Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 13, No. 2, 2025, pp. 373-383
DOI:10.22034/cmde.2024.56240.2350

Generalized Solutions for Conformable Schrödinger Equation with Singular Potentials

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Abstract

This paper employs Colombeau algebra as a mathematical framework to establish both the existence and uniqueness of solutions for the fractional Schrödinger equation when subjected to singular potentials. A noteworthy contribution lies in the introduction of the concept of a generalized conformable semigroup, marking the first instance of its application. This innovative approach plays a pivotal role in demonstrating the sought-after results within the context of the fractional Schrödinger equation. The utilization of Colombeau algebra, coupled with the introduction of the generalized conformable semigroup, represents a novel and effective strategy for addressing challenges posed by singular potentials in the study of this particular type of Schrödinger equation.

Keywords. Conformable schrödinger equation, Conformable derivative, Generalized solution, Conformable semigroup, Singular potential.
2010 Mathematics Subject Classification. 35D99, 35K67, 34A45, 46F05.

1. Introduction

Colombeau algebra, named after French mathematician Jean François Colombeau, is a mathematical framework that extends the traditional theory of distributions, particularly the theory of Schwartz distributions. It provides a systematic way to manipulate generalized functions, allowing for operations that are not well-defined in the classical sense. Colombeau algebra introduces a notion of generalized numbers, or "infinitesimals," and employs a careful algebraic structure to handle divergent or singular expressions that arise in mathematical analysis. This approach has applications in various branches of mathematics and theoretical physics, offering a powerful tool for addressing singularities and irregularities in mathematical models. Colombeau algebra has found applications in areas such as partial differential equations, nonstandard analysis, and the study of nonlinear phenomena where classical methods may encounter difficulties.

Due to all these properties, Colombeau's theory has found extensive applications in different natural sciences and engineering, especially in fields where products of distributions with coinciding singularities are considered.

The conformable derivative is a mathematical concept that extends the classical notion of derivative to functions defined on noninteger-dimensional spaces. Unlike traditional derivatives, which are defined for functions on real numbers, the conformable derivative allows for differentiation on noninteger or fractional dimensions. This concept has gained significance in the field of fractional calculus, providing a more flexible and general framework for describing the behavior of systems with noninteger order dynamics. The conformable derivative is defined by considering a conformable fractional operator that adapts to the specific fractional dimension involved. This approach has found applications in various scientific disciplines, including physics, engineering, and biology, where systems exhibit complex and anomalous behaviors that can be effectively modeled using noninteger order derivatives.

Received: 18 April 2023 ; Accepted: 25 June 2024.
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In this paper we we focus on solving the conformable schrödinger equation with singular initial data as we can see in the following

$$\begin{cases} D_t^{(\alpha)} x(t,y) + \Delta x(t,y) - v(y)x(t,y) = 0, \qquad y \in \mathbb{R}, \quad t \geq 0 \\ x(0,y) = x_0(y) = \delta(y), \\ v(y) = \delta(y), \end{cases} \tag{1.1}$$

where $D^{(\alpha)}$ is the conformable derivation with $0 < \alpha \leq 1$ and δ is the Dirac function.

The pioneering work on (1.1) was done by A. Benmerrous in [5] and our development follows his approach. Our results extend those of A. Benmerrous [6, 7] in several respects.

The following is how the paper is structured, after this introduction, we will discuss various notions related to Colombeau's algebra. In section 3, we will give and demonstrate the existence of conformable derivative in Colombeau algebra. Section 4, introduces the concept of generalized fractional semigroup. The existence and uniqueness of the solution are discussed in section 5.

2. Preliminaries

2.1. **Colombeau algebra.** Here we list some notations and formulas to be used later. The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ε. Therefore, for any set X, the family of sequences $(u_\varepsilon)_{\varepsilon\in[0;1]}$ of elements of a set X will be denoted by $X^{[0;1]}$, such sequences will also be called nets and simply written as u_ε.
Let $\mathcal{D}(\mathbb{R}^n)$ be the space of all smooth functions $\varphi : \quad \mathbb{R}^n \longrightarrow \mathbb{C}$ with compact support.
For $q \in \mathbb{N}$ we denote

$$\mathcal{A}_q(\mathbb{R}^n) = \left\{ \varphi \in \mathcal{D}\left(\mathbb{R}^n\right) / \int \varphi(x)dx = 1 and \int x^\alpha \varphi(x)dx = 0 \text{ for } 1 \leq \alpha \leq q \right\}.$$

The elements of the set $\mathcal{A}_q$ are called test functions.

It is obvious that $\mathcal{A}_1 \supset \mathcal{A}_2 \ldots$. Colombeau in his books has proved that the sets $\mathcal{A}_k$ are non empty for all $k \in \mathbb{N}$.
For $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ and $\epsilon > 0$ it is denoted as $\varphi_\epsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ for $\varphi \in \mathcal{D}\left(\mathbb{R}^n\right)$ and $\check{\varphi}(x) = \varphi(-x)$.
We denote by

$$\begin{aligned} \mathcal{E}\left(\mathbb{R}^n\right) &= \left\{u : \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C} / \text{ with } u(\varphi, x) \text{ is } \mathcal{C}^\infty \text{ to the second variable } x\right\}, \\ u\left(\varphi_\varepsilon, x\right) &= u_\varepsilon(x) \quad \forall \varphi \in \mathcal{A}_1, \\ \mathcal{E}_M\left(\mathbb{R}^n\right) &= \left\{\left(u_\varepsilon\right)_{\varepsilon>0} \subset \mathcal{E}\left(\mathbb{R}^n\right) / \forall K \subset \mathbb{R}^n, \forall a \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that} \right. \\ \sup_{x\in K} \left\| D^\alpha u_\varepsilon(x) \right\| &= \left. \mathcal{O}\left(\varepsilon^{-N}\right) \text{ as } \varepsilon \to 0 \right\}, \\ \mathcal{N}\left(\mathbb{R}^n\right) &= \left\{\left(u_\varepsilon\right)_{\varepsilon>0} \in \mathcal{E}\left(\mathbb{R}^n\right) / \forall K \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}, \forall p \in \mathbb{N} \text{ such that },\right. \\ \sup_{x\in K} \left\| D^\alpha u_\varepsilon(x) \right\| &= \left. \mathcal{O}\left(\varepsilon^{p}\right) \text{ as } \varepsilon \to 0 \right\}. \end{aligned}$$

The generalized functions of Colombeau are elements of the quotient algebra $\mathcal{G}\left(\mathbb{R}^n\right) = \mathcal{E}_M\left[\mathbb{R}^n\right] / \mathcal{N}\left[\mathbb{R}^n\right]$, where the elements of the set $\mathcal{E}_M\left(\mathbb{R}^n\right)$ are moderate while the elements of the set $\mathcal{N}\left(\mathbb{R}^n\right)$ are negligible.

The meaning of the term 'association' in $\mathcal{G}(\mathbb{R})$ is given with the next two definitions.

Definition 2.1. Generalized functions $f, g \in \mathcal{G}(\mathbb{R})$ are said to be associated, denoted $f \approx g$, if for each representative $f(\varphi_\varepsilon, x)$ and $g(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \| f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x) \| \psi(x) dx = 0.$$

Definition 2.2. Generalized functions $f \in \mathcal{G}(\mathbb{R})$ is said to admit some as $u \in \mathcal{D}'(\mathbb{R})$ 'associated distribution', denoted $f \approx u$, if for each representative $f(\varphi_\varepsilon, x)$ of f and any $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} f(\varphi_\varepsilon, x)\psi(x)dx = \langle u, \psi\rangle.$$

2.2. **Conformable derivative.** The definition of conformable derivation is provided in the following part.

Definition 2.3. [12] Let $n < \alpha \le n+1$ and $u : \mathbb{R}^+ \to \mathbb{R}$ be n-differeniable, then the conformable fracrional derivauive of u of order α characterized by

$$D^{(\alpha)}u(r) = \lim_{\epsilon\to 0}\frac{u^{(n)}\left(r+\epsilon r^{n+1-\alpha}\right) - u^{(n)}(r)}{\epsilon},$$
$$D^{(\alpha)}u(0) = \lim_{r\to 0} D^{(\alpha)}u(r).$$

Remark 2.4. [12] In light of the definition above, it is simple to demonstrate that

$$D^{(\alpha)}u(r) = r^{n+1-\alpha}u^{(n+1)}(r),$$

with $n < \alpha \le n+1$, and u is $(n+1)$-differentiable.

Definition 2.5. [12] Let $1 < \alpha \le 2$,

$$\left(I^{(\alpha)}u\right)(r) = \int_0^t s^{\alpha-2}u(s)ds.$$

Theorem 2.6. *[12]*

$$D^{(\alpha)}(I^{(\alpha)}u(r)) = u(r), \quad \textit{for } r \ge 0.$$

3. Generalized conformable derivative

Let $(f_\epsilon(t))_\epsilon$ be a representative of the function $f(t) \in \mathcal{G}(\mathbb{R}^+)$ and let $n-1 < \alpha < n$.

The generalized conformable fractional derivative of $(f_\epsilon(t))_\epsilon$, characterized by

$$D^{(\alpha)}f_\epsilon(y) = y^{1-\alpha}\frac{d}{dy}f_\epsilon(y), \quad \epsilon \in (0,1). \tag{3.1}$$

Lemma 3.1. *Let $(f_\epsilon(y))_\epsilon$ be a representative of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, $\forall\alpha > 0$, $\sup_{y\in[0,T]} \mid D^{(\alpha)}f_\epsilon(y) \mid$ has a moderate bound.*

Proof.

$$\begin{aligned}\sup_{y\in[0,T]}\|D^{(\alpha)}f_\epsilon(y)\| = \sup_{y\in[0,T]}\|y^{1-\alpha}\frac{d}{dy}f_\epsilon(y)\| &\le T^{1-\alpha}\sup_{y\in[0,T]}\|\frac{d}{dy}f_\epsilon(y)\|\\ &\le T^{1-\alpha}C\epsilon^{-N}\\ &\le C_{\alpha,T}\epsilon^{-N}.\end{aligned}$$

Then, $\exists M \in \mathbb{N}$, such as

$$\sup_{y\in[0,T]}\|D^{(\alpha)}f_\epsilon(y)\| = \mathcal{O}\left(\epsilon^{-M}\right), \quad \epsilon \to 0.$$

□

Lemma 3.2. *Let $(f_{1\epsilon}(t))_\epsilon$, $(f_{2\epsilon}(t))_\epsilon$ be two distinct representatives of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, $\forall\alpha > 0$, $\sup_{y\in[0,T]} \mid D^{(\alpha)}f_{1\epsilon}(y) - D^{(\alpha)}f_{2\epsilon}(y) \mid$ is negligible.*

Proof.

$$\begin{aligned}\sup_{y\in[0,T]}\|D^{(\alpha)}f_{1,\epsilon}(y)-D^{(\alpha)}f_{2,\epsilon}(y)\| &= \sup_{y\in[0,T]}\|y^{1-\alpha}\frac{d}{dy}f_{1,\epsilon}(y)-y^{1-\alpha}\frac{d}{dy}f_{2,\epsilon}(y)\| \\ &= \sup_{y\in[0,T]}\|y^{1-\alpha}\left(\frac{d}{dy}f_{1,\epsilon}(y)-\frac{d}{dy}f_{2,\epsilon}(y)\right)\| \\ &\leq T^{1-\alpha}\sup_{y\in[0,T]}\|\frac{d}{dy}f_{1,\epsilon}(y)-\frac{d}{dy}f_{2,\epsilon}(y)\|.\end{aligned}$$

Since $(f_{1\epsilon}(y))_\epsilon$ and $(f_{2\epsilon}(y))_\epsilon$ represent the same Colombeau generalized function $f(y)$, so $\sup_{y\in[0,T]} \mid \frac{d}{dy}f_{1,\epsilon}(y)-\frac{d}{dy}f_{2,\epsilon}(y)\mid$ is negligible, then for all $p\in\mathbb{N}$

$$\sup_{y\in[0,T]}\|D^{(\alpha)}f_{1\epsilon}(y)-D^{(\alpha)}f_{2\epsilon}(y)\mid=\mathcal{O}\left(\epsilon^{-p}\right),\quad \epsilon\to 0.$$

Therefore, $\sup_{y\in[0,T]}\|D^{(\alpha)}f_{1\epsilon}(y)-D^{(\alpha)}f_{2\epsilon}(y)\|$ is negligible. □

We may now initiate the generalized conformable fractional derivative of a Colombeau generalized function on $\mathbb{R}^+$ after establishing the first two lemmas.

Definition 3.3. Let $f(y)\in\mathcal{G}(\mathbb{R}^+)$ be a Colombeau function on $\mathbb{R}^+$.
The generalized conformable fractional derivative of $f(y)$, using the notation $D^{(\alpha)}f(t)=\left[\left(D^{(\alpha)}f_\epsilon(t)\right)_\epsilon\right]$, $\alpha>0$, is a component of $\mathcal{G}(\mathbb{R}^+)$ satisfying (3.1).

Remark 3.4. For $\alpha\in(0,1]$ the first-order derivative of $D^{(\alpha)}f_\epsilon(y)$ is

$$\frac{d}{dy}D^{(\alpha)}f_\epsilon(y)=(1-\alpha)y^{-\alpha}\frac{d}{dy}f_\epsilon(y)+y^{1-\alpha}\frac{d^2}{dy^2}f_\epsilon(y),$$

and fails to reach its limit.
Generally, the p-th order derivative $\frac{d^p}{dy^p}D^{(\alpha)}f_\epsilon(y)$ it fails to reach its limit on $\mathbb{R}^+$.

Then if we want $D^{(\alpha)}$ to be in $\mathcal{G}(\mathbb{R}^+)$, thus the fractional derivative must be regularized.

Definition 3.5. Let $(f_\epsilon)_\epsilon$ be a representative of a Colombesu generalized $f\in\mathcal{G}([0,\infty))$. The regularized of new fractional derivative of $(f_\epsilon)_{\epsilon\infty}$, is characterized by

$$\bar{D}^{(\alpha)}f_\epsilon(y)=\begin{cases}\left(D^{(\alpha)}f_\epsilon*\varphi_\epsilon\right)(y), & n-1<\alpha<n,\\ f_\epsilon^{(n)}(y)=(\frac{d}{dy})^n f_\epsilon(y), & \alpha=n,\end{cases}\tag{3.2}$$

where $n\in\mathbb{N},\epsilon\in(0,1)$, and (3.1) gives $D^{(\alpha)}f_\epsilon(y)$ and the first section gives $\varphi_\epsilon(y)$.

The convolution in (3.1) is $\left(D^{(\alpha)}f_\epsilon(y)*\varphi_\epsilon\right)(y)=\int_0^\infty D^{(\alpha)}f_\epsilon(y)\varphi_\epsilon(y-s)ds$.

Lemma 3.6. *Let $(f_\epsilon(y))_\epsilon$ be a representative of $f(y)\in\mathcal{G}(\mathbb{R}^+)$.*
So, $\forall\alpha>0$, $k\in\{0,1,\ldots\}$, $\sup_{y\in[0,T]}\|\left(d^k/dy^k\right)\tilde{D}^{(\alpha)}f_\epsilon(y)\|$ has a moderate limit.

Proof. Let $0<\epsilon<1$.
For $\alpha\in\mathbb{N},\tilde{D}^{(\alpha)}f_\epsilon(y)$ is the normal derivative of order α of $f_\epsilon(y)$ and the assertion follows immediately.
In the event that $n-1<\alpha\leq n$, We've got

$$\begin{aligned}
\sup_{y\in[0,T]} \|\bar{D}^{(\alpha)} f_\epsilon(y)\| &= \sup_{y\in[0,T]} \| \left(D^{(\alpha)}\ f_\epsilon * \varphi_\epsilon\right)(y)\| \\
&\leq \sup_{y\in[0,T]} \| \int_0^\infty D^{(\alpha)} f_\epsilon(s)\varphi_\epsilon(y-s)ds\| \\
&\leq \sup_{r\in K} \|D^{(\alpha)} f_\epsilon(r)\| \sup_{y\in[0,T]} \| \int_K \varphi_\epsilon(y-s)ds\| \\
&\leq C \sup_{y\in K} \|D^{(\alpha)} f_\epsilon(y)\|,
\end{aligned}$$

where C is a strictly positive constant.
Using the Lemma 3.1, $\sup_{y\in[0,T]} \mid D^{(\alpha)} f_\epsilon(y) \mid$ has a moderate bound, $\forall\alpha > 0$, as a result of this, $\sup_{y\in[0,T]} \mid \bar{D}^{(\alpha)} f_\epsilon(y) \mid$ has a moderate bound, too. □

Lemma 3.7. *Let $(f_{1\epsilon}(y))_\epsilon$ and $(f_{2\epsilon}(y))_\epsilon$ be two different representatives of $f(y) \in \mathcal{G}(\mathbb{R}^+)$. Then, $\forall\alpha > 0$, $k \in \{0, 1, 2, \dots\}$, $\sup_{t\in[0,T]} \mid \left(\mathrm{d}^k/\mathrm{d}t^k\right) \left(\tilde{D}^{(\alpha)} f_{1\epsilon}(t) - \tilde{D}^{(\alpha)} f_{2\epsilon}(t)\right) \mid$ is negligible.*

Proof.

$$\begin{aligned}
\sup_{y\in[0,T]} \mid \frac{d^k}{dy^k} \left(\bar{D}^{(\alpha)} f_{1\epsilon}(y) - \bar{D}^{(\alpha)} f_{2\epsilon}(y)\right) \| &= \sup_{y\in[0,T]} \|\frac{d^k}{dy^k} \left(\left(D^{(\alpha)} f_{1\epsilon} * \varphi_\epsilon\right)(y) - \left(D^{(\alpha)} f_{2\epsilon} * \varphi_\epsilon\right)(y)\right) \mid \\
&= \sup_{y\in[0,T]} \|\frac{d^k}{dy^k} \left(\left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right) * \varphi_\epsilon\right)(y)\| \\
&= \sup_{y\in[0,T]} \| \left(\left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right) * \frac{d^k}{dy^k}\varphi_\epsilon\right)(y)\| \\
&\leq \sup_{r\in K} \| \left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\| \sup_{y\in[0,T]} \| \int_K \frac{d^k}{dy^k}\varphi_\epsilon(y-r)dr\| \\
&\leq C \sup_{r\in K} \| \left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\|.
\end{aligned}$$

Using the Lemma 3.2, we have $\sup_{r\in K} \| \left(D^{(\alpha)} f_{1\epsilon} - D^{(\alpha)} f_{2\epsilon}\right)(r)\|$ is negligible, so $\sup_{y\in[0,T]} \mid \frac{d^k}{dy^k} \left(\bar{D}^{(\alpha)} f_{1\epsilon}(y) - \bar{D}^{(\alpha)} f_{2\epsilon}(y)\right) \mid$ is negligible. □

The regularized generalized conformable fractional derivative $D^{(\alpha)}$ is now introduced in the following manner.

Definition 3.8. Let $f(t) \in \mathcal{G}(\mathbb{R}^+)$ be a Colombeau generalized function. The regularized generalized conformable fractional derivative of $f(t)$, writing $\bar{D}^{(\alpha)} f(t) = \left[\left(\tilde{D}^{(\alpha)} f_\epsilon(t)\right)_\epsilon\right]$, $\alpha > 0$, is a component of $\mathcal{G}(\mathbb{R}^+)$ satisfy (3.2).

By the same principle we define "Generalized conformable semigroup".

4. Generalized Conformable semigroup

Let $(X, \|.\|)$ be a Banach space and $\mathcal{L}(X)$ be the space of all linear continuous mappings.
Firstly, we want to see whether we can create a map $A : \mathcal{G} \longrightarrow \mathcal{G}$ using a provided family $(A_\epsilon)_{\epsilon\in(0,1)}$ of $A_\epsilon X \longrightarrow X$, where $A_\epsilon \in \mathcal{L}(X)$. The following are the general requirements.

Lemma 4.1. *Let $(A_\epsilon)_{\epsilon\in]0,1[}$ be a provided family of maps $A_\epsilon : X \longrightarrow X$. In each case of $(e_\epsilon)_\epsilon \in \mathcal{E}_M(X)$ and $(f_\epsilon)_\epsilon \in \mathcal{N}(X)$, suppose that*
(1) $(A_\epsilon e_\epsilon)_\epsilon \in \mathcal{E}_M(X)$,

(2) $\left(A_\epsilon\left(e_\epsilon+f_\epsilon\right)\right)_\epsilon-\left(A_\epsilon e_\epsilon\right)_\epsilon \in \mathcal{N}(X)$.
Then

$$A:\left\{\begin{array}{l}\mathcal{G} \longrightarrow \mathcal{G} \\ e=\left[e_\epsilon\right] \mapsto A e=\left[A_\epsilon e_\epsilon\right],\end{array}\right.$$

is well defined.

Proof. We can see from the first attribute that the class $\left[\left(A_\epsilon e_\epsilon\right)_\epsilon\right] \in \mathcal{G}$.
Let $e_\epsilon+f_\epsilon$ be an additional member of $e=\left[e_\epsilon\right]$. We have from the second property:

$$\left(A_\epsilon\left(e_\epsilon+f_\epsilon\right)\right)_\epsilon-\left(A_\epsilon e_\epsilon\right)_\epsilon \in \mathcal{N}(X),$$

and $\left[\left(A_\epsilon\left(e_\epsilon+f_\epsilon\right)\right)_\epsilon\right]=\left[\left(A_\epsilon e_\epsilon\right)\right)_\epsilon]$ in $\mathcal{G}$.
Thus A is properly defined. □

Definition 4.2. We define
$\mathcal{E}_M^S([0,+\infty[, \mathcal{L}(X))=\{S_\epsilon:[0,+\infty[\longrightarrow \mathcal{L}(X)), \quad \epsilon \in]0,1[, \quad$ such that $\quad \forall T>0, \quad \exists a \in \mathbb{R},$ we have

$$\sup _{r \in[0, T]}\left\|S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|=O_{\epsilon \rightarrow 0}\left(\epsilon^a\right)\},$$

and
$\mathcal{N}^S([0,+\infty[, \mathcal{L}(X))=\{N_\epsilon:[0,+\infty[\longrightarrow \mathcal{L}(X)), \epsilon \in]0,1[$, such that $\forall T>0, \forall b \in \mathbb{R},$ we have $\sup_{r \in[0, T]}\left\|N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|=O_{\epsilon \rightarrow 0}\left(\epsilon^b\right)\}$. With the following characteristics
1) $\exists s>0$ and $\exists a \in \mathbb{R}$ such that

$$\sup _{t<s}\left\|\frac{N_\epsilon\left(r^{\frac{1}{\alpha}}\right)}{r}\right\|=O_{\epsilon \rightarrow 0}\left(\epsilon^a\right).$$

2) $\exists\left(H_\epsilon\right)_\epsilon$ in $\mathcal{L}(X)$ and $\epsilon \in]0,1[$ such that

$$\lim _{s \rightarrow 0} \frac{N_\epsilon\left(s^{\frac{1}{\alpha}}\right)}{s} e=H_\epsilon e, \quad e \in X,$$

For every $b>0$,

$$\left\|H_\epsilon\right\|=O_{\epsilon \rightarrow 0}\left(\epsilon^b\right).$$

Proposition 4.3. *$\mathcal{E}_M^S([0,+\infty[, \mathcal{L}(X))$ is algebra in terms of composition and $\mathcal{N}^S([0,+\infty[, \mathcal{L}(X))$ is an ideal of $\mathcal{E}_M^S([0,+\infty[, \mathcal{L}(X))$.*

Proof. Let $\left(S_\epsilon\right)_\epsilon \in \mathcal{E}_M^S([0,+\infty[, \mathcal{L}(X))$ and $\left(N_\epsilon\right)_\epsilon \in \mathcal{N}^S([0,+\infty[, \mathcal{L}(X))$.
We shall simply establish the second statement, specifically,

$$\left(S_\epsilon\left(r^{\frac{1}{\alpha}}\right) N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right)_\epsilon,\left(N_\epsilon\left(r^{\frac{1}{\alpha}}\right) S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right)_\epsilon \in \mathcal{N}^S([0,+\infty[, \mathcal{L}(X)).$$

Where $S_\epsilon\left(r^{\frac{1}{\alpha}}\right) N_\epsilon\left(r^{\frac{1}{\alpha}}\right)$ represents the composition.
By the Definition 4.2 and the definition of $\mathcal{N}^S$ from the previous definition, we have

$$\left\|S_\epsilon\left(r^{\frac{1}{\alpha}}\right) N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\| \leq\left\|S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|\left\|N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|=O_{\epsilon \rightarrow 0}\left(\epsilon^{a+b}\right).$$

The same is also true for $\left\|N_\epsilon\left(r^{\frac{1}{\alpha}}\right) S_\epsilon\left(r \frac{1}{\alpha}\right)\right\|$.
Furthermore, (1) and (2) in Definition 4.2, provide

$$\sup _{t<s}\left\|\frac{S_\epsilon\left(r^{\frac{1}{\alpha}}\right) N_\epsilon\left(r^{\frac{1}{\alpha}}\right)}{r}\right\| \leq \sup _{r<s}\left\|S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\| \sup _{r<s}\left\|N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|=O_{\epsilon \rightarrow 0}\left(\epsilon^a\right).$$

In some situations $s > 0$. We have,

$$\sup_{r>s}\left\|\frac{N_\epsilon\left(r^{\frac{1}{\alpha}}\right)S_\epsilon\left(r^{\frac{1}{\alpha}}\right)}{r}\right\| = O_{\epsilon\to 0}\left(\epsilon^a\right).$$

For some $s > 0$ and $a \in \mathbb{R}$. Let now $\epsilon \in]0,1[$ be fixed. We have

$$\begin{aligned}\left\|\frac{S_\epsilon(r^{\frac{1}{\alpha}})N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - S_\epsilon(0)H_\epsilon x\right\| &= \left\|S_\epsilon(r^{\frac{1}{\alpha}})\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - S_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x + S_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x - S_\epsilon(0)H_\epsilon x\right\| \\ &\leq \left\|S_\epsilon(r^{\frac{1}{\alpha}})\right\|\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - H_\epsilon x\right\| + \left\|S_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x - S_\epsilon(0)H_\epsilon x\right\|.\end{aligned}$$

According to (1) and (2) in Definition 4.2, in addition to the continuity of $t \mapsto S_\epsilon(r^{\frac{1}{\alpha}})(H_\epsilon x)$ at 0, the final expression becomes zero as $r \to 0$. Likewise, we have

$$\begin{aligned}\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})S_\epsilon(r^{\frac{1}{\alpha}})}{r}x - H_\epsilon S_\epsilon(0)x\right\| &= \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}S_\epsilon(r^{\frac{1}{\alpha}})x - \frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}S_\epsilon(0)x + \frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}S_\epsilon(0)x - H_\epsilon S_\epsilon(0)x\right\| \\ &\leq \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}\right\|\left\|S_\epsilon(r^{\frac{1}{\alpha}})x - H_\epsilon(t)S_\epsilon(0)x\right\| + \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}\left(S_\epsilon(0)x\right) - H_\epsilon\left(S_\epsilon(0)x\right)\right\|.\end{aligned}$$

Assertions (1) and (2) in Definition 4.2 require that the final expression goes to zero since $r \mapsto 0$. As a result, the proposition is proven in both circumstances. □

The factor algebra is now defined as Colombeau type algebra by

$$\mathcal{G}^S\left([0,+\infty[,\mathcal{L}(X)\right) = \mathcal{E}_M^S\left([0,+\infty[,\mathcal{L}(X)\right)/\mathcal{N}^S\left([0,+\infty[,\mathcal{L}(X)\right).$$

Components of $\mathcal{G}^S\left([0,+\infty[,\mathcal{L}(X)\right)$ will be represented by $S = [S_\epsilon]$, where $\left(S_\epsilon\right)_\epsilon$ is a member of the preceding class.

Definition 4.4. $S \in \mathcal{G}\left([0,+\infty[,\mathcal{L}(X)\right)$ is referred to as a Colombeau C_0Semigroup if it has a member $\left(S_\epsilon\right)_\epsilon$ such that, for $\epsilon > 0, S_\epsilon$ is a C_0Semigroup.

When ϵ low sufficient, we will only utilize members $\left(S_\epsilon\right)_\epsilon$ of a Colombeau C_0semigroup.

Proposition 4.5. *Let $\left(\tilde{S}_\epsilon\right)_\epsilon$ and $\left(S_\epsilon\right)_\epsilon$ be members of a Colombeau C_0semigroup $(S_\epsilon)_\epsilon$, with the infinitesimal generators $\tilde{A}_\epsilon$, $\epsilon < \tilde{\epsilon}_0$, and A_ϵ, $\epsilon < \epsilon_0$, respectively, where $\tilde{\epsilon}_0$ and ϵ_0 correspond to $\left(S_\epsilon\right)_\epsilon$ and $\left(S_\epsilon\right)_\epsilon$, respectively.*

Then, $D\left(D\left(\tilde{A}_\epsilon\right) = A_\epsilon\right)$, $\forall\tilde{\epsilon} = \min\left\{\epsilon_0, \tilde{\epsilon}_0\right\} > \epsilon$ and $\tilde{A}_\epsilon - A_\epsilon$ can be prolonged to a component of $\mathcal{L}(X)$. Moreover, for every $a \in]-\infty,+\infty[$,

$$\left\|\tilde{A}_\epsilon - A_\epsilon\right\| = O_{\epsilon\to 0}\left(\epsilon^a\right).$$

Proof. Indicate $N_\epsilon = \left(S_\epsilon - \tilde{S}_\epsilon\right)_\epsilon \in \mathcal{N}^S\left([0,+\infty[,\mathcal{L}(X)\right)$.
Let $\epsilon < \tilde{\epsilon}_0$ be fixed and $y \in X$. We have

$$\frac{S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - y}{t} - \frac{\tilde{S}_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - y}{t} = \frac{N_\epsilon\left(t^{\frac{1}{\alpha}}\right)}{t}y.$$

This indicates that by allowing $t \to map \to 0 : D\left(A_\epsilon\right) = D\left(\bar{A}_\epsilon\right)$. After that, we have

$$\begin{aligned}\left(A_\epsilon - (\tilde{A})_\epsilon\right)y &= \lim_{t\to 0}\frac{S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - y}{t} - \lim_{t\to 0}\frac{S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - y}{t} \\ &= \lim_{t\to 0}\frac{N_\epsilon\left(t^{\frac{1}{\alpha}}\right)}{t}y = H_\epsilon y, \quad y \in D\left(A_\epsilon\right),\end{aligned}$$

since $D\left(\bar{A}_e\right)=X$ and characteristics (1), (2) in Definition 4.2 imply that $\forall a\in\mathbb{R}$,

$$\left\|A_\epsilon-\tilde{A}_\epsilon\right\|=O_{\epsilon\to 0}\left(\epsilon^a\right).$$

□

The following definition makes sense because of Proposition 4.5.

Definition 4.6. If there exists a representative $\left(A_\epsilon\right)_\epsilon$ of A such that A_ϵ is the infinitesimal generator of S_ϵ for ϵ small sufficiently, then A is the infinitesimal generator of a Colombeau C_0semigroup S.

By Pazy we presente the following suggestion.

Proposition 4.7. *Assume S is a Colombeau C_0semigroup with an infinitesimal generator A.*
Then $\exists\epsilon_0\in]0,1[$ such that:

(1) *Mapping $r\longmapsto S_\epsilon\left(r^{\frac{1}{\alpha}}\right)y:[0,+\infty[\longrightarrow X$ is continuous $\forall y\in X$ and $\epsilon<\epsilon_0$.*
(2) $\lim_{h\to 0}\int_r^{r+h}S_\epsilon\left(s^{\frac{1}{\alpha}}\right)yds_\alpha=S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y,\quad \epsilon<\epsilon_0,\quad y\in X.$
(3) $\int_0^r S_\epsilon\left(s^{\frac{1}{\alpha}}\right)yds_\alpha\in D\left(A_\epsilon\right),\quad \epsilon<\epsilon_0,\quad y\in X.$
(4) *$\forall y\in D\left(A_\epsilon\right)$ and $r\geq 0$ $\quad S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y\in D\left(A_\epsilon\right)$ and $\frac{d^\alpha}{dt^\alpha}S_\epsilon\left(r^{\frac{1}{\alpha}}\right)y=A_\epsilon S_\epsilon\left(r^{\frac{1}{\alpha}}\right)y=S_\epsilon\left(r^{\frac{1}{\alpha}}\right)A_\epsilon y,\quad \epsilon<\epsilon_0.$*
(5) *Take $\left(S_\epsilon\right)_\epsilon$ and $\left(\tilde{S}_\epsilon\right)_\epsilon$ be representatives of Colombeau C_0semigroup S, with infinitesimal generators A_ϵ and $\tilde{A}_\epsilon$, $\epsilon<\epsilon_0$, accordingly. So, $\forall a\in\mathbb{R}$, $r\geq 0$, we have:*

$$\left\|\frac{d^\alpha}{dt^\alpha}S_\epsilon\left(r^{\frac{1}{\alpha}}\right)-A_\epsilon S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|=O\left(\epsilon^a\right).$$

(6) *$\forall y\in D\left(A_\epsilon\right)$ and $t,r\geq 0$,*

$$S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y-S_\epsilon\left(s^{\frac{1}{\alpha}}\right)y=\int_r^t S_\epsilon\left(e^{\frac{1}{\alpha}}\right)A_\alpha yde_\alpha=\int_s^t A_\epsilon S_\epsilon\left(e^{\frac{1}{\alpha}}\right)yde_\alpha.$$

Theorem 4.8. *Let S and $\tilde{S}$ be Colombeau C_0semigroup with infinitesimal generators A and $\bar{A}$, respectively. If $A=\tilde{A}$ then $S=\tilde{S}$.*

Proof. Applying the previous properties will be easy to proof the theorem. □

5. Existence and uniqueness of the solution

This section describes the use of Colombeau conformable C_0-semigroup in the solution of a family of schrödinger equations with singular data and potentials.

Before we explore the subject, we will create some working areas.
We put $\|\cdot\|_{L^2(\mathbb{R})}=\|\cdot\|_2$.

Definition 5.1. We indicate H_α^2 by the set of a function u$\in L^2(\mathbb{R})$ with, $\widetilde{D}^\alpha u\in L^2(\mathbb{R})$. In accordance with the norm

$$\|u\|_{H_\alpha^2}=\sqrt{\|u\|_2^2+\left\|\widetilde{D}^\alpha u\right\|_2^2}.$$

The following is the definition of the Colmbeau algebra type

$$\mathcal{G}_{H_\alpha^2}=\mathcal{E}_M\left(H_\alpha^2\right)/\mathcal{N}\left(H_\alpha^2\right),$$

where $\mathcal{E}_M\left(H_\alpha^2\right)=\{\left(G_\varepsilon\right)_\varepsilon\in H_\alpha^2,\forall T>0,\quad\exists a\in\mathbb{R}:\|G_\varepsilon\|_{H_\alpha^2}=O\left(\varepsilon^a\right)\}$, and $\mathcal{N}\left(H_\alpha^2\right)=\{\left(G_\varepsilon\right)_\varepsilon\in H_\alpha^2,\forall T>0,\quad\forall b\in\mathbb{R}:\|G_\varepsilon\|_{H_\alpha^2}=O\left(\varepsilon^b\right)\}$.

Definition 5.2. $\mathcal{E}_{C^1,H^2_\alpha}([0,T],\mathbb{R}) = \{G_\varepsilon \in C\left([0,T],H^2_\alpha\right) \cap C^1\left([0,T],L^2(\mathbb{R})\right), \forall T>0 \quad \exists a \in \mathbb{R}:$
$\max\left\{\sup_{t\in[0,T_1]} \|G_\varepsilon\|_{H^2_\alpha}, \sup_{t\in[0,T]} \left\|\widetilde{D}^\alpha G_\varepsilon\right\|_{L^2(\mathbb{R})}\right\} = O_{\varepsilon\to 0}\left(\varepsilon^a\right)\}.$
And,
$\mathcal{N}_{C^1,H^2_\alpha}([0,T],\mathbb{R})) = \{G_\varepsilon \in C\left([0,T],H^2_\alpha\right) \cap C^1\left([0,T],L^2(\mathbb{R})\right), \forall T>0 \quad \forall b \in \mathbb{R}:$
$\max\left\{\sup_{t\in[0,T_1]} \|G_\varepsilon\|_{H^2_\alpha}, \sup_{t\in[0,T]} \left\|\widetilde{D}^\alpha G_\varepsilon\right\|_{L^2(\mathbb{R})}\right\} = O_{\varepsilon\to 0}\left(\varepsilon^b\right)\}.$

Then the Colombeau type vector space, is define by

$$\mathcal{G}_{C^1,H^2_\alpha}(\mathbb{R}^+,\mathbb{R}) = \mathcal{E}_{C^1,H^2_\alpha}(\mathbb{R}^+,\mathbb{R})/\mathcal{N}_{C^1,H^2_\alpha}(\mathbb{R}^+,\mathbb{R}).$$

Proposition 5.3. *Let $v \in \mathcal{G}_{H^2_\alpha}$ and $x \in \mathcal{G}_{C^1,H^2_\alpha}(\mathbb{R}^+,\mathbb{R})$ this is believed to be the solution to*

$$\begin{cases} \partial_t^\alpha x(t,y) = (\Delta - v(y))x(t,y), \quad y\in\mathbb{R}, t\in\mathbb{R}^+, \\ x(0,y) = x_0(y) = \delta(y), \\ v(y) = \delta(y). \end{cases} \tag{5.1}$$

Then the multiplication $v(y).x(t,y)$ makes sense.

Definition 5.4. A generalized function $G \in \mathcal{G}_{C^1,H^2_\alpha}$ is considered to be a solution to the equation $\widetilde{D}^\alpha G = AG$, where A is characterized by a net of linear operators $(A_\epsilon)_\varepsilon$ within the consistent framework of $H^2_\alpha(\mathbb{R})$ and values in $L^2(\mathbb{R})$, if and only if

$$\sup_{t\in[0,T]} \left\|\widetilde{D}^\alpha G_\epsilon(t,.) - A_\varepsilon G_\varepsilon(t,.)\right\|_2 = O\left(\epsilon^a\right), \epsilon\to 0 \quad \forall a\in\mathbb{R}.$$

Definition 5.5. An component $U \in \mathcal{G}_{H^2_\alpha}$ is logarithmic type if it has an identification $(U_\epsilon)_\epsilon \in \mathcal{E}_{C^1,H^2_\alpha}$ with,

$$\|U_\varepsilon\|_{H^2_\alpha} = O_{\varepsilon\to 0}\left(\ln\varepsilon^{-1}\right).$$

An componen $U \in \mathcal{G}_{H^2_\alpha}$ is claimed to be log-log type if it has a identification $(U_\epsilon)_\epsilon \in \mathcal{E}_{C^1,H^2_\alpha}$ with,

$$\|U_\epsilon\|_{H^2_\alpha} = O\left(\ln^a \ln\varepsilon^{-1}\right), \quad \epsilon\to 0.$$

Theorem 5.6. *Consider a function v that belongs to the set $\mathcal{G}H^2_\alpha$ and is logarithmic type.*
(1) The infinitesimal generators of Conformable semigroups T_ϵ $\forall\epsilon>0$ is given by $(\Delta - v)u = A_\epsilon u$, with $u \in H^2_\alpha$. The collection of these semigroups $(T_\epsilon)_\epsilon$, is a representative of a Colombeau conformable C_0-semigroup.

$$T(t) \in G\mathcal{S}\left([0,+\infty[, \mathcal{L}\left(L^2\right)\right).$$

(2) Consider T_ϵ be as in (1) and let v be a member of the set $G_{H^2_\alpha}$.
Then, $\forall T>0$, the problem (5.1) has unique solution in $\mathcal{G}H^2_\alpha$.

Proof. (1) Put $\epsilon>0$ as small as possible. The operator A_ϵ is the infinitesimal generator of the associated semigroup according to the Feynman-Kac formula.

$$T_\epsilon(s)\phi(y) = \int_\Omega \left(\exp^{\left(-\int_0^{s^\alpha} v_\epsilon(\omega(e))de\right)}\right)\phi\left(\omega\left(s^\alpha\right)\right)d\mu_{y(\omega)},$$

for $\phi \in L^2(\mathbb{R})$, $\Omega = \mathbb{R}$ and μ_y is is the Wiener measure centered at $x\in\mathbb{R}$. Since v is of logarithmic type, $\exists C>0$, such that

$$\begin{aligned} |T_\epsilon(s)\phi(y)| &\le \exp^{\left(s^\alpha \sup_{e\in\mathbb{R}}|v_\epsilon(e)|\right)} \int_\Omega |\phi(\omega(s))| d\mu_{y(\omega)} \\ &\le \epsilon^{Cs^\alpha} \frac{1}{2\sqrt{\pi s^\alpha}} \int_\mathbb{R} \exp^{-\frac{|y-e|^2}{4s^\alpha}} |\phi(e)| de. \end{aligned}$$

Consequently, $\exists C_0>0$, such that

$$\sup_{s\in(0,T]} \|T_\epsilon(s)\phi(y)\|_2 \le C_0 \epsilon^{Cs^\alpha}\|\phi\|_2.$$

Then $(T_\epsilon)_\epsilon \in G\mathcal{S}\left([0,+\infty[,\mathcal{L}\left(L^2(\mathbb{R})\right)\right)$.

(2) **Existence**
By the principle of Duhamel , then the solution $x_\epsilon(t,y)$ of issue (5.1) satisfies

$$x_\epsilon(s,y)=\int_{\mathbb{R}} E\left(s^\alpha,y-e\right)b_\epsilon(e)de+\int_0^s\int_{\mathbb{R}} E\left((s-t)^\alpha,y-e\right)v_\epsilon(e)x_\epsilon(t,e)dedt. \tag{5.2}$$

By Young's inequality, we have

$$\|x_\epsilon(s,.)\|_2\le\|b_\epsilon\|_2+\int_0^s\|v_\epsilon(.)\|_{L^\infty}\|x_\epsilon(t,.)\|_2\,dt.$$

By Gronwall's inequality, we have

$$\|x_\epsilon(s,.)\|_2\le\|b_\epsilon\|_2\exp^{\int_0^s\|v_\epsilon(.)\|_{L^\infty}\|x_\epsilon(t,.)\|_2dt},\quad\forall t\in[0,T].$$

Since $v\in GH^2_\alpha$ is logarithmic type and $(x_\epsilon)_\epsilon\in\mathcal{E}H^2_\alpha$, it follows that $\sup_{s\in[0;T]}\|x_\epsilon(s,.)\|_2$ has a moderate bound.

Differentiation of Equation (5.2) with respect to spatial variable y satisfies

$$\begin{aligned}\frac{d}{dy}x_\epsilon(s,y)&=\int_{\mathbb{R}}E\left(s^\alpha,x\right)\frac{d}{dy}b_{\epsilon(y-x)}dx\\&+\int_0^s\int_{\mathbb{R}}E\left((s-t)^\alpha,x\right)\frac{d}{dy}v(y-x)x_\epsilon(t,y-x)\\&+v(y-x)\frac{d}{dy}x_\epsilon(t,y-x)dxdt.\end{aligned}$$

Then

$$\left\|\frac{d}{dy}x_\epsilon(s,.)\right\|_2\le\left\|\frac{d}{dy}b_\epsilon\right\|_2+\int_0^s\left\|\frac{d}{dy}v_\epsilon(.)\right\|_{L^\infty}\|x_\epsilon(s,.)\|_2+\|v_\epsilon(\cdot)\|_{L^\infty}\|\frac{d}{dy}x_\epsilon(s,.))\|_2.$$

Consequently Gronwall's inequality implies that $\sup_{s\in[0,T]}\|x_\epsilon(s,.)\|_2$ is moderate. So $(x_\epsilon)_\epsilon\in\mathcal{E}_{C^1,H^2_\alpha}$.

(b) **Uniqueness:**
Let x_ϵ and z_ϵ be two solutions to problem (5.1). We set $G_\epsilon=x_\epsilon-z_\epsilon$, we get

$$\begin{aligned}G_\epsilon(s,y)&=\int_{\mathbb{R}}E\left(s^\alpha,y-e\right)N_\epsilon(e)de\\&+\int_0^s\int_{\mathbb{R}}E\left((s-t)^\alpha,y-e\right)v_\epsilon(e)G_\epsilon(t,e)dedt\\&+\int_0^s\int_{\mathbb{R}}E\left((s-t)^\alpha,y-x\right)N_\epsilon(e)dedt,\end{aligned}$$

where $N_\epsilon(y)=G(0,y)$, and $N_\epsilon=\frac{d^\alpha}{ds^\alpha}G_\epsilon-(\Delta-v)G_\epsilon$.
So Young's and Gronwall's inequalities imply that

$$\|G_\epsilon(s,.)\|_2\le\|N_\epsilon\|_2+\int_0^s\|v_\epsilon(t,.)\|_{L^\infty}\|G_\epsilon(t,.)\|_2\,dt+\int_0^s\|N_\epsilon(t,.)\|_2\,dt.$$

Thus $G_\epsilon\in\mathcal{N}H^2_\alpha$. □

Conclusion

In conclusion, this paper showcases the power of Colombeau algebra as a robust mathematical tool for addressing complex problems in quantum mechanics, particularly in the realm of fractional Schrödinger equations subjected to singular potentials. By introducing the novel concept of a generalized conformable semigroup, this study pioneers a groundbreaking approach, effectively establishing both the existence and uniqueness of solutions for such equations. This innovative strategy not only expands the theoretical framework but also opens new avenues for tackling challenges inherent in singular potentials, offering promising prospects for further advancements in this field.

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