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A novel hybrid approach to approximate fractional sub-diffusion equation

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Abstract

This article introduces a new numerical hybrid approach based on an operational matrix and spectral technique to solve Caputo fractional sub-diffusion equations. This method transform the model into a set of nonlinear algebraic equation system. Chebyshev polynomials are used as basis function. The study includes theoretical analysis to demonstrate the convergence and error bounds of the proposed method. Two test problems are conducted to illustrate the method's accuracy. The results indicate the efficiency of the proposed method.

Keywords. Fractional sub-diffusion equation, Operational matrix, Spectral collocation method.
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1. Introduction

In recent decades, there has been a growing recognition of the significant applications of fractional calculus in various engineering disciplines. This growing recognition and use of fractional calculus principles has promoted the emergence of fractional partial differential equations (FPDEs) as an important area of research. As the applications of fractional calculus concepts have widened, the need to solve and comprehend FPDEs has increased in importance. Meanwhile, fractional sub-diffusion equations (FSDEs) are a powerful tool for modeling anomalous diffusion processes, where particles spread in a way that deviates from the classical Fickian diffusion model. These equations find applications in various scientific fields, including physics (describing charge transport in disordered materials), finance (modeling stock price fluctuations), and engineering (characterizing pollutant transport in porous media) [15]. Obtaining analytical solutions for most FSDEs is quite challenging. As a result, significant efforts have focused on developing numerical methods to solve these equations. In this study, we focus on FSDEs as follows:

$$\frac{\partial^\eta f(x,t)}{\partial t^\eta} = B\frac{\partial^2 f(x,t)}{\partial x^2} + Q(x,t), \quad B>0,\ \eta\in(0,1),\ x\in(0,l),\ t\in(0,\tau], \tag{1.1}$$

with initial

$$f(x,0)=f_1(x),\ \mathrm{x}\in[0,l], \tag{1.2}$$

and boundary condition

$$\begin{aligned} f(0,t)&=f_2(t),\ \mathrm{t}\in(0,\tau],\\ f(l,t)&=f_3(t),\ \mathrm{t}\in(0,\tau], \end{aligned} \tag{1.3}$$

where η and B represent the fractional order and the diffusion coefficient, respectively. η is described in the Caputo sense as follows:

$$\frac{\partial^\eta f(x,t)}{\partial t^\eta} = \frac{1}{\Gamma(n-\eta)}\int_0^t (t-s)^{n-\eta-1}\frac{\partial^n f(x,s)}{\partial s^n}ds, \qquad n-1<\eta\le n, t>0, n\in\mathbb{N}. \tag{1.4}$$

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Recent research has explored various methods to approximate solutions for this type of equation. For instance, Dehghan et al. [6] proposed a fully discrete method to approximate the time fractional reaction-sub-diffusion equation. They used a time discrete scheme based on a finite difference scheme and the meshless Galerkin method to approximate the spatial derivatives. Hooshmandasl et al. [11] proposed an efficient Galerkin method for solving FSDE. They used fractional-order Legendre functions (FLFs) as basis function. Sweilam et al. [22] investigated a straightforward numerical technique for solving two significant types of FSDE, commonly observed in spiny neuronal dendrites and chemical reactions.
The spectral method has emerged as a powerful tool for numerically approximating solutions to various fractional integral and differential equations [16, 18, 31]. Its popularity arises from its versatility in handling problems over finite and infinite intervals to achieve rapid convergence with minimal grid points, leading to computational efficiency [17]. This method offers a flexible framework for approximating various fractional equations and their boundary conditions. The spectral method's strength lies in its accuracy, efficiency, and adaptability, making it a valuable tool for understanding and solving complex fractional systems [14].
The intrinsic difficulties associated with fractional equations, such as non-locality, singularity, and intricate spectral behavior, make their numerical approximation a complex task [2]. A significant advantage of the spectral method is its ability to efficiently and accurately tackle problems with intricate geometries and boundary conditions. Various orthogonal polynomials, including Jacobi, Legendre, and Chebyshev polynomials, serve as bases for the spectral method [24, 25, 27]. Existing studies showcase the application of the spectral method in specific fractional calculus problems. Zayernouri et al. introduced a spectral collocation technique utilizing fractional Lagrange interpolation for time fractional partial differential equations (FPDEs) [29]. Zaky et al. proposed the tau method as another approach to approximate solutions to the fractional diffusion equation [28]. These examples demonstrate the spectral method's versatility in addressing various fractional calculus problems, showcasing its ability to handle intricate geometries and boundary conditions [5]. Overall, the spectral method establishes itself as an efficient tool for solving FPDEs, offering flexibility, rapid convergence, and the ability to handle complex geometries and boundary conditions.
Several recent studies have explored the spectral method in conjunction with operational matrices as a practical approach for solving FPDEs over finite and infinite intervals [12, 30]. Operational matrices are powerful tools used to approximate solutions of FPDEs. This approach represents fractional derivatives in terms of algebraic matrices, allowing the transformation of FPDEs into a system of algebraic equations [10, 19, 32]. The operational matrix method provides an efficient and accurate numerical scheme for solving FPDEs [1, 23, 26]. These matrices enable the computation of approximate solutions with reduced complexity, making them valuable when analytical solutions are unavailable or computationally expensive [20].
This article presents a novel method for solving Caputo fractional sub-diffusion equations. Our approach combines operational matrices with spectral techniques based on the shifted Chebyshev polynomials (SCPs). This method effectively transforms the fractional model into a system of algebraic equations, enabling the computation of efficient and accurate solutions. Furthermore, we estimate the corresponding error bound and apply the proposed technique to two sample problems to validate its effectiveness. The remainder of this article is structured as follows:
Section 2 provides an overview of essential concepts in fractional calculus and mathematical preliminaries, including the properties of Chebyshev polynomials and their derivative computation. Section 3 presents the proposed numerical algorithm for approximating the solution of the Caputo fractional sub-diffusion equation. In section 4, the error bound of the proposed method is computed. Section 5 showcases numerical experiments to demonstrate the method's efficiency. Finally, a concluding remark summarizes the key findings of the paper.

2. Mathematical preliminaries

In this part, some basic fractional calculus ideas and fundamental principles are reviewed [26].

Definition 2.1. Following are the formulas used to determine the Riemann-Liouville fractional integral of order $\eta > 0$.

$$\begin{aligned} &\mathrm{I}^0 v(t) = v(t) \\ &\mathrm{I}^\eta v(t) = \frac{1}{\Gamma(\eta)} \int_0^t v(s)(t-s)^{(\eta-1)} ds = \frac{1}{\Gamma(\eta)} t^{\eta-1} * v(t), \quad \eta > 0, t > 0, \end{aligned} \tag{2.1}$$

where $\Gamma(.)$ and $*$ represent the gamma function and the convolution operator, respectively.

Definition 2.2. The Caputo fractional derivative of order η is determined as follows:

$$D^\eta v(t) = I^{n-\eta} D^n v(t) = \frac{1}{\Gamma(n-\eta)} \int_0^t (t-s)^{n-\eta-1} \frac{d^n}{ds^n} v(s) ds, \quad n-1 < \eta \leq n,\, t > 0, \tag{2.2}$$

where D^n is the classical differential operator of order n. The following apply to the Caputo derivation:

$$D^\eta t^i = \begin{cases} 0, & if\, i \in \mathbb{N}_0\, and\, i < \lceil \eta \rceil, \\ \frac{\Gamma(i+1)}{\Gamma(i+1-\eta)} t^{(i-\eta)}, & if\, i \in \mathbb{N}_0\, and\, i \geq \lceil \eta \rceil\, or\, i \notin \mathbb{N}\, and\, i > \lfloor \eta \rfloor. \end{cases} \tag{2.3}$$

One of the characteristics of the Caputo derivative is its linearity, i.e.

$$D^\eta (\textstyle\sum_{i=0}^n a_i v_i(t)) = \sum_{i=0}^n a_i D^\eta v_i(t), \tag{2.4}$$

where $a_i, i = 0, 1, \ldots, n$ are constants.

Definition 2.3. The Legendre weighted Sobolev space on $\Omega = I_x \times I_t = [0, l] \times [0, \tau]$ is defined as:

$$B^n(\Omega) = \left\{ f : \frac{\partial^k f}{\partial t^k} \in L^2(\Omega), k = 0, 1, \ldots, n \right\}, \tag{2.5}$$

with

$$(f, u)_{B^n} = \sum_{k=0}^n \left(\frac{\partial^k f}{\partial t^k}, \frac{\partial^k u}{\partial t^k} \right), \|f\|_{B^n} = (f, f)_{B^n}^{\frac{1}{2}}.$$

It is worth mentioning that $H^n(\Omega)$ is a subspace of (2.5), i.e.:

$$\|f\|_{B^n} \leq c \|f\|_{H^n}, n \geq 0.$$

Definition 2.4. The Hilbert space $\mathrm{H}^{(\mathrm{s,r})}(\Omega)$ on $\Omega = I_x \times I_t = [0, l] \times [0, \tau]$ is defined as

$$\mathrm{H}^{\mathrm{s,r}}(\Omega) = \mathrm{H}^{\mathrm{r}}(\mathrm{I_t}, \mathrm{H}^{\mathrm{s}}(\mathrm{I_x})) = \left\{ \mathrm{f} \in \mathrm{L}^2(\Omega) \mid \frac{\partial^{\mathrm{i+j}} \mathrm{f(x,t)}}{\partial \mathrm{x^i} \partial \mathrm{t^j}} \in \mathrm{L}^2(\Omega), 0 \leq \mathrm{i} \leq \mathrm{s}, 0 \leq \mathrm{j} \leq \mathrm{r} \right\}, \tag{2.6}$$

Such that the norm and the inner product are determined as follows:

$$\begin{aligned} &\|\mathrm{f}\|_{\mathrm{H}^{s,r}} = \left(\sum_{\mathrm{i}=0}^{s} \sum_{\mathrm{j}=0}^{\mathrm{r}} \left\| \frac{\partial^{\mathrm{i+j}} \mathrm{f(x,t)}}{\partial \mathrm{x^i} \partial \mathrm{t^j}} \right\|_{L^2(\omega)}^2 \right)^{\frac{1}{2}}, \\ &\langle \mathrm{f}, \mathrm{g} \rangle_{\mathrm{s,r}} = \sum_{\mathrm{i}=0}^{s} \sum_{\mathrm{j}=0}^{\mathrm{r}} \int_0^\tau \int_0^l \frac{\partial^{\mathrm{i+j}} \mathrm{f(x,t)}}{\partial \mathrm{x^i} \partial \mathrm{t^j}} \frac{\partial^{\mathrm{i+j}} \mathrm{g(x,t)}}{\partial x^{\mathrm{i}} \partial \mathrm{t}^j} dx dt. \end{aligned} \tag{2.7}$$

2.1. **Chebyshev polynomials.** The Chebyshev polynomial of degree j is determined as follows:

$$\begin{aligned} &G_0(t) = 1,\, G_1(t) = t, \\ &G_{j+1}(t) = 2t G_j(t) - G_{j-1}(t), \end{aligned} \tag{2.8}$$

where $j = 0, 1, 2, \ldots$ and $t \in [-1, 1]$. SCPs is defined in the interval $[0, \tau]$ by using a change of variable $z = \frac{2t}{\tau} - 1$. We denote $G_j(\frac{2t}{\tau} - 1)$ as $G_{\tau,j}(t)$. The analytical expression for $G_{\tau,j}(t)$ is provided by:

$$G_{\tau,j}(t) = \tau \sum_{k=0}^{j} (-1)^{j-k} \frac{(j+k-1)! 2^{2k}}{(j-k)!(2k)! \tau^k} t^k, j = 1, 2, 3, \ldots, N, \tag{2.9}$$

where $G_{\tau,j}(0) = (-1)^j$ and $G_{\tau,j}(\tau) = 1$.
Let $\omega_\tau(t) = \frac{1}{\sqrt{t(\tau-t)}}$ is considered as the weight function for the following space:

$$L^2_\omega[0,\tau] := \left\{ f \mid f : [0,\tau] \to \mathbb{R} \quad such\, that \quad \int_0^\tau \omega_\tau(t) f^2(t) dt < \infty \right\}, \tag{2.10}$$

The set $\{G_{\tau,j}(t)\}_{j=0}^{\infty}$ consists of classical Chebyshev polynomials, determine a complete orthogonal system on $L^2_\omega[0,\tau]$ [13]. If $f(t) \in L^2_\omega[0,\tau]$ then:

$$f(t) = \sum_{j=0}^{\infty} a_j G_{\tau,j}(t), \tag{2.11}$$

such that

$$a_j = \frac{4}{\tau\pi\epsilon_j} \int_0^\tau f(t) G_{\tau,j}(t) \omega_\tau(t) dt, \quad j = 0,1,2,\dots \quad . \tag{2.12}$$

where

$$\epsilon_j = \begin{cases} 2, & j = 0, \\ 1, & j \geq 1. \end{cases} \tag{2.13}$$

In practical applications, a finite set of equations, as given by Eq. (2.11), is typically regarded as:

$$f_A(t) = \sum_{j=0}^{N} a_j G_{\tau,j}(t) = A^T \psi_{\tau,N}(t), \tag{2.14}$$

where

$$\begin{aligned} &\psi_{\tau,N}(t) = [G_{\tau,0}(t), G_{\tau,1}(t), \dots, G_{\tau,N}(t)], \\ &A^T = [a_0, \dots, a_N]. \end{aligned} \tag{2.15}$$

Next, we can determine the derivative of $\psi_{\tau,N}(t)$ through the following relation:

$$\frac{d\psi_{\tau,N}(t)}{dt} = \mathbf{D}^{\mathbf{1}}_{\tau,\mathbf{N}} \psi_{\tau,N}(t), \tag{2.16}$$

such that $\mathbf{D}^{\mathbf{1}}_{\tau,\mathbf{N}}$ represents operational matrix of derivative as

$$\mathbf{D}^{\mathbf{1}}_{\tau,\mathbf{N}} = (d_{is}) = \begin{cases} \frac{4i}{\epsilon_s \tau}, & s = 0,1,\dots, i = s + c, \begin{cases} c = 1,3,5,\dots,N-1, & \text{N is even} \\ c = 1,3,5,\dots,N, & \text{N is odd} \end{cases} \\ 0, & otherwise, \end{cases} \tag{2.17}$$

and ϵ_s is considered as Eq. (2.13). By utilizing SCPs, any continuous function $f(x,t)$ is expressed in the following manner:

$$f(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \hat{k}_{ij} G_{l,i}(x) G_{\tau,j}(t), \quad (x,t) \in [0,l] \times [0,\tau]. \tag{2.18}$$

After truncating both dimensions, we obtained

$$f_{M,N}(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} \hat{k}_{ij} G_{l,i}(x) G_{\tau,j}(t) = \psi^T_{l,M}(x) \mathbf{K} \psi_{\tau,N}(t). \tag{2.19}$$

where

$$\begin{aligned} &\psi_{l,M}(x) = [G_{l,0}(x), G_{l,1}(x), \dots, G_{l,M}(x)]^{\mathrm{T}}, \\ &\psi_{\tau,N}(t) = [\mathrm{G}_{\tau,0}(t), G_{\tau,1}(t), \dots, G_{\tau,N}(t)]^T. \end{aligned} \tag{2.20}$$

and the coefficient matrix $\mathbf{K}$ is obtained by

$$\mathbf{K}=\begin{pmatrix} \hat{k}_{00} & \hat{k}_{01} & & \hat{k}_{0N} \\ \hat{k}_{10} & \hat{k}_{11} & \cdots & \hat{k}_{1N} \\ & \vdots & \ddots & \vdots \\ \hat{k}_{M0} & \hat{k}_{M1} & \cdots & \hat{k}_{MN} \end{pmatrix}, \tag{2.21}$$

where

$$\begin{aligned} &\hat{\mathrm{k}}_{\mathrm{ij}}=\frac{1}{h_i h_j}\int_0^{\tau}\int_0^{l} f(x,t)G_{l,i}(x)G_{\tau,j}(t)\omega_{\tau}(t)\omega_l(x)dxdt, \\ &\omega_l(x)=\frac{1}{\sqrt{x(l-x)}},\ \omega_{\tau}(t)=\frac{1}{\sqrt{t(\tau-t)}}, \\ &h_i=\frac{\epsilon_i}{2}\pi, \qquad h_j=\frac{\epsilon_j}{2}\pi. \end{aligned} \tag{2.22}$$

ϵ_i, ϵ_i for $i=0,1,\dots,M$, $j=0,1,\dots,N$ are considered as Eq. (2.13).
The following theorem states that the fractional derivative of SCPs can be determined using an operational matrix form.

Theorem 2.5. *Let $\psi_{\tau,N}(t)$ denote the shifted Chebyshev vector defined in Equation (2.20), and η represent the fractional derivative order. The following relationship holds:*

$$D^{\eta}\psi_{\tau,N}(t)\simeq \mathbf{D}^{\eta}_{\tau,\mathbf{N}}\psi_{\tau,N}(t). \tag{2.23}$$

In this case, $\mathbf{D}^{\eta}_{\tau,\mathbf{N}}$ represents the operational matrix of the fractional derivative, which has dimensions $(N+1)\times(N+1)$ and we have:

$$\mathbf{D}^{\eta}_{\tau,\mathbf{N}}=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ d_{\eta}(\lceil\eta\rceil,0) & d_{\eta}(\lceil\eta\rceil,1) & d_{\eta}(\lceil\eta\rceil,2) & \cdots & d_{\eta}(\lceil\eta\rceil,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ d_{\eta}(p,0) & d_{\eta}(p,1) & d_{\eta}(p,2) & \cdots & d_{\eta}(p,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ d_{\eta}(N,0) & d_{\eta}(N,1) & d_{\eta}(N,2) & \cdots & d_{\eta}(N,N) \end{pmatrix}, \tag{2.24}$$

where

$$d_{\eta}(p,q)=\sum_{c=\lceil\eta\rceil}^{p}\frac{(-1)^{p-c}2p(p+c-1)!\Gamma\left(c-\eta+\frac{1}{2}\right)}{\epsilon_q\tau^{\eta}\Gamma\left(c+\frac{1}{2}\right)(p-c)!\Gamma(c-\eta-q+1)\Gamma(c+q-\eta+1)}. \tag{2.25}$$

where $p=\lceil\eta\rceil,...,N$ and $q=\lceil\eta\rceil,...,N$ and ϵ_q is considered as Eq. (2.13). Not that the first $\lceil\eta\rceil$ are zeros in Eq. (2.24) and $\lceil\eta\rceil$ denotes ceiling function.

Proof. [9].

□

Moreover, similar to Eq. (2.17), we can compute $\mathbf{D}^{\mathbf{1}}_{\mathbf{l},\mathbf{M}}(\mathbf{x})$. It is obvious that

$$\mathbf{D}^{\mathbf{m}}_{\mathbf{l},\mathbf{M}}(\mathbf{x})=\left(\mathbf{D}^{\mathbf{1}}_{\mathbf{l},\mathbf{M}}(\mathbf{x})\right)^{\mathbf{m}} \quad m\in\mathbb{N}. \tag{2.26}$$

3. The proposed method

This section outlines a suggested approach for estimating the solution of the model (1.1)-(1.3). First of all, we rewrite Eq. (1.1)-(1.3) as follows:

$$\frac{\partial^\eta f(x,t)}{\partial t^\eta} - B\frac{\partial^2 f(x,t)}{\partial x^2} = f(x,0) - f_1(x) + Q(x,t), \tag{3.1}$$

with boundary condition

$$\begin{aligned} f(0,t) &= f_2(t),\ \mathrm{t} \in (0,\tau], \\ f(l,t) &= f_3(t),\ \mathrm{t} \in (0,\tau]. \end{aligned} \tag{3.2}$$

By substituting operational matrix define in Eqs. (2.19), (2.23) and (2.26) in Eq. (3.1), we obtain

$$\psi^T_{l,M}(x)\left(\mathbf{K}\mathbf{D}^\eta_{\tau,\mathbf{N}} - B\left(\mathbf{D}^{\mathbf{2}}_{\mathbf{l},\mathbf{M}}\right)^T\mathbf{K}\right)\psi_{\tau,N}(t) = \psi^T_{l,M}(x)\mathbf{K}\psi_{\tau,N}(0) - f_1(x) + Q(x,t), \tag{3.3}$$

We consider the roots of $\mathrm{G_{l,M-1}(x)}$ and $\mathrm{G_{\tau,N+1}(t)}$ as collocation nodes, denoted by $\mathrm{x_i}$ for $\mathrm{i} = 0,1,\dots,\mathrm{M}-2$ and $\mathrm{t_j}$ for $\mathrm{j} = 0,1,\dots,\mathrm{N}$ respectively. we obtain a system of algebraic equations with a total order of $(\mathrm{M}-1)\times(\mathrm{N}+1)$ by utilizing these collocation nodes in Eq. (3.3), i.e.:

$$\psi^T_{l,M}(x_i)\left(\mathbf{K}\mathbf{D}^\eta_{\tau,\mathbf{N}} - B\left(\mathbf{D}^{\mathbf{2}}_{\mathbf{l},\mathbf{M}}\right)^T\mathbf{K}\right)\psi_{l,M}(t_j) = \psi^T_{l,M}(x_i)\mathbf{K}\psi_{\tau,N}(0) - f_1(x_i) + Q(x_i,t_j). \tag{3.4}$$

It is necessary $2(\mathrm{N}+1)$ additional equations to ensure a unique solution for the system (3.4). Therefore, we use Eq. (2.19) in the boundary condition, i.e.:

$$\begin{aligned} &\psi^T_{l,M}(0)\,\mathbf{K}\psi_{\tau,N}(t) = f_2(t_j), \\ &\psi^T_{l,M}(l)\,\mathbf{K}\psi_{\tau,N}(t) = f_3(t_j), \end{aligned} \tag{3.5}$$

where $j = 0,1,\dots,N$. By associating Eq. (3.5) with Eq. (3.4) yields $(M+1)\times(N+1)$ nonlinear system equation which is solved by the Newton method. Then, $\mathbf{K}$ is applied in Eq. (2.19) to compute approximation of $f(x,t)$.

4. Error bound

In this part, we demonstrate an error bound for the proposed method. Let $\Omega = I_x \times I_t = [0,l]\times[0,\tau]$ and $P_{M,N}(\Omega) = span\{G_{l,i}(x)G_{\tau,j}(t), i = 1,\dots,M, j = 1,\dots,N\}$. $\Pi_{M,N}f$ is defined from $L^2(\Omega)$ into $P_{M,N}(\Omega)$ in the following manner

$$(\Pi_{M,N}f - f, u) = 0, \forall u \in P_{M,N}(\Omega),$$

in other words,

$$(\Pi_{M,N}f)(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}G_{l,i}(x)G_{\tau,j}(t).$$

Indeed, $\Pi_{M,N}f$ represents the best approximation of f out of $P_{M,N}(\Omega)$ [21]. Initially, it is necessary to proof the following Theorems 4.1, 4.2, and 4.3.

Theorem 4.1. *For any $f \in H^s(\Omega)$, we have*

$$\|D^q_x(f - \Pi_M f)\|_{L^2(\Omega)} \le C_1\sqrt{\frac{(M-s+1)!}{(M-q+1)!}}(\mathrm{M}+s)^{\frac{q-s}{2}}\|\partial^s_x f\|_{H^s}, \quad 0 \le q < s \le M+1, q \in \mathbb{N} \tag{4.1}$$

where C_1 is a constant and $q \in \mathbb{N}$ and $H^s(\Omega)$ is considered as defined in Definition 2.3. Therefore,

$$\le C_1\sqrt{\frac{(M-s+1)!}{(M-q+1)!}}(M+s)^{\frac{q-s}{2}}\|f\|_{H^s}. \tag{4.2}$$

and

$$\|D^q_x(f - \Pi_M f)\|_{L^2(\Omega)} \le C_1\sqrt{\frac{(M-s+1)!}{(M-q+1)!}}(M+s)^{\frac{q-s}{2}}\|f\|_{H^s}. \tag{4.3}$$

Proof. [21] □

Theorem 4.2. *Suppose* $\Pi_{M,N}f$ *is the projection of* f *upon* $P_{M,N}$ *as*

$$(\Pi_{M,N}f)(x,t) = f_{M,N}(x,t). \tag{4.4}$$

there are constants C_2 *and* C_3 *for any function* $f \in L_2(\Omega)$ *such that*

$$\|f - \Pi_{M,N}f\|_{L^2(\Omega)} \le C_2\sqrt{\frac{(M-s+1)!}{(M+1)!}}(M+s)^{-\frac{s}{2}}\|f\|_{H^{s,0}} + C_3\sqrt{\frac{(N-r+1)!}{(N+1)!}}(N+r)^{-\frac{r}{2}}\|f\|_{H^{0,r}}, \tag{4.5}$$

where $0 \le s \le M+1$ *and* $0 \le r \le N+1, s, r \in \mathbb{N}$. $H^{s,0}$ *and* $H^{0,r}$ *are considered as definition 2.4.*

Proof. [3]. □

Theorem 4.3. *Assume that* $r \in \mathbb{N}$, $n_\eta - 1 < \eta \le n_\eta = \lceil \eta \rceil$, $n_\eta < r \le N+1$ *and* $f \in L^2(\Omega)$ *then*

$$\|D_t^\eta f - D_t^\eta(\Pi_{M,N}f)\|_{L^2(\Omega)} \le \frac{C_\eta\sqrt{\frac{(N-r+1)!}{(N-n_\eta+1)!}}(N+r)^{\frac{(n_\eta-r)}{2}}}{\Gamma(n_\eta-\eta+1)}\|f\|_{H^{0,r}}, \tag{4.6}$$

Where C_3 *is constant. Also,* $H^{0,r}$ *is considered as Definition 2.4.*

Proof. Employing Eq. (2.1), and the Following Eq. [3]

$$\|f * g\|_{L^2(\Omega)} \le \|f\|_1\|g\|_{L^2(\Omega)}. \tag{4.7}$$

we obtain

$$\begin{aligned}\|D_t^\eta f - D_t^\eta(\Pi_{M,N}f)\|^2_{L^2(\Omega)} &= \left\|I^{n_\eta-\eta}\left(D_t^{n_\eta}f - D_t^{n_\eta}(\Pi_{M,N}f)\right)\right\|^2_{L^2(\Omega)},\\ &= \|\frac{1}{t^{1+\eta-n_\eta}\Gamma(n_\eta-\eta)} * \left(D_t^{n_\eta}f - D_t^{n_\eta}(\Pi_{M,N}f)\right)\|^2_{L^2(\Omega)},\\ &\le \|\frac{1}{t^{1+\eta-n_\eta}\Gamma(n_\eta-\eta)}\|_1^2 \left\|D_t^{n_\eta}f - D_t^{n_\eta}(f)\right\|^2_{L^2(\Omega)},\end{aligned} \tag{4.8}$$

According to [3] and Theorem 4.1,

$$\le \left(\frac{1}{\Gamma(n_\eta-\eta+1)}\right)^2\left(C_\eta\sqrt{\frac{(N-r+1)!}{(N-n_\eta+1)!}}(N+r)^{\frac{(n_\eta-r)}{2}}\right)^2\|f\|^2_{H^{0,r}}. \tag{4.9}$$

□

Assume $f_{M,N}(x,t)$ denotes the numerical solution obtained by the proposed method. The error between $f(x,t)$ and $f_{M,N}(x,t)$ is obtained as follows:

$$e_{M,N}(x,t) = f_{M,N}(x,t) - f(x,t).$$

To establish the convergence, it is essential to demonstrate that $f_{M,N}(x,t)$ converges towards $f(x,t)$ as $M, N \to \infty$. By substituting $f_{M,N}(x,t)$ into Eq. (3.1), we have:

$$\frac{\partial^\eta f_{M,N}(x,t)}{\partial t^\eta} = B\frac{\partial^2 f_{M,N}(x,t)}{\partial x^2} + Q(x,t) + f_{M,N}(x,0) - f_1(x) + R_{M,N}(x,t), \tag{4.10}$$

where $R_{M,N}(x,t)$ called the residual function and

$$f_{M,N}(0,t) = f_2(t), \quad f_{M,N}(l,t) = f_3(t), \tag{4.11}$$

By subtraction Eq. (4.10) from Eq. (3.1). We obtain

$$R_{M,N}(x,t) = \frac{\partial^\eta e_{M,N}(x,t)}{\partial t^\eta} - B\frac{\partial^2 e_{M,N}(x,t)}{\partial x^2}, \tag{4.12}$$

with boundary conditions

$$e_{M,N}(0,t) = 0, \quad e_{M,N}(l,t) = 0, \tag{4.13}$$

By taking norm on the both side of Eq. (4.12) and using the triangle inequality

$$\|R_{M,N}(x,t)\|_{L^2(\Omega)} \leq \left\| \frac{\partial^\eta e_{M,N}(x,t)}{\partial t^\eta} \right\|_{L^2(\Omega)} + \left\| B \frac{\partial^2 e_{M,N}(x,t)}{\partial x^2} \right\|_{L^2(\Omega)}. \tag{4.14}$$

According to Theorems 4.1 and 4.3, we obtain

$$\|R_{M,N}(x,t)\|_{L^2(\Omega)} \leq \frac{C_\eta \sqrt{\frac{(N-r+1)!}{(N-n_\eta+1)!}} (N+r)^{\frac{(n_\eta - r)}{2}}}{\Gamma(n_\eta - \eta + 1)} \|f\|_{H^{0,r}} + BC \sqrt{\frac{(M-s+1)!}{(M-2+1)!}} (M+s)^{\frac{2-s}{2}} \|f\|_{H^{s,0}}, \tag{4.15}$$

where C_η and C are constants. when $M, N \to \infty$, we have $\|\mathrm{R_{M,N}}\| \to 0$. Therefore, $f_{M,N}(x,t)$ approaches to $f(x,t)$ as $M, N \to \infty$.

5. Numerical examples

All computations were performed by applying MATLAB 2023 on a Laptop with $12th$ Gen Intel® Core$^{\text{TM}}$i7 and 16GB, DDR4 memory. Also, the maximum absolute errors (MAEs) are derived as

$$e^{M,N} := \max \{ |f(x_i,t_j) - f_{M,N}(x_i,t_j)| \} \quad i = 0,1,\ldots,M, \quad j = 0,1,\ldots,N. \tag{5.1}$$

Example 5.1. The sub-diffusion equation is considered as follows:

$$\frac{\partial^\eta f(x,t)}{\partial t^\eta} = \frac{\partial^2 f(x,t)}{\partial x^2} + \sin \pi x \left(\Gamma(1+\eta) + \pi^2 t^\eta \right), \quad \eta \in (0,1],\ x \in (0,1),\ t \in (0,1], \tag{5.2}$$

and

$$\begin{aligned} f(x,0) &= 0, \quad x \in [0,1], \\ f(0,t) &= 0, \quad t \in (0,1], \\ f(1,t) &= 0, \quad t \in (0,1], \end{aligned} \tag{5.3}$$

The exact solution of Eqs. (5.2)-(5.3) is

$$\mathrm{f(x,t)} = \mathrm{t}^\eta \sin \pi \mathrm{x}. \tag{5.4}$$

The numerical errors $e^{M,N}$ for $\mathrm{M} = \mathrm{N}$, $\mathrm{M} = 5,7,9,11$ and $\eta = 0.1, 0.3, 0.5, 0.7, 0.9$ are tabulated in Table 1. The absolute error plotted in Figure 1. In Figure 2(a) and Figure 2(b), the contour plots of the exact and the approximate solution are illustrated, respectively.

Table 1. The numerical errors $e^{M,N}$ for various values of M and N as the number of collocation points for Example 5.1.

$M = N$	5	7	9	11
$\eta = 0.1$	$4.7725E-06$	$1.2185E-06$	$4.7165E-07$	$3.62001E-07$
$\eta = 0.3$	$3.5222E-06$	$8.1345E-07$	$2.9182E-07$	$8.8966E-08$
$\eta = 0.5$	$4.9956E-06$	$1.0398E-06$	$3.0072E-07$	$1.0682E-07$
$\eta = 0.7$	$9.6609E-06$	$1.4863E-06$	$3.7532E-07$	$1.2641E-07$
$\eta = 0.9$	$3.3110E-06$	$4.3095E-07$	$7.5592E-08$	$2.6987E-08$

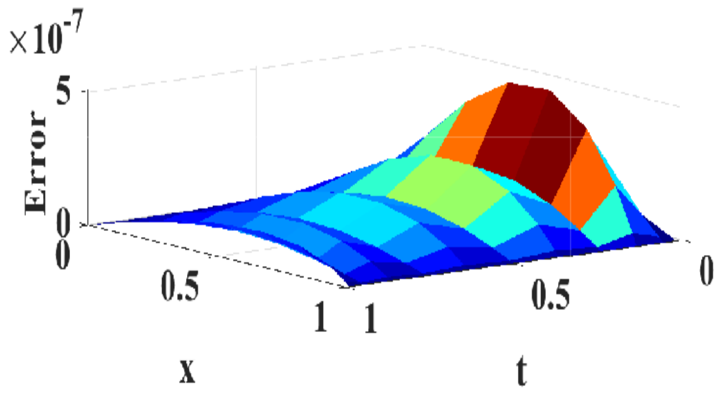

FIGURE 1. The graph of absolute error, with N = M = 7 and $\eta = 0.75$, for Example 5.1.

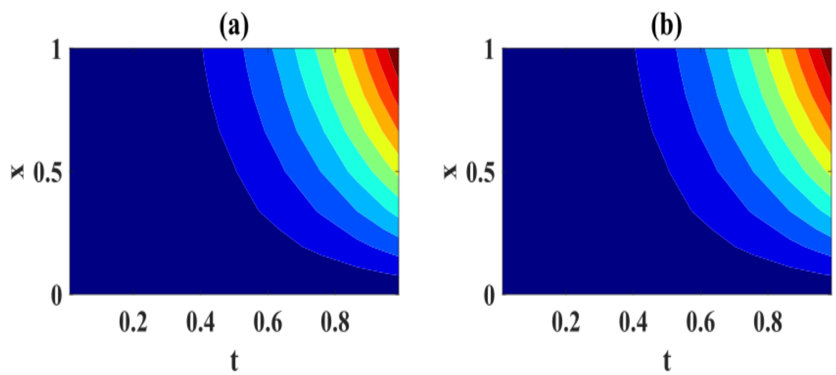

FIGURE 2. Contour plots for (a) the exact and (b) the approximate solution, N = M = 7 and $\eta = 0.75$ for Example 5.1.

Example 5.2. The following fractional sub-diffusion equation is considered as

$$\frac{\partial^\eta f(x,t)}{\partial t^\eta} = \frac{\partial^2 f(x,t)}{\partial x^2} + e^x\left(\Gamma(2+\eta)t - t^{1+\eta}\right), \quad \eta \in (0,1],\, x \in (0,1),\, t \in (0,1], \tag{5.5}$$

and

$$\begin{array}{ll} f(x,0) = 0, & x \in [0,1], \\ f(0,t) = t^{1+\eta}, & \mathrm{t} \in (0,1], \\ f(1,t) = e^1 t^{1+\eta}, & t \in (0,1]. \end{array} \tag{5.6}$$

The exact solution of Eqs. ((5.5)-(5.6)) is

$$f(x,t) = e^x t^{1+\eta}.$$

The numerical errors $e^{M,N}$ for $M = N$, $M = 5,7,9,11$ and $\eta = 0.1, 0.3, 0.5, 0.7, 0.9$ are tabulated in Table 2. The absolute error is plotted in the Figure 3. In Figure 4(a) and Figure 4(b), the contour plots of the exact and the approximate solution are illustrated, respectively.

TABLE 2. The numerical errors $e^{M,N}$ for various values of M and N as the number of collocation points for Example 5.2.

$M = N$	5	7	9	11
$\eta = 0.1$	$2.8389E-04$	$1.5035E-04$	$9.2124E-05$	$7.5181E-05$
$\eta = 0.3$	$2.4838E-04$	$1.1691E-04$	$6.4991E-05$	$3.7091E-05$
$\eta = 0.5$	$3.0098E-05$	$5.0472E-06$	$4.0842E-06$	$3.5394E-06$
$\eta = 0.7$	$9.2509E-05$	$2.2176E-05$	$6.5376E-06$	$2.7481E-06$
$\eta = 0.9$	$2.2265E-05$	$6.7575E-06$	$4.7999E-06$	$2.4070E-06$

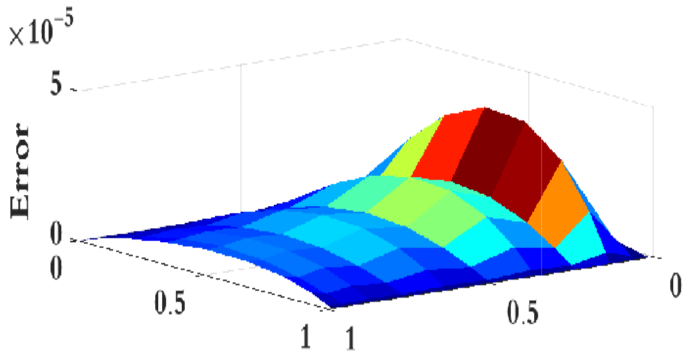

FIGURE 3. The graph of absolute error, with $N = M = 7$ and $\eta = 0.75$, for Example 5.2.

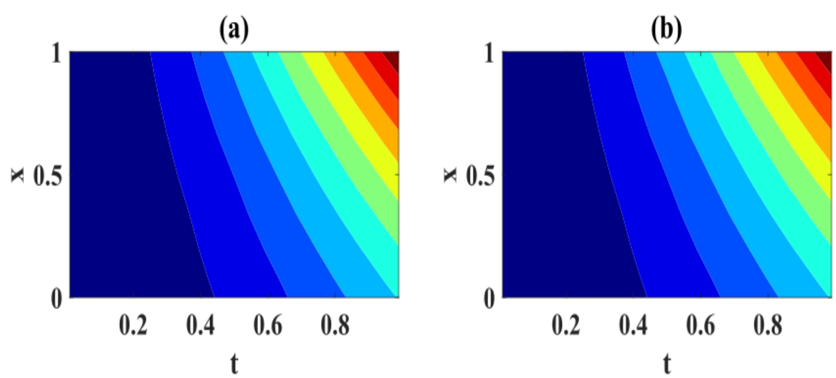

FIGURE 4. Contour plots for (a) the exact and (b) the approximate solution, N = M = 7 and $\eta = 0.75$ for Example 5.2.

6. CONCLUSION

This study introduces a novel numerical hybrid method based on the spectral method and operational matrix for solving Caputo's type fractional sub-diffusion equations. The theoretical convergence analysis demonstrates the method's convergence properties. Moreover, the numerical results obtained from various test problems validate the accuracy of the method. This study contributes to the field of fractional calculus by providing a novel method for solving sub-diffusion equations. Its accuracy and effectiveness make it an effective tool for researchers in related fields, allowing them to explore the application of this method to more complex problems and extend its capabilities to other types of fractional differential equations.

References

[1] S. Abbasbandy, S. Kazem, M. S. Alhuthali, and H. H. Alsulami, *Application of the operational matrix of fractional-order Legendre functions for solving the time-fractional convection–diffusion equation*, Appl. Math. Comput., *266* (2015), 31–40.

[2] I. G. Ameen, M. A. Zaky, and E. H. Doha, *Singularity preserving spectral collocation method for nonlinear systems of fractional differential equations with the right-sided Caputo fractional derivative*, Math. Comput. Simul., *392* (2021).

[3] A. Baseri, S. Abbasbandy, and E. Babolian, *A collocation method for fractional diffusion equation in a long time with Chebyshev functions*, Appl. Math. Comput., *322* (2018), 55–65.

[4] L. Beghin and C. Ricciuti, *Lévy Processes Linked to the Lower-Incomplete Gamma Function*, Fractal. Fract., *5* (2021).

[5] H. Chen, S. Lü, and W. Chen, *Spectral methods for the time fractional diffusion-wave equation in a semi-infinite channel*, Comput. Math. Appl., *71* (2016), 1818–1830.

[6] M. Dehghan, M. Abbaszadeh, and A. Mohebbi, *Error estimate for the numerical solution of fractional reaction-sub-diffusion process based on a meshless method*, J. Comput. Appl. Math., *280* (2015), 14–36.

[7] M. D'Elia, C. Glusa, T. Abdeljawad, and N. Mlaiki, *A fractional model for anomalous diffusion with increased variability: Analysis, algorithms and applications to interface problems*, Numer. Methods. Partial. Differ. Equ., *38* (2022), 2084–2103.

[8] C. S. Drapaca, *The Impact of Anomalous Diffusion on Action Potentials in Myelinated Neurons*, Fractal Fract., *5* (2021).

[9] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, *A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order*, Comput. Math. appl., *62* (2011), 2364–2373.

[10] H. Hassani and Z. Avazzadeh, *Novel operational matrices for solving 2-dim nonlinear variable order fractional optimal control problems via a new set of basis functions*, Appl. Numer. Math., *166* (2021), 26–39.

[11] M. R. Hooshmandasl, M. H. Heydari, and C. Cattani, *Numerical solution of fractional sub-diffusion and time-fractional diffusion-wave equations via fractional-order Legendre functions*, Eur. Phys. J. Plus., *131* (2016).

[12] M. Jani, E. Babolian and B. Dambaru, *A Petrov-Galerkin spectral method for the numerical simulation and analysis of fractional anomalous diffusion*, Int. J. Comput. Math., *44* (2020), 2021–2032.

[13] F. Kamali and H. Saeedi, *Generalized fractional-order Jacobi functions for solving a nonlinear systems of fractional partial differential equations numerically*, Math. Methods Appl. Sci., *41* (2017), 3155–3174.

[14] F. Lin and H. Qu, *A Runge-Kutta Gegenbauer spectral method for nonlinear fractional differential equations with Riesz fractional derivatives*, Int. J. Comput. Math., *96* (2018), 417–435.

[15] N. Mendes, M. Chhay, J. Berger, and D. Dutykh, *Numerical Methods for Diffusion Phenomena in Building Physics*, Springer Cham, 2019.

[16] M. Molavi-Arabshahi, J. Rashidinia, and S. Tanoomand, *An efficient spectral collocation method based on the generalized Laguerre polynomials to multi-term time fractional diffusion-wave equations*, AIP Advances, *14*(2) (2024), DOI: 10.1063/5.0187493.

[17] M. Molavi-Arabshahi, J. Rashidinia, and M. Yousefi, *An efficient approach for solving the fractional model of the human T-cell lymphotropic virus I by the spectral method*, Journal of Mathematical Modeling, *11*(3) (2023), 463–477.

[18] N. Rajagopala, S. Balajia, R. Seethalakshmia, and V. S. Balajib, *A new numerical method for fractional order Volterra integro-differential equations*, Ain. Shams. Eng. J., *11* (2020), 171–177.

[19] J. Rashidinia, T. Eftekhari, and K. Maleknejad, *Numerical solutions of two-dimensional nonlinear fractional Volterra and Fredholm integral equations using shifted Jacobi operational matrices via collocation method*, J. King. Saud. Univ. Sci., *33* (2021).

[20] J. Rashidinia and E. Mohmedi, *Approximate solution of the multi-term time fractional diffusion and diffusion-wave equations*, Comp. Appl. Math., *39* (2020).

[21] J. Shen, T. Tang, and L. Wang, *Spectral methods, Algorithms, analysis and applications*, Springer, 2010.

[22] N. H. Sweilam, S. M. Ahmed, and M. Adel, *A simple numerical method for two-dimensional nonlinear fractional anomalous sub-diffusion equations*, Math. Method. Appl. Sci., *5* (2020), 2914–2933.

[23] I. Talib, F. Jarad, M. U. Mirza, A. Nawaz, and M. B. Riaz, *A generalized operational matrix of mixed partial derivative terms with applications to multi-order fractional partial differential equations*, Alex. Eng. J., *61* (2013), 135–145.

[24] L. Z. Wu, S. R. Zhu, and J. Peng, *Application of the Chebyshev spectral method to the simulation of groundwater flow and rainfall-induced landslides*, Appl. Math. Model., *80* (2020), 408–425.

[25] C. Yang and J. Hou, *Jacobi spectral approximation for boundary value problems of nonlinear fractional pantograph differential equations*, Numer. Algor., *86* (2021), 1089–1108.

[26] X. J. Yang, *General Fractional Derivatives: Theory, Methods and Applications*, CRC Press Taylor & Francis Group, 2019.

[27] X. Yu, L. Li, and Z. Wang, *Efficient space-time Legendre rational spectral method for parabolic problems in unbounded domains*, Appl. Numer. Math., *170* (2021), 39–54.

[28] M. A. Zaky, *A Legendre spectral quadrature tau method for the multi-term time-fractional diffusion equations*, Comp. Appl. Math., *37* (2018), 3525–3538.

[29] M. Zayernouri and G. Karniadakis, *Fractional spectral collocation method*, SIAM J. Sci. Comput., *36* (2014), 40–62.

[30] B. Zhang, W. Bu, and A. Xiao, *Efficient difference method for time-space fractional diffusion equation with Robin fractional derivative boundary condition*, Numer. Algor., *88* (2021), 1965–1988.

[31] T. Zhang, H. Li, and Z. Wang, *Efficient space-time Jacobi rational spectral methods for second order time-dependent problems on unbounded domains*, Appl. Numer. Math., *176* (2022), 159–181.

[32] M. Zheng, Z. Jin, F. Liu, and V. Anh, *Matrix transfer technique for anomalous diffusion equation involving fractional Laplacian*, Appl. Numer. Math., *172* (2022), 242–258.