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Numerical study of astrophysics equations using Bessel collocation methods of first Kind

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Abstract

A hybrid computational procedure of Newton Raphson method and orthogonal collocation method has been applied to study the behavior of non linear astrophysics equations. The non linear Lane Emden equation has been discretized using orthogonal collocation method using n^{th}-order Bessel polynomial as $J_n(\xi)$ as base function. The system of collocation equations has been solved numerically using Newton Raphson method. Numerical examples have been discussed to check the reliability and efficiency of the scheme. Numerically calculated results have been compared to the exact values as well as the values already given in the literature to check the compatibility of the scheme. Error analysis has been studied by calculating the absolute error, $L_2 - norm$ and $L_\infty - norm$. Computer codes have been prepared using MATLAB.

Keywords. Astrophysics equations, Orthogonal collocation, Newton Rephson method, Polytropic fluid.
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1. Introduction

Differential equations are often used to describe the law of nature and the physical world. These equations play a significant role in physics, engineering, biology and economics etc. Lane-Emden equation is an example of such equations which have many applications in stellar structure, thermal explosions, isothermal gas spheres, radioactive cooling, thermionic currents and the thermal behavior of a spherical cloud of gas. In astrophysics, geometrical and beam optics, the LaneEmden equation is a dimensionless form of Poisson's equation for the gravitational potential of simple models of a star [7, 13, 14, 26].

Therefore, a variety of computational methods have been developed to solve these differential equations. Number of mathematicians and scientists expressed their interest towards second order differential equations. One such example is Lane-Emden equation named after two astrophysicist Jonathan Lane and Robert Emden. The Lane-Emden equation has many phenomena in physics, quantum mechanics and astrophysics [4, 5, 21, 24, 32] and it describes the density profile of a gaseous star.

Polytropes and isothermal spheres provide information regarding spherical galaxies and stars. In astrophysics, the Lane-Emden equation is a singular, linear as well as non linear boundary value problem which is considered as dimensionless form of Poisson's equation for the gravitational potential of newtonian self gravitating, specially symmetric, polytropic fluid [4]. Due to the singularity behaviour at the origin, Lane-Emden equation is taken as a a challenging problem by mathematicians and physicist.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\theta^m(\xi), \tag{1.1}$$

$$\frac{1}{\xi^2}\left(2\xi\frac{d\theta(\xi)}{d\xi}\right) + \xi^2\frac{d^2\theta(\xi)}{d\xi^2} = -\theta^m(\xi), \tag{1.2}$$

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where m is a real constant, ξ is a dimensionless radius and θ is related to the density. In the polytropic fluid, the index m is defined as a polytropic index.
By simplifying Eq. (1.1) the following equation is obtained as:

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) = -\theta^m(\xi), \tag{1.3}$$

in general form Eq. (1.3) can be written as:

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) = g(\xi, \theta(\xi)). \tag{1.4}$$

The general form of Lane-Emden equation of first and second kind are described as:

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) + f(\xi)\theta^p(\xi) = 0, \tag{1.5}$$

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) + g(\xi)e^{q\theta} = 0, \tag{1.6}$$

where p and q are taken to be real constants and the function $f(\xi)$ and $g(\xi)$ are arbitrary real valued functions. If isothermal fluids are used instead of taking polytropic fluids then Lane-Emden equation reduces to the Emden-Chandershekher equation [21]. An American astrophysicist Subrmanyam Chandersekhar has introduced the Chandershekher's white dwarf equation. It was introduced on the basis of gravitational potential of completely degenerate white dwarf. The Chandershekher's white dwarf equation is defined as:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) + (\theta^2 - C)^{\frac{3}{2}} = 0, \tag{1.7}$$

where θ represents the density of white dwarf and C is any real constant and is related to the density of white dwarf at center. For $C = 0$ Eq. (1.7) reduces to the Lane-Emden equation with $m = 3$.
Consider the following generalized Lane Emden equation:

$$\theta''(\xi) = f(\xi, \theta(\xi), \theta'(\xi)); a \leq \xi \leq b, \tag{1.8}$$

with general boundary conditions

$$a_1\theta(a) + a_2\theta'(a) = c_1; a_i \geq 0, i = 1, 2,$$
$$b_1\theta(b) + b_2\theta'(b) = c_2; b_i \geq 0, i = 1, 2,$$

where a_1,a_2,b_1,b_2,c_1 and c_2 are arbitrary real constants and $f(\xi, \theta(\xi), \theta'(\xi))$ is either linear or non linear functional equation of ξ,θ and θ'.
In present study orthogonal collocation method with Bessel polynomials as base functions has been followed to numerically discretize the Lane-Emden equation. This method is called Bessel collocation method (BCM). In section 2, the technique of Bessel collocation has been explained in detail. In section 3, the application of Bessel collocation with Newton Raphson method has been discussed. Convergence analysis has been discussed in section 4 and the validity of numerical technique has been studied through examples.

2. Description of Method

The collocation method belongs to the general class of approximate methods known as weighted residual methods. In this method the residual is set orthogonal to the weight function. In orthogonal collocation, the trial function $\theta(\xi)$ is represented in a series of known polynomials with unknown coefficients [1, 15, 27]. The residual is set equal to zero at the collocation points.
On the basis of implementation of trial function, the collocation technique can be classified into three categories. Collocation method [1] is useful to find the numerical solution of functional equations. The numerical solution $\theta(\xi)$ is obtained by using the trial function which satisfies the functional equation $L_v(\theta) = 0$ at collocation points [27]. If the trial function satisfies the differential equation $\mathbf{L_V}\mathbf{l}(\theta) = 0$ with operator$\mathbf{L_V}$ over the volume V, then it is termed

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as interior collocation. If the trial function satisfies the boundary $\mathbf{L_B}(\theta) = 0$, where B is the boundary adjoining the volume V then it is termed as boundary collocation. If the trail function satisfies neither the equation nor the boundary and is adjusted to both then it is termed as mixed collocation [10, 11, 27, 29].

The choice of base function is the first important step in the technique of collocation. In the present study Bessel polynomials of order $'n'$ has been chosen as trial function and the technique is called Bessel collocation method (BCM) [17, 22, 28]. It was first introduced by Yzba [4, 30] to solve the system of differential equations. In this method, a trial function is introduced with Bessel polynomials $J_i(\xi)$ where $i = 1, 2, 3...n$ and $\xi \in (a, b)$, as base functions [9, 17, 25, 28]. This trial function is fitted to both the differential equation and the boundary conditions to discretize the problem. The residual is set equal to zero at the collocation points.

During the study of problems in dynamic astronomy to solve the Kepler's problem, a German astronomer F.W. Bessel in 1824 introduced Bessel polynomials which are the solution of a second order boundary value problem. In hypergeometric form as given in [3, 25] Bessel function can be written as:

$$J_n(\xi) = \frac{\xi^n}{2^n n!} {}_0F_1(-; n+1; -\frac{1}{4}\xi^2). \tag{2.1}$$

The first order derivative of Bessel function is defined as:

$$\frac{d}{d\xi}(\xi^n J_n(\xi)) = \xi^n J_{n-1}(\xi),$$

$$\frac{d}{d\xi}(\xi^{-n} J_n(\xi)) = -\xi^{-n} J_{n+1}(\xi).$$

The Bessel coefficients also follow from the power series expansion for small values of ξ

$$\lim_{\xi\to 0} {}_0F_1(-; n+1; -\frac{1}{4}\xi^2) = 1,$$

$$\lim_{\xi\to 0} \xi^{-n} J_n(\xi) = \frac{1}{2^n n!},$$

which shows that as $\xi \to 0$, the Bessel coefficient $J_n(\xi)$ approaches to $\frac{1}{2^n n!}$.

In the proposed method numerical approximation of second order differential equation is expressed in terms of Bessel series as:

$$\theta(\xi) = \sum_{i=1}^{n+1} J_i(\xi) d_i \;\; a \le \xi \le b, \tag{2.2}$$

where $d_i' s$ are unknown constants an d$J_i(\xi)$ are i^{th} order Bessel polynomials. $\theta(\xi)$ is considered as an approximate numerical solution of Eq. (1.8). To simplify Eq. (2.2), the Bessel polynomials can be rewritten as given by [4, 8, 29–32]::

$$J(\xi) = T(\xi)E^T, \tag{2.3}$$

where $T(\xi)$ is defined as:

$$T(\xi) = \begin{bmatrix} T_1(\xi) & T_2(\xi) & T_3(\xi) & ... & T_{n+1}(\xi) \end{bmatrix},$$

which can be generalized as:

$$T_i(\xi) = \xi^{i-1},$$

and $J(\xi)$ is defined as:

$$J(\xi) = \begin{bmatrix} J_1(\xi) & J_2(\xi) & J_3(\xi) & ... & J_{n+1}(\xi) \end{bmatrix}.$$

Eq.(2.3) is simplified by [30] by introducing a square matrix E of order n+1. if n is an odd number then transpose of E is given by [30] as defined below:

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$$E^T = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \dots & 0 & 0 \\ \frac{-1}{1!1!2^2} & 0 & \frac{1}{0!2!2^2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(-1)^{\frac{n-1}{2}}}{(\frac{n-1}{2})!(\frac{n-1}{2})!2^{n-1}} & 0 & \frac{(-1)^{\frac{n-3}{2}}}{(\frac{n-3}{2})!(\frac{n+1}{2})!2^{n-1}} & \dots & \frac{1}{0!(n-1)!2^{n-1}} & 0 \\ 0 & \frac{(-1)^{\frac{n-1}{2}}}{(\frac{n-1}{2})!(\frac{n+1}{2})!2^{n}} & 0 & \dots & 0 & \frac{1}{0!n!2^n} \end{bmatrix}.$$

For n being an even number, transpose of E as given by [30] is defined as:

$$E^T = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \dots & 0 & 0 \\ \frac{-1}{1!1!2^2} & 0 & \frac{1}{0!2!2^2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{(-1)^{\frac{n-2}{2}}}{(\frac{n-2}{2})!(\frac{n}{2})!2^{n-1}} & 0 & \dots & \frac{1}{0!(n-1)!2^{n-1}} & 0 \\ \frac{(-1)^{\frac{n}{2}}}{(\frac{n}{2})!(\frac{n}{2})!2^{n}} & 0 & \frac{(-1)^{\frac{n-2}{2}}}{(\frac{n-2}{2})!(\frac{n+2}{2})!2^{n}} & \dots & 0 & \frac{1}{0!n!2^n} \end{bmatrix}.$$

By using the above matrices, approximate function $\theta(\xi)$ can be represented as:

$$\theta(\xi) = \sum_{i=1}^{n+1} T_i(\xi) E^T d_i. \tag{2.4}$$

The generalization of Eq. (2.4) can be defined as:

$$\theta(\xi) = J(\xi) d,$$

where d is $d = [d_1 \;\; d_2 \;\; d_3 \dots d_{n+1}]$.

3. Collocation Points

The next step is the choice of collocation points. It is an important part of the collocation technique. In this study, instead of taking the uniform points, the zeros of orthogonal polynomials such as Jacobi polynomials have been taken as collocation points. Legendre and Chebyshev polynomials are special cases of Jacobi polynomials and the zeros of these orthogonal polynomials are preferably taken as collocation points. Runge's divergence formula also states that non uniform collocation points give less error as compared to the uniform collocation points.

Theorem 3.1. *[23]: If $\mathcal{Q}_n(\xi)$ form a simple set of real polynomials and $w(\xi) > 0$ on $a \leq \xi \leq b$, the necessary and sufficient condition that the set $\mathcal{Q}_n(\xi)$ is orthogonal with respect to the $w(\xi)$ over the interval $a \leq \xi \leq b$ is that:*

$$\int_a^b w(\xi) x^k \mathcal{Q}_n(\xi) d\xi = 0, \qquad k = 0, 1, 2, 3, ..., (n-1).$$

Theorem 3.2. *[23]: If the simple set of real polynomials $\mathcal{Q}_n(\xi)$ is orthogonal with respect to the weight function $w(\xi) > 0$ on the interval $a \leq \xi <\leq$, then the zeros of $\mathcal{Q}_n(\xi)$ are distinct and lie in the interval $a \leq \xi \leq b$.*

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Since $\mathcal{Q}_n(\xi)$ is a polynomial of degree n, then it has exactly n roots, multiplicity counted, such that the roots are distinct and all lie in $a \le \xi \le b$.

The zeros of Legendre polynomials have been taken as collocation points. The zeros of Legendre polynomials have been calculated using the following recurrence relation:

$$P_i(\xi) = (\xi - 0.5)P_{i-1}(\xi) - \frac{(i-1)^2}{4(2i-3)(2i-1)}P_{i-2}(\xi), \quad i = 1, 2, \dots, n+1.$$

The details of these polynomials is given elsewhere[1, 25].
At j^{th} collocation point, Eq. (2.4) can be written as:

$$\theta_j = \sum_{i=1}^{n+1} T_i(\xi_j)E^T d_i,$$

where, $\theta_j = \theta(\xi_j)$. In matrix form Eq. (2.4) can be written as:

$$\theta(\xi) = \sum_{i=1}^{n+1} \xi^{i-1}E^T X^{-1}\theta_i, \tag{3.1}$$

where $X = \xi_j^{i-1}E^T$. Hence, Eq. (3.1) can be written as:

$$\theta(\xi) = \sum_{i=1}^{n+1} \xi^{i-1}E^T X^{-1}\theta_i. \tag{3.2}$$

The discretized form of first and second order derivatives at j^{th} collocation point are described as:

$$\begin{aligned} \frac{d\theta}{d\xi} &= \sum_{i=1}^{n+1}(i-1)\xi^{i-2}E^T X^{-1}\theta_i, \\ \frac{d^2\theta}{d\xi^2} &= \sum_{i=1}^{n+1}(i-1)(i-2)\xi^{i-3}E^T X^{-1}\theta_i. \end{aligned} \tag{3.3}$$

In simplified form, the discretized form of first and second order derivative at j^{th} collocation point can be written as:

$$\theta_j^{'}(\xi) = \sum_{i=1}^{n+1} A_{ji}\theta_i, \tag{3.4}$$

and

$$\theta_j^{''}(\xi) = \sum_{i=1}^{n+1} B_{ji}\theta_i. \tag{3.5}$$

After applying the collocation principle, the linear as well as non linear differential equation converts into algebraic equation. By substituting Eq. (3.2), Eq. (3.4) and Eq. (3.5) in Eq. (1.8), following system of algebraic equations is formed:

$$\sum_{i=1}^{n+1} B_{ji}\theta_i = f\left(\xi_j, \theta_j, \sum_{i=1}^{n+1} A_{ji}\theta_i\right), \ a \le \xi \le b,$$

$$\sum_{i=1}^{n+1} B_{ji}\theta_i - f\left(\xi_j, \theta_j, \sum_{i=1}^{n+1} A_{ji}\theta_i\right) = 0, \ a \le \xi \le b. \tag{3.6}$$

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This system of n+1 algebraic equations can be solved numerically by using iterative methods. To solve the system of algebraic equations, Newton Raphson method has been followed [12, 20]. System of algebraic equations defined by Eq. (3.6) can be generalized as:

$$\begin{aligned} f_1(\theta_1(\xi),\theta_2(\xi),\theta_3(\xi),...\theta_{n+1}(\xi)) &= 0 \\ f_2(\theta_1(\xi),\theta_2(\xi),\theta_3(\xi),...\theta_{n+1}(\xi)) &= 0 \\ f_3(\theta_1(\xi),\theta_2(\xi),\theta_3(\xi),...\theta_{n+1}(\xi)) &= 0 \\ \vdots \quad & \quad \vdots \\ f_{n+1}(\theta_1(\xi),\theta_2(\xi),\theta_3(\xi),...\theta_{n+1}(\xi)) &= 0 \end{aligned}$$

These n+1 algebraic equations depends upon n+1 variables $\theta_1(\xi)$, $\theta_2(\xi)$, $\theta_3(\xi)$, ... $\theta_{n+1}(\xi)$. These variables represent $\theta(\xi_1)$, $\theta(\xi_2)$, $\theta(\xi_3)$, ... $\theta(\xi_{n+1})$, respectively. As the derivatives exist for all algebraic equations, Newton Raphson method can be followed to discretize the system of equations. For the solution to these algebraic equations, a function $M(\theta)$ has been defined as:

$$M(\theta) = \begin{bmatrix} f_1(\theta_1(\xi),\theta_2(\xi),\theta_3(\xi),...\theta_{n+1}(\xi)) \\ f_2(\theta_1(\xi),\theta_2(\xi),\theta_3(\xi),...\theta_{n+1}(\xi)) \\ f_3(\theta_1(\xi),\theta_2(\xi),\theta_3(\xi),...\theta_{n+1}(\xi)) \\ \vdots \\ f_{n+1}(\theta_1(\xi),\theta_2(\xi),\theta_3(\xi),...\theta_{n+1}(\xi)) \end{bmatrix},$$

Now, the Newton Raphson method for the system of algebraic equations can be applied by choosing initial value approximation Q_1^m, Q_2^m, Q_3^m,.....Q_{n+1}^m for θ_1, θ_2, θ_3, ... θ_{n+1}, respectively. Afterwards, Jacobian matrix is computed to solve the system of equations iteratively.

$$Q^{m+1} = Q^m + h^m, \tag{3.7}$$

where Q^m and Q^{m+1} can be represented in m^{th} and $(m+1)^{th}$ terms as:

$$Q^m = \begin{bmatrix} \theta_1^m(\xi) \\ \theta_2^m(\xi) \\ \theta_3^m(\xi) \\ \vdots \\ \theta_{n+1}^m(\xi) \end{bmatrix},$$

and

$$Q^{m+1} = \begin{bmatrix} \theta_1^{m+1}(\xi) \\ \theta_2^{m+1}(\xi) \\ \theta_3^{m+1}(\xi) \\ \vdots \\ \theta_{n+1}^{m+1}(\xi) \end{bmatrix},$$

where m=0,1,2,3...until the convergence is achieved. h^m in Eq. (3.7) can be calculated as

$$\mathcal{J}^m h^m = -R^m,$$

$$h^m = -[\mathcal{J}^m]^{-1} R^m, \tag{3.8}$$

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where R^m is

$$R^m = \begin{bmatrix} f_1(\theta_1^m(\xi), \theta_2^m(\xi), \theta_3^m(\xi)...\theta_{n+1}^m(\xi)) \\ f_2(\theta_1^m(\xi), \theta_2^m(\xi), \theta_3^m(\xi)...\theta_{n+1}^m(\xi)) \\ f_3(\theta_1^m(\xi), \theta_2^m(\xi), \theta_3^m(\xi)...\theta_{n+1}^m(\xi)) \\ \vdots \\ f_{n+1}(\theta_1^m(\xi), \theta_2^m(\xi), \theta_3^m(\xi)...\theta_{n+1}^m(\xi)) \end{bmatrix},$$

and the jacobian $\mathcal{J}$ can be calculated as

$$\mathcal{J} = \mathcal{J}(\theta_1, \theta_2...\theta_{n+1}) = DM(\theta),$$

$$\mathcal{J} = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1(\xi)} & \frac{\partial f_1}{\partial \theta_2(\xi)} & \frac{\partial f_1}{\partial \theta_3(\xi)} & \cdots & \frac{\partial f_1}{\partial \theta_{n+1}(\xi)} \\ \frac{\partial f_2}{\partial \theta_1(\xi)} & \frac{\partial f_2}{\partial \theta_2(\xi)} & \frac{\partial f_2}{\partial \theta_3(\xi)} & \cdots & \frac{\partial f_2}{\partial \theta_{n+1}(\xi)} \\ \frac{\partial f_3}{\partial \theta_1(\xi)} & \frac{\partial f_3}{\partial \theta_2(\xi)} & \frac{\partial f_3}{\partial \theta_3(\xi)} & \cdots & \frac{\partial f_3}{\partial \theta_{n+1}(\xi)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_{n+1}}{\partial \theta_1(\xi)} & \frac{\partial f_{n+1}}{\partial \theta_2(\xi)} & \frac{\partial f_{n+1}}{\partial \theta_3(\xi)} & \cdots & \frac{\partial f_{n+1}}{\partial \theta_{n+1}(\xi)} \end{bmatrix},$$

here $\mathcal{J}$ is an $(n+1)\times(n+1)$ matrix. By substituting assumed initial approximation θ_1^m, θ_2^m ... θ_{n+1}^m in above jacobian,

$$\mathcal{J}^m = \mathcal{J}(\theta_1^m, \theta_2^m...\theta_{n+1}^m).$$

$\mathcal{J}^m$ is an $(n+1)\times(n+1)$ matrix, h^m is an $(n+1)\times 1$ matrix and M^m is an $(n+1)\times 1$ matrix. Substitute the calculated solution obtained from first iteration in place of assumed initial values at different collocation points in next iterative solution. Continue the above procedure until Q^{m+1} which contains the numerical values $\theta(\xi)$ at different collocation points will not change further by calculating next iteration. Then the values $\theta(\xi)$ are taken as solution that is close to the exact solution.

4. Convergence Analysis

Theorem 4.1. *[23]: If $\{\mathcal{Q}_n(\xi)\}$ represents for simple set of polynomials and if $\mathcal{Y}(\xi)$ is a polynomial of degree m then there exist constants a_k such that*

$$\mathcal{Y}(\xi) = \sum_{k=0}^{m} a_k \mathcal{Q}_k(\xi).$$

The $a_k's$ are function of k and any parameter involved in $\mathcal{Y}(\xi)$.

Theorem 4.2. *[22]: There exists a unique polynomial $P_n(\xi)$ of degree n which assumes prescribed values at $n+1$ distinct points $\xi_0 < \xi_1 < \ldots < \xi_n$.*

Theorem 4.3. *[22]:- Given any interval $a \le \xi \le b$, real number $\varepsilon > 0$ and any real valued continuous function $f(\xi)$ on $a \le \xi \le b$, then there exists a polynomial $P(\xi)$ such that:*

$$\| \, f(\xi) - P(\xi) \, \| < \varepsilon.$$

To check the convergence of the numerical results, it is convenient to use norms [2, 16, 18, 19]. To know the accuracy of approximated solution obtained from proposed method based on finding norms such as $\|\theta\|_2$ and$\|\theta\|_\infty$ and the error analysis based on absolute error, $E_a = \|\theta - \theta_h\|$, $\|\theta - \theta_h\|_2$ and $\|\theta - \theta_h\|_\infty$, where θ represents analytic solution and θ_h represents approximate solution [17].
For the general term for L_p norm

$$\|\theta\|_p = \left[\sum_{i=1}^{n+1} \left| w_i(x)(\theta_h)_i^2 \right| \right]^{\frac{1}{p}}.$$

L_p norm with the value of p such that $1 \le$ p $< \infty$, is said to converge to the exact solution if $\|\theta - \theta_h\|_p \longrightarrow 0$ as $n \longrightarrow \infty$.

In this paper L_2 and L_∞ norms have been calculated by using weight functions [15, 19]:

$$\|\theta\|_2 = \sqrt{\sum_{i=1}^{n+1} |w_i(\xi)\theta_i^2|},$$

and the L_2 norm for error has been calculated as:

$$\|\theta - \theta_h\|_2 = \sqrt{\sum_{i=1}^{n+1} |w_i(\xi)(\theta - \theta_h)_i^2|}.$$

L_∞-norm is also known as the maximum norm and can be written as

$$\|\theta\|_\infty = \max |\theta_i|; i = 1, 2, 3 ... n+1,$$

and to check the maximum error, the norm can be calculated as

$$\|\theta - \theta_h\|_\infty = \max |(\theta - \theta_h)_i|;\ \ i = 1, 2, 3 ... n+1.$$

5. Numerical Examples

Example 5.1. *Consider a linear Lane-Emden equation*

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{1}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) = \left(\frac{8}{8-\xi^2}\right)^2,$$

including boundary conditions $\theta(1) = 0$ and $\theta'(0) = 0$ with the exact solution $\theta(\xi) = 2log\left(\frac{7}{8-\xi^2}\right)$ [6]. In Table 1 numerical values have been compared to the exact values in terms of absolute error. A good comparison has been observed between the two and absolute error is found to be of order 10^{-10}. The comparison of results with the exact solutions and with the solutions obtained by B-Spline approach [6] shows that the results obtained by using Bessel collocation method are better than B-spline method. In Figure 1 comparison of numerical and exact values have been shown graphically and a good match has been found between the two.

Example 5.2. *Consider the linear Lane-Emden equation*

$$-\frac{d^2\theta(\xi)}{d\xi^2} - \frac{2}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) + (1-\xi^2)\theta(\xi) - \xi^4 + 2\xi^2 - 7 = 0,$$

with boundary conditions $\theta'(0) = 0$ and $\theta(1) = 0$. The exact solution to this problem is $\theta(\xi) = 1 - \xi^2$ [6]. The absolute error, L_2 and L_∞ norms with respect to weight function have been calculated and compared with the exact solution and the results already present in the literature [6]. In Table 2 the values of θ obtained by Bessel collocation method using shifted Legendre collocation points and by B-spline method [6] has been presented. The absolute error is found to be of order 10^{-16} in case of Bessel collocation method. In Figure 2 numerical values have been compared to exact values and a good match have been found between the two.

Example 5.3. *Consider the Lane Emden equation*

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{1}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) + \theta(\xi) = 0,$$

with boundary conditions $\theta'(0) = 0$ and $\theta(1) = 1$. The exact solution to this problem is $\frac{J_0(\xi)}{J_0(1)}$ and is given in [6].The absolute error, L_2 and L_∞ norms with respect to weight function have been calculated and compared with the exact solution and the results already present in the literature [6]. In Table 3 the comparison of absolute error, i.e., $\|\theta - \theta_h\|$ obtained by Bessel collocation method using shifted Legendre collocation points and B-spline method [6] has been presented. It is observed that Bessel collocation approach gives better results as compared to B-spline method. In Figure 3 numerical values have been compared to exact values and a good match has been found between the two.

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Example 5.4. *Consider the Lane Emden equation*

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) - 4\theta(\xi) = -2;\ 0 < \xi \leq 1,$$

with boundary conditions $\theta'(0) = 0$ and $\theta(1) = 5.5$. The exact solution to this problem is $\theta(\xi) = 0.5 + \frac{5.sinh(2\xi)}{\xi.sinh(2)}$ [6]. The absolute error, L_∞ and L_2 norms with respect to weight function have been calculated and compared with the exact solution and the results already present in the literature [6]. In Table 4 the absolute error, i.e., $\|\theta - \theta_h\|$ obtained by Bessel collocation method using shifted Legendre collocation points and B-spline method [6] has been presented. It is observed that Bessel collocation approach gives better results as compared to B-spline method. In Table 4, absolute error is found to be of order 10^{-7} in case of Bessel collocation method and is of order 10^{-5} in case of B-spline method which shows the efficiency of Bessel collocation method. In Figure 4 numerical values have been compared to exact values and a good match have been found between the two.

Example 5.5. *Consider the non linear Lane-Emden equation*

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{1}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) + e^{\theta(\xi)} = 0,$$

with boundary conditions $\theta'(0) = 0$, $\theta(1) = 0$. The exact solution to this problem is $\theta(\xi) = 2ln\left(\frac{\alpha+1}{\alpha\xi^2+1}\right)$ where $\alpha = 3 - 2\sqrt{2}$ [4]
The absolute error, L_2 norm and L_∞ norm with respect to the weight function have been calculated. In Table 5 the values of θ obtained using Bessel collocation method has been compared to the exact values at node points. The absolute error is found to be of order 10^{-6}. In Figure 5 numerical values have been compared to the exact values graphically and a good match has been observed between exact and numerical values.

Example 5.6. *Consider a non linear Lane-Emden differential equation*

$$\frac{d^2\theta(\xi)}{d\xi^2} + \frac{2}{\xi}\left(\frac{d\theta(\xi)}{d\xi}\right) - 6\theta(\xi) - 4\theta(\xi)ln(\theta(\xi)) = 0; 0 \leq \xi \leq 1,$$

including boundary conditions $\theta(0) = 1$ and $\theta'(0) = 0$. This problem has exact solution $\theta(\xi) = exp(\xi^2)$ [4]. In Table 6 the numerical values have been compared to the exact values in terms of absolute error at different collocation points. The absolute error varies from 10^{-6} to 10^{-2}. Due to the presence of logarithmic non linear term, the above problem becomes more stiff and consumes more computational time. In Figure 6 numerical values obtained from Bessel collocation method has been compared to exact values and a good match has been observed between the two.

6. Conclusion

Bessel collocation method with Legendre collocation points has been successfully implemented on six Lane Emden equations of linear and non linear type. Rate of convergence has been checked by calculating Eucledian and maximum norms. Numerical values obtained from Bessel collocation method have been compared with B-spline method and are found to be better than the latter. By calculating absolute error, $\|\theta - \theta_h\|_\infty$ and $\|\theta - \theta_h\|_2$with respect to the weight function, it is found that the numerical approach is stable and the results obtained by this approach are consistent and convergent.

Uncorrected Proof

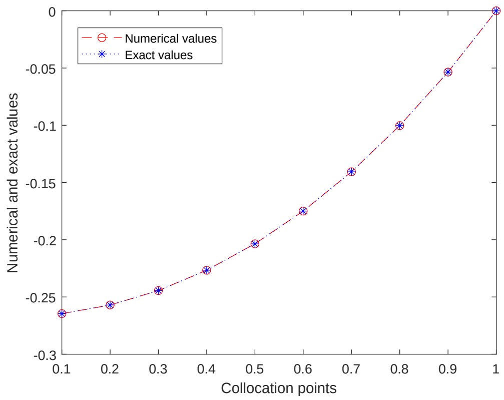

FIGURE 1. Graphical representation of $\theta(\xi)$ with respect to ξ for Example 5.1.

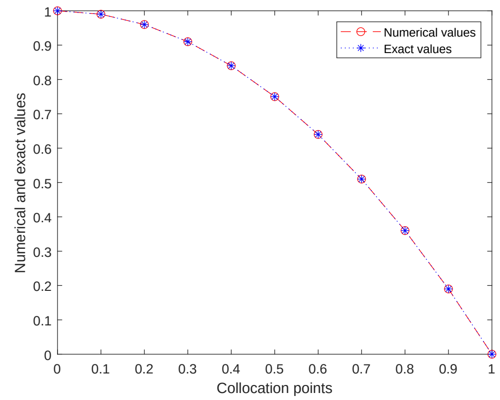

FIGURE 2. Graphical representation of $\theta(\xi)$ with respect to ξ for Example 5.2.

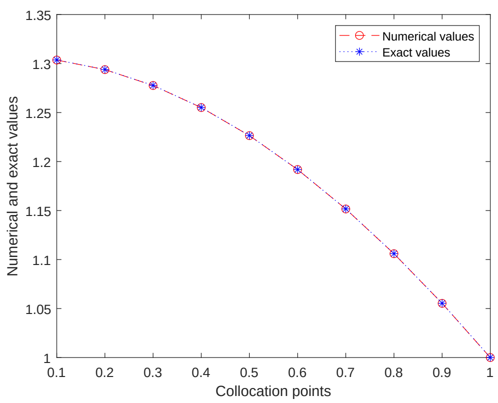

FIGURE 3. Graphical representation of $\theta(\xi)$ with respect to ξ for Example 5.3.

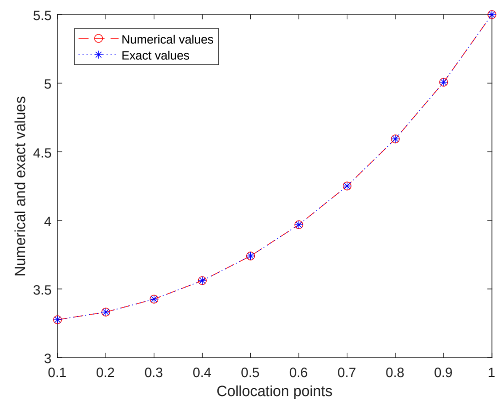

FIGURE 4. Graphical representation of $\theta(\xi)$ with respect to ξ for Example 5.4.

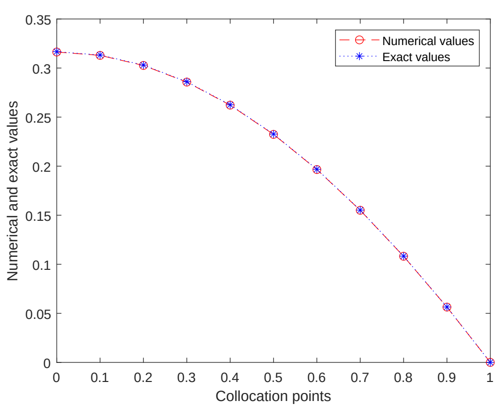

FIGURE 5. Graphical representation of $\theta(\xi)$ with respect to ξ for Example 5.5.

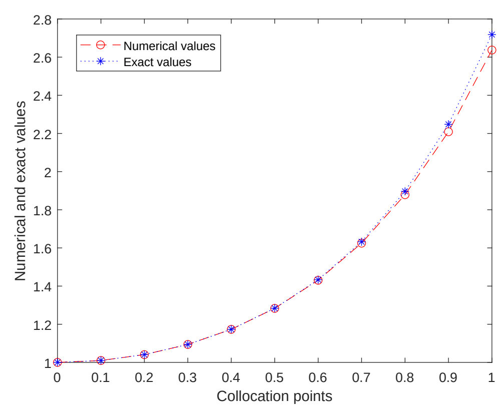

FIGURE 6. Graphical representation of $\theta(\xi)$ with respect to ξ for Example 5.6.

Table 1. Comparison of numerical values from Bessel collocation method and B-spline method in terms of absolute error for Example 5.1.

ξ	Exact solution	Numerical solution by Bessel collocation method	Absolute error by Bessel collocation method	Absolute error by B-spline method [6]	Absolute error by [32]
0.1	-0.26456122145	-0.26456122137	7.8420e-11	5.7786e-06	
0.2	-0.25703770160	-0.25703770155	5.6279e-11	2.9840e-07	3.8810e-09
0.3	-0.24443526545	-0.24443526576	3.1375e-10	5.7346e-06	
0.4	-0.22665737061	-0.22665737059	1.9153e-11	5.6294e-06	3.9877e-09
0.5	-0.20356538862	-0.20356538827	3.4701e-10	4.6114e-06	
0.6	-0.17497490825	-0.17497490836	1.0925e-10	4.0918e-06	4.0502e-09
0.7	-0.14065063344	-0.14065063379	3.4585e-10	3.3666e-06	
0.8	-0.10029956737	-0.10029956718	1.8948e-10	2.4326e-06	4.0959e-09
0.9	-0.05356204535	-0.05356204532	3.1992e-11	9.5464e-07	
L_2	-	-	7.2475e-11	-	-
L_∞	-	-	1.1853e-10	-	-

Table 2. Comparison of numerical values obtained from Bessel collocation method and B-spline method for Example 5.2.

ξ	Exact solution	Numerical solution by Bessel collocation method	Absolute error by Bessel collocation method	Numerical solution by B-spline method [6]
0.1	0.990000	0.990000	6.6613e-16	0.990000
0.2	0.960000	0.960000	7.7716e-16	0.960000
0.3	0.910000	0.910000	5.5511e-16	0.910000
0.4	0.840000	0.840000	4.4409e-16	0.840000
0.5	0.750000	0.750000	4.4409e-16	0.750000
0.6	0.640000	0.640000	2.2204e-16	0.640000
0.7	0.510000	0.510000	1.1102e-16	0.510000
0.8	0.360000	0.360000	2.2204e-16	0.360000
0.9	0.190000	0.190000	5.8287e-16	0.190000
L_2	-	-	7.7716e-16	-
L_∞	-	-	5.7152e-16	-

Uncorrected Proof

TABLE 3. Comparison of numerical values obtained by Bessel collocation and B-Spline method for Example 5.3.

ξ	Exact solution	Numerical solution by Bessel collocation method	Absolute error by Bessel collocation method	Absolute error by B-spline method [6]
0.1	1.303691	1.303587	1.0426e-04	1.1200e-04
0.2	1.293919	1.293816	1.0305e-04	1.1000e-04
0.3	1.277714	1.277613	1.0136e-04	1.0800e-04
0.4	1.255198	1.255098	9.9817e-04	1.0600e-04
0.5	1.226539	1.226441	9.8012e-05	1.0400e-04
0.6	1.191850	1.191855	5.2282e-05	0.0000e-00
0.7	1.151690	1.151599	9.1002e-05	9.5000e-05
0.8	1.106059	1.105972	8.6821e-05	9.0000e-05
0.9	1.055394	1.055314	8.0044e-05	8.2000e-05
L_2	-	-	1.0426e-04	-
L_∞	-	-	2.7126e-04	-

TABLE 4. Comparison of numerical values with exact solution and absolute error obtained from Bessel collocation method with other techniques for Example 5.4.

ξ	Exact solution	Numerical solution by Bessel collocation method	Absolute error by Bessel collocation method	Absolute error by B-spline method [6]	Absolute error by [32]
0.1	3.2756238169	3.2756238165	3.792318e-10	7.38165e-05	
0.2	3.3313215814	3.3313215813	8.760059e-11	7.35813e-05	1.7208e-09
0.3	3.4256414199	3.4256414206	7.032503e-10	7.14206e-05	
0.4	3.5608635375	3.5608635373	2.228844e-10	6.95373e-05	1.8712e-09
0.5	3.7402713692	3.7402713683	8.952803e-10	6.53683e-05	
0.6	3.9682461449	3.9682461451	2.212768e-10	5.91451e-05	2.1233e-09
0.7	4.2503934670	4.2503934677	6.833876e-10	5.14677e-05	
0.8	4.5937058612	4.5937058607	4.837366e-10	3.88607e-05	2.5085e-09
0.9	5.0067664244	5.0067664243	9.475620e-11	2.24243e-05	
L_2	-	-	2.434153e-10	-	-
L_∞	-	-	3.893299e-10	-	-

Uncorrected Proof

Table 5. Comparison of numerical values with exact solution for Example 5.5.

ξ	Exact solution	Numerical solution by Bessel collocation method	Absolute error by Bessel collocation method
0.1	0.313269401	0.313265850	3.5502e-06
0.2	0.303017975	0.303015423	2.5519e-06
0.3	0.286046949	0.286047265	3.1677e-07
0.4	0.262528148	0.262531127	2.9790e-06
0.5	0.232692979	0.232696784	3.8053e-06
0.6	0.196824158	0.196826806	2.6480e-06
0.7	0.155247456	0.155248107	6.5095e-07
0.8	0.108323430	0.108322763	6.6700e-07
0.9	0.056439164	0.056438603	5.6162e-07
L_2	-	-	1.7341e-06
L_∞	-	-	3.3788e-06

Table 6. Comparison of numerical values with exact solution for Example 5.6.

ξ	Exact solution	Numerical solution by Bessel collocation method	Absolute error by Bessel collocation method
0.1	1.01005013	1.01005017	3.6909e-08
0.2	1.04080766	1.04081077	3.1108e-06
0.3	1.09413785	1.09417428	3.6427e-05
0.4	1.17329862	1.17351087	2.1225e-04
0.5	1.28317623	1.28402541	8.4918e-04
0.6	1.43064131	1.43332941	2.6881e-03
0.7	1.62505284	1.63231622	7.2634e-03
0.8	1.87894995	1.89648088	1.7531e-02
0.9	2.20898683	2.24790798	3.8921e-02
L_2	-	-	8.1095e-02
L_∞	-	-	3.3856e-02

Uncorrected Proof

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