $\tau = 0.35 > \tau_0 = 0.32$ .



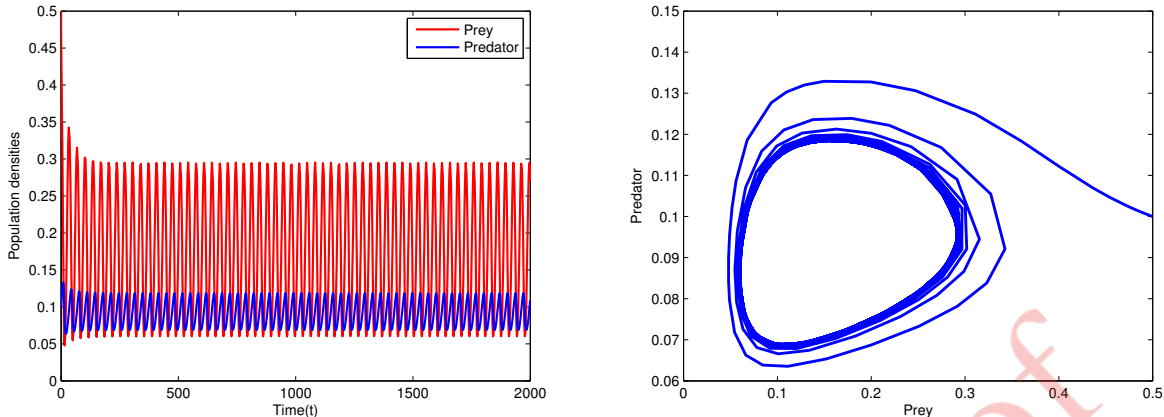


FIGURE 3. The paths and phase diagrams of the framework (2.3)-(2.4) with  $\tau = 0.5 > \tau_0 = 0.32$ .

**Example 5.2.** We consider the following parameters for the various fear effects on population growth  $r = 0.0972, k = 0.072, \alpha = 0.0954, s = 0.0549, n = 0.8721, a = 0.0989, \beta = 0.549$  with initial conditions  $u_0 = 0.5, v_0 = 0.10$ . The fear effect trajectories (Figures 4, and 5) show how fear has a significant impact on how many predator-prey species there are. The trajectory is more skewed the lower the fear cost provided by the simulation model in Equations (2.1)-(2.2). This implies that expansion will result in continual stability as the cost of anxiety decreases. On the other side, more erratic growth will result in stability the higher the cost of worry. The model's stability is still taken into account by the estimated parameter coefficients. It also appears that the cost of dread has certain manifestations that restrain population expansion. Low fear of prey reduces fluctuations in the expansion of the predator population. This is a practical thing to think about under ecological circumstances. In the meantime, the population of prey is moving.

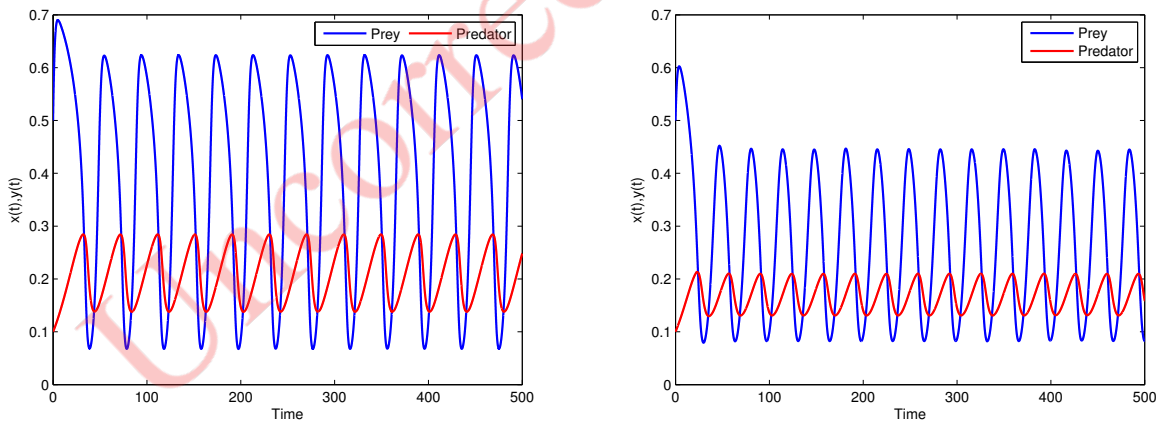


FIGURE 4. The trajectories of fear level  $\delta = 0.002$  and  $\delta = 0.954$ .



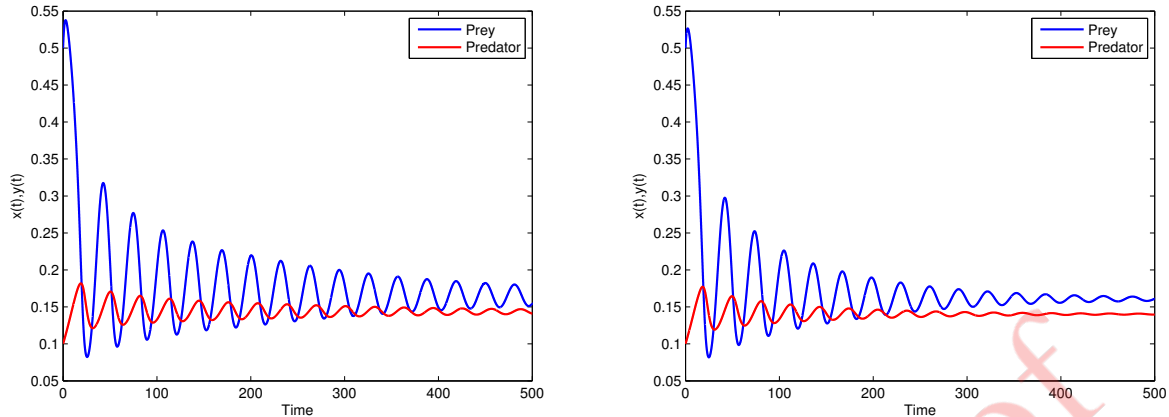


FIGURE 5. The trajectories of fear level  $\delta = 1.891$  and  $\delta = 2.0972$ .

## 6. CONCLUSIONS AND DISCUSSION

In this study, we explore the behaviour of a predator-prey model that takes into account the influence of fear on the reproductive and mortality rates of prey species. In order to enhance the accuracy of the dynamics, the model involves both a gestation latency and intraspecific rivalry among predator species. We look at the framework's positivity and boundedness in the first stage, where positivity denotes that species persist and boundedness can be understood like a usual constraint towards enlargement because of restricted resources. The impact of delay on the model's stability is further investigated by varying the delay value, which allows us to find the circumstances under which the framework is resilient.

We build the Hopf-bifurcation diagrams (Figures 1, 2, and 3) by choosing the proper parameter values; if  $\tau$  is increased continually, a threshold  $\tau_0$  can be established. The system is stable if  $\tau < \tau_0$ , otherwise the system deviates from stability, leading to a Hopf-bifurcation at  $\tau = \tau_0$ . These results demonstrate the critical function that latency performs within the system. It has come to our attention, via numerical simulations (Figures 4, 5), that fear could potentially lead to a reduction in species diversity and even the extinction of certain species. Fear therefore exerts a substantial influence on the intricacies of predator-prey schemes. Future improvements to the scheme can accommodate a two-prey, one-predator mechanism, resulting in may be crucial for maintaining the biodiversity and creating group structure.

**Conflict of Interest:** The authors declare no conflict of interest.

**Funding:** Not applicable.

**Author contributions:** Both authors participated equally and extensively to the creation of this work, including typing, reading, and approval of the final text.

## ACKNOWLEDGMENT

The authors are grateful to the anonymous referees for their careful reading, valuable comments and helpful suggestions, which have helped them to improve the presentation of this work significantly.

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Uncorrected Proof

