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Lie symmetry analysis, and exact solutions to the time–fractional Black–Scholes equation of the Caputo–type

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Abstract

In this study, the Lie symmetry analysis and exact solutions are investigated to the fractional Black–Scholes(B-S) equations of the Caputo–type modeling the pricing options under the absence of arbitrage and self-financing portfolio assumptions. A class of exact invariant and solitary solutions are given to B-S equations. Some examples are presented in which we use the obtained reductions to find their exact solutions.

Keywords. Lie symmetry analysis, Time–fractional Black–Scholes equation, Caputo fractional derivative, Invariant solutions.
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Uncorrected Proof

1. Introduction

Differential equations with the fractional order describe the phenomena for which the memory and hereditary properties play crucial roles. Many systems modeled with the help of fractional calculus display rich fractional dynamical behavior, such as viscoelastic systems, colored noise, boundary layer effects in ducts, electromagnetic waves, fractional kinetics, and electrode-electrolyte polarization, etc. (see, e.g.,[9, 15, 19] and references cited therein). Due to these applications, many researchers have generalized the existing methods of solving integer-order differential equations into fractional ones. However, the non–local nature of fractional derivatives causes these generalizations not to be a straightforward task. Several definitions have been proposed for the fractional derivatives. The Caputo derivative and the Riemann-Liouville ones are mostly used. From a practical point of view, the Caputo derivative is more important because of the initial value of fractional differential equation with the Caputo derivative is the same as that of integer-order differential equation. However, the Riemann–Liouville approach needs initial conditions with the limit values of the same fractional derivatives at $t = 0$.
Over the past few years, FDEs have been studied extensively, from theoretical and numerical points of view. To obtain exact solutions for differential equations, the Lie symmetry analysis is an effective method that has been applied to many equations [20, 21, 31, 36, 41]. This method has been employed to FDEs of the Riemann–Liouville type by researchers. In [5, 14, 30, 34, 39, 40] the authors applied the classical Lie symmetry analysis for differential equations involving fractional Riemann–Liouville derivatives. One of the possible extensions of the Lie symmetry method is the nonclassical method, which has been applied to the Riemann–Liouville FDEs in [3, 23–27].
To derive exact solutions and conservation laws for FDEs of the Caputo-type, we also can use the Lie symmetry analysis. In [12], the authors have given the prolongation formula in the case of $0 < \alpha < 1$. We consider the Black-Sholes(B-S) equation, which models the pricing of the European call, put and double barrier options. Based on the specific forms of the payoffs and rebates, double barrier options can have many different forms. If the price change of the option in the financial market is considered as a fractal transmission system, we obtain the fractional B-S equation with the Caputo-type derivative[7, 18]. In the current study, we apply the results above to obtain invariants, exact solutions of the fractional B-S with the Caputo-type. There are some methods applied to the fractional B-S equations,

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which give the analytical solutions of them in the form of a convolution of some special functions or an infinite series with an integral. These representations make them difficult to use. We infer to spline collocation methods [1, 33, 37], homotopy analysis methods [10, 35], Touchard wavelet technique [28], or via the method of separation of variables [7]. Methods based on the classical and non-classical Lie group analysis have been used to find the exact solutions of the fractional (B-S) equation with the Riemann-Liouville type by the authors in [6, 8]. The main means to apply this method is using the exact prolongation formula for fractional operators of the Riemann-Liouville type, which has been given by the authors in [12]. Other researchers have generalized and applied this finding to multiple classes of FDEs [5, 14, 30, 34, 39, 40]. Lie symmetry analysis has not been applied to the fractional (B-S) equation with the Caputo-type yet. Using the results obtained here, we derive invariants of the (B-S) equation, which leads us to have a class of exact solutions. Also we obtain the solitary wave solutions of the time–fractional (B–S) equation [2, 17, 32]. The structure of the present study is as follows. In section 2 we first, give a brief background on the Lie symmetry analysis. In section 3, we arrive at the invariants of the B-S equation with fractional order. Utilizing the results of this section, we derive some exact solutions for B-S equation in section 4. We present and classify the conservation laws of the (B-S) equation in section 5.

2. Preliminaries

We first give a brief review of the fractional calculus and the Lie symmetry analysis, which will be needed in the sequel.

Definition 2.1. [9] Let $m \in \mathbb{N}$ and $m-1 < \alpha < m$. The Riemann-Liouville and Caputo fractional derivatives of order α of a function f are defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} D_t^m \int_0^t (t-s)^{m-\alpha-1} f(s) ds,$$
$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D_s^m f(s) \mathrm{d}s,$$

respectively, provided the right-hand sides integrals are finite for $t \in (0,T)$ where $D_t^m f(t) = \frac{d^m f}{dt^m}$.

Proposition 2.2. *[9](Leibniz's formula for Riemann-Liouville fractional derivative). Let $\alpha > 0$ and assume that f and g are analytic on $(-h,h)$ with some $h > 0$. Then,*

$$D_t^\alpha (fg)(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_t^k f(t) D_t^{\alpha-k} g(t),$$

for $0 < t < h/2$.

Proposition 2.3. *Let f be analytic in $(-h,h)$ for some $h > 0$, and $m-1 < \alpha < m$, then*

$${}^C D_t^\alpha f(t) = \sum_{k=m}^{\infty} \binom{\alpha-m}{k-m} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} D_t^k f(t),$$

for $0 < x < h/2$.

Proof. The proof is similar to the case of the Riemann-Liouville fractional derivative given in [9]. Recalling Definition 2.1

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D_s^m f(s) \mathrm{d}s,$$

we employ the Taylor's expansion of $D_s^m f(s)$ with respect to s around $s = t$, and substitute the result into the formula above, and compute the integral, thus we deduce

$${}^C D_t^\alpha f(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{m+k-\alpha}}{k!(m+k-\alpha)\Gamma(m-\alpha)} D_t^{m+k} f(t).$$

Uncorrected Proof

Since $\frac{(-1)^k}{k!(m+k-\alpha)\Gamma(m-\alpha)} = \binom{\alpha-m}{k}\frac{1}{\Gamma(k+m+1-\alpha)}$, then it implies the desired result. $\square$

Proposition 2.4. *[9] Let $m \geq 1$ and $m-1 < \alpha < m$. Assume that f is such that both $D_t^\alpha f$ and ${}^C D_t^\alpha f$ exist. Then*

$$ {}^C D_t^\alpha f(t) = D_t^\alpha f(t) - \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} D_t^k f(t)|_{t=0}. $$

Proposition 2.5. *Let $m \geq 1$ and $m-1 < \alpha < m$. Let $u = u(x,t)$, and $f = f(x,t,u)$ be analytic functions, and ${}^C\mathcal{D}_t^\alpha f$ denote the total derivative of f, then we have*

$$ \begin{aligned} {}^C\mathcal{D}_t^\alpha f = \ & {}^C D_t^\alpha f + f_u \, {}^C D_t^\alpha u - u \, {}^C D_t^\alpha f_u + \sum_{k=1}^{\infty} \binom{\alpha}{k} D_t^k f_u D_t^{\alpha-k} u \\ & + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left[f_u D_t^k u\big|_{t=0} - D_t^k (u f_u)\big|_{t=0} \right] + \nu, \end{aligned} $$

with $f_u = \frac{\partial f}{\partial u}$ and $D_t^m = \frac{\partial^m}{\partial t^m}$, where

$$ \nu = \sum_{n=m}^{\infty} \sum_{j=2}^{n} \sum_{k=2}^{j} \sum_{r=0}^{k-1} \binom{\alpha-m}{n-m} \binom{n}{j} \binom{k}{r} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \frac{1}{k!} (-u)^r D_t^j u^{k-r} D_t^{n-j} \left(D_u^k f \right). $$

Proof. Using Proposition 2.3, and the definition of the total derivative $\mathcal{D}_t^n f$, we have

$$ \begin{aligned} {}^C\mathcal{D}_t^\alpha f(x,t,u(x,t)) &= \sum_{n=m}^{\infty} \binom{\alpha-m}{n-m} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \mathcal{D}_t^n f(x,t,u(x,t)) \\ &= \sum_{n=m}^{\infty} \binom{\alpha-m}{n-m} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{r=0}^{k} \binom{n}{j} \binom{k}{r} \frac{1}{k!} (-u)^r D_t^j u^{k-r} D_t^{n-j} \left(D_u^k f \right). \end{aligned} $$

By rearranging the indexes, recalling Proposition 2.3, and some technical calculations we deduce

$$ \begin{aligned} &= {}^C D_t^\alpha f - u \sum_{n=m}^{\infty} \binom{\alpha-m}{n-m} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} D_t^n f_u \\ &\quad + \sum_{n=m}^{\infty} \binom{\alpha-m}{n-m} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \sum_{j=0}^{n} \binom{n}{j} D_t^j u D_t^{n-j} (f_u) + \nu, \end{aligned} $$

where ν is as in the Proposition. Now using the classical Leibniz's formula, we conclude

$$ \begin{aligned} &= {}^C D_t^\alpha f - u \sum_{n=m}^{\infty} \binom{\alpha-m}{n-m} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} D_t^n f_u + \sum_{n=m}^{\infty} \binom{\alpha-m}{n-m} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} D_t^n (u f_u) + \nu \\ &= {}^C D_t^\alpha f - u \, {}^C D_t^\alpha f_u + {}^C D_t^\alpha (u f_u) + \nu, \end{aligned} $$

where for the practical use, in view of Proposition 2.4, we obtain

$$ = {}^C D_t^\alpha f - u \, {}^C D_t^\alpha f_u + D_t^\alpha (u f_u) - \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} D_t^k (u f_u)|_{t=0} + \nu. $$

By employing the Leibniz's formula for the Riemann-Liouville fractional derivative, we have

$$ \begin{aligned} &= {}^C D_t^\alpha f - u \, {}^C D_t^\alpha f_u + f_u D_t^\alpha u + \sum_{k=1}^{\infty} \binom{\alpha}{k} D_t^k f_u D_t^{\alpha-k} u \\ &\quad - \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} D_t^k (u f_u)|_{t=0} + \nu. \end{aligned} $$

Uncorrected Proof

Recalling Proposition 2.4 yields

$$= {}^C D_t^\alpha f - u\, {}^C D_t^\alpha f_u + f_u\, {}^C D_t^\alpha u + f_u \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} D_t^k u|_{t=0}$$
$$+ \sum_{k=1}^{\infty} \binom{\alpha}{k} D_t^k f_u D_t^{\alpha-k} u - \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} D_t^k (u f_u)|_{t=0} + \nu,$$

which is our desired result. □

We now consider the one-parameter Lie group of transformations on an open subset $M \subset \mathbb{R}^2 \times \mathbb{R}$

$$\bar{x} = x + \varepsilon \xi(x,t,u) + O(\varepsilon^2), \qquad \bar{t} = t + \varepsilon \tau(x,t,u) + O(\varepsilon^2),$$
$$\bar{u} = u + \varepsilon \varphi(x,t,u) + O(\varepsilon^2), \tag{2.1}$$

with the infinitesimal generator

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u}, \tag{2.2}$$

where ε is the group parameter. The one-parameter Lie group of transformations is prolonged to the Caputo fractional derivative ${}^C D_t^\alpha u(x,t)$ for $0 < \alpha < 1$ with respect to t as follows[12]:

$${}^C D_{\bar{t}}^\alpha \bar{u}(\bar{x},\bar{t}) = {}^C D_t^\alpha u(x,t) + \varepsilon \varphi_C^{(\alpha,t)} + O(\varepsilon^2), \tag{2.3}$$

where

$$\varphi_C^{(\alpha,t)} = {}^C \mathcal{D}_t^\alpha \varphi - {}^C \mathcal{D}_t^\alpha (\xi u_x) + \xi\, {}^C D_t^\alpha u_x - \mathcal{D}_t^\alpha (\tau u_t) + \tau D_t^\alpha u_t.$$

It is worth noticing that in Definition 2.1, the lower limit of the integral is fixed, then it should be invariant concerning the group of transformation (2.1); i.e.

$$\tau(x,0,u(x,0)) = 0, \qquad x \in \mathbb{R}. \tag{2.4}$$

3. Lie symmetry analysis to the fractional Black-Scholes equation of Caputo-type derivative

In this section, we present a complete classification of Lie symmetries for the time-fractional B-S equation described in [7]. Let $C(S,\tau)$ denote the price of an option with S the underlying asset and τ the current time. Then by the (B-S) model, the option C should verify the following equation

$$\frac{\partial^\alpha C(S,\tau)}{\partial \tau^\alpha} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,\tau)}{\partial S^2} + (r-\delta) S \frac{\partial C(S,\tau)}{\partial S} - rC(S,\tau) = 0,$$
$$0 < \alpha \le 1, \qquad 0 \le \tau < T, \qquad x \in \mathbb{R}^+, \tag{3.1}$$

where T is the maturity, r is the risk-free rate, σ is the volatility, and δ is the dividend yield. The fractional derivative in this equation is a modified right Riemann–Liouville derivative defined as

$$\frac{\partial^\alpha C(S,\tau)}{\partial \tau^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_\tau^T (z-\tau)^{-\alpha} \left[C(S,z) - C(S,T)\right] dz, \quad 0 < \alpha < 1.$$

To help the solution process, using a change of the variables $t = T - \tau$, $x = \ln S$ and displaying $u(x,t) = C(e^x, T-t)$, Eq. (3.1) is reduced to the following equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{2}\sigma^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \left(r - \delta - \frac{1}{2}\sigma^2\right) \frac{\partial u(x,t)}{\partial x} - ru(x,t), \tag{3.2}$$

where $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the modified Riemann–Liouville derivative defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-z)^{-\alpha} \left[u(x,z) - u(x,0)\right] dz, \quad 0 < \alpha < 1.$$

Uncorrected Proof

It can be shown that for $0 < \alpha < 1$ the fractional derivative appearing in (3.2) is the Caputo-type fractional derivative, see [7], thus we deduce

$$B-S: {}^{C}D_t^{\alpha}u(x,t)-\frac{1}{2}\sigma^2\frac{\partial^2 u(x,t)}{\partial x^2}-\left(r-\delta-\frac{1}{2}\sigma^2\right)\frac{\partial u(x,t)}{\partial x}+ru(x,t)=0. \tag{3.3}$$

The one-parameter group G of transformations (2.1) with infinitesimal generator (2.2) is called a group admitted by (3.3) if and only if

$${}^{C}Pr^{(\alpha,t)}V(B-S)|_{B-S=0}=0,$$

where

$${}^{C}Pr^{(\alpha,t)}V = V+\varphi^{x}\frac{\partial}{\partial u_x}+\varphi^{xx}\frac{\partial}{\partial u_{xx}}+\varphi_C^{(\alpha,t)}\frac{\partial}{\partial\left({}^{C}D_t^{\alpha}u\right)},$$

with $\varphi_C^{(\alpha,t)}$ as in (2.3) and

$$\varphi^{x}=\mathcal{D}_x\varphi-\mathcal{D}_x(\xi)u_x-\mathcal{D}_x(\tau)u_t,$$
$$\varphi^{xx}=\mathcal{D}_x(\varphi^{x})-\mathcal{D}_x(\xi)u_{xx}-\mathcal{D}_x(\tau)u_{xt}.$$

Applying ${}^{C}Pr^{(\alpha,t)}$ to (3.3), we find infinitesimal criterion

$$\varphi_C^{(\alpha,t)}-\frac{1}{2}\sigma^2\varphi^{xx}-\left(r-\delta-\frac{1}{2}\sigma^2\right)\varphi^{x}+r\varphi\big|_{B-S=0}=0. \tag{3.4}$$

Substituting $\varphi_C^{(\alpha,t)},\varphi^{xx},\varphi^{x}$ into Eq. (3.4), then replacing ${}^{C}D_t^{\alpha}u$ by $\frac{1}{2}\sigma^2u_{xx}+(r-\delta-\frac{1}{2}\sigma^2)u_x-ru$, simplifying and collecting coefficients of the various monomials in the partial derivatives of u, we find the determining equations for the symmetry group of the fractional B-S equation as follows:

$$\begin{aligned}
&{}^{C}D_t^{\alpha}\varphi-ru\varphi_u-u\,{}^{C}D_t^{\alpha}\varphi_u+r\alpha u\tau_t+\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\big[\varphi_u(x,t,u)\\
&\qquad-\varphi_u(x,0,u)\big]u(x,0)-\frac{1}{2}\sigma^2\varphi_{xx}-\left(r-\delta-\frac{1}{2}\sigma^2\right)\varphi_x+r\varphi=0,\\
&\left(r-\delta-\frac{1}{2}\sigma^2\right)(\alpha\tau_t-\xi_x)+\frac{1}{2}\sigma^2(2\varphi_{xu}-\xi_{xx})=0,\quad \frac{1}{2}\sigma^2\xi_{uu}=0,\\
&\frac{1}{2}\sigma^2(\varphi_{uu}-2\xi_{xu})-\left(r-\delta-\frac{1}{2}\sigma^2\right)\xi_u=0,\quad \frac{1}{2}\sigma^2\alpha\tau_t-\sigma^2\xi_x=0,\\
&\frac{1}{2}(\alpha-1)\sigma^2\tau_u=0,\quad (\alpha-1)\left(r-\delta-\frac{1}{2}\sigma^2\right)\tau_u-\sigma^2\tau_{xu}=0,\\
&\frac{1}{2}\sigma^2\tau_{xx}+\left(r-\delta-\frac{1}{2}\sigma^2\right)\tau_x+\alpha ru\tau_u=0,\quad \frac{1}{2}\sigma^2\tau_{uu}=0,\quad \sigma^2\tau_x=0,\\
&\frac{3}{2}\sigma^2\xi_u=0,\qquad \sigma^2\tau_u=0,\quad \xi\,{}^{C}\mathcal{D}_t^{\alpha}u_x-{}^{C}\mathcal{D}_t^{\alpha}(\xi u_x)=0,\\
&\binom{\alpha}{k}D_t^{k}\varphi_u-\binom{\alpha}{k+1}\mathcal{D}_t^{k+1}\tau=0,\quad k\in\mathbb{N},
\end{aligned}$$

an over-determined system of equations for the unknowns ξ, τ,and φ. By solving the system above, infinitesimal generators for the fractional B-S equation (3.3) are stated as follows:

- if σ, r and δ are arbitrary constants,

$$V_1=\frac{\partial}{\partial x},\qquad V_2=u\frac{\partial}{\partial u},\qquad V_3=K(x,t)\frac{\partial}{\partial u},$$

where K is an arbitrary function and satisfies the fractional Black-Scholes equation (3.3).

Uncorrected Proof

- if $2\sigma^2 r + (r-\delta-\frac{1}{2}\sigma^2)^2 = 0$

$$V_4 = 4t\frac{\partial}{\partial t} + 2\alpha x\frac{\partial}{\partial x} + \frac{\sigma^2 - 2(r-\delta)}{\sigma^2}\alpha x u\frac{\partial}{\partial u}.$$

At this stage, we suppose $C(S,\tau)$ be the price of a double barrier option such that the price change of the option in the financial market is considered as a fractal transmission system. Then $C(S,\tau)$ should satisfy the following system

$$\begin{aligned}
&\frac{\partial^\alpha C(S,\tau)}{\partial \tau^\alpha} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,\tau)}{\partial S^2} + (r-\delta)S\frac{\partial C(S,\tau)}{\partial S} - rC(S,\tau) = 0,\\
&C(S_1,\tau) = p(\tau),\\
&C(S_2,\tau) = q(\tau),\\
&C(S,T) = A(S),
\end{aligned}$$

where $0<\alpha<1,\ 0\le\tau<T$ and $S_1<S<S_2$. Here $p(\tau), q(\tau)$ are the rebates paid when the corresponding barrier, $S=S_1, S=S_2$, is hit, and $A(S)$ is the payoff function. In the following examples, we intend to obtain the exact solution of the similar problems using the infinitesimal generators of the Eq. (3.3).

Example 3.1. We consider the pricing of double barrier option above with the given $\sigma,\delta,r,\ S_1=1, S_2=\infty,\ p(\tau)=0$, $q(\tau)=\infty$, and $A(S)=S^k-1$ with $k=-\frac{2(r-\delta)}{\sigma^2}+1$. The change of variables yields that $u(x,t)=C(e^x,T-t)$ verifies the following initial and boundary value problem:

$$\begin{aligned}
&{}^C D_t^\alpha u(x,t) - \frac{1}{2}\sigma^2\frac{\partial^2 u(x,t)}{\partial x^2} - \left(r-\delta-\frac{1}{2}\sigma^2\right)\frac{\partial u(x,t)}{\partial x} + ru(x,t) = 0,\ x>0, \qquad (3.5)\\
&u(x,0) = e^{kx}-1,\\
&u(0,t) = 0, \qquad \lim_{x\to\infty} u(x,t) = \infty.
\end{aligned}$$

Because of the initial value, we use the one-parameter group generated by $V=V_1+kV_2$ with the invariant solution $u_1(x,t)=e^{kx}f(t)$, and also the group generated by V_1 with the invariant solution $u_2(x,t)=f(t)$. Setting

$$u(x,t) = u_1(x,t)+u_2(x,t) = \left(e^{kx}-1\right)f(t),$$

and substituting this solution into Eq. (3.5), we get the reduced fractional ordinary differential equation

$${}^C D_t^\alpha f(t) + rf(t) = 0.$$

The solution of this equation can be written in terms of the Mittag–Leffler functions

$$f(t) = E_\alpha(-rt^\alpha),$$

thus

$$u(x,t) = \left(e^{kx}-1\right)E_\alpha(-rt^\alpha),$$

is the invariant solution of the problem (3.5). For example, for different values of σ, r, δ, we obtain the exact solution of the problem (3.5);

- if $\sigma=0.4, r=0.03$ and $\delta=0.07$

$$u(x,t) = \left(e^{1.5x}-1\right)E_\alpha(-0.03t^\alpha),$$

- if $\sigma=0.1, r=0.06$ and $\delta=0$

$$u(x,t) = \left(e^{-11x}-1\right)E_\alpha(-0.06t^\alpha),$$

- if $\sigma=0.25, r=0.05$ and $\delta=0$

$$u(x,t) = \left(e^{-0.6x}-1\right)E_\alpha(-0.05t^\alpha).$$

Notice that for $\alpha=1$, $u(x,t)=\left(e^{kx}-1\right)e^{-rt}$ is the exact solution of corresponding B-S equation with the integer-order.

Uncorrected Proof

Example 3.2. We consider the pricing of double barrier option

$$
\begin{aligned}
&{}^{C}D_t^{\alpha}u(x,t)-\frac{1}{2}\sigma^2\frac{\partial^2 u(x,t)}{\partial x^2}-\left(r-\delta-\frac{1}{2}\sigma^2\right)\frac{\partial u(x,t)}{\partial x}+ru(x,t)=0, x\in\mathbb{R}, \qquad (3.6)\\
&u(x,0)=e^{kx},\\
&\lim_{x\to-\infty}u(x,t)=0,\quad \lim_{x\to+\infty}u(x,t)=\infty,
\end{aligned}
$$

where $k>0$. Because of the initial value, we use the one-parameter group generated by $V=V_1+kV_2$ with the invariant solution $u(x,t)=e^{kx}f(t)$, and substitute this solution into (3.6), to get the reduced fractional ordinary differential equation

$$
{}^{C}D_t^{\alpha}f(t)-\beta f(t)=0,
$$

where $\beta=\frac{1}{2}\sigma^2k^2+\left(r-\delta-\frac{1}{2}\sigma^2\right)k-r$. The solution of this equation can be written in terms of Mittag–Leffler functions

$$
f(t)=E_{\alpha}(\beta t^{\alpha}),
$$

thus

$$
u(x,t)=e^{kx}E_{\alpha}(\beta t^{\alpha}),
$$

is the invariant solution for (3.6). We note that for $\alpha=1$, $u(x,t)=e^{kx+\beta t}$ is the exact solution of the corresponding B-S equation with integer-order. The change of variables leads to the option price $C(S,\tau)=S^kE_{\alpha}\left(\beta(T-\tau)^{\alpha}\right)$. Here an invariant solution is a solution of the differential equation which is also an invariant surface of a group admitted by the differential equation; that means it is mapped to itself by the group of transformation.

4. Exact and solitary wave solutions of the time–fractional Black–Scholes equation

4.1. **Exact and general solutions extracted from invariant surfaces.** In this subsection, using some of the invariance obtained from the Lie symmetries, we represent more exact and general solutions of the fractional B-S equation. This idea has been employed to find the new exact solutions of some FDEs in [25]. If there exists a class of solutions $g(x,t,u)=c$ to (3.3) admitting the group generated by V_3 with $K(x,t)=\theta(x)$, then $g(\bar{x},\bar{t},\bar{u})=\bar{c}$ defines a solution too and by Theorem 2.2.7-3 in [4], we have $V_3g(x,t,u)=1$. Thus we can write $\theta(x)g_u=1$, so by solving this equation and according to the definition of g, we obtain

$$
u(x,t)=c\theta(x)-\phi(x,t),
$$

where $\phi(x,t)$ is an arbitrary function. For example if $\phi(x,t)=h(x)f(t)$, then

$$
u(x,t)=c\theta(x)-h(x)f(t),
$$

substituting this solution into (3.3) and setting

$$
\frac{1}{2}\sigma^2h''+\left(r-\delta-\frac{1}{2}\sigma^2\right)h'-(r+\gamma)h=0, \qquad (4.1)
$$

the reduced fractional ordinary differential equation becomes

$$
{}^{C}D_t^{\alpha}f(t)-\gamma f(t)=0.
$$

Then the invariant solution of (3.3) is

$$
u(x,t)=c\theta(x)-h(x)E_{\alpha}(\gamma t^{\alpha}),
$$

where $\theta(x)$ satisfies the following equation

$$
\frac{1}{2}\sigma^2\theta''+\left(r-\delta-\frac{1}{2}\sigma^2\right)\theta'-r\theta=0. \qquad (4.2)
$$

Uncorrected Proof

After solving second order equations with constant coefficients (4.1) and (4.2), we obtain the new exact solution of the fractional B-S equation.

4.2. **Solitary wave solutions.** In this subsection, applying the fractional complex transformation, we obtain the solitary wave solutions of the fractional B-S equation. The fractional wave transformation, which reduces the fractional partial differential equations into ordinary differential equations (ODEs), is in the form

$$u(x,t) = u(\eta), \qquad \eta = kx + \frac{wt^{\alpha}}{\Gamma(\alpha+1)}, \tag{4.3}$$

where k and w are nonzero arbitrary constants. By applying the transformation (4.3), Eq. (3.3) is transformed into the following ordinary differential equation

$$\frac{1}{2}\sigma^2 k^2 u'' + \left(rk - \delta k - \frac{1}{2}\sigma^2 k - w\right) u' - ru = 0, \tag{4.4}$$

where the prime denotes the derivation with respect to η. By assuming $\Delta = \left(rk - \delta k - \frac{1}{2}\sigma^2 k - w\right)^2 + 2\sigma^2 k^2 r$, we obtain the solutions of Eq. (4.4) in the following cases:

- If $\Delta > 0$, then by assuming $m_i = \frac{-\left(rk-\delta k-\frac{1}{2}\sigma^2 k-w\right)\pm\sqrt{\Delta}}{\sigma^2 k^2}$, $i = 1, 2$, we have
  $u(\eta) = c_1 e^{m_1\eta} + c_2 e^{m_2\eta}$.
- If $\Delta = 0$, then by assuming $m = \frac{-\left(rk-\delta k-\frac{1}{2}\sigma^2 k-w\right)}{\sigma^2 k^2}$,, we have
  $u(\eta) = c_1 e^{m\eta} + c_2 \eta e^{m\eta}$.
- If $\Delta < 0$, then by assuming $p \pm qi = \frac{-\left(rk-\delta k-\frac{1}{2}\sigma^2 k-w\right)\pm\sqrt{\Delta}}{\sigma^2 k^2}$, we have
  $u(\eta) = e^{p\eta}\left(c_1 \cos q\eta + c_2 \sin q\eta\right).$

where c_1 and c_2 are arbitrary smooth functions. Inserting $\eta = kx + \frac{wt^{\alpha}}{\Gamma(\alpha+1)}$, we obtain the solitary wave solutions of Eq. (3.3) as follows

- If $\Delta > 0$
  $u(x,t) = c_1 e^{m_1\left(kx+\frac{wt^{\alpha}}{\Gamma(\alpha+1)}\right)} + c_2 e^{m_2\left(kx+\frac{wt^{\alpha}}{\Gamma(\alpha+1)}\right)}.$
- If $\Delta = 0$
  $$u(x,t) = c_1 e^{m\left(kx+\frac{wt^{\alpha}}{\Gamma(\alpha+1)}\right)} + c_2\left(kx + \frac{wt^{\alpha}}{\Gamma(\alpha+1)}\right) e^{m\left(kx+\frac{wt^{\alpha}}{\Gamma(\alpha+1)}\right)}.$$
- If $\Delta < 0$
  $$u(x,t) = e^{p\left(kx+\frac{wt^{\alpha}}{\Gamma(\alpha+1)}\right)}\left[c_1 \cos q\left(kx+\frac{wt^{\alpha}}{\Gamma(\alpha+1)}\right) + c_2 \sin q\left(kx+\frac{wt^{\alpha}}{\Gamma(\alpha+1)}\right)\right].$$

5. Conservation laws for the fractional B-S equation of the Caputo-type

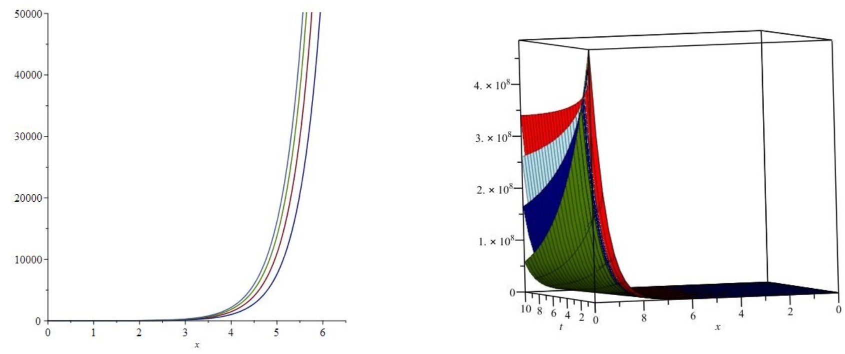
In the current section, we construct the conservation laws of Eq. (3.3). A vector $F = (F^x, F^t)$ is called a conserved vector for Eq. (3.3), if it fulfils the conservation equation

$$\mathcal{D}_x(F^x) + \mathcal{D}_t(F^t)\big|_{B-S=0} = 0, \tag{5.1}$$

where $F^t = F^t(t, x, u, \ldots)$ and $F^x = F^x(t, x, u, \ldots)$. Equation (5.1) is called a conservation law for Eq.(3.3). The new conservation theorem proposed by Ibragimov [16] provides a method to construct conservation laws for differential equations. Based on this theorem, the Lagrangian formal for Eq. (3.3) can be introduced as

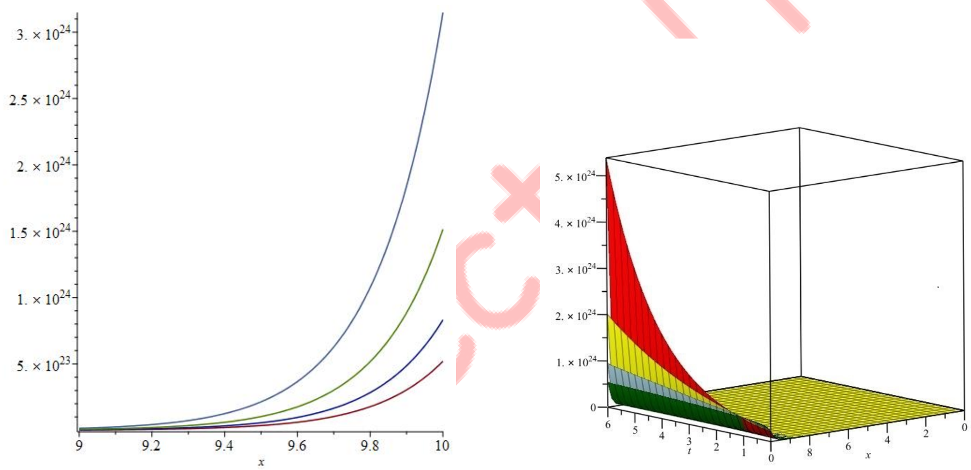
$$\mathcal{L} = v(x,t)\left[{}^{C}D_t^{\alpha} u - \frac{1}{2}\sigma^2 u_{xx} - \left(r - \delta - \frac{1}{2}\sigma^2\right) u_x + ru\right],$$

Uncorrected Proof



(a) For the time $t = 5$. (b) For $0 < x < 10$, $0 < t < 10$.

FIGURE 1. The plot of $u(x,t)$ obtained by Example 3.2 for different values of α with parameters $k = 2$, $\sigma = 0.15$, $r = 0.025$, $\delta = 0.13$



(a) For the time $t = 5$. (b) for $0 < x < 10$, $0 < t < 6$.

FIGURE 2. The plot of solitary wave solutions for different values of α with parameters $k = 10, w = 1$, $\sigma = 0.2$, $r = 0.04$, $\delta = 0.02$.

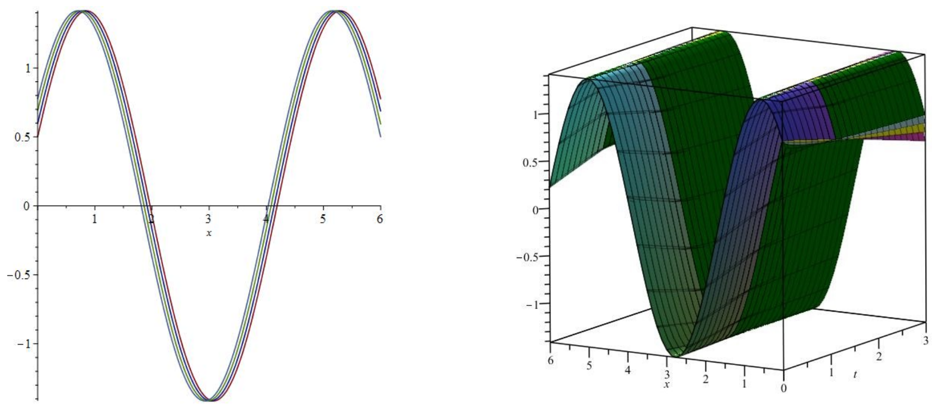
where $v(x,t)$ is a new dependent variable. The Euler-Lagrange operator is presented as follows:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (\mathcal{D}_t^\alpha)^* \frac{\partial}{\partial\,({}^C D_t^\alpha u)} - \mathcal{D}_x \frac{\partial}{\partial u_x} + \sum_{k=2}^{\infty} (-1)^k \mathcal{D}_{i_1} \mathcal{D}_{i_2} \ldots \mathcal{D}_{i_k} \frac{\partial}{\partial u_{i_1, i_2, \ldots, i_k}},$$

where $(\mathcal{D}_t^\alpha)^*$ is the adjoint operator of ${}^C D_t^\alpha u$ and is defined by

$$(\mathcal{D}_t^\alpha)^* f(t) = {}_t\mathcal{D}_T^\alpha f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} D_t^m \int_t^T (s-t)^{m-\alpha-1} f(s)ds, \quad m-1 < \alpha < m,$$

Uncorrected Proof



(a) For the time $t = 5$. (b) For $0 < x < 6$, $0 < t < 3$.

FIGURE 3. The plot of solitary wave solutions for different values of α with parameters $k = 5, w = -0.5$, $\sigma = 0.2$, $r = -0.04$, $\delta = 0.04$.

(see [22] for more details). The adjoint equation to Eq. (3.3) is given by

$$\frac{\delta \mathcal{L}}{\delta u} = 0, \tag{5.2}$$

or, in the equivalent form

$$(\mathcal{D}_t^\alpha)^* v - \frac{1}{2}\sigma^2 v_{xx} + \left(r - \delta - \frac{1}{2}\sigma^2\right) v_x + rv = 0. \tag{5.3}$$

We also have fundamental identity [11, 16, 22]

$$^{C}Pr^{(\alpha,t)}V + \mathcal{D}_t(\tau)I + \mathcal{D}_x(\xi)I = W\frac{\delta}{\delta u} + \mathcal{D}_t(N^t) + \mathcal{D}_x(N^x), \tag{5.4}$$

where I is the identity operator, N^t and N^x represent Noether's operators and $W = \varphi - \tau u_t - \xi u_x$. For the time–fractional Caputo-type derivative, the operator N^t is written in the form [22]

$$N^t = \tau I + \sum_{k=0}^{m-1} \mathcal{D}_t^k(W)\, {}_t\mathcal{D}_T^{\alpha-1-k} \frac{\partial}{\partial\,({}^C D_t^\alpha u)} - J\left(\mathcal{D}_t^m(W), \frac{\partial}{\partial\,({}^C D_t^\alpha u)}\right), \tag{5.5}$$

where the integral operator J is defined by

$$J(f, g) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_t^T (s-\nu)^{m-\alpha-1} f(\nu) g(s)\, ds\, d\nu.$$

For Eq. (3.3), the operator N^x can be constructed by the following formula

$$N^x = \xi I + W\left(\frac{\partial}{\partial u_x} - \mathcal{D}_x \frac{\partial}{\partial u_{xx}}\right) + \mathcal{D}_x(W)\frac{\partial}{\partial u_{xx}}. \tag{5.6}$$

For any infinitesimal generator V admitted by Eq. (3.3) and its solution, we conclude

$$^{C}Pr^{(\alpha,t)}V\mathcal{L} + \mathcal{D}_t(\tau)\mathcal{L} + \mathcal{D}_x(\xi)\mathcal{L}\big|_{B-S=0} = 0.$$

Therefore, in view of (5.2), Eq. (5.4) leads to the conservation law

$$\mathcal{D}_t(N^t\mathcal{L}) + \mathcal{D}_x(N^x\mathcal{L})\big|_{B-S=0} = 0.$$

Uncorrected Proof

Now, we present the components of conserved vectors for Eq. (3.3). Using Eqs. (5.5) and (5.6), we have

$$\begin{aligned} F^t = N^t\mathcal{L} &= \tau\mathcal{L} + W_i\ {}_t\mathcal{D}_T^{\alpha-1}\frac{\partial \mathcal{L}}{\partial\left({}^C D_t^{\alpha} u\right)} - J\left(\mathcal{D}_t(W_i), \frac{\partial \mathcal{L}}{\partial\left({}^C D_t^{\alpha} u\right)}\right) \\ &= W_i\ {}_t\mathcal{D}_T^{\alpha-1} v - J\left(\mathcal{D}_t(W_i), v\right), \end{aligned}$$

and

$$\begin{aligned} F^x = N^x\mathcal{L} =& \xi\mathcal{L} + W_i\left(\frac{\partial \mathcal{L}}{\partial u_x} - \mathcal{D}_x\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) + \mathcal{D}_x(W_i)\frac{\partial \mathcal{L}}{\partial u_{xx}} \\ =& W_i\left[\frac{1}{2}\sigma^2 v_x - \left(r - \delta - \frac{1}{2}\sigma^2\right)v\right] - \frac{1}{2}\sigma^2\mathcal{D}_x(W_i)v, \end{aligned}$$

where $v(x,t)$ is an arbitrary nontrivial solution of Eq. (5.3) and for different infinitesimal generator V_i, $i = 1, \dots, 11$, of Eq. (3.3), W_i are defined as follows:

$$\begin{aligned} &W_1 = -u_x, \qquad W_2 = u, \qquad W_3 = K(x,t), \\ &W_4 = \frac{\sigma^2 - 2(r-\delta)}{\sigma^2}\alpha x u - 4tu_t - 2\alpha x u_x. \end{aligned}$$

6. Conclusion

The fractional differential equations of the Caputo type arise in a number of applications such as regularized long wave (RLW) equations, which describe the nature of ion-acoustic waves in plasma and shallow water waves in oceans; and the Caputo fractional Korteweg–de Vries equations that have applications in wide areas including crystal lattice with acoustic waves, waves in a density-stratified ocean having long internal shallow-water waves with weakly nonlinear restoring forces [13, 29, 38]. The fractional operators are non-local, therefore technical difficulties are found in applying the Lie symmetry analysis to obtain the exact solutions of these equations. The authors in [12] obtained the exact prolongation formula for fractional operators of the Caputo type for $0 < \alpha < 1$. We applied the Lie symmetry analysis to the Caputo-type B-S equation arising from the double barrier option pricing. We constructed the symmetries of this equation, and showed that the symmetry analysis of this equation gives rise to some interesting solutions in various forms. From a theoretical point of view, we have given a class of invariance to (3.3), which can be used in the literature. In particular, we concentrated on a fractional B–S model governing a double barrier option with arbitrary parameters, and we arrived at the option pricing $C(S,\tau)$ at the underlying asset S and the time τ, in example 3.2. The explicit expression of the C were given, and graphically depicted for various values of the fractional parameters $\alpha = 0.25, 0.5, 0.75, 1$ in Fig. 1(b). In Fig. 1(a), we represented the values of the options graphically, when we fixed the time $T - \tau = 5$. We notice that the order α has an effect in the option pricing. We can observe that the decay in the cost of the option are seen, when the order of fractional derivative increases. At last, we derived an adjoint equation of the Caputo-type fractional B-S equation, and utilizing it many conservation laws of the equation were exhibited. Using the methods based on the Lie symmetry analysis to fractional differential equations, we arrive at a wide class of exact solutions of a given equation. In addition to this advantage of these methods, there are some disadvantages as well. For the given B-S equation, we directly, using a particular reduction, derived the solitary solutions, Fig. 2 and 3. However we could not obtain the solitary solutions by employing the Lie symmetry analysis. In fact according to (4.3) the corresponding infinitesimal generator, which leads to these solutions is in the form $V = wt^{\alpha-1}\frac{\partial}{\partial x} - k\Gamma(\alpha)\frac{\partial}{\partial t}$, where w, k are constants. Recalling the infinitesimal generator (2.2), that means the $\xi = wt^{\alpha-1}$ and is dependent on the variable t. Regarding the complicated calculations in the given procedure to obtain the invariance of a given equation, we imposed the constraint $\xi = \xi(x)$. Due to this constraint we did not obtain the solitary solutions as invariant solution of the B-S equation.

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