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Some existence and nonexistence results for a class of Kirchhoff-double phase systems in bounded domains

Nguyen Thanh Chung [1,*] **and Zohreh Naghizadeh**[2]

[1]Department of Mathematics, Quang Binh University, 312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam.

[2]Faculty of Sciences, Department of Mathematics, University of Science and Technology of Mazandaran, P.O. Box 48518-78195, Behshahr, Iran.

Abstract

In this paper, the existence and nonexistence of multiple solutions for a class of Kirchhoff-double phase systems depending on one parameter in bounded domains are considered. Our main tools are essentially based on variational techniques. To our best knowledge, there seems to be few results on Kirchhoff-double phase type systems in the existing literature.

Keywords. Existence and nonexistence, Kirchhoff-double phase sytems, Musielak-Orlicz space, Variational technique.

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1. Introduction

A significant attention in recent years has been focused on addressing problems including double phase operator, since they are basically due to applications as a models in the field of mathematical physics and engineering such as strongly anisotropic materials and elasticity theory, see [7, 24, 25] and the references are cited there. In this paper, we deal with the following Kirchhoff-double phase systems of the form

$$\begin{cases} -M_1\left[\displaystyle\int_\Omega \Big(\frac{|\nabla u_1|^{p_1}}{p_1}+w_1(x)\frac{|\nabla u_1|^{q_1}}{q_1}\Big)dx\right](\Delta_{p_1}u_1+\Delta_{q_1}^{w_1}u_1)=\lambda F_{u_1}(x,u_1,u_2) & \text{in } \Omega,\\ -M_2\left[\displaystyle\int_\Omega \Big(\frac{|\nabla u_2|^{p_2}}{p_2}+w_2(x)\frac{|\nabla u_2|^{q_2}}{q_2}\Big)dx\right](\Delta_{p_2}u_2+\Delta_{q_2}^{w_2}u_2)=\lambda F_{u_2}(x,u_1,u_2) & \text{in } \Omega,\\ u_1=u_2=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ represents a bounded domain with Lipschitz boundary $\partial\Omega$, $1 < p_i < q_i < N$, $\frac{q_i}{p_i} < 1 + \frac{1}{N}$ for $i = 1, 2$, $w_i : \overline{\Omega} \to [0, \infty)$ is supposed to be Lipschitz continuous and λ denotes a parameter. Here, $\Delta_{p_i}u_i + \Delta_{q_i}^{w_i}u_i$ represents the double phase operator given by

$$\Delta_{p_i}u_i+\Delta_{q_i}^{w_i}u_i=\operatorname{div}\Big(|\nabla u_i|^{p_i-2}\nabla u_i+w_i(x)|\nabla u_i|^{q_i-2}\nabla u_i\Big),\ i=1,2,$$

where $(F_{u_1}, F_{u_2}) = \nabla F$ means the gradient of F and two Kirchhoff functions $M_1, M_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are supposed to be continuous with some additional conditions.

Let us recall some prior results that propel us to the present investigation. In [18], Liu and Dai considered the following double phase problem

$$\begin{cases} -\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u+a(x)|\nabla u|^{q-2}\nabla u\Big)=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

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∗ Corresponding author. Email: ntchung82@yahoo.com.

where Ω is a bounded domain with Lipschitz boundary in $\mathbb{R}^N$, $N \geq 2$ and $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$, $a : \overline{\Omega} \to [0, +\infty)$ is a Lipschitz continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Using variational methods, the authors studied the existence and multiplicity of solutions to problem (1.2) in the case when f is q-superlinear at infinity. After that, there are many papers working on this topic, we refer to [6, 12, 13, 15, 21].

In [6], Cao et al. established existence and nonexistence of solutions for the double phase problem depending on two parameters via critical point theorems due to Ricceri. Based on variational and topological tools such as truncation arguments and genus theory, the existence of solutions to critical double phase problems were considered by Farkas et al. in [12]. Marino and Winkert considered in [21] a class of double phase problems involving convection terms and obtained some existence and uniqueness results by applying the theory of pseudomonotone operators. In [13], Feng et al. considered a class of double phase systems with convex nonlinearities of the form

$$\begin{cases} -\Delta_p u - \operatorname{div}(\eta(x)|\nabla u|^{q-2}\nabla u) = (t_1+1)|u|^{t_1-1}u|v|^{t_2+1}, & x \in \Omega, \\ -\Delta_p v - \operatorname{div}(\eta(x)|\nabla v|^{q-2}\nabla v) = (t_2+1)|u|^{t_1+1}|v|^{t_2-1}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

There, using the Nehari manifold argument, the authors prove the existence of nontrivial solutions to problem (1.3). In [15], Guarnotta et al. considered a class of variable exponent double phase systems with nonlinear boundary conditions and obtained some existence results by using the sub-supersolution method.

Notice that problem (1.1) contains integrals over Ω, so the first two equations here are no longer pointwise identities. For this reason, it is often called nonlocal problem. Problems of this type model several physical and biological systems, where the unknown function u describes a process which depends on the average of itself, such as the population density, see [8]. Moreover, problem (1.1) is related to the stationary of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.4}$$

presented by Kirchhoff in 1883, see [17]. Equation (1.4) is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. Problems involving Kirchhoff type operators have been studied by many authors in recent years, we refer to [9–11, 16, 19, 20].

Together with Kirchhoff type problems involving p-Laplace operators and $p(x)$-Laplace operators, we can find some results on Kirchhoff-double phase problems, see [1, 3, 14]. In [1] and [14], the authors studied Kirchhoff-double phase problems with superlinear terms and obtained some existence and multiplicity results. Arora et al. considered in [3] double phase Kirchhoff problems with singular nonlinearity by using the fibering method in form of the Nehari manifold.

Motivated by the papers mentioned above, our goal is to obtain some existence and nonexistence results for Kirchhoff-double phase system (1.1) with sublinear terms. It should be noticed that the results introduced here are also extensions from the study of boundary-value problems involving p-Laplace equations of gradient form (see [4, 22]). As far as we know, there are relatively few results even on Kirchhoff-double phase problems and our obtained results have not been investigated in existing literature.

The rest of this paper is constructed in such a way: Section 2 presents the fundamental properties of our working space. Nonexistence result and existence of at least two distinct, nonnegative, nontrivial solutions are presented in section 3, whereas the proofs of this results are provided in sections 4 and 5, respectively. Eventually, section 6 sums up the main conclusions.

Along this paper, there exist some notations:

- (•) $\rightharpoonup$ and $\to$ stand for weakly and strongly convergence, respectively.
- (•) $L^r(\Omega)$, with $r \in [1, \infty)$ denotes a Lebesgue space and the norm of $L^r(\Omega)$ is denoted by $\|.\|_r$.
- (•) $C, C', \tilde{C}, \hat{C}$ denote various positive constants.

2. Preliminaries

In order to study double phase systems, we need to introduce our working space and recalling some facts about it. Define the functions $\mathcal{H}_i : \Omega \times [0, \infty) \to [0, \infty)$ by

$$\mathcal{H}_i(x, t) = t^{p_i} + w_i(x)t^{q_i},$$

where $1 < p_i < q_i < N$, $\frac{q_i}{p_i} < 1 + \frac{1}{N}$ and $w_i : \overline{\Omega} \to [0, \infty)$ is Lipschitz continuous for $i = 1, 2$. Consider $\rho_{\mathcal{H}_i}(u) := \int_\Omega \mathcal{H}_i(x, |u|)dx$. The Musielak-Orlicz Lebesgue space is described as

$$L^{\mathcal{H}_i}(\Omega) = \Big\{u|u : \Omega \to \mathbb{R} \text{ is measurable}, \ \rho_{\mathcal{H}_i}(u) < +\infty\Big\},$$

including the norm

$$\|u\|_{\mathcal{H}_i} := \inf\Big\{\sigma > 0 : \rho_{\mathcal{H}_i}(\frac{u}{\sigma}) \le 1\Big\}.$$

By [5, Proposition 2.13] we arrive at the following relation between $\|u\|_{\mathcal{H}_i}$ and $\rho_{\mathcal{H}_i}$.

Lemma 1. If $u \in L^{\mathcal{H}_i}(\Omega)$, then for $i = 1, 2$, we have

$$\min\Big\{\|u\|^{p_i}_{\mathcal{H}_i}, \|u\|^{q_i}_{\mathcal{H}_i}\Big\} \le \rho_{\mathcal{H}_i}(u) \le \max\Big\{\|u\|^{p_i}_{\mathcal{H}_i}, \|u\|^{q_i}_{\mathcal{H}_i}\Big\}.$$

The Musielak-Orlicz Sobolev space is described as

$$W^{1,\mathcal{H}_i}(\Omega) := \Big\{u \in L^{\mathcal{H}_i}(\Omega) : |\nabla u| \in L^{\mathcal{H}_i}(\Omega)\Big\}$$

including the norm

$$\|u\|_{1,\mathcal{H}_i} := \|\nabla u\|_{\mathcal{H}_i} + \|u\|_{\mathcal{H}_i},$$

where $\|\nabla u\|_{\mathcal{H}_i} = \||\nabla u|\|_{\mathcal{H}_i}$ and $i = 1, 2$. We characterize $W_i = W^{1,\mathcal{H}_i}_0(\Omega)$ as the complement of $C^\infty_0(\Omega)$ concerning the norm

$$\|u\|_{W_i} := \|\nabla u\|_{\mathcal{H}_i}.$$

Our working space W is described as

$$W = W_1 \times W_2$$

with

$$\|(u_1, u_2)\|_W = \|u_1\|_{W_1} + \|u_2\|_{W_2},$$

which displays a separable and reflexive Banach space (see [5, Proposition 2.12]). Similar to Proposition 2.16 in [5], we arrive at the following embedding lemma.

Lemma 2. For any $r_i \in [1, p^*_i]$ $(i = 1, 2)$, the embedding $W_i \hookrightarrow L^{r_i}(\Omega)$ is continuous; the embedding is compact if $r_i \in [1, p^*_i)$, where $p^*_i = \frac{Np_i}{N-p_i}$ is the critical exponent.

From now on, we denote by C_{r_i} the best constant for which one has

$$\|u_i\|_{r_i} \le C_{r_i}\|u_i\|_{W_i}, \ \forall u_i \in W_i, \tag{2.1}$$

and for any r_i, r_j $(i, j = 1, 2)$, we denote

$$C_{r_i,r_j} = \max\{C^{r_i}_{r_i}, C^{r_j}_{r_j}\}. \tag{2.2}$$

Lemma 3 (see [21]). Let $\mathcal{A} : W_1 \times W_2 \to W^*_1 \times W^*_2$ be the operator defined by

$$\begin{aligned}\langle \mathcal{A}(u_1, u_2), (\varphi_1, \varphi_2)\rangle = &\int_\Omega (|\nabla u_1|^{p_1-2}\nabla u_1 + w_1(x)|\nabla u_1|^{q_1-2}\nabla u_1)\nabla\varphi_1 dx \\ &+ \int_\Omega (|\nabla u_2|^{p_2-2}\nabla u_2 + w_2(x)|\nabla u_2|^{q_2-2}\nabla u_2)\nabla\varphi_2 dx\end{aligned}$$

for all $u_i, \varphi_i \in W_i, \ i = 1, 2$, where $\langle ., .\rangle$ denotes the duality pairing among W_i and its dual space W^*_i.

Then, $\mathcal{A}$ is of type $(S)_+$, namely, if $(u^{(k)}_1, u^{(k)}_2) \rightharpoonup (u_1, u_2)$ in W and $\overline{\lim}_{k\to\infty}\langle \mathcal{A}(u^{(k)}_1, u^{(k)}_2), (u^{(k)}_1, u^{(k)}_2) - (u_1, u_2)\rangle \le 0$ it follows that $(u^{(k)}_1, u^{(k)}_2) \to (u_1, u_2)$ strongly in W.

3. Statement of main results

In order to describe our main results, we make the subsequent assumptions:

(**M**) For $i = 1, 2$, the function $M_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and there are numbers $m_i, m_i' > 0$ obeying the following relationship
$$m_i \leq M_i(t) \leq m_i' \ \forall i = 1, 2,$$
for all $t \geq 0$.

(**F0**) $F(x, 0, 0) = 0$ for a.e.$x \in \Omega$, $F(x, t, s) = F(x, 0, s)$ for all $t \leq 0$, $s \in \mathbb{R}$ and a.e.$x \in \Omega$, $F(x, t, s) = F(x, t, 0)$ for all $t \in \mathbb{R}$, $s \leq 0$ and a.e. $x \in \Omega$.

(**F1**) There exists $C > 0$ such that the following two conditions are fulfilled:
$$|F_t(x, t, s)| \leq C\Big[\min\{|t|^{p_1-1}, |t|^{q_1-1}\} + \min\{|s|^{\frac{q_2}{q_1'}}, |s|^{\frac{q_2}{p_1'}}, |s|^{\frac{p_2}{q_1'}}, |s|^{\frac{p_2}{p_1'}}\}\Big],$$
$$|F_s(x, t, s)| \leq C\Big[\min\{|t|^{\frac{q_1}{q_2'}}, |s|^{\frac{p_1}{q_2'}}, |t|^{\frac{q_1}{p_2'}}, |t|^{\frac{p_1}{p_2'}}\} + \min\{|s|^{q_2-1}, |s|^{p_2-1}\}\Big],$$
where p_i' and q_i' are the conjugate variables to p_i and q_i, respectively, that is, $\frac{1}{p_i} + \frac{1}{p_i'} = \frac{1}{q_i} + \frac{1}{q_i'} = 1$ for $i = 1, 2$.

We say that a function h complies the property $(*)_p$ if
$$h(t, s) \leq \tilde{C}(|t|^{p_1} + |s|^{p_2})$$
and h complies the property $(*)_q$ if
$$h(t, s) \leq \hat{C}(|t|^{q_1} + |s|^{q_2}),$$
where $\tilde{C}, \hat{C} > 0$ are independent of h.

Let K_1 and K_2 denote two functions verifying property $(*)_p$ and $(*)_q$, respectively. We suppose the following assumptions on the treatment of F at infinity and at origin:

(**F2**) $\overline{\lim}_{|(t,s)| \to \infty} \frac{F(x,t,s)}{K_1(t,s)} \leq 0$, uniformly for $x \in \Omega$;

(**F3**) $\overline{\lim}_{|(t,s)| \to 0} \frac{F(x,t,s)}{K_2(t,s)} \leq 0$, uniformly for $x \in \Omega$;

(**F4**) There exist $C_1, C_2 > 0$ with $F(x, t, s) > 0$ for a.e. $x \in \Omega$ and $(t, s) \in (0, C_1] \times (0, C_2]$.

We point out that the condition (**F2**) means that the nonlinear term F considered in this work is sublinear at infinity.

Definition 1. By a weak solution of system (1.1) we mean $(u_1, u_2) \in W$ obeying the following relationship
$$\begin{aligned}
&M_1\left[\int_\Omega \Big(\frac{|\nabla u_1|^{p_1}}{p_1} + w_1(x)\frac{|\nabla u_1|^{q_1}}{q_1}\Big)dx\right]\int_\Omega (|\nabla u_1|^{p_1-2}\nabla u_1 + w_1(x)|\nabla u_1|^{q_1-2}\nabla u_1)\nabla\varphi_1 dx\\
&+ M_2\left[\int_\Omega \Big(\frac{|\nabla u_2|^{p_2}}{p_2} + w_2(x)\frac{|\nabla u_2|^{q_2}}{q_2}\Big)dx\right]\int_\Omega (|\nabla u_2|^{p_2-2}\nabla u_2 + w_2(x)|\nabla u_2|^{q_2-2}\nabla u_2)\nabla\varphi_2 dx\\
&\quad = \lambda \int_\Omega \Big[F_{u_1}(x, u_1, u_2)\varphi_1 + F_{u_2}(x, u_1, u_2)\varphi_2\Big]dx
\end{aligned}$$
for all $(\varphi_1, \varphi_2) \in W$.

Now, we state the main results of this paper.

Theorem 1 (Nonexistence result)**.** Assume that

$(\mathbf{M})'$ $M_i(t) \geq m_i \ \forall i = 1, 2$.

$(\mathbf{F1})'$ There exists $C > 0$ such that the subsequent two conditions are obeyed:
$$|F_t(x, t, s)| \leq C\Big[\min\{|t|^{p_1-1}, |t|^{q_1-1}\} + \min\{|s|^{\frac{q_2}{q_1'}}, |s|^{\frac{q_2}{p_1'}}, |s|^{\frac{p_2}{q_1'}}, |s|^{\frac{p_2}{p_1'}}\}\Big],$$
$$|F_s(x, t, s)| \leq C\Big[\min\{|t|^{\frac{q_1}{q_2'}}, |s|^{\frac{p_1}{q_2'}}, |t|^{\frac{q_1}{p_2'}}, |t|^{\frac{p_1}{p_2'}}\} + \min\{|s|^{q_2-1}, |s|^{p_2-1}\}\Big]$$
for all $t, s \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then, there is $\lambda_* > 0$ such that for every $\lambda < \lambda_*$, system (1.1) does not have any nontrivial weak solution in W.

Theorem 2 (Multiplicity result)**.** Assume that $(\mathbf{M}), (\mathbf{F0}) - (\mathbf{F4})$ are complied. Then, there exists $\lambda^* > 0$ such that for every $\lambda \geq \lambda^*$, system (1.1) has at least two distinct nontrivial nonnegative weak solutions in W.

4. Proof of Theorem 1

Assume that (u_1, u_2) is a weak solution to system (1.1), then we need to consider the following four cases. Case 1. If $\|(u_1, u_2)\|_W < 1$, then, we have $\|u_1\|_{W_1} < 1$, $\|u_2\|_{W_2} < 1$. By $(\mathbf{M})'$ and Lemma 1, we infer that

$$\begin{aligned}
\min\{m_1, m_2\}\Big(\|u_1\|_{W_1}^{q_1} + \|u_2\|_{W_2}^{q_2}\Big) &\leq M_1\left[\int_\Omega \Big(\frac{|\nabla u_1|^{p_1}}{p_1} + w_1(x)\frac{|\nabla u_1|^{q_1}}{q_1}\Big)dx\right]\rho_{\mathcal{H}_1}(u_1) \\
&\quad + M_2\left[\int_\Omega \Big(\frac{|\nabla u_2|^{p_2}}{p_2} + w_2(x)\frac{|\nabla u_2|^{q_2}}{q_2}\Big)dx\right]\rho_{\mathcal{H}_2}(u_2) \\
&= M_1\left[\int_\Omega \Big(\frac{|\nabla u_1|^{p_1}}{p_1} + w_1(x)\frac{|\nabla u_1|^{q_1}}{q_1}\Big)dx\right] \\
&\quad \times \int_\Omega (|\nabla u_1|^{p_1} + w_1(x)|\nabla u_1|^{q_1})dx \\
&+ M_2\left[\int_\Omega \Big(\frac{|\nabla u_2|^{p_2}}{p_2} + w_2(x)\frac{|\nabla u_2|^{q_2}}{q_2}\Big)dx\right] \\
&\quad \times \int_\Omega (|\nabla u_2|^{p_2} + w_2(x)|\nabla u_2|^{p_2})dx \\
&= \lambda \int_\Omega \Big(F_{u_1}(x, u_1, u_2)u_1 + F_{u_2}(x, u_1, u_2)u_2\Big)dx.
\end{aligned} \tag{4.1}$$

Using $(\mathbf{F1})'$ and Young's inequality, we have

$$\begin{aligned}
\int_\Omega \Big(F_{u_1}(x, u_1, u_2)u_1 + F_{u_2}(x, u_1, u_2)u_2\Big)dx &\leq C\int_\Omega (|u_1|^{q_1-1} + |u_2|^{\frac{q_2}{q_1'}})|u_1|dx \\
&\quad + C\int_\Omega (|u_1|^{\frac{q_1}{q_2'}} + |u_2|^{q_2-1})|u_2|dx \\
&\leq C\int_\Omega (|u_1|^{q_1} + |u_2|^{q_2})dx + C\int_\Omega \Big(\frac{|u_1|^{q_1}}{q_1} + \frac{|u_2|^{q_2}}{q_1'}\Big)dx \\
&\quad + C\int_\Omega \Big(\frac{|u_1|^{q_1}}{q_2'} + \frac{|u_2|^{q_2}}{q_2}\Big)dx \\
&\leq C'\int_\Omega (|u_1|^{q_1} + |u_2|^{q_2})dx.
\end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we get

$$\min\{m_1, m_2\}\Big(\|u_1\|_{W_1}^{q_1} + \|u_2\|_{W_2}^{q_2}\Big) \leq \lambda C'\int_\Omega (|u_1|^{q_1} + |u_2|^{q_2})dx. \tag{4.3}$$

On the other hand, since $1 < p_i < q_i < N$ and $\frac{q_i}{p_i} < 1 + \frac{1}{N}$, we deduce that $Nq_i - p_i < Np_i$ and hence $Nq_i - p_iq_i < Nq_i - p_i < Np_i$. Thus $q_i < \frac{Np_i}{N-p_i}$, that is, $q_i < p_i^*$ for $i = 1, 2$. So, in view of Lemma 2, we obtain

$$\|u_1\|_{q_1}^{q_1} \leq C_{q_1}^{q_1}\|u_1\|_{W_1}^{q_1}, \ \|u_2\|_{q_2}^{q_2} \leq C_{q_2}^{q_2}\|u_2\|_{W_2}^{q_2}. \tag{4.4}$$

Let C_{q_1,q_2} be the same constant specified by (2.2), we arrive at

$$\frac{\|u_1\|_{q_1}^{q_1} + \|u_2\|_{q_2}^{q_2}}{\|u_1\|_{W_1}^{q_1} + \|u_2\|_{W_2}^{q_2}} \leq C_{q_1,q_2}.$$

So, by (4.3), we deduce that

$$\lambda \geq \frac{\min\{m_1, m_2\}}{C' C_{q_1,q_2}}.$$

Now, by choosing $\lambda_*^1 = \frac{\min\{m_1,m_2\}}{C'C_{q_1,q_2}}$ in this case, we obtain the desired result.

Case 2. If $\|u_1\|_{W_1} \geq 1$ and $\|u_2\|_{W_2} < 1$. Following the same methods in the proof of case 1 (by replace q_1 by p_1) and by choosing $\lambda_*^2 = \frac{\min\{m_1,m_2\}}{C'C_{p_1,q_2}}$, we obtain the result.

Case 3. If $\|u_1\|_{W_1} < 1$ and $\|u_2\|_{W_2} \geq 1$, similar to the above proof process and by choosing $\lambda_*^3 = \frac{\min\{m_1,m_2\}}{C'C_{q_1,p_2}}$, we obtain the result.

Case 4. If $\|u_1\|_{W_1}, \|u_2\|_{W_2} \geq 1$, by choosing $\lambda_*^4 = \frac{\min\{m_1,m_2\}}{C'C_{p_1,p_2}}$, we obtain the result.

Now, taking $\lambda_* = \min_{1\leq i\leq 4} \lambda_*^i$, system (1.1) does not have any nontrivial weak solution in W for $\lambda < \lambda^*$ and we complete the proof of Theorem 1.

5. Proof of Theorem 2

We are going to apply the mountain pass theorem to prove Theorem 2 (see [2, 23]). For each $\lambda \in \mathbb{R}$, we introduce the Euler functional $\Phi_\lambda : W \to \mathbb{R}$ which is associated with system (1.1) defined as

$$\Phi_\lambda(u_1, u_2) = I(u_1, u_2) - \lambda J(u_1, u_2),$$

where

$$I(u_1, u_2) = \widehat{M_1}(L_1(u_1)) + \widehat{M_2}(L_2(u_2))$$

with $\widehat{M_i}(t) = \int_0^t M_i(\tau)d\tau$ for $i = 1, 2$ and

$$L_i(u_i) = \int_\Omega \Big(\frac{|\nabla u_i|^{p_i}}{p_i} + w_i(x)\frac{|\nabla u_i|^{q_i}}{q_i}\Big)dx,$$

and

$$J(u_1, u_2) = \int_\Omega F(x, u_1, u_2)dx$$

for every $(u_1, u_2) \in W$. By (**F1**), the functional Φ_λ is of class C^1 and the Gâteaux derivative of Φ_λ is characterized by

$$\begin{aligned}\langle \Phi_\lambda'(u_1, u_2), (\varphi_1, \varphi_2)\rangle &= M_1(L_1(u_1))\langle L_1'(u_1), \varphi_1\rangle + M_2(L_2(u_2))\langle L_2'(u_2), \varphi_2\rangle \\ &\quad - \lambda \int_\Omega \Big(F_{u_1}(x, u_1, u_2)\varphi_1 + F_{u_2}(x, u_1, u_2)\varphi_2\Big)dx, \end{aligned} \tag{5.1}$$

for every $(\varphi_1, \varphi_2) \in W$ with

$$\langle L_1'(u_i), \varphi_i\rangle = \int_\Omega (|\nabla u_i|^{p_i-2} + w_i(x)|\nabla u_1|^{q_i-2})\nabla u_i \nabla\varphi_i dx$$

for every $\varphi_i \in W_i$.

Lemma 4. For every $\lambda \in \mathbb{R}$, Φ_λ is sequentially weakly lower semicontinuous functional on W.

Proof. Let $\{(u_1^{(k)}, u_2^{(k)})\} \subseteq W$ be a sequence that converges weakly to $(u_1, u_2) \in W$. By the results can be found in [6], we get

$$\liminf_{k\to\infty} L_i(u_i^{(k)}) \geq L_i(u_i) \ \forall i = 1, 2.$$

On the other hand, the function $t \to \widehat{M}_i(t)$ for $i = 1, 2$ is continuous and monotone. So, we arrive at

$$\begin{aligned}\liminf_{k\to\infty} I(u_1^{(k)}, u_2^{(k)}) &= \liminf_{k\to\infty} \left[\widehat{M}_1(L_1(u_1^{(k)})) + \widehat{M}_2(L_2(u_2^{(k)}))\right] \\ &\geq \liminf_{k\to\infty} \widehat{M}_1(L_1(u_1^{(k)})) + \liminf_{k\to\infty} \widehat{M}_2(L_2(u_2^{(k)})) \\ &\geq \widehat{M}_1\Big(\liminf_{k\to\infty} L_1(u_1^{(k)})\Big) + \widehat{M}_2\Big(\liminf_{k\to\infty} L_2(u_2^{(k)})\Big) \\ &\geq \widehat{M}_1(L_1(u_1)) + \widehat{M}_2(L_2(u_2)) = I(u_1, u_2).\end{aligned}$$

So, I is sequentially weakly lower semicontinuous on W. Next, using Hölder's inequality and (**F1**), we infer that

$$\begin{aligned}&|J(u_1^{(k)}, u_2^{(k)}) - J(u_1, u_2)| \\ &= \left|\int_\Omega [F(x, u_1^{(k)}, u_2^{(k)}) - F(x, u_1, u_2)]dx\right| \\ &\leq \int_\Omega \left|\frac{\partial F}{\partial u_1}(x, u_1 + \delta_{1,k}(u_1^{(k)} - u_1), u_2 + \delta_{2,k}(u_2^{(k)} - u_2)\right| |u_1^{(k)} - u_1|dx \\ &\quad + \int_\Omega \left|\frac{\partial F}{\partial u_2}(x, u_1 + \delta_{1,k}(u_1^{(k)} - u_1), u_2 + \delta_{2,k}(u_2^{(k)} - u_2)\right| |u_2^{(k)} - u_2|dx \\ &\leq C\int_\Omega \Big(1 + |u_1 + \delta_{1,k}(u_1^{(k)} - u_1)|^{q_1-1} + |u_2 + \delta_{2,k}(u_2^{(k)} - u_2)|^{\frac{q_2}{q_1'}}\Big)|u_1^{(k)} - u_1|dx \\ &\quad + C\int_\Omega \Big(1 + |u_1 + \delta_{1,k}(u_1^{(k)} - u_1)|^{\frac{q_1}{q_2'}} + |u_2 + \delta_{2,k}(u_2^{(k)} - u_2)|^{q_2-1}\Big)|u_2^{(k)} - u_2|dx \\ &\leq C\left(|\Omega|^{\frac{1}{q_1'}} + \|u_1 + \delta_{1,k}(u_1^{(k)} - u_1)\|_{q_1}^{q_1-1} + \|u_2 + \delta_{2,k}(u_2^{(k)} - u_2)\|_{q_2}^{\frac{q_2}{q_1'}}\right)\|u_1^{(k)} - u_1\|_{q_1} \\ &\quad + C\left(|\Omega|^{\frac{1}{q_2'}} + \|u_1 + \delta_{1,k}(u_1^{(k)} - u_1)\|_{q_1}^{\frac{q_1}{q_2'}} + \|u_2 + \delta_{2,k}(u_2^{(k)} - u_2)\|_{q_2}^{q_2-1}\right)\|u_2^{(k)} - u_2\|_{q_2}, \qquad (5.2)\end{aligned}$$

where $\delta_k = (\delta_{1,k}, \delta_{2,k})$ with $0 \leq \delta_{1,k}(x), \delta_{2,k}(x)) \leq 1$ for each $x \in \Omega$.

Moreover, by Lemma 2, the embedding $W \hookrightarrow L^{q_1}(\Omega) \times L^{q_2}(\Omega)$ is compact, so $(u_1^{(k)}, (u_2^{(k)}) \to (u_1, u_2)$ in W, i.e, $u_1^{(k)} \to u_1$ in $L^{q_1}(\Omega)$ and $u_2^{(k)} \to u_2$ in $L^{q_2}(\Omega)$. Besides, by the boundedness of two sequences $\|u_1 + \delta_{1,k}(u_1^{(k)} - u_1)\|_{q_1}$ and $\|u_2 + \delta_{2,k}(u_2^{(k)} - u_2)\|_{q_2}$, we arrive at

$$J(u_1^{(k)}, u_2^{(k)}) \to J(u_1, u_2) \text{ as } k \to \infty,$$

which means that J is sequentially weakly continuous on W. Thus, Φ_λ is sequentially weakly lower semicontinuous on W and we obtained the desired result. □

Lemma 5. Φ_λ denotes a coercive functional on W, that is to say, $\lim_{\|(u,v)\|\to+\infty} \Phi_\lambda(u, v) = +\infty$.

Proof. In view of (**F1**) and (**F2**), there exists $C_\lambda = C(\lambda) > 0$ such that for a.e. $x \in \Omega$ and for every $(s, t) \in \mathbb{R}^2$ we have

$$\lambda F(x, t, s) \leq \frac{\min\{m_1, m_2\}}{2\max\{q_1, q_2\}\tilde{C}C_{p_1,p_2}} K_1(t, s) + C_\lambda,$$

where $\tilde{C}$ is the same constant specified by $(\star)_p$. Invoking Lemma 1, Lemma 2 and condition (**M**), for every $(u_1, u_2) \in W$ with $\|(u_1, u_2)\|_W \geq 1$, we obtain

$$\begin{aligned}\Phi_\lambda(u_1, u_2) &\geq \frac{m_1}{q_1}\rho_{\mathcal{H}_1}(u_1) + \frac{m_2}{q_2}\rho_{\mathcal{H}_2}(u_2) - \int_\Omega \left(\frac{\min\{m_1, m_2\}}{2\max\{q_1, q_2\}\tilde{C}C_{p_1,p_2}}K_1(u_1, u_2) + C_\lambda\right)dx \\ &\geq \frac{m_1}{q_1}\|u_1\|_{W_1}^{p_1} + \frac{m_2}{q_2}\|u_2\|_{W_2}^{p_2} - \frac{m_1}{2q_1C_{p_1}^{p_1}}\int_\Omega |u_1|^{p_1}dx - \frac{m_2}{2q_2C_{p_2}^{p_2}}\int_\Omega |u_2|^{p_2}dx - C_\lambda|\Omega| \\ &\geq \left(\frac{m_1}{q_1}\|u_1\|_{W_1}^{p_1} - \frac{m_1}{2q_1}\|u_1\|_{W_1}^{p_1}\right) + \left(\frac{m_2}{q_2}\|u_2\|_{W_2}^{p_2} - \frac{m_2}{2q_2}\|u_2\|_{W_2}^{p_2}\right) - C_\lambda|\Omega| \\ &= \frac{m_1}{2q_1}\|u_1\|_{W_1}^{p_1} + \frac{m_2}{2q_2}\|u_2\|_{W_2}^{p_2} - C_\lambda|\Omega|,\end{aligned}$$

which implies that $\Phi_\lambda(u_1, u_2) \to \infty$ as $\|(u_1, u_2)\|_W \to +\infty$. □

Lemma 6. If $(u_1, u_2) \in W$ is a weak solution of system (1.1), then $u_1 \geq 0$ and $u_2 \geq 0$ in Ω.

Proof. In view of (**F0**), if $\bar{t} < 0$ then $F(x, t, s) = F(x, \bar{t}, s) = F(x, 0, s)$ for all $t < 0$, all $s \in \mathbb{R}$, $x \in \Omega$ and thus,

$$F_t(x, \bar{t}, s) = \lim_{t\to\bar{t}} \frac{F(x, t, s) - F(x, \bar{t}, s)}{t - \bar{t}} = 0 \text{ for } x \in \Omega.$$

Similarly, if $\bar{s} < 0$ then $F_s(x, t, \bar{s}) = 0$ for $x \in \Omega$ and all $t \in \mathbb{R}$.

Now, if (u_1, u_2) is a weak solution of system (1.1), invoking Lemma 1 and (**M**), we deduce that

$$\begin{aligned}0 &= \langle \Phi_\lambda'(u_1, u_2), (u_1, u_2)^-\rangle \\ &= M_1(L_1(u_1))\langle L_1'(u_1), \bar{u_1}\rangle + M_2(L_2(u_2))\langle L_2'(u_2), \bar{u_2}\rangle \\ &\qquad - \lambda\int_\Omega \Big[F_{u_1}(x, u_1, u_2)\bar{u_1} + F_{u_2}(x, u_1, u_2)\bar{u_2}\Big]dx \\ &\geq m_1\int_\Omega \Big(|\nabla\bar{u_1}|^{p_1} + w_1(x)|\nabla\bar{u_1}|^{q_1}\Big)dx + m_2\int_\Omega \Big(|\nabla\bar{u_2}|^{p_2} + w_2(x)|\nabla\bar{u_2}|^{q_2}\Big)dx \\ &= m_1\rho_{\mathcal{H}_1}(u_1) + m_2\rho_{\mathcal{H}_2}(u_2) \\ &\geq m_1\min\{\|\bar{u_1}\|_{W_1}^{p_1}, \|\bar{u_1}\|_{W_1}^{q_1}\} + m_2\min\{\|\bar{u_2}\|_{W_2}^{p_2}, \|\bar{u_2}\|_{W_2}^{q_2}\},\end{aligned}$$

where $\bar{u_i} = \min\{u_i(x), 0\}$. So, for $i = 1, 2$, we have $u_i(x) \geq 0$ for a.e. $x \in \Omega$. □

Via Lemmas 4–6 and the direct method in the calculus of variations (see [23]), Φ_λ admits a global minimizer $(u_1^*, u_2^*) \in W$, which is a least energy of system (1.1). The subsequent lemma implies that (u_1^*, u_2^*) is nontrivial.

Lemma 7. There is $\lambda^* > 0$ so that $\inf_W \Phi_\lambda < 0$ for each $\lambda \geq \lambda^*$ and hence $(u_1^*, u_2^*) \not\equiv 0$.

Proof. Let C_1, C_2 be as in (**F4**) and $(u_1, u_2) \in C^1(\Omega) \times C^1(\Omega)$ with $u_i(x) > 0$ for all $x \in \Omega$ and $i = 1, 2$. Then $(u_1, u_2) \in W$ and $\int_\Omega F(x, u_1, u_2)dx > 0$. Taking into account (**M**), we conclude that

$$\begin{aligned}\Phi_\lambda(u_1, u_2) &\leq \frac{m_1'}{p_1}\rho_{\mathcal{H}_1}(u_1) + \frac{m_2'}{p_2}\rho_{\mathcal{H}_2}(u_2) - \lambda\int_\Omega F(x, u_1, u_2)dx \\ &\leq \frac{m_1'}{p_1}\max\{\|u_1\|_{W_1}^{p_1}, \|u_1\|_{W_1}^{q_1}\} + \frac{m_2'}{p_2}\max\{\|u_2\|_{W_2}^{p_2}, \|u_2\|_{W_2}^{q_2}\} - \lambda\int_\Omega F(x, u_1, u_2)dx\end{aligned}$$

Thus, by choosing

$$\lambda^* = 2\frac{\frac{m_1'}{p_1}\max\{\|u_1\|_{W_1}^{p_1}, \|u_1\|_{W_1}^{q_1}\} + \frac{m_2'}{p_2}\max\{\|u_2\|_{W_2}^{p_2}, \|u_2\|_{W_2}^{q_2}\}}{\int_\Omega F(x, u_1, u_2)dx},$$

we have $\Phi_\lambda(u_1, u_2) < 0$ for each $\lambda \geq \lambda^*$ and hence $(u_1^*, u_2^*) \not\equiv 0$. □

Our next goal is to attain the second weak solution by using the mountain pass theorem (see [2]).

Lemma 8. There exist two positive constants $\alpha, \beta > 0$ with $\|(u_1^*, u_2^*)\|_W > \alpha$ such that $\Phi_\lambda(u_1, u_2) \geq \beta$ for any $(u_1, u_2) \in W$ with $\|(u_1, u_2)\|_W = \alpha$.

Proof. In virtue of (**F1**) and (**F3**), we get

$$\lambda F(x,t,s) \leq \frac{\min\{m_1, m_2\}}{2\max\{q_1, q_2\}\hat{C}C_{q_1,q_2}} k_2(t,s) + C_\lambda(|t|^{\sigma_1} + |t|^{\sigma_2}),\ \forall (t,s) \in \mathbb{R}^2, x \in \Omega,$$

where $p_i < q_i < \sigma_i < p_i^*$, for $i = 1, 2$ and $\hat{C}$ is the same constant specified by $(\star)_q$.

In view of Lemma 1, Lemma 2 and (**M**), for every $(u_1, u_2) \in W$, we obtain

$$\begin{aligned}
\Phi_\lambda(u_1, u_2) \geq\ & \frac{m_1}{q_1}\rho_{\mathcal{H}_1}(u_1) + \frac{m_2}{q_2}\rho_{\mathcal{H}_2}(u_2) - \int_\Omega \frac{\min\{m_1, m_2\}}{2\max\{q_1, q_2\}\hat{C}C_{q_1,q_2}} k_2(u_1, u_2)dx \\
& - C_\lambda \int_\Omega (|u_1|^{\sigma_1} + |u_2|^{\sigma_2})dx \\
\geq\ & \frac{m_1}{q_1}\|u_1\|_{W_1}^{q_1} - \frac{m_1}{2q_1 C_{q_1}^{q_1}} \int_\Omega |u_1|^{q_1} dx - C_\lambda \int_\Omega |u_1|^{\sigma_1} dx \\
& + \frac{m_2}{q_2}\|u_2\|_{W_2}^{q_2} - \frac{m_2}{2q_2 C_{q_2}^{q_2}} \int_\Omega |u_2|^{q_2} dx - C_\lambda \int_\Omega |u_2|^{\sigma_2} dx \\
\geq\ & \left(\frac{m_1}{q_1}\|u_1\|_{W_1}^{q_1} - \frac{m_1}{2q_1}\|u_1\|_{W_1}^{q_1} - C_\lambda C_{\sigma_1}^{\sigma_1}\|u_1\|_{W_1}^{\sigma_1}\right) \\
& + \left(\frac{m_2}{q_2}\|u_2\|_{W_2}^{q_1} - \frac{m_2}{2q_2}\|u_2\|_{W_2}^{q_2} - C_\lambda C_{\sigma_2}^{\sigma_2}\|u_2\|_{W_2}^{\sigma_2}\right) \\
=\ & \left(\frac{m_1}{2q_1}\|u_1\|_{W_1}^{q_1} - C_\lambda C_{\sigma_1}^{\sigma_1}\|u_1\|_{W_1}^{\sigma_1}\right) + \left(\frac{m_2}{2q_2}\|u_2\|_{W_2}^{q_1} - C_\lambda C_{\sigma_2}^{\sigma_2}\|u_2\|_{W_2}^{\sigma_2}\right).
\end{aligned}$$

Since $q_i < \sigma_i < p_i^*$, there are constants $\alpha, \beta > 0$ such that $\Phi_\lambda(u_1, u_2) \geq \beta$ for each $(u_1, u_2) \in W$ with $\|(u_1, u_2)\|_W = \alpha$. □

Lemma 9. If (**M**) and (**F1**) hold. Then Φ_λ satisfies the (PS) condition for all $\lambda > 0$, namely, any sequence $\{(u_1^{(k)}, u_2^{(k)})\}$ verifying

$$\left|\Phi_\lambda(u_1^{(k)}, u_2^{(k)})\right| \leq C,\ \Phi_\lambda'(u_1^{(k)}, u_2^{(k)}) \to 0 \text{ in } W^* \text{ as } n \to \infty, \tag{5.3}$$

admits a convergent subsequence in W, where W^* denotes the dual space of W.

Proof. Let $\{(u_1^{(k)}, u_2^{(k)})\}$ be a sequence verifying (5.3). In view of Lemma 5, $\{(u_1^{(k)}, u_2^{(k)})\}$ is bounded. In virtue of the reflexivity of W, for a subsequence, still denoted by $\{(u_1^{(k)}, u_2^{(k)})\}$, we have $(u_1^{(k)}, u_2^{(k)}) \rightharpoonup (u_1, u_2)$ weakly. So, we get

$$\lim_{n\to\infty} \langle \Phi_\lambda'(u_1^{(k)}, u_2^{(k)}), (u_1^{(k)}, u_2^{(k)}) - (u_1, u_2)\rangle = 0. \tag{5.4}$$

Using (**F1**) and Hölder's inequality, we infer that

$$\begin{aligned}
\int_\Omega |F_{u_1}(x, u_1^{(k)}, u_2^{(k)})||u_1^{(k)} - u_1|dx &\leq C\int_\Omega \left(1 + |u_1^{(k)}|^{q_1-1} + |u_2^{(k)}|^{\frac{q_2}{q_1'}}|\right)\ |u_1^{(k)} - u_1|dx \\
&\leq C\left(|\Omega|^{\frac{1}{q_1'}} + \|u_1^{(k)}\|_{q_1}^{q_1-1} + \|u_2^{(k)}\|_{q_2}^{\frac{q_2}{q_1'}}\right)\|u_1^{(k)} - u_1\|_{q_1}
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\int_\Omega |F_{u_2}(x,u_1^{(k)},u_2^{(k)})||u_2^{(k)}-u_2|dx &\le C\int_\Omega \left(1+|u_1^{(k)}|^{\frac{q_1}{q_2'}}+|u_2^{(k)}|^{q_2-1}\right)\,|u_2^{(k)}-u_2|dx\\ &\le C\left(|\Omega|^{\frac{1}{q_2'}}+\|u_1^{(k)}\|_{q_1}^{\frac{q_1}{q_2'}}+\|u_2^{(k)}\|_{q_2}^{q_2-1}\right)\|u_2^{(k)}-u_2\|_{q_2}. \end{aligned}\tag{5.6}$$

Since the embedding $W \hookrightarrow L^{q_1}(\Omega)\times L^{q_2}(\Omega)$ is compact, invoking (5.5) and (5.6), we get

$$\lim_{k\to\infty}\langle J'(u_1^{(k)},u_2^{(k)}),(u_1^{(k)},u_2^{(k)})-(u_1,u_2)\rangle = 0. \tag{5.7}$$

Combining (5.4) and (5.7) and using (**M**), we arrive at

$$\lim_{k\to\infty}\left[\int_\Omega(|\nabla u_1^{(k)}|^{p_1-2}+w_1(x)|\nabla u_1^{(k)}|^{q_1-2})\nabla u_1^{(k)}(\nabla u_1^{(k)}-\nabla u_1)dx \right.$$
$$\left. +\int_\Omega(|\nabla u_2^{(k)}|^{p_2-2}+w_2(x)|\nabla u_2^{(k)}|^{q_2-2})\nabla u_2^{(k)}(\nabla u_2^{(k)}-\nabla u_2)dx\right]=0.$$

In view of Lemma 3, $(u_1^{(k)},u_2^{(k)})\to(u_1,u_2)$ strongly in W and we obtain the desired result. □

Now, we demonstrate the proof of Theorem 2.

Proof of Theorem 2. In terms of Lemmas 4–9 and the mountain pass theorem, there is a second weak solution $(\hat{u_1},\hat{u_2})\in W$ of system (1.1) with $\Phi_\lambda(\hat{u_1},\hat{u_2})>0$. Besides, $(\hat{u_1},\hat{u_2})$ is nontrivial and $(\hat{u_1},\hat{u_2})\neq(u_1^*,u_2^*)$ because $\Phi_\lambda(\hat{u_1},\hat{u_2})>0>\Phi_\lambda(u_1^*,u_2^*)$. This concludes the proof.

6. Conclusions

We consider a class of Kirchhoff-double phase systems in bounded domains. The essential contribution of this paper is the study of the existence and nonexistence of solutions to the problem by using variational techniques. To our best knowledge, there seems to be few results on Kirchhoff-double phase type systems in the existing literature.

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References

[1] M. E. Ahmadi, M. Berrajaa, and A. Ayoujil, *Existence of two solutions for Kirchhoff-double phase problems with a small perturbation without (AR)-condition*, Contin. Dyn. Syst. Ser. S., 2023.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., *14*(4) (1973), 349–381.

[3] R. Arora, A. Fiscella, T. Mukherjee, and P. Winkert, *On double phase Kirchhoff problems with singular nonlinearity*, Adv. Nonlinear Anal., *12* (2023), 20220312.

[4] J. F. Bonder, *Multiple positive solutions for quasilinear elliptic problems with sign-changing nonlinearities*, Abstract Appl. Anal., *2004*(12) (2004), 1047–1055.

[5] A. C. Blanco, L. Gasiński, P. Harjulehto, and P. Winkert, *A new class of double phase variable exponent problems: Existence and uniqueness*, J. Differential Equations., *323* (2022), 182–288.

[6] F. X. Cao, B. Ge, and Y. W. Shou, *Existence and nonexistence of solutions for the double phase problem, Results in Mathematics*, J. Differential Equations., *76*(3) (2021), 132.

[7] L. Cherfils and Y. Ill'yasov, *On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian*, Commun. Pure Appl. Anal., *4*(1) (2005), 9–22.

[8] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal., *30*(7) (1997), 4619–4627.

[9] N. T. Chung, *Existence of positive solutions for a class of Kirchhoff type systems involving critical exponents*, Filomat., *33*(1) (2019), 267–280.
[10] N. T. Chung and Z. Naghizadeh, *Multiplicity of solutions for a class of fourth-order elliptic equations of $p(x)$-Kirchhoff type*, Math. Slovaca., *71*(6) (2021), 1441–1458.
[11] P. Chen and Y. Zhang, *Existence and multiplicity results for Kirchhoff-type problem with sublinear nonlinearity*, Appl. Math. Letters., *114* (2021), 106900.
[12] C. Farkas, A. Fiscella, and P. Winkert, *On a class of critical double phase problems*, J. Math. Anal. Appl., *515* (2022), 126420.
[13] Y. Feng, S. Shang, and Z. Bai, *Existence of the solution for a double phase system with convex nonlinearities*, J. Nonlinear Math. Phys., *31*(1) (2024), 15.
[14] A. Fiscella and A. Pinamonti, *Existence and multiplicity results for Kirchhoff-type problems on a double-phase setting*, Mediterr. J. Math., *20* (2023), 23.
[15] U. Guarnotta, R. Livrea, and P. Winkert, *The sub-supersolution method for variable exponent double phase systems with nonlinear boundary conditions*, Rend. Lincei Mat. Appl., *34*(3) (2023), 617–639.
[16] M.K. Hamdani, A. Harrabi, F. Mtiri, and D.D. Repovš, *Existence and multiplicity results for a new $p(x)$-Kirchhoff problem*, Nonlinear Anal. (TMA), *190* (2020), 111598.
[17] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, Germany, 1883.
[18] W. Liu and G. Dai, *Existence and multiplicity results for double phase problem*, J. Differential Equations., *265* (2018), 4311–4334.
[19] T. F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal., *63* (2005), 1967–1977.
[20] K. Perera and Z. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differential Equations., *221*(1) (2006), 246–255.
[21] G. Marino and P. Winkert, *Existence and uniqueness of elliptic systems with double phase operators and convection terms*, J. Math. Anal. Appl., *492*(1) (2020), 124423.
[22] K. Perera, *Multiple positive solutions for a class of quasilinear elliptic boundary-value problems*, Electronic J. Differ. Equ., *2003*(07) (2003), 1–5.
[23] M. Struwe, *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, *34*(4) (2008).
[24] V. V. Zhikov, *On Lavrentiev's phenomenon*, Russian J. Math. Phys., *3*(0) (1995), 249–269.
[25] V. V. Zhikov, *On some variational problems*, Russian J. Math. Phys., *5*(1) (1997), 249–269.