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Exploring high-frequency waves and soliton solutions of fluid turbulence through relaxation medium modeled by vakhnenko-parkes equation

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Abstract

One of the most important natural phenomena that has been studied extensively in engineering, oceanography, meteorology and other fields is called fluid turbulence (FT). FT stands for irregular flow of fluid. Scientists detected models to describe this phenomenon, among these models is the (3+1)-dimensional Vakhnenko-Parkes (VP) equation. In this research, the high-frequency waves' dynamical behavior through the relaxation medium is explored by considering two semi-analytic methods, the $\left(\frac{G'}{G}\right)$ and the tanh-coth (TC) expansion methods. Nineteen different solutions have been detected and some of these solutions have been illustrated graphically. Figures show a range of degenerate, periodic, and complex propagating soliton wave solutions.

Keywords. (G'/G)-expansion method, Tanh-coth method, Vakhnenko-Parkes equation, Fluid turbulence, Nonlinear partial differential equations.
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1. Introduction

Most phenomena in life, whether physical, biological, geological, or meteorological have applications in areas such as plasma physics, nonlinear optics, chemical reactions, electric networks, wave propagation, and fluid turbulence all of which are described by nonlinear partial differential equations (NLPDEs). The fluid turbulence is a common occurrence in various engineering and natural contexts. In engineering, the importance of turbulence appears in the design of pipelines, ships, and aircraft. In industrial procedures, it is critical for efficient mixing and chemical reactions in reactors. Furthermore, the turbulent flow is crucial for the mixing of water in rivers and oceans, as well as for climate dynamics. One of the models that describes this issue is the equation governing the propagation of high-frequency waves in a relaxing medium, known as the (3+1)-dimensional Vakhnenko-Parkes equation (see [24]). There are several analytical methods for solving nonlinear partial differential equations, including Lie-symmetry analysis (see [18, 27]), the Jacobi elliptic function method (see [5]), the Riccati equation method (see [2]), the inverse scattering transformation method (see [3]), the Darboux transformation method (see [1]), the Backlund transformation method (see [28]), the tanh-coth method (see [7, 25]), the (G'/G) expansion [13] and similarity methods for partial differential equations of integer or fractional order (see [14, 15, 19, 20]). Moreover, some numerical techniques have also been employed (see [16, 17]). Vakhnenko (see [24]) derived a nonlinear evolution equation imitation the propagation of short waves in a relaxing medium given by:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+u=0. \tag{1.1}$$

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Victor et al presented a model equation for relaxing high-rate processes in active barotropic media given by (see [12]):

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial t}+u\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\right)u+u=0. \tag{1.2}$$

In 2019 Wazwaz developed a (3+1)-dimensional Vakhnenko-Parkes equation in the following manner (see [26]):

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t}+u\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\right)u+u=0, \tag{1.3}$$

where, u is a differentiable function in the variables $x,\ y,\ z$, and t, which governing the propagation of high-frequency waves in a relaxing medium. Khater et al investigated the solitary wave solutions of VP equation by applying the Khater II method (see [8]. Khan and Akbar found the general solution by using the $exp\ (-\ \Phi\ (\xi))$-expansion method (see [9]). Kumar obtained some exact solutions of VP equation with power law nonlinearity (see [10]). Wazwaz derived multiple real and multiple complex soliton solutions for VP and the modified Vakhnenko-Parkes (MVP) using the simplified Hirota's method (see [26]). Roshid et al. found some solitary wave solutions of this equation through the exp-function method and the $exp\ (-\ \Phi\ (\xi))$-expansion method (see [21]). Kumar and Nikita Mann utilized three methods to obtain the precise exact solitary wave solutions for the VP equation, the generalized Kudryashov method (GKM), the generalized exponential rational function method (GERFM), and the generalized Riccati equation mapping method (GREMM) (see [11]).

In this study, two distinct methods are applied, namely tanh-coth and $\left(\frac{G'}{G}\right)$ expansion methods, to find the analytic solutions of the (3+1)-dimensional VP equation. The paper is arranged as follows. Section 2, presents several analytic solutions of the VP equation via the tanh-coth method. In section 3, the $\left(\frac{G'}{G}\right)$ expansion method is utilized. The paper ends with conclusions in section 4.

2. Tanh-coth method for the (3+1)-dimensional Vakhnenko-Parkes equation

In this section, the traveling wave solutions of the (3+1)-dimensional VP equation are constructed through considering the tanh-coth method. Özkan et al. (see [22]) converted (1.3) into the following potential form:

$$(u_{xxt}+u_{xxy}+u_{xxz}+u_xu_t+u_xu_y+u_xu_z+u_x=0). \tag{2.1}$$

Analyzing (2.1) using the tanh-coth method (see [7, 25]) fundamentally about reducing the PDE to an ODE by employing the traveling wave transformation given as:

$$\varsigma=\ Dx+Ay+Bz-Ct. \tag{2.2}$$

Substituting Eq. (2.2) into (2.1) simplifies it to:

$$qD^2u'''+Dq(u^{'})^2+Du'=0, \tag{2.3}$$

with

$$q=A+B-C. \tag{2.4}$$

The derivative symbol represents the derivatives with respect to ς. The remaining steps of the method is summarized as:

Firstly, introducing the independent variable:

$$Y=\tanh(\psi\varsigma) \tag{2.5}$$

where, ς is defined in (2.2) and ψ is considered the wave number. Assume that the solution of (2.3) can be represented by the following finite expansion:

$$u(\psi\varsigma)=s(y)=\sum_{k=0}^{M}a_ky^k+\sum_{k=1}^{M}b_ky^{-k}, \tag{2.6}$$

where, M is a positive integer that can be determined by balancing the highest order nonlinear terms with the linear terms of highest order through a scheme as follows:

$$\begin{cases} u \to M, \\ u^n \to nM, \\ u^{'} \to M+1, \\ u^{(n)} \to M+n. \end{cases} \tag{2.7}$$

Substituting (2.5) into (2.3) and then collecting all coefficients of each power of y^k, $0 \le k \le nM$, in the resulting equation where these coefficients must vanish. This gives a system of algebraic equations involving the parameters $a_k,\ b_k, \psi,\ A,\ D,\ B$, and C. Two cases of solutions will be discussed as follows:

Case 1:

Balancing u''' with $(u^{'})^2$ in (2.3) results in:

$$M+3 = 2(M+1), \quad \to \qquad M = 1. \tag{2.8}$$

Substituting from (2.8) into (2.6):

$$u(\psi\varsigma) = \ S(Y) \ = \ \sum_{k=0}^{1} a_k y^k + \sum_{k=1}^{1} b_k y^{-k}. \tag{2.9}$$

The following values of S and its derivatives are obtained:

$$\begin{cases} s = a_0 + a_1 y + b_1 y^{-1}, \\ s^{'} = a_1 - b_1 y^{-2}, \\ s^{''} = 2b_1 y^{-3}, \\ s^{'''} = -6b_1 y^{-4}. \end{cases} \tag{2.10}$$

Substituting (2.10) into (2.3) while taking into account (2.5), gathering the coefficients of each y^k power, and setting the sum to zero yields the subsequent algebraic equations:

Coefficient of y^0:

$$-2\psi^3 a_1 q D^2 + \psi^2 D\left({b_1}^2 + {a_1}^2 + 4a_1 b_1\right) q + D\psi\left(a_1 + b_1\right) = 0. \tag{2.11}$$

Coefficient of y^2:

$$2\psi^3 D^2 q\left(4a_1 + 3b_1\right) - 6b_1\psi^3 q D^2 - 2\psi^2 D q\left(a_1 b_1 + {a_1}^2\right) - a_1 D\psi = 0. \tag{2.12}$$

Coefficient of y^4:

$$-6a_1\psi^3 D^2 q + {a_1}^2 D q = 0. \tag{2.13}$$

Coefficient of y^{-2}:

$$8b_1\psi^3 D^2 q + \psi^2 D q\left(-2{b_1}^2 - 2a_1 b_1\right) - b_1 D\psi = 0. \tag{2.14}$$

Coefficient of y^{-4}:

$$-6b_1 D^2 q\psi^3 + D{b_1}^2 q\psi^2 = 0. \tag{2.15}$$

Solving the algebraic system (2.11)-(2.15) using Maple package confers three different sets of solutions of the system which result in three different solutions of the VP Equation (2.1) as follows:

Set 1:

$$a_1 = \frac{-3}{4(\left(-\frac{16BD\psi^2 - 16CD\psi^2 + 1}{8\psi D}\right) + 2B\psi - 2C\psi)}, \quad b_1 = 6D\psi, \quad A = \frac{16BD\psi^2 - 16CD\psi^2 + 1}{-16\psi^2 D}, \tag{2.16}$$

$$u_1 = a_0 + 6D\psi \tanh\left(\psi\left(Dx + \frac{16BD\psi^2 - 16CD\psi^2 + 1}{-16\psi^2 D} y + Bz - ct\right)\right)^{-1} \tag{2.17}$$

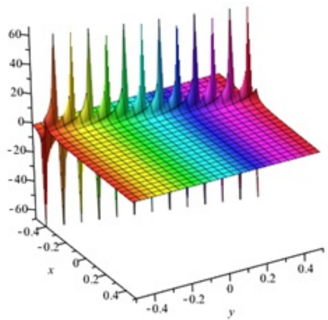

FIGURE 1. Propagating wave u_2 *at* $A = -\frac{1}{4}$, $B = 1$, $C = 1$, $D = 1$, $\psi = 1$ $\lambda = 1$, $a_0 = 1$, $a_1 = 0$, $b_1 = 6$, $z = 0.1$, *and* $t = 0.2$.

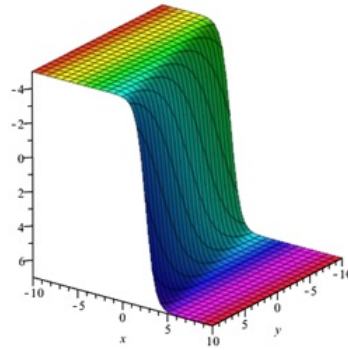

FIGURE 2. Kink wave u_3 *at* $A = -\frac{1}{4}$, $B = 1$, $C = 1$, $D = 1$, $\psi = 1$ $\lambda = 1$, $a_0 = 1$, $a_1 = 6$, $z = 0.1$, *and* $t = 0.2$.

$$+ \frac{-3}{4\left(\left(-\frac{16BD\psi^2-16CD\psi^2+1}{8\psi D}\right)+2B\psi-2C\psi\right)}\tanh\left(\psi\left(Dx+\frac{16BD\psi^2-16CD\psi^2+1}{-16\psi^2 D}y+Bz-ct\right)\right).$$

Set 2:

$$a_1 = 0, \qquad b_1 = 6D\psi, \qquad A = \frac{4BD\psi^2 - 4CD\psi^2 + 1}{-4D}, \tag{2.18}$$

$$u_2 = a_0 + 6D\psi \tanh\left(\psi(Dx + \frac{4BD\psi^2 - 4CD\psi^2 + 1}{-4D}\ y + Bz - ct)\right)^{-1}. \tag{2.19}$$

Set 3:

$$a_1 = 6D\psi, \qquad b_1 = 0, \qquad A = \frac{4BD\psi^2 - 4CD\psi^2 + 1}{-4D}, \tag{2.20}$$

$$u_3 = a_0 + 6D\psi \tanh\left(\psi(Dx + \frac{4BD\psi^2 - 4CD\psi^2 + 1}{-4D}y + Bz - ct)\right). \tag{2.21}$$

The multi-Soliton wave u_2 is illustrated in Figure 1 at:

$$A = -\frac{1}{4},\ B = 1,\ C = 1,\ D = 1,\ \psi = 1,\ \lambda = 1,\ a_0 = 1,\ a_1 = 0,\ b_1 = 6,\ z = 0.1,\ \text{and}\ t = 0.2.$$

Also, Figure 2 shows the kink wave solutionu_3 at:

$$A = -\frac{1}{4},\ B = 1,\ C = 1,\ D = 1,\ \psi = 1,\ \lambda = 1,\ a_0 = 1,\ a_1 = 6,\ z = 0.1,\ \text{and}\ t = 0.2.$$

Case 2:

Equation (2.1) can be reduced in order through considering:

$$u^{'} = v. \tag{2.22}$$

Substituting Eq. (2.22) into (2.1) confers:

$$qD^2v'' + Dqv^2 + Dv = 0. \tag{2.23}$$

Applying the TC method to solve this equation, where balancing v'' and v^2 results in; $M=2$, then Eq. (2.5) becomes:

$$u(\psi\varsigma) \;=\; S(Y) \;=\; \sum_{k=0}^{2} a_k y^k + \sum_{k=1}^{2} b_k y^{-k}, \tag{2.24}$$

with the following obtained forms of s and its derivatives

$$\begin{cases} s = a_0 + a_1 y + a_2 y^2 + b_1 y^{-1} + b_2 y^{-2}, \\ s' = a_1 + 2a_2 y - b_1 y^{-2} - 2b_2 y^{-3}, \\ s'' = 2a_2 + 2b_1 y^{-3} + 6b_2 y^{-4}. \end{cases} \tag{2.25}$$

Inserting (2.25) into (2.23), considering (2.5), collecting all coefficients of each power of y^k and equating by zero lead to:
Coefficient of y^{-1}:

$$-2\psi^2 b_1 q D^2 + 2a_0 b_1 D q + 2a_1 b_2 D q + b_1 D = 0. \tag{2.26}$$

Coefficient ofy^{-2}:

$$-8\psi^2 b_2 q D^2 + 2a_0 b_2 D q + {b_1}^2 D q + b_2 D = 0. \tag{2.27}$$

Coefficient of y^{-3}:

$$2\psi^2 b_1 q D^2 + 2b_1 b_2 D q = 0. \tag{2.28}$$

Coefficient of y^{-4}:

$$6\psi^2 b_2 q D^2 + {b_2}^2 D q = 0. \tag{2.29}$$

Coefficient ofy^0:

$$2\psi^2 a_2 q D^2 + 2\psi^2 b_2 q D^2 + {a_0}^2 D q + 2a_1 b_1 D q + 2a_2 b_2 D q + a_0 D = 0. \tag{2.30}$$

Coefficient of y^1:

$$-2\psi^2 a_1 q D^2 + 2\psi^2 b_2 q D^2 + 2a_0 a_1 D q + 2a_2 b_1 D q + a_1 D = 0. \tag{2.31}$$

Coefficient of y^2:

$$-8\psi^2 a_2 q D^2 + 2a_0 a_2 D q + {a_1}^2 D q + a_2 D = 0. \tag{2.32}$$

Coefficient of y^3:

$$2\psi^2 a_1 q D^2 + 2a_1 a_2 D q = 0. \tag{2.33}$$

Coefficient of y^4:

$$6\psi^2 a_2 q D^2 + {a_2}^2 D q = 0. \tag{2.34}$$

Solving the algebraic system (2.26)-(2.34) using Maple software confers different sets of solutions of the system which result in six different solutions of the VP Equation (2.1) as: Set 4:

$$\begin{aligned} a_0 &= \frac{-3}{2(\frac{-16BD\psi^2+16CD\psi^2-1}{8\psi^2 D} + 2B - 2C)}, a_1 = 0,\ a_2 = -6D\psi^2,\ b_1 = 0,\ b_2 = -6D\psi^2, \\ A &= \frac{-16BD\psi^2 + 16CD\psi^2 - 1}{16\psi^2 D}, \end{aligned} \tag{2.35}$$

$$\begin{aligned} v_1 = {} & \frac{-3}{2\left(\frac{-16BD\psi^2+16CD\psi^2-1}{8\psi^2 D} + 2B - 2C\right)} - 6D\psi^2 \tanh\left(\psi(Dx - \frac{16BD\psi^2 - 16CD\psi^2 + 1}{16\psi^2 D}\, y + Bz - ct)\right)^2 \\ & - 6D\psi^2 \tanh\left(\psi(Dx - \frac{16BD\psi^2 - 16CD\psi^2 + 1}{16\psi^2 D}\, y + Bz - ct)\right)^{-2}. \end{aligned} \tag{2.36}$$

Verifying (2.22) to get the VP solution in the form:

$$\begin{aligned} u_4 =& \frac{-3}{2(\frac{-16BD\psi^2+16CD\psi^2-1}{8\psi^2 D}+2B-2C)}\varsigma + \frac{6D\psi^2}{\psi}\tanh(\psi\varsigma)) + \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)-1)}{2\psi} \\ &- \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)+1)}{2\psi} + \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)-1)}{2\psi} - \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)+1)}{2\psi} \\ &+ \frac{6D\psi^2}{\psi\tanh(\psi\varsigma))}. \end{aligned} \tag{2.37}$$

Set 5:

$$\begin{aligned} a_0 =& \frac{-1}{2(\frac{-16BD\psi^2+16CD\psi^2-1}{8\psi^2 D}+2B-2C)},\ a_1 = 0,\ a_2 = -6D\psi^2,\ b_1 = 0,\ b_2 = -6D\psi^2, \\ A =& \frac{-16BD\psi^2+16CD\psi^2+1}{16\psi^2 D}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} v_2 =& \frac{-1}{2(\frac{-16BD\psi^2+16CD\psi^2-1}{8\psi^2 D}+2B-2C)} - 6D\psi^2(\tanh(\psi(Dx - \frac{16BD\psi^2-16CD\psi^2-1}{16\psi^2 D}y + Bz - ct))^2 \\ &- 6D\psi^2(\tanh(\psi(Dx - \frac{16BD\psi^2-16CD\psi^2-1}{16\psi^2 D}y + Bz - ct))^{-2}, \end{aligned} \tag{2.39}$$

$$\begin{aligned} u_5 =& \frac{-1}{2\left(\frac{-16BD\psi^2+16CD\psi^2-1}{8\psi^2 D}+2B-2C\right)}\varsigma + \frac{6D\psi^2}{\psi}\tanh(\psi\varsigma)) + \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)-1)}{2\psi} \\ &- \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)+1)}{2\psi} + \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)-1)}{2\psi} - \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)+1)}{2\psi} \\ &+ \frac{6D\psi^2}{\psi\tanh(\psi\varsigma))}. \end{aligned} \tag{2.40}$$

Set 6:

$$\begin{aligned} a_0 =& \frac{-3}{(\frac{-4BD\psi^2+4CD\psi^2-1}{2\psi^2 D}+2B-2C)},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = -6D\psi^2, \\ A =& \frac{-4BD\psi^2+4CD\psi^2-1}{4\psi^2 D}, \end{aligned} \tag{2.41}$$

$$\begin{aligned} v_3 =& -6D\psi^2\tanh(\psi(Dx - \frac{4BD\psi^2-4CD\psi^2+1}{4\psi^2 D}y + Bz - ct))y^{-2} \\ &+ \frac{-3}{(\frac{-4BD\psi^2+4CD\psi^2-1}{2\psi^2 D}+2B-2C)}, \end{aligned} \tag{2.42}$$

$$\begin{aligned} u_6 =& \frac{-3}{\left(\frac{-4BD\psi^2+4CD\psi^2-1}{2\psi^2 D}+2B-2C\right)}\varsigma + \frac{6D\psi^2 * ln(\tanh(\psi\varsigma)-1)}{2\psi} \\ &+ \frac{-6D\psi^2 * ln(\tanh(\psi\varsigma)+1)}{2\psi} + \frac{6D\psi^2}{\psi\tanh(\psi\varsigma))}. \end{aligned} \tag{2.43}$$

Set 7:

$$a_0 = \frac{1}{(\frac{-4BD\psi^2+4CD\psi^2+1}{2\psi^2 D} + 2B - 2C)},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = -6D\psi^2,$$
$$A = \frac{-4BD\psi^2 + 4CD\psi^2 - 1}{4\psi^2 D} \tag{2.44}$$

$$v_4 = -6D\psi^2 \tanh\left(\psi\left(Dx - \frac{4BD\psi^2 - 4CD\psi^2 + 1}{4\psi^2 D}y + Bz - ct\right)\right)y^{-2}$$
$$+ \frac{1}{(\frac{-4BD\psi^2+4CD\psi^2+1}{2\psi^2 D} + 2B - 2C)}, \tag{2.45}$$

$$u_7 = \frac{1}{(\frac{-4BD\psi^2+4CD\psi^2+1}{2\psi^2 D} + 2B - 2C)}\varsigma + \frac{6D\psi^2 * ln(\tanh(\psi\varsigma) - 1)}{2\psi}$$
$$- \frac{6D\psi^2 * ln(\tanh(\psi\varsigma) + 1)}{2\psi} + \frac{6D\psi^2}{\psi \tanh(\psi\varsigma))}. \tag{2.46}$$

Set 8:

$$a_0 = \frac{-3}{(\frac{-4BD\psi^2+4CD\psi^2-1}{2\psi^2 D} + 2B - 2C)},\ a_1 = 0,\ a_2 = -6D\psi^2,\ b_1 = 0,\ b_2 = 0,$$
$$A = \frac{-4BD\psi^2 + 4CD\psi^2 - 1}{4\psi^2 D}, \tag{2.47}$$

$$v_5 = -6D\psi^2 \tanh\left(\psi\left(Dx - \frac{4BD\psi^2 - 4CD\psi^2 + 1}{4\psi^2 D}y + Bz - ct\right)\right)y^{2}$$
$$+ \frac{-3}{(\frac{-4BD\psi^2+4CD\psi^2-1}{2\psi^2 D} + 2B - 2C)} \tag{2.48}$$

$$u_8 = \frac{-3}{(\frac{-4BD\psi^2+4CD\psi^2-1}{2\psi^2 D} + 2B - 2C)}\varsigma + \frac{6D\psi^2 * ln(\tanh(\psi\varsigma) - 1)}{2\psi}$$
$$- \frac{6D\psi^2 * ln(\tanh(\psi\varsigma) + 1)}{2\psi} + \frac{6D\psi^2 \tanh(\psi\varsigma))}{\psi}. \tag{2.49}$$

Set 9:

$$a_0 = \frac{1}{(\frac{-4BD\psi^2+4CD\psi^2+1}{2\psi^2 D} + 2B - 2C)},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = -6D\psi^2,$$
$$A = \frac{-4BD\psi^2 + 4CD\psi^2 + 1}{4\psi^2 D}, \tag{2.50}$$

$$v_6 = -6D\psi^2 \tanh(\psi(Dx - \frac{4BD\psi^2 - 4CD\psi^2 - 1}{4\psi^2 D}y + Bz - ct))y^{-2} + \frac{1}{(\frac{-4BD\psi^2+4CD\psi^2+1}{2\psi^2 D} + 2B - 2C)}, \tag{2.51}$$

$$u_9 = \frac{1}{(\frac{-4BD\psi^2+4CD\psi^2+1}{2\psi^2 D} + 2B - 2C)}\varsigma + \frac{6D\psi^2 * ln(\tanh(\psi\varsigma) - 1)}{2\psi}$$
$$- \frac{6D\psi^2 * ln(\tanh(\psi\varsigma) + 1)}{2\psi} + \frac{6D\psi^2 \tanh(\psi\varsigma))}{\psi}. \tag{2.52}$$

Plots of solutions u_4, u_6, and u_8 are presented in Figures 3-5.

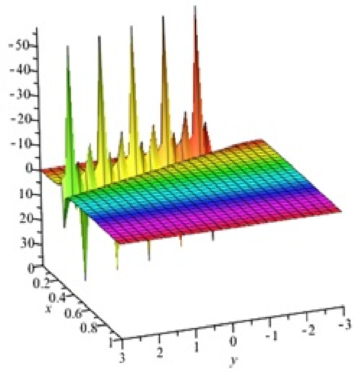

FIGURE 3. Soliton wave solution u_4 at $B = 1,\ C = 1,\ D = 1,\ \psi = 1,\ \lambda = 1,\ z = 0.1,\ t = 0.2,\ a_0 = 12, a_2 = -6,\ b_2 = -6\ and\ A = -1/16$.

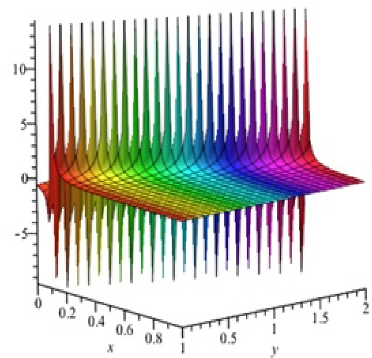

FIGURE 4. Propagating wave u_6 at $B = 1,\ C = 1,\ D = 1,\ \psi = 1,\ \lambda = 1,\ z = 0.1,\ t = .2,\ a_0 = 6,\ b_2 = -6\ , and\ A = -0.25$.

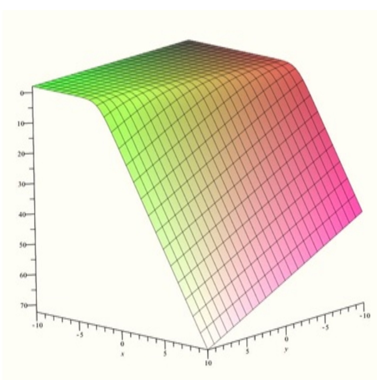

FIGURE 5. Typical breaking soliton wave solution u_8 at $B = 1,\ C = 1,\ D = 1,\ \psi = 1,\ \lambda = 1,\ z = 0.1,\ t = 0.2,\ a_0 = 6,\ a_2 = -6, and\ A = -0.25$.

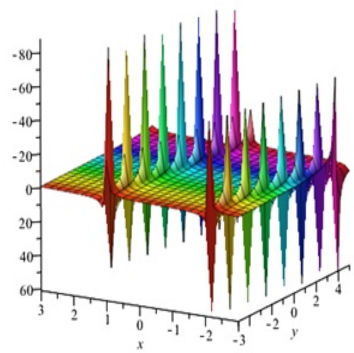

FIGURE 6. Degenerate wave solution u_{11}, for $A = 0.1, B = 1, C = 1, D = 1, \psi = 2, \lambda = 2, a_0 = 1, a_1 = 6, z = 1, t = 2, c_1 = 1, and c_2 = 2$.

3. G'/G EXPANSION METHOD OF THE (3+1)-DIMENSIONAL VP EQUATION

The $\frac{G'}{G}$ expansion method mainly starts from the reduced Equation (2.3), based on the assumption described in [4], where the solution of the ODE is given by:

$$u(\varsigma) = \sum_{i=0}^{M} a_i \left(\frac{G'}{G}\right)^i, \qquad a_m \neq 0, \tag{3.1}$$

where a_i(i = 0, 1, 2, . . . , m) are constants, while $G = G(\varsigma)$ will satisfy the following second-order linear differential equation:

$$G''(\varsigma) + \lambda G'(\varsigma) + \mu G(\varsigma) = 0, \tag{3.2}$$

where λ and μ are constants (see [6]). The positive integer M is determined by balancing the highest order nonlinear terms with the linear terms of highest order aforementioned in (2.7). Solutions of Eq. (3.1) (see [23]) are:

$$\left(\frac{G'}{G}\right)=\begin{cases}\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1 sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\ \varsigma)\ +c_2 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\ \varsigma)}{c_1 cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\ \varsigma)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\ \varsigma\right)}\right), & \lambda^2-4\mu>0,\\ \frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1 sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\ \varsigma)+c_2 cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\varsigma)}{c_1 cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\ \varsigma)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\varsigma\right)}\right), & \lambda^2-4\mu<0,\\ \frac{-\lambda}{2}+\frac{c_2}{c_1+c_2\varsigma}, & \lambda^2-4\mu=0.\end{cases} \tag{3.3}$$

By substituting Eq. (3.1) into Eq. (2.3), utilizing Eq. (2.7), and consolidating all terms with the same power of $\frac{G'}{G}$, Eq. (2.3) can be transformed into a polynomial in powers of $\frac{G'}{G}$. Equating each coefficient of the resulting polynomial to zero, a set of algebraic equations is obtained for a_i, λ ,μ, A, D, B, and C. Two cases of solutions will be discussed by solving the algebraic equations system as follows:

Case 1: $M=1$, as shown in Eq. (2.8), then according to (3.1):

$$u=a_0+\frac{G'}{G}a_1. \tag{3.4}$$

By substituting Eq. (3.4) into Eq. (2.3) utilizing Eq. (3.2), obtain the following system of algebraic equations:

$$\begin{cases}\left(\frac{G'}{G}\right)^0: -a_1\mu D+a_1^2\mu^2 qD+qD^2\left(-a_1\lambda^2-2a_1\mu^2\right)=0,\\ \left(\frac{G'}{G}\right): -a_1\lambda D+2qa_1^2\mu\lambda D+qD^2\left(-a_1\lambda^3-8a_1\lambda\mu\right)=0,\\ \left(\frac{G'}{G}\right)^2: -a_1 D+qD\left(2a_1^2\mu+a_1^2\lambda^2\right)+qD^2\left(-7a_1\lambda^2-8a_1\mu\right)=0,\\ \left(\frac{G'}{G}\right)^3: 2qa_1^2\lambda D+qD^2-12a_1\lambda=0,\\ \left(\frac{G'}{G}\right)^4: qa_1^2\ D-6qa_1D^2=0.\end{cases} \tag{3.5}$$

The solutions of the system (3.5) with Maple package are:

Set 10:

$$a_0=a_0,\quad a_1=6D,\ A=-\frac{4BD\mu^3-4CD\mu^3-BD\lambda\mu^2+CD\lambda\mu^2-\ \mu^2}{D\left(4\mu^3-\lambda\mu^2\right)}, \tag{3.6}$$

At $\mu=2$, $\lambda=3$, $a_0=1$, $a_1=6$, $c_1=1$, and $c_2=2$,

$$u_{11}=1+6\left(\frac{-3}{2}+\frac{1}{2}\left(\frac{sinh(\frac{1}{2}\ \varsigma)\ \ +2cosh(\frac{1}{2}\ \varsigma)}{cosh(\frac{1}{2}\ \varsigma)+2\sinh\left(\frac{1}{2}\ \varsigma\right)}\right)\right). \tag{3.7}$$

At μ =2, $\lambda=2$, $a_0=1$, $a_1=6$, $c_1=1$, and $c_2=2$,

$$u_{12}=1+6\left(1+\left(\frac{-sin(\ \varsigma)\ \ +2cos(\ \varsigma)}{cos(\ \varsigma)+2\sin(\ \varsigma)}\right)\right). \tag{3.8}$$

At $\mu=1$, $\lambda=2$, $a_0=1$, $a_1=6$, $c_1=1$, and $c_2=2$,

$$u_{13}=1+6(-1+\frac{2}{1+2\varsigma}). \tag{3.9}$$

Case 2: Solution of Eq. (3.1) with $M=2$

$$u=a_0+\frac{G'}{G}a_1+a_2\left(\frac{G'}{G}\right)^2. \tag{3.10}$$

The following system of algebraic equations is obtained as:

$$\begin{cases} \left(\frac{G'}{G}\right)^0 : a_0 D + Dqa_0^2 + qD^2\left(a_1\mu\lambda + 2a_2\mu^2\right) = 0, \\ \left(\frac{G'}{G}\right) : Da_1 + 2qDa_0a_1 + qD^2\left(6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2\right) = 0, \\ \left(\frac{G'}{G}\right)^2 : Da_2 + qD\left(2a_0a_2 + a_1^2\right) + qD^2\left(8a_2\psi + 3a_1\lambda + 4a_2\lambda^2\right) = 0, \\ \left(\frac{G'}{G}\right)^3 : 2qDa_1a_2 + qD^2\left(2a_1 + 10a_2\lambda\right) = 0, \\ \left(\frac{G'}{G}\right)^4 : qDa_2^2 + qD^2 6a_2 = 0. \end{cases} \tag{3.11}$$

The solutions of system (3.11) using Maple package are given by:
Set 11:

$$\begin{aligned} &a_0 = \frac{1}{36}\frac{432D^3\mu\lambda - {a_1}^3}{Da_1}\ , \quad a_1 = a_1, \quad a_2 = -6D, \\ &A = \frac{-864BD^3\mu\lambda + 864CD^3\mu\lambda - B{a_1}^3 + C{a_1}^3 + 36Da_1}{864D^3\mu\lambda + {a_1}^3}\ , \quad \mu = \frac{-6D\mu\lambda}{a_1} \quad \textit{and}\ \lambda = \frac{a_1}{-6D} \end{aligned} \tag{3.12}$$

$$v_1 = a_0 + a_1 * \frac{G'}{G} - 6D\left(\frac{G'}{G}\right)^2. \tag{3.13}$$

For $\lambda^2 - 4\mu < 0$, $\mu = 1$, and $\lambda = 1$,

$$\begin{aligned} u_{14} = &\frac{1}{36}\frac{432D^3\mu\lambda - {a_1}^3}{Da_1}\varsigma - \frac{1}{2}a_1\lambda\varsigma + a_1\ln\left(\cos\left(\frac{\sqrt{3}}{2}\varsigma\right) + 2\sin(\frac{\sqrt{3}}{2}\varsigma)\right) \\ &- \frac{30D\lambda^2}{4\sqrt{3} * (1 + 2\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)} + \frac{30D\mu}{\sqrt{3}(1 + 2\tan\left(\frac{\sqrt{3}}{2}\varsigma\right))} + 6D\lambda\ln(1 + 2\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)) \\ &- 3D\lambda\ln(\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)^2 + 1) - \frac{6D\lambda^2\tan^{-1}\left(\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)\right)}{\sqrt{3}} - \frac{6D\lambda^2\tan^{-1}\left(\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)\right)}{\sqrt{3}} \\ &+ \frac{12D\mu\tan^{-1}\left(\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)\right)}{\sqrt{3}}. \end{aligned} \tag{3.14}$$

At $\lambda^2 - 4\mu > 0$, $\psi = 1$, and $\lambda = -6$,

$$\begin{aligned} u_{15} = &a_0\varsigma - \frac{1}{2}a_1\lambda\varsigma + a_1\ln\left(\cosh\left(2\sqrt{2}\varsigma\right) + 2\sinh\left(2\sqrt{2}\varsigma\right)\right) \\ &+ \frac{3a_2\lambda^2\tanh\left(\sqrt{2}\varsigma\right)}{4\sqrt{2}\left(\tanh\left(\sqrt{2}\varsigma\right)^2 + 4\tanh\left(\sqrt{2}\varsigma\right) + 1\right)} - \frac{12a_2\mu\tanh\left(\sqrt{2}\varsigma\right)}{4\sqrt{2}\left(\tanh\left(\sqrt{2}\varsigma\right)^2 + 4\tanh\left(\sqrt{2}\varsigma\right) + 1\right)} \\ &+ a_2\lambda\ln\left(\tanh\left(\sqrt{2}\varsigma\right) - 1\right) - \frac{a_2\lambda^2\ln\left(\tanh\left(\sqrt{2}\varsigma\right) - 1\right)}{4\sqrt{2}} + \frac{2a_2\mu\ln\left(\tanh\left(\sqrt{2}\varsigma\right) - 1\right)}{4\sqrt{2}} \\ &- a_2\lambda\ln\left(\tanh\left(\sqrt{2}\varsigma\right)^2 + 4\tanh\left(\sqrt{2}\varsigma\right) + 1\right) + a_2\lambda\ln\left(\tanh\left(\sqrt{2}\varsigma\right) + 1\right) \\ &+ \frac{a_2\lambda^2\ln\left(\tanh\left(\sqrt{2}\varsigma\right) + 1\right)}{4\sqrt{2}} - \frac{a_2\mu\ln\left(\tanh\left(\sqrt{2}\varsigma\right) + 1\right)}{2\sqrt{2}}. \end{aligned} \tag{3.15}$$

At $\lambda^2 - 4\mu \quad = 0$, $\mu = 1$, and $\lambda = 2$,

$$u_{16} = a_0\varsigma - a_1\varsigma + a_1 \ln(c_2\varsigma + c_1) + a_2\varsigma - \frac{a_2 c_2}{c_2\varsigma + c_1} - 2a_2 \ln(c_2\varsigma + c_1). \tag{3.16}$$

Set 12:

$$a_0 = 36\frac{D^2\mu\lambda}{a_1} \quad , \quad a_1 = a_1 \quad , \; a_2 = -6D \; , \mu = \frac{-6D\psi\lambda}{a_1}, \lambda = \frac{a_1}{-6D} \quad ,$$
$$A = -\frac{864BD^3\mu\lambda - 864CD^3\mu\lambda + Ba_1{}^3 - Ca_1{}^3 + 36Da_1}{864D^3\mu\lambda + a_1{}^3}, \tag{3.17}$$
$$v_2 = 36\frac{D^2\mu\lambda}{a_1} + a_1 * \frac{G'}{G} - 6D(\frac{G'}{G})^2 .$$

At $\lambda^2 - 4\mu < 0$, $\mu = 1$, and $\lambda = 1$,

$$\begin{aligned} u_{17} = {} & 36\frac{D^2\mu\lambda}{a_1}\varsigma - \frac{1}{2}a_1\lambda\varsigma + a_1 \ln\left(\cos\left(\frac{\sqrt{3}}{2}\varsigma\right) + 2\sin(\frac{\sqrt{3}}{2}\varsigma)\right) - \frac{30D\lambda^2}{4\sqrt{3} * (1 + 2\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)} \\ & + \frac{30D\mu}{\sqrt{3}(1 + 2\tan\left(\frac{\sqrt{3}}{2}\varsigma\right))} + 6D\lambda \ln(1 + 2\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)) - 3D\lambda \ln(\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)^2 + 1) \\ & - \frac{6D\lambda^2 \tan^{-1}\left(\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)\right)}{\sqrt{3}} - \frac{6D\lambda^2 \tan^{-1}\left(\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)\right)}{\sqrt{3}} + \frac{12D\mu \tan^{-1}\left(\tan\left(\frac{\sqrt{3}}{2}\varsigma\right)\right)}{\sqrt{3}}. \end{aligned} \tag{3.18}$$

At $\lambda^2 - 4\mu > 0$, $\mu = 1$, and $\lambda = -6$,

$$\begin{aligned} u_{18} = {} & a_0\varsigma - \frac{1}{2}a_1\lambda\varsigma + a_1 \ln\left(\cosh\left(2\sqrt{2}\varsigma\right) + 2\sinh\left(2\sqrt{2}\varsigma\right)\right) \\ & + \frac{3a_2\lambda^2 \tanh(\sqrt{2}\varsigma)}{4\sqrt{2}\left(\tanh(\sqrt{2}\varsigma)^2 + 4\tanh(\sqrt{2}\varsigma) + 1\right)} - \frac{12a_2\mu \tanh(\sqrt{2}\varsigma)}{4\sqrt{2}\left(\tanh(\sqrt{2}\varsigma)^2 + 4\tanh(\sqrt{2}\varsigma) + 1\right)} \\ & + a_2\lambda \ln\left(\tanh\left(\sqrt{2}\varsigma\right) - 1\right) - \frac{a_2\lambda^2 \ln(\tanh(\sqrt{2}\varsigma) - 1)}{4\sqrt{2}} + \frac{2a_2\mu \ln(\tanh(\sqrt{2}\varsigma) - 1)}{4\sqrt{2}} \\ & - a_2\lambda \ln\left(\tanh\left(\sqrt{2}\varsigma\right)^2 + 4\tanh\left(\sqrt{2}\varsigma\right) + 1\right) + a_2\lambda \ln\left(\tanh\left(\sqrt{2}\varsigma\right) + 1\right) \\ & + \frac{a_2\lambda^2 \ln(\tanh(\sqrt{2}\varsigma) + 1)}{4\sqrt{2}} - \frac{a_2\mu \ln(\tanh(\sqrt{2}\varsigma) + 1)}{2\sqrt{2}}. \end{aligned} \tag{3.19}$$

At $\lambda^2 - 4\mu = 0$, $\mu = 1$, and $\lambda = 2$,

$$u_{19} = a_0\varsigma - a_1\varsigma + a_1 \ln(c_2\varsigma + c_1) + a_2\varsigma - \frac{a_2 c_2}{c_2\varsigma + c_1} - 2a_2 \ln(c_2\varsigma + c_1). \tag{3.20}$$

4. Conclusion

The (3+1)-dimensional Vakhnenko–Parkes equation has been investigated by using two distinct schemes namely $\left(\frac{G'}{G}\right)$ and tanh-coth expansion methods. Two different cases are discussed according to the values of the parameter M. Several soliton wave solutions contain some kink and multi solitons are obtained through the tanh-coth expansion method that are represented in Equations (2.17),(2.19), and (2.21) for $M = 1$ and in Equations (2.37), (2.40), (2.43), (2.46), (2.49), and (2.52) for $M = 2$. On the other side, several explicit solutions contain some hyperbolic, trigonometric, and rational functions are obtained by the $\left(\frac{G'}{G}\right)$ expansion methods which are represented in Equations (3.7)-(3.9) for M=1 and in Equations (3.14)-(3.16) and (3.18)-(3.20) for M=2. Some of the obtained results would be useful

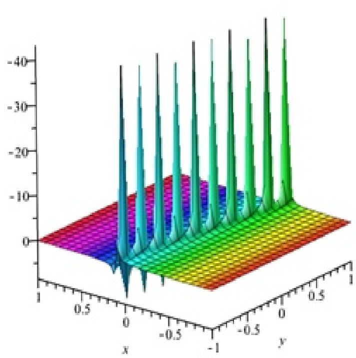

FIGURE 7. Soliton wave solution u_{12} for $A = 0.2, B = 1, C = 1, D = 1, \psi = 2, \lambda = 3, a_0 = 1, a_1 = 6, z = 1, t = 2, c_1 = 1, and c_2 = 2$.

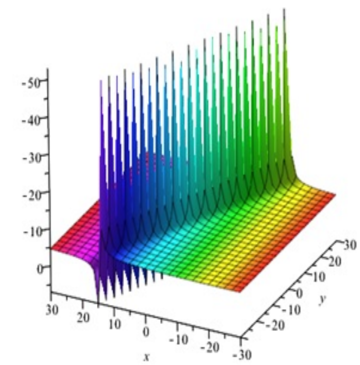

FIGURE 8. Soliton wave solution u_{13} for $A = 0.5,\ B = 1,\ C = 1,\ D = 1,\ \psi = 1,\ \lambda = 2\ ,\ a_0 = 1,\ a_1 = 6,\ z = 1,\ t = 2, c_1 = 1,\ and\ c_2 = 2$.

to investigate multiple physical applications. Many of these solutions are essential for understanding the behavior of high frequency waves in relaxation mediums. Such results are tremendously recommended in advanced research and innovation.

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This section should come before the References and should be unnumbered. Funding information may also be included here. plots of some of these solutions are depicted in Figures 6-8.

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