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A higher-order kernel approach for linear fourth-order boundary value problems

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Abstract

This paper aims at finding high-order convergent numerical approach to solve fourth-order linear boundary value problems (BVPs). By employing the good property of reproducing kernel functions (RKFs), a new collocation technique is proposed. The present approach can give highly accurate numerical solutions to fourth-order BVPs. Some numerical experiments are performed and compared with other approaches to indicate the validity of the proposed technique.

Keywords. Kernel functions, Linear fourth-order BVPs, Higher-order method.

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1. Introduction

We are concerned with the fourth-order BVPs as follows:

$$\begin{cases} v''''(x) + a_3(x)v'''(x) + a_2(x)v''(x) + a_1(x)v'(x) + a_0v(x) = g(x), \quad 0 < x < 1, \\ v(0) = \alpha_1, v(1) = \alpha_2, v''(0) = \beta_1, v''(1) = \beta_2, \end{cases} \tag{1.1}$$

where $a_0(x),\ a_1(x),\ a_2(x),\ a_3(x)$, and $g(x) \in C[0,1]$.

By employing suitable function transformation, the solution of the above equation can be easily reduced to the solution of the following BVPs with homogeneous boundary conditions

$$\begin{cases} y''''(x) + a_3(x)y'''(x) + a_2(x)y''(x) + a_1(x)y'(x) + a_0y(x) = f(x), \quad 0 < x < 1, \\ y(0) = y(1) = 0, y''(0) = y''(1) = 0. \end{cases} \tag{1.2}$$

Fourth-order BVPs have extensive applications in biology, physics, and engineering. The existence, uniqueness, and solvability were discussed in [6, 7, 26]. Due to their significant applications, it is required to develop effective numerical techniques for such problems. Based on Quasi-Newton's method and the kernel technique, Xu, Lin, and Wang [27] proposed a new numerical approach for fourth-order BVPs. Zha, Li, and Yi [29] developed an effective finite element technique for fourth-order BVPs. Costabile and Napoli [9] introduced a general collocation technique for high even-order differential equations. The Variational iteration technique, the B-Spline approach and other numerical approaches were presented for fourth-order BVPs in [10, 19–21, 24].

The reproducing kernel Hilbert spaces (RKHSs) theory has superiority in function approximation and numerical solutions of operator equations (see, e.g., [1–5, 8, 10–18, 22, 23, 25, 28, 30]). In [10], Geng proposed an orthogonalization-based reproducing kernel approach for nonlinear fourth-order BVPs. However, its convergence order is low. In the work, based on the reproducing kernel functions (RKFs) in RKHS $W^4[0,1]$, a novel numerical technique will be proposed for linear fourth-order BVPs. It has a higher convergence order and its convergence order can reach $O(h^8)$.

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2. Numerical approach

As the foundation of our numerical approach, the related theory on RKHS shall be introduced.

Definition 2.1. A Hilbert space H defined on E is called RKHS if there exists a function $K : E \times E \to \mathbb{R}$ with the following properties:

(1) $K(\cdot,\ s) \in H, for\ all\ s \in E,$

(2) $v(s) = (v(\cdot),\ K(\cdot,\ s)), for\ all\ s \in E\ and\ all\ v \in H.$

The function $K(x, y)$ is called the RKF of space H. The space $W^4[0, 1] = \{$Real value function $y(x) \in C^3[0, 1] : y^{(4)} \in L^2[0, 1], y'''$ is absolutely continuous, and $y(0) = y''(0) = y(1) = y''(1) = 0\}$. The inner product of the space is given by

$$(y_1(x), y_2(x))_4 = \sum_{i=0}^{1} y_1^{(i)}(0)y_2^{(i)}(0) + \sum_{i=0}^{1} y_1^{(i)}(1)y_2^{(i)}(1) + \int_0^1 y_1^{(4)} y_2^{(4)} dx. \tag{2.1}$$

Theorem 2.2. *Space $W^4[0, 1]$ is an RKHS, and its RKF is*

$$K(x, s) = \begin{cases} \eta(x, s), & s \le x, \\ \eta(s, x), & s > x, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} \eta(x, s) = \frac{1}{55099714320} s(&723s^6(-6x^7 + 21x^6 - 21x^5 + 7x^3 + 15120x - 15121) + 7s^5x(2169x^6 - 15152x^5 \\ &+ 30273x^4 - 40333x^2 + 23043) + 21s^4x(-723x^6 + 10091x^5 - 25212x^4 + 3682807x^2 - 10932483x \\ &+ 7265520) + 7s^2x(723x^6 - 40333x^5 + 11048421x^4 - 54662415x^3 + 72693364x^2 - 29039760) \\ &+ 5040x(2169x^6 - 15152x^5 + 30273x^4 - 40333x^2 + 23043)). \end{aligned}$$

Proof. If $u \in W^4[0, 1]$, then

$$u(0) = u(1) = 0,\ u''(0) = u''(1) = 0, \tag{2.3}$$

and therefore

$$\begin{aligned} (u(y), K(x, y))_4 = {}& u'(0)[\frac{\partial K(x, y)}{\partial y}|_{y=0} - \frac{\partial^6 K(x, y)}{\partial y^6}|_{y=0}] + u'(1)[\frac{\partial K(x, y)}{\partial y}|_{y=1} + \frac{\partial^6 K(x, y)}{\partial y^6}|_{y=1}] \\ &+ u'''(1)\frac{\partial^4 K(x, y)}{\partial y^4}|_{y=1} - u'''(0)\frac{\partial^4 K(x, y)}{\partial y^4}|_{y=0} + \int_0^1 u(y)\frac{\partial^8 K(x, y)}{\partial y^8} dy. \end{aligned} \tag{2.4}$$

In addition,

$$K(x, 0) = K(x, 1) = 0,\ \frac{\partial^2 K(x, y)}{\partial y^2}|_{y=0} = 0,\ \frac{\partial^2 K(x, y)}{\partial y^2}|_{y=1} = 0. \tag{2.5}$$

Let

$$\begin{aligned} &\tfrac{\partial K(x,y)}{\partial y}|_{y=0} - \tfrac{\partial^6 K(x,y)}{\partial y^6}|_{y=0} = 0,\ \tfrac{\partial K(x,y)}{\partial y}|_{y=1} + \tfrac{\partial^6 K(x,y)}{\partial y^6}|_{y=1} = 0, \\ &\tfrac{\partial^4 K(x,y)}{\partial y^4}|_{y=1} = 0,\ \tfrac{\partial^4 K(x,y)}{\partial y^4}|_{y=0} = 0. \end{aligned} \tag{2.6}$$

Then (2.4) is reduced to

$$(u(y), K(x, y))_4 = \int_0^1 u(y)\frac{\partial^8 K(x, y)}{\partial y^8} dy. \tag{2.7}$$

Due to the reproducing property of $K(x, y)$, it is required that

$$\frac{\partial^8 K(x, y)}{\partial y^8} = \delta(y - x), \tag{2.8}$$

where $\delta(\cdot)$ is the Dirac delta function. Therefore,

$$K(x,y)=\begin{cases}\alpha_1+\alpha_2y+\alpha_3y^2+\alpha_4y^3+\alpha_5y^4+\alpha_6y^5+\alpha_7y^6+\alpha_8y^7, & y\le x,\\ \beta_1+\beta_2y+\beta_3y^2+\beta_4y^3+\beta_5y^4+\beta_6y^5+\beta_7y^6+\beta_8y^7, & y>x.\end{cases} \tag{2.9}$$

From (2.8), we have

$$\begin{aligned}&K(x,x+0)=K(x,x-0),\quad \frac{\partial K(x,y)}{\partial y}|_{x+0}=\frac{\partial K(x,y)}{\partial y}|_{x+0},\quad \frac{\partial^2 K(x,y)}{\partial y^2}|_{x-0}=\frac{\partial^2 K(x,y)}{\partial y^2}|_{x+0},\\ &\frac{\partial^3 K(x,y)}{\partial y^3}|_{x-0}=\frac{\partial^3 K(x,y)}{\partial y^3}|_{x+0},\quad \frac{\partial^4 K(x,y)}{\partial y^4}|_{x-0}=\frac{\partial^4 K(x,y)}{\partial y^4}|_{x+0},\quad \frac{\partial^5 K(x,y)}{\partial y^5}|_{x-0}=\frac{\partial^5 K(x,y)}{\partial y^5}|_{x+0},\\ &\frac{\partial^6 K(x,y)}{\partial y^6}|_{x-0}=\frac{\partial^6 K(x,y)}{\partial y^6}|_{x+0},\quad \frac{\partial^7 K(x,y)}{\partial y^7}|_{x+0}-\frac{\partial^7 K(x,y)}{\partial y^7}|_{x-0}=1.\end{aligned} \tag{2.10}$$

Then the unknown constants $\alpha_i,\beta_i, i=1,2,\ldots,8$ in (2.9) can be solved by the linear system (2.5), (2.6), and (2.10). □

It is known that RKF $K(x,s)\in C^6[0,1]$ for a fixed $s\in[0,1]$, and it is piecewise polynomials of degree seven, that is, it is a spline of degree seven.

By using the basis functions yielded via the kernel function $K(x,s)$ and choosing suitable collocation nodes, a new highly accurate collocation technique will be proposed. Let $T_N: 0=x_1<x_2<\ldots<x_N=1$ be a partition of $[0,1]$, $\sigma_l=(x_l,x_{l+1})$, $h_l=x_{l+1}-x_l$, and $h=\max\limits_l h_l$. Denote by $\pi_n(x)$ the space of polynomials of degree $\le n$. By employing the RKF $K(x,s)$, we establish the following basis functions:

$$\psi_l(x)=\begin{cases}K(x,x_{l+1}), & 1\le l\le N-2,\\ \frac{\partial K(x,s)}{\partial s}|_{s=x_{i-N+2}}, & N-1\le l\le 2(N-1),\\ \frac{\partial^2 K(x,s)}{\partial s^2}|_{s=x_{i-2N+3}}, & 2N-1\le l\le 3N-4,\\ \frac{\partial^3 K(x,s)}{\partial s^3}|_{s=x_{i-3N+4}}, & 3N-3\le l\le 4(N-1).\end{cases}$$

Put $S_N=Span\{\psi_j(x), j=1,2,\cdots,4N-4\}$. Obviously,

$$S_N=\{v(x)\in C^3[0,1]\mid v|_{\sigma_i}\in\pi_7, i=1,2,\cdots,N-1\}.$$

$4N-4$ collocation nodes are needed, because the dimension of S_N is $4N-4$.

The detailed collocation nodes are as follows

$$Z_{4N-4}=\{z_{li}=x_l+\mu_ih_l\mid i=1,2,,3,4, 1\le l\le N-1\},$$

where

$$\mu_1=\frac{1}{2}\left(1-\sqrt{\frac{1}{35}\left(15+2\sqrt{30}\right)}\right),\qquad \mu_2=\frac{1}{2}\left(1-\sqrt{\frac{1}{35}\left(15-2\sqrt{30}\right)}\right),$$
$$\mu_3=\frac{1}{2}\left(1+\sqrt{\frac{1}{35}\left(15-2\sqrt{30}\right)}\right),\qquad \mu_4=\frac{1}{2}\left(1+\sqrt{\frac{1}{35}\left(15+2\sqrt{30}\right)}\right),$$

are Gauss-Legendre nodes on $[0,1]$.

Find the approximate solution of (1.2) in space S_N. Its representation is

$$y_N(x)=\sum_{l=1}^{4N-4}c_l\psi_l(x). \tag{2.11}$$

Define linear operator as

$$Ly(x)=y''''(x)+a_3(x)y'''(x)+a_2(x)y''(x)+a_1(x)y'(x)+a_0y(x).$$

Let $y_N(x)$ satisfy (1.2) at points in Z_{4N-4}. This gives

$$Ly_N(s) = \sum_{l=1}^{4N-4} c_l L\psi_l(s) = f(s),\ s \in Z_{4N-4}. \tag{2.12}$$

$\{c_l\}$ can be obtained by the above linear system.

Consequently, the approximate solution $y_N(x)$ to (1.2) is determined. The steps of solving (1.2) are given as follows:

(1) Select proper RKF $K(x,y)$ and nodes $x_i, 1 \le i \le N$.
(2) Construct basis functions $\psi_l(x), 1 \le l \le 4N-4$.
(3) Represent the approximate solution by basis functions $\psi_l(x)$, $y_N(x) = \sum\limits_{l=1}^{4N-4} c_l\psi_l(x)$.
(4) Select proper collocation nodes $z_{li}, 1 \le l \le N, 1 \le i \le 4$.
(5) Let $y_N(x)$ satisfy (1.2) at points z_{li}.
(6) Obtain the undetermined coefficients $\{c_l\}$ in $y_N(x)$.

3. Analysis of convergence order

In this section, the convergence order of our numerical approach will be introduced.

Theorem 3.1. *Assume that $f(x) \in C^8[0,1]$. We have*

$$\| y_N - y \|_\infty \le c\, h^8,$$

where $\| f \|_\infty = \max\limits_{x\in[0,1]} |f(x)|$.

Proof. (1.2) can be rewrited as

$$y(x) = \int_0^1 G(x,t)f(t)dt,$$

where $G(x,s)$ is the Green's function. Observe that

$$y_N(x) - y(x) = \int_0^1 G(x,s)[Ly_N(s) - Ly(s)]ds = \int_0^1 G(x,s)[Ly_N(s) - f(s)]ds. \tag{3.1}$$

Denoting $Ly_N(s)$ by $f_N(s)$, (3.1) is simplified to

$$y_N(x) - y(x) = \int_0^1 G(x,s)[f_N(s) - f(s)]ds.$$

From the fact that there are four collocation points z_{l1}, z_{l2}, z_{l3}, and z_{l4} on subinterval $[x_l, x_{l+1}]$, we have $Ly_N(z_{li}) = f(z_{li})$, $i = 1,2,3,4$, that is, $f_N(z_{li}) = f(z_{li})$, $i = 1,2,3,4$. Then

$$f_N(s) - f(s) = \frac{f_N''''(\tau) - f''''(\tau)}{4!}\prod_{i=1}^{4}(s - z_{li}) = F_l''''(\tau)\prod_{i=1}^{4}(s - z_{li}), \tag{3.2}$$

where $\tau \in [x_l, x_{l+1}]$. Put $H_l(x,s) = G(x,s)F_l''''(\tau)$. Hence,

$$\int_{x_l}^{x_{l+1}} G(x,s)[f_N(s) - f(s)]ds = \int_{x_l}^{x_{l+1}} H_l(x,s)\prod_{i=1}^{4}(s - z_{li})ds.$$

The Taylor's expansion of $H_l(x,s)$ at z_{l1} is

$$\begin{aligned} H_l(x,s) = {} & H_l(x,z_{l1}) + \frac{\partial H_l}{\partial s}(x,z_{l1})(s - z_{l1}) + \frac{1}{2}\frac{\partial^2 H_l}{\partial s^2}(x,\tau)(s - z_{l1})^2 \\ & + \frac{1}{3!}\frac{\partial^3 H_l}{\partial s^3}(x,\tau)(s - z_{l1})^3 + \frac{1}{4!}\frac{\partial^4 H_l}{\partial s^4}(x,\tau)(s - z_{l1})^4. \end{aligned}$$

Since $z_{li}, i = 1,2,3,4$, are the Gauss nodes in $[x_l, x_{l+1}]$, for any polynomials of degree seven $q(s)$, the Gaussian quadrature rule $\sum_{i=1}^{4} \omega_i g(z_{li})$ satisfies

$$\int_{x_l}^{x_{l+1}} q(s)ds = \sum_{i=1}^{4} \omega_i q(z_{li}).$$

Furthermore, we have

$$\int_{x_l}^{x_{l+1}} H_l(x,s)\prod_{i=1}^{4}(s-z_{li})ds = \frac{1}{4!}\int_{x_l}^{x_{l+1}} \frac{\partial^4 H_l}{\partial s^4}(x,\tau)(s-z_{l1})^5(s-z_{l2})(s-z_{l3})(s-z_{l4})ds. \tag{3.3}$$

Putting $s = th_l + x_l$, it follows that

$$\int_{x_l}^{x_{l+1}} \frac{\partial^4 H_l}{\partial s^4}(x,\tau)(s-z_{l1})^5(s-z_{l2})(s-z_{l3})(s-z_{l4})ds = h_l^9\int_0^1 \frac{\partial^4 H_l}{\partial s^4}((x,\overline{\tau}))\varpi(t)dt, \tag{3.4}$$

where $\varpi(t) = (t-\mu_1)^5(t-\mu_2)(t-\mu_3)(t-\mu_4)$.

It follows from $f \in C^8[0,1]$ that $\int_0^1 [\frac{\partial^4 H_l}{\partial s^4}(x,\overline{\tau})]^2 ds$ is bounded, that is,

$$\int_0^1 [\frac{\partial^4 H_l}{\partial s^4}(x,\overline{\tau})]^2 ds \le \beta.$$

The use of Schwarz's inequality gives

$$|\int_0^1 \frac{\partial^4 H_l}{\partial s^4}((x,\overline{\tau}))(t-\mu_1)^5(t-\mu_2)(t-\mu_3)(t-\mu_4)dt|^2 \le \int_0^1 [\frac{\partial^4 H_l}{\partial s^4}(x,\overline{\tau})]^2 dt\int_0^1 [\varpi(t)]^2 dt \le \frac{1}{44100}\beta. \tag{3.5}$$

Then

$$|\int_{x_l}^{x_{l+1}} H_l(x,\tau)\prod_{i=1}^{4}(s-z_{li})ds| \le \frac{1}{4!}\sqrt{\frac{1}{44100}\beta}\, h^9. \tag{3.6}$$

Therefore,

$$\| y_N - y \|_\infty \le \sum_{l=1}^{N-1} |\int_{x_l}^{x_{l+1}} H_l(x,\tau)\prod_{i=1}^{4}(s-z_{li})ds| \le \frac{N-1}{4!}\sqrt{\frac{1}{44100}\beta}\, h^9 \le c\, h^8, \tag{3.7}$$

with a positive constant c. □

4. Numerical tests

Three numerical examples are performed to illustrate the accuracy and convergence order of the present approach. The software package Mathematica 12.0 is used. In the following experiments, we select $x_l = \frac{l-1}{N-1}$ for $l = 1,2,\ldots,N$.

Test 4.1. Consider the fourth-order BVP used in [9, 20]

$$\begin{cases} v''''(x) - v''(x) - 2v(x) = f(x), \\ v(0) = 1,\ v(1) = 0, \\ v''(0) = -1,\ v''(1) = -2e, \end{cases} \tag{4.1}$$

where $f(x) = (x-3)e^x$. Its exact solution is $v(x) = (1-x)e^x$. Tables 1 and 2 show the numerical results of our technique and techniques in [9, 20]. The convergence order is also listed in Table 2. The absolute errors of the obtained approximate solution and its derivatives are depicted in Figures 1 and 2.

Table 1. Results of absolute errors for Test 4.1.

x	Method in [20]	Method in [9]	PM(N=11)
0.1	2.00×10^{-10}	6.33×10^{-09}	1.11×10^{-16}
0.2	7.00×10^{-10}	7.50×10^{-09}	2.22×10^{-16}
0.3	1.35×10^{-09}	3.12×10^{-10}	1.11×10^{-16}
0.4	2.00×10^{-09}	3.91×10^{-09}	1.11×10^{-16}
0.5	2.50×10^{-09}	9.89×10^{-10}	1.11×10^{-16}
0.6	2.72×10^{-09}	1.75×10^{-09}	1.11×10^{-16}
0.7	2.21×10^{-09}	2.91×10^{-09}	0
0.8	1.80×10^{-09}	1.01×10^{-08}	1.11×10^{-16}
0.9	7.25×10^{-10}	7.80×10^{-09}	2.77×10^{-17}

Table 2. Maximum absolute errors of the numerical solution and its derivative for Test 4.1.

N	E_N	E'_N	Convergence order
3	6.00×10^{-09}	4.80×10^{-08}	–
5	2.80×10^{-11}	4.40×10^{-10}	7.74
9	1.18×10^{-13}	3.50×10^{-12}	7.89
17	5.60×10^{-16}	2.90×10^{-14}	7.72

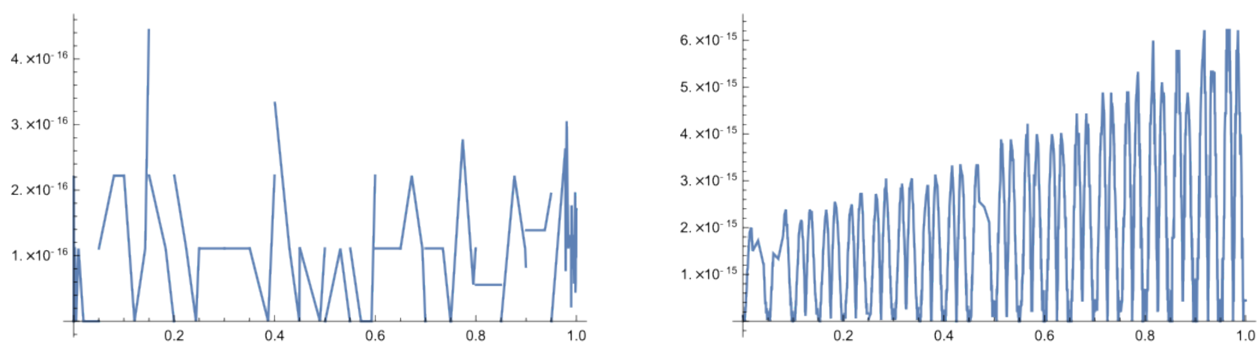


Figure 1. Absolute errors $|v_{21}(x) - v(x)|$ (left) and $|v'_{21}(x) - v'(x)|$ (right) for Test 4.1.

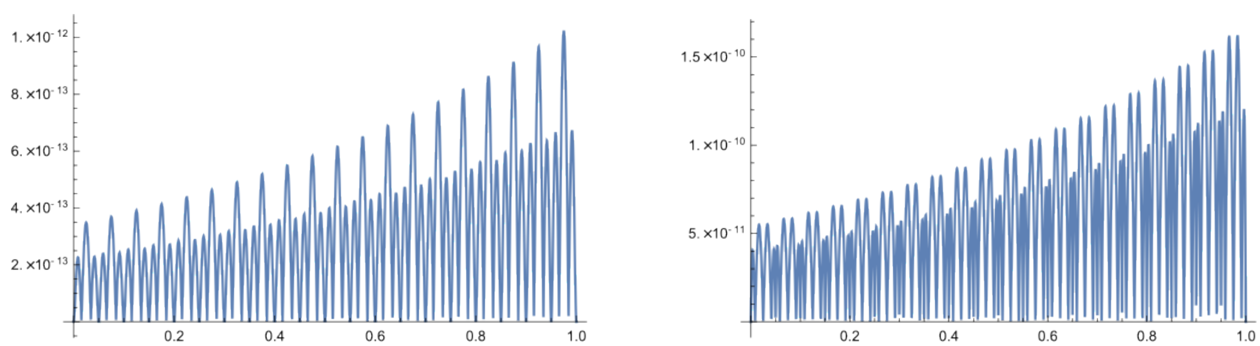
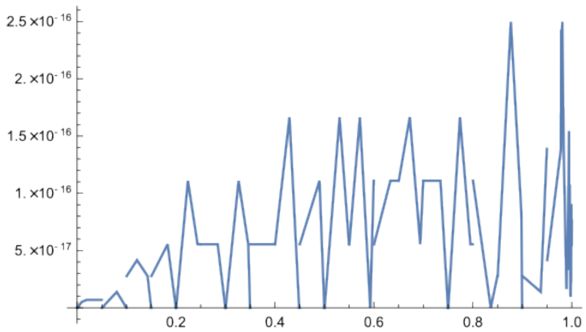


Figure 2. Absolute errors $|v''_{21}(x) - v''(x)|$ (left) and $|v'''_{21}(x) - v'''(x)|$ (right) for Test 4.1.



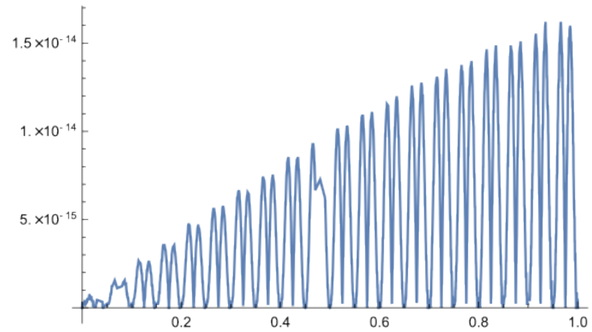


FIGURE 3. Absolute errors $|v_{21}(x) - v(x)|$ (left) and $|v_{21}'(x) - v'(x)|$ (right) for Test 4.2.

Test 4.2. Consider the fourth-order BVP used in [24]

$$\begin{cases} v''''(x) - v(x) = f(x), \\ v(0) = 1,\ v(1) = 0, \\ v''(0) = 0,\ \ v''(1) = -2\sin 1 - 4\cos 1, \end{cases} \tag{4.2}$$

where $f(x) = -4(3\sin x + 2x\cos x)$. Its exact solution is $v(x) = (x^2 - 1)\sin x$. Tables 3 and 4 show the numerical results of our technique and techniques in [24]. The convergence order is also listed in Table 2. The absolute errors of the obtained approximate solution and its derivatives are depicted in Figures 3 and 4.

TABLE 3. Results of absolute errors for Test 4.2.

x	Method in [24]	PM(N=11)
0.1	1.52×10^{-06}	2.35×10^{-16}
0.2	2.91×10^{-06}	4.72×10^{-16}
0.3	4.05×10^{-06}	6.66×10^{-16}
0.4	4.92×10^{-06}	7.77×10^{-16}
0.5	5.00×10^{-06}	8.88×10^{-16}
0.6	4.50×10^{-06}	8.32×10^{-16}
0.7	3.75×10^{-06}	8.32×10^{-16}
0.8	2.62×10^{-06}	6.10×10^{-16}
0.9	1.31×10^{-06}	4.72×10^{-16}

TABLE 4. Maximum absolute errors of the numerical solution and its derivative for Test 4.2.

N	E_N	E_N'	Convergence order
3	1.85×10^{-08}	1.40×10^{-07}	–
5	7.80×10^{-11}	1.15×10^{-09}	7.89
9	3.20×10^{-13}	9.50×10^{-12}	7.93
17	1.25×10^{-15}	7.50×10^{-14}	8.00

Test 4.3. Our approach can deal with fourth-order BVPs with other boundary conditions. Consider a special fourth-order BVP in [10, 19]

$$\begin{cases} v''''(x) - (1+\mu)v'(x) + \mu v(x) = 0.5\mu x^2 - 1, \\ v(0) = 1,\ v(1) = 1.5 + \sinh 1, \\ v'(0) = 1,\ v'(1) = 1 + \cosh 1, \end{cases} \tag{4.3}$$

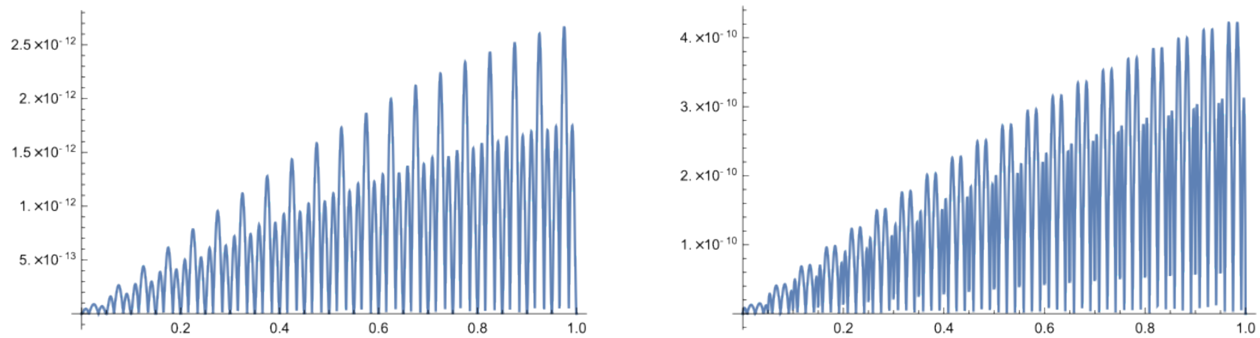


FIGURE 4. Absolute errors $|v''_{21}(x) - v''(x)|$ (left) and $|v'''_{21}(x) - v'''(x)|$ (right) for Test 4.2.

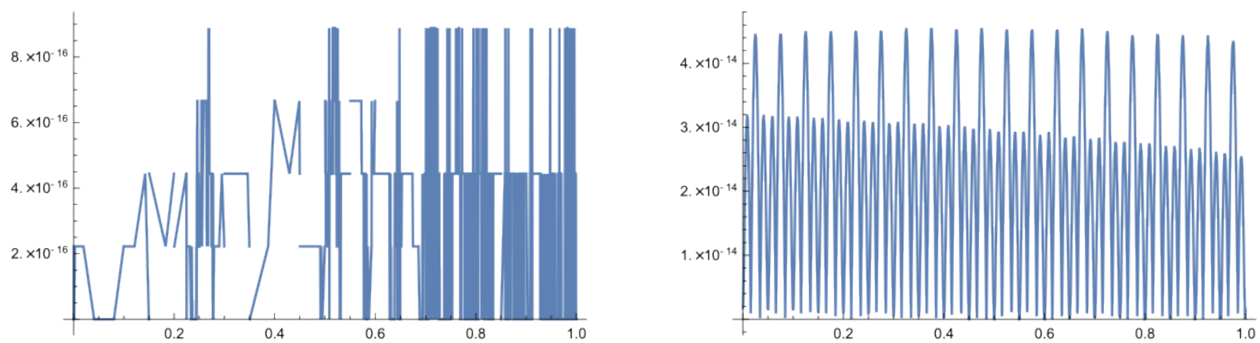


FIGURE 5. Absolute errors $|v_{21}(x) - v(x)|$ (left) and $|v'_{21}(x) - v'(x)|$ (right) with $\mu = 10^6$ for Test 4.3.

where parameter μ is an arbitrary constant. Its exact solution is $v(x) = 1 + \sinh x + 0.5x^2$, which is independent of the parameter μ. Table 5 shows the numerical results of our technique and the approach in [10]. The absolute errors of the obtained approximate solution and its derivatives are depicted in Figures 5 and 6. The maximum absolute errors of the numerical solution and its derivative for $\mu = 10^6$ are listed in Table 6. In [10], the problem was solved in RKHS W^5, while our approach is performed in the simpler RKHS W^4, And our method has higher accuracy.

TABLE 5. Absolute errors of different approaches with $\mu = 10^6$ for Test 4.3.

x	DTM in [19]	Method in [10]	PM(N=11)
0.1	1.50×10^{-10}	2.70×10^{-09}	2.88×10^{-15}
0.2	3.70×10^{-08}	2.40×10^{-09}	5.55×10^{-15}
0.3	9.00×10^{-07}	1.90×10^{-09}	7.54×10^{-15}
0.4	8.50×10^{-06}	9.60×10^{-09}	8.88×10^{-15}
0.5	4.80×10^{-05}	1.70×10^{-08}	8.65×10^{-15}
0.6	1.90×10^{-04}	2.40×10^{-08}	8.65×10^{-15}
0.7	6.40×10^{-04}	2.70×10^{-08}	7.99×10^{-15}
0.8	1.70×10^{-03}	2.40×10^{-08}	5.32×10^{-15}
0.9	4.20×10^{-03}	1.20×10^{-08}	2.66×10^{-15}

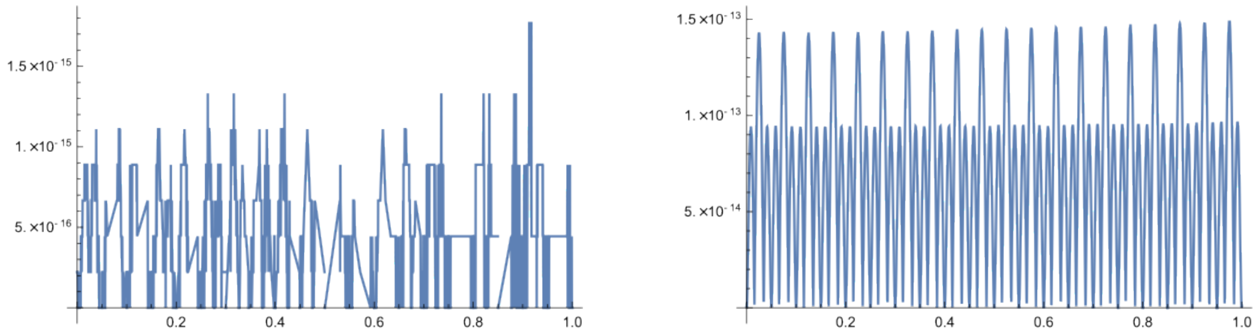


FIGURE 6. Absolute errors $|v_{21}(x) - v(x)|$ (left) and $|v'_{21}(x) - v'(x)|$ (right) with $\mu = 10^7$ for Test 4.3.

TABLE 6. Maximum absolute errors of the numerical solution and its derivative for Test 4.3.

N	E_N	E'_N	Convergence order
3	1.43×10^{-08}	$\times10^{-07}$	–
5	3.50×10^{-12}	9.44×10^{-11}	11.99
9	2.37×10^{-13}	1.46×10^{-11}	7.25
17	2.70×10^{-15}	1.98×10^{-13}	6.46

5. Conclusion

By employing RKFs, spline basis functions are constructed, and then an effective collocation approach is proposed for fourth-order BVPs. Numerical results illustrate that our approach has a higher accuracy. Compared with the method in [10], the present approach uses RKHS with lower regularity and has higher convergence order.

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