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Limit cycles in piecewise smooth differential systems of focus-focus and saddle-saddle dynamics

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Abstract

In this paper, we obtained the Poincare return maps for the planar piecewise linear differential systems of the type focus-focus. Normal forms for planar piecewise smooth systems with two zones of the type focus-focus and saddle-saddle, separated by a straight line and with a center at the origin, are obtained. Upper bounds for the number of limit cycles bifurcated from the period annulus of these normal forms due to perturbation by polynomial functions of any degree are established.

Keywords. Piecewise linear differential system, Piecewise smooth differential system, Limit cycle, Picard Fuch equations, Poincare return map, Melnikov function.

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1. Introduction

Piecewise-smooth differential systems have applications in various fields, such as automatic control, neural networks, electrical engineering, economics, and ecosystems. In the last decade, vast research has been done on bifurcations of such systems induced by discontinuity loci. The study of the bifurcation of limit cycles for planar piecewise smooth differential systems, from period annulus as well as from equilibrium point, is attracting many researchers. In [23], authors derived the first-order Melnikov function for a planar piecewise smooth Hamiltonian system, which can be used to study the number of limit cycles for such systems. The First-order Melnikov function for planar piecewise smooth integrable systems is derived in [11], and it is used to find the number of limit cycles bifurcated from the period annulus of a planar piecewise smooth integrable differential system with a cusp singularity when the system is perturbed in the class of polynomial functions. In [7], a lower bound for the maximum number of limit cycles in the focus-saddle and node-saddle cases are obtained. A piecewise linear differential system with two zones when one of the linear differential systems has a center, either real or virtual, has at most two limit cycles; for instance, see [8]. The number of crossing periodic orbits for the saddle-focus dynamics is discussed in [16]. In [6], the problem of the maximum number of limit cycles of a piecewise linear system of focus-focus type is studied. In this paper, we present the Poincaré full return maps for piecewise differential systems of the types focus-focus, center-center, and center-focus.

A survey of known bifurcations for a family of Filippov systems is presented in [3], with bifurcations and qualitatively different phase portraits of equilibria into zones are discussed. In [4], piecewise planar systems with crossing limit cycles are studied through canonical forms.

The maximal number of limit cycles for planer piecewise linear refracting systems studied in [5, 9, 10, 13, 18–20]. According to Lum-Chau's conjecture, a planar continuous piecewise differential system placed in two zones and a straight line functioning as a separation boundary can have at most one limit cycle, that, if it exists, can be attracting or repelling. A new simple proof of Lum-Chau's conjecture is presented in [22].

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Using the Picard-Fuchs equations, the upper bound for the number of limit cycles that bifurcate from the period annulus is obtained in [12]. In [21], the authors studied the canonical forms of an n-dimensional piecewise linear system.

In this article, we discuss normal forms of piecewise linear differential systems of the types focus-focus and saddle-saddle. Moreover, we have obtained an upper bound for the number of limit cycles that are bifurcated from the period annulus due to polynomial perturbation.

The paper is organized as follows:

In section 2, some preliminary results are presented. Section 3 characterizes piecewise linear (PWL) systems (2.2) of the Focus-Focus (FF) type with having a period annulus. Section 4 is about the normal forms of piecewise smooth (PWS) systems (2.1), having origin as a center and hence a period annulus around the equilibrium point $(0,0)$. Section 5 discusses the number of limit cycles bifurcated from the period annulus of the normal forms derived in section 4 as a result of perturbation in the class of polynomial functions.

We derive the first-order Melnikov functions when the system is perturbed. The number of limit cycles for the perturbed system is equal to the number of zeros of the first-order Melnikov function. Upper bounds on the number of zeros of the first-order Melnikov function can be computed using the Chebyshev criterion or Picard-Fuchs equations. Moreover, upper bounds for the number of zeros are obtained, and these bounds are strict.

2. Preliminaries

We briefly present some preliminaries of a piecewise smooth planar differential system $Z(x,y)$ with a straight line separation boundary $\Sigma = \{(x,y) \in \mathbb{R}^2 \colon y = 0\}$.

If $Z^+(x,y) = (Z_1^-(x,y), Z_2^+(x,y))$ is a smooth vector field on the upper half plane $\Sigma^+ = \{(x,y) \in \mathbb{R}^2 \colon y > 0\}$ and $Z^-(x,y) = (Z_1^-(x,y), Z_2^-(x,y))$ is a smooth vector field on the lower half plane $\Sigma^- = \{(x,y) \in \mathbb{R}^2 \colon y < 0\}$, then a planar piecewise smooth differential system Z is defined as

$$Z(x,y) = \begin{cases} Z^+(x,y), & \text{if } y \in \Sigma^+ \cup \Sigma, \\ Z^-(x,y), & \text{if } y \in \Sigma^- \cup \Sigma. \end{cases} \tag{2.1}$$

In particular, a piecewise linear differential system (PWL) with a straight line separation boundary Σ, is given by

$$Z(x,y) = \begin{cases} (a^+x + b^+y + \alpha^+, c^+x + d^+y + \beta^+), & \text{if } (x,y) \in \Sigma^+ \cup \Sigma, \\ (a^-x + b^-y + \alpha^-, c^-x + d^-y + \beta^-), & \text{if } (x,y) \in \Sigma^- \cup \Sigma. \end{cases} \tag{2.2}$$

Note that Z is multivalued on Σ. Hence, we cannot define a solution $(x(t), y(t))$ of the system (2.1), when $(x(0), y(0)) \in \Sigma$. If a solution of the system (2.1), which starts at a point in Σ^+ or Σ^-, remains in the same region forever, then it is a full orbit of Z^+ or Z^-, respectively. To define an orbit that arrives at Σ, we can have a vector field

$$Z_\mu(x,0) = \mu Z^+(x,0) + (1-\mu)Z^-(x,0), \;\; 0 \le \mu \le 1. \tag{2.3}$$

According to the Filippov convention, the vector field (2.3) on Σ is formed by a convex combination.

We partition Σ into two parts:

(1) Crossing region: $\Sigma^c = \{(x,0) \in \Sigma \colon Z_2^+(x,0) \cdot Z_2^-(x,0) > 0\}$,
(2) Sliding region: $\Sigma^s = \{(x,0) \in \Sigma \colon Z_2^+(x,0) \cdot Z_2^-(x,0) \le 0\}$.

If $(x,0) \in \Sigma^c$, then both of the vector fields Z^+ and Z^- are transversals to Σ, with their normal components having the same sign and an orbit of Z passing through $(x,0)$ is obtained by concatenating orbits of Z^+ and Z^- in a natural way. If $(x,0) \in \Sigma^s$, then either the normal components of vector fields Z^+ and Z^- at $(x,0)$ have opposite signs, or at least one of them vanishes. In this paper, we assume that the orbit slides along Σ according to the Filippov vector field (2.3). Therefore, the normal component of the vector field Z_μ on Σ^s is zero. Hence, we have

$$\mu Z_2^+(x,0) + (1-\mu)Z_2^-(x,0)) = 0 \text{ for all } (x,0) \in \Sigma^s.$$

Thus, for all $(x,0) \in \Sigma^s$,

$$\mu(x) = \frac{Z_2^-(x,0)}{Z_2^+(x,0) - Z_2^-(x,0)}, \text{ and}$$
$$Z_\mu(x,0) = \left(\frac{Z_1^+(x,0)Z_2^-(x,0) - Z_1^-(x,0)Z_2^+(x,0)}{Z_2^+(x,0) - Z_2^-(x,0)},\ 0\right),$$

when $Z_2^+(x,0) \neq Z_2^-(x,0)$.

If $Z_2^+(x,0) = Z_2^-(x,0)$, then $Z_2^+(x,0) = Z_2^-(x,0) = 0$, so that

$$Z_\mu(x,0) = (\mu Z_1^+(x,0) + (1-\mu)Z_1^-(x,0), 0),$$

which is not uniquely determined.

Assume that each of the vector fields Z^+ and Z^- has a single equilibrium point, say (x_0^+, y_0^+) and (x_0^-, y_0^-), respectively. We have the following situations, according to the locations of equilibrium points $(x_0^\pm, y_0^\pm)$:

(1) Both Z^+ and Z^- have the same equilibrium point, i.e., $x_0^+ = x_0^-$ and $y_0^+ = y_0^-$.
(2) Either Z^+ or Z^- have an equilibrium point on Σ, i.e., $y_0^+ = 0$ or $y_0^- = 0$.
(3) Both Z^+ and Z^- have a real equilibrium point but not on boundary Σ, i.e., $y_0^+ > 0$ and $y_0^- < 0$.
(4) Both Z^+ and Z^- have virtual equilibrium but not on boundary Σ, i.e., $y_0^+ < 0$ and $y_0^- > 0$.

Equilibrium points $(x_0^\pm, y_0^\pm)$ for $Z^\pm$ are one of the types: Focus or center (F), Saddle (S), or Node (N).

System (2.1) is classified into six types according to the type of equilibrium points in two zones; FF, FS, FN, SS, SN and NN.

Poincare full-return map P of Z can be obtained from the composition of half-return maps P^+ and P^- of Z^+ and Z^-, respectively, is given by $P(\rho) = P^- \circ P^+(\rho), \rho \in I \subseteq \Sigma$.

If $P(\rho) = \rho$ for all ρ in some interval I, then all orbits passing through the points $(\rho, 0), \rho \in I$, are closed, so that the orbits are periodic, and hence the system (2.1) has a period annulus.

3. Piecewise linear system of type focus focus

In this section, first, we will discuss the half-return maps of a linear system with a focus at (x_0, y_0) defined on some region on the x-axis.

Let a, b, c, d, α and β be constants, and let $\mu = a + d$, $\delta = ad - bc$ and $\Delta = 4\delta - \mu^2$. If $\delta \neq 0$ and $\Delta > 0$, then the linear system

$$(\dot{x}, \dot{y}) = (ax + by + \alpha, cx + dy + \beta), \tag{3.1}$$

has a focus or a center at

$$(x_0, y_0) = \left(\frac{b\beta - d\alpha}{ad - bc}, \frac{c\alpha - a\beta}{ad - bc}\right),$$

and does not have any other equilibrium point. Further, if $\mu = 0$, then (x_0, y_0) is a center. The orientation of orbits of system (3.1) is determined by the signs of constants b and c, for instance, see [17].

At the point $(x_0 + \varepsilon, y_0)$, the vector field (3.1) is given by

$$(\dot{x}, \dot{y}) = (ax_0 + by_0 + \alpha + a\varepsilon, cx_0 + dy_0 + \beta + c\varepsilon) = \varepsilon(a, c).$$

Assume $0 < \varepsilon \ll 1$. An orbit of a system (3.1) is oriented anticlockwise if and only if the orbit through $(x_0 + \varepsilon, y_0)$ must forward above $y > y_0$ so that the component along the y-axis of the tangent vector $\varepsilon(a, c)$ is positive. When we evaluate the vector field (3.1) at a point $(x_0, y_0 + \varepsilon)$, the orbit is oriented anticlockwise if and only if the x-component of the tangent vector $\varepsilon(b, d)$ is negative. Note that the solutions of the system exhibit rotation if and only if $bc < 0$, and the sense of rotation is determined by the sign of c or b. Thus, the solutions of the system exhibit rotation if and only if $bc < 0$ and are oriented anticlockwise if and only if either $c > 0$ or $b < 0$, for instance, see [17].

In the following lemma, we find an expression for the half-return map for the system (3.1) with the center. The expression for the half-return map obtained is independent of the orientation of orbits when $y_0 = 0$.

Lemma 3.1. *In* (3.1)*, let* $\mu = 0,\ \delta > 0$*. Suppose that* P *defines a half-return map for the system* (3.1)*, which maps the point* $(\rho, 0)$ *to the point* $(P(\rho), 0)$*. Then*

$$P(\rho) = \begin{cases} -\rho - \dfrac{2(a\alpha + b\beta)}{a^2 + bc} = -\rho + 2x_0, & \text{if } y_0 = a\beta - c\alpha = 0, \\ -\rho - \dfrac{2\beta}{c} = -\rho + 2\left(x_0 - \frac{a}{c}y_0\right), & \text{if } y_0 = a\beta - c\alpha \neq 0. \end{cases} \tag{3.2}$$

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $f(t) = \cos(\sqrt{\delta}t)$, and $g(t) = \sin(\sqrt{\delta}t)$. Then eigenvalues of A are $\pm i\sqrt{\delta}$. Suppose $v = v_1 + iv_2$ is an eigenvector of A associated to the eigenvalue $i\sqrt{\delta}$, where $v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ and $v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \in \mathbb{R}^2$. So that, $(A - i\sqrt{\delta}I)(v_1 + iv_2) = 0$ and hence,

$$\frac{1}{\sqrt{\delta}}A = [-v_2, v_1][v_1, v_2]^{-1}.$$

Then the general solution of (3.1) is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \left(f(t)[v_1, v_2] + g(t)[-v_2, v_1]\right)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \frac{1}{\delta}\begin{bmatrix} b\beta + a\alpha \\ c\alpha - a\beta \end{bmatrix},$$

where c_1 and c_2 are arbitrary constants. Applying the initial conditions $x(0) = \rho$ and $y(0) = 0$, we get

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [v_1, v_2]^{-1}\left(\begin{bmatrix} \rho \\ 0 \end{bmatrix} - \frac{1}{\delta}\begin{bmatrix} b\beta + a\alpha \\ c\alpha - a\beta \end{bmatrix}\right).$$

Hence, an orbit (a solution), $\Gamma(\rho)$, of the system that starts at $(\rho, 0)$, is given by

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \left(f(t)I + g(t)[-v_2, v_1][v_1, v_2]^{-1}\right) + \left(\begin{bmatrix} \rho \\ 0 \end{bmatrix} - \frac{1}{\delta}\begin{bmatrix} b\beta + a\alpha \\ c\alpha - a\beta \end{bmatrix}\right) + \frac{1}{\delta}\begin{bmatrix} b\beta + a\alpha \\ c\alpha - a\beta \end{bmatrix} \\ &= \left(f(t)I + \frac{g(t)}{\sqrt{\delta}}A\right) + \left(\begin{bmatrix} \rho \\ 0 \end{bmatrix} - \frac{1}{\delta}\begin{bmatrix} b\beta + a\alpha \\ c\alpha - a\beta \end{bmatrix}\right) + \frac{1}{\delta}\begin{bmatrix} b\beta + a\alpha \\ c\alpha - a\beta \end{bmatrix}. \end{aligned} \tag{3.3}$$

Let T be the smallest positive time required for $\Gamma(\rho)$ to reach the point $(P(\rho), 0)$ from $(\rho, 0)$. Then,

$$\begin{bmatrix} P(\rho) \\ 0 \end{bmatrix} - \frac{1}{\delta}\begin{bmatrix} b\beta + a\alpha \\ c\alpha - a\beta \end{bmatrix} = \left(f(T)I + \frac{g(T)}{\sqrt{\delta}}A\right) + \left(\begin{bmatrix} \rho \\ 0 \end{bmatrix} - \frac{1}{\delta}\begin{bmatrix} b\beta + a\alpha \\ c\alpha - a\beta \end{bmatrix}\right). \tag{3.4}$$

Thus,

$$\begin{aligned} P(\rho) - x_0 =& f(T)(\rho - x_0) + g(T)\frac{a(\rho - x_0) - by_0}{\sqrt{\delta}}, \text{ and} \\ y_0 =& -f(T)y_0 + g(T)\frac{c(\rho - x_0) + ay_0}{\sqrt{\delta}}. \end{aligned}$$

If $y_0 = 0$, then $g(T) = 0$, $f(T) = \pm 1$ and hence $P(\rho) - x_0 = \pm(\rho - x_0)$, which implies that

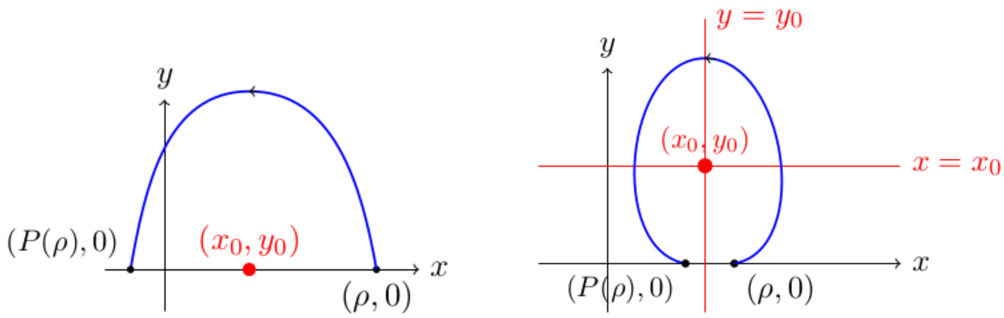
$$P(\rho) = \rho \text{ or } P(\rho) = -\rho + \frac{2(b\beta - a\alpha)}{a^2 + bc}.$$

Hence,

$$P(\rho) = -\rho + 2x_0 = -\rho + \frac{2(b\beta - d\alpha)}{ad - bc}.$$

Assume that $y_0 \neq 0$. The system of Equations (3.4) is linear in $f(T)$ and $g(T)$ with determinant of its coefficient matrix,

$$D = \frac{c(\rho - x_0)^2 + 2ay_0(\rho - x_0) - by_0^2}{\sqrt{\delta}},$$



(a) $y_0 = 0$.

(b) $y_0 \neq 0$.

FIGURE 1. Half-return map for center.

which is a quadratic polynomial in ρ and its discriminant, $-4y_0^2\delta$, is negative. Hence, the system of Equations (3.4) has a unique solution,

$$\begin{pmatrix} f(T) \\ g(T) \end{pmatrix} = \begin{pmatrix} \dfrac{c(\rho - x_0)(P(\rho) - x_0) + ay_0((P(\rho) + \rho - 2x_0)) - by_0^2}{c(\rho - x_0)^2 + 2ay_0(\rho - x_0) - by_0^2} \\ \dfrac{y_0\sqrt{\delta}\,(-P(\rho) + \rho)}{(c(\rho - x_0)^2 + 2ay_0(\rho - x_0) - by_0^2)} \end{pmatrix}. \tag{3.5}$$

Now substituting $x_0 = \dfrac{b\beta + a\alpha}{\delta}, y_0 = \dfrac{c\alpha - a\beta}{\delta}$ and $\delta = -a^2 - bc$ in (3.5), we get

$$f(T) = \frac{Pbc^2\rho + \left(Pa^2\rho + \beta\,(P+\rho)\,b - \alpha^2\right)c + \left((P+\rho)\,a^2 + 2a\alpha + b\beta\right)\beta}{b\rho^2c^2 + (a^2\rho^2 + 2b\beta\rho - \alpha^2)\,c + 2\left(a^2\rho + a\alpha + \frac{b\beta}{2}\right)\beta},$$

$$\text{and } g(T) = \frac{(a\beta - c\alpha)\,(P-\rho)\sqrt{-a^2 - bc}}{\left(\rho\,(c\rho + 2\beta)\,a^2 + 2\beta a\alpha + (c\rho + \beta)^2\,b - c\alpha^2\right)}.$$

Since $f^2(T) + g^2(T) = 1$, we have

$$\begin{aligned} &\left(b\beta^2 + \left(2P(\rho)a^2 + 2Pbc + 2a\alpha\right)\beta + c\left(P^2(\rho)a^2 + P^2(\rho)bc - \alpha^2\right)\right) \\ &\left(b\beta^2 + \left(2a^2\rho + 2bc\rho + 2a\alpha\right)\beta + c\left(a^2\rho^2 + bc\rho^2 - \alpha^2\right)\right) \\ &= \left(b\beta^2 + \left(2a^2\rho + 2bc\rho + 2a\alpha\right)\beta + c\left(a^2\rho^2 + bc\rho^2 - \alpha^2\right)\right)^2. \end{aligned} \tag{3.6}$$

Solving the Equation (3.6) for P, the Poincaré half-return map is given by

$$P(\rho) = -\rho - 2\frac{\beta}{c} = -\rho + 2\left(x_0 - \frac{a}{c}y_0\right).$$

□

In the following lemma, we obtain an expression of the half-return map defined on the x-axis when (x_0, y_0) is a focus but not a center of the system (3.1). We assume that the solutions of the system (3.1) are oriented in an anticlockwise direction, and equilibrium point (x_0, y_0) of the system (3.1) does not lie in the lower part of the plane, i.e., $y_0 \geq 0$. If the solutions of the system are clockwise oriented, then by a change of variable, t to $-t$, we can transform the system (3.1) such that solutions are anticlockwise oriented, and we get the same expression of the half-return map. We find an expression of the half-return P which maps the point $(\rho, 0)$ to the point $(P(\rho), 0)$, provided $\rho > x_0$. Observe that if $y_0 = 0$, we get the simple expression for the half-return map, whereas we get a rather complicated expression for the half-return map when $y_0 \neq 0$.

Lemma 3.2. *In system (3.1), let $\mu = a + d \neq 0,\ \delta = ad - bc,\ \Delta = 4\delta - \mu^2 > 0,$ $bc < 0, c > 0$ or $b < 0,\ f(t) = e^{\mu t/2}\cos(\sqrt{\Delta}t/2)$ and $g(t) = e^{\mu t/2}\sin(\sqrt{\Delta}t/2)$. Suppose that P is the half-return map for the system (3.1), which maps the point $(\rho, 0)$ to the point $(P(\rho), 0)$. Then, for $y_0 \leq 0$, we have*

$$P(\rho) = x_0 - e^{\frac{\mu\pi}{\sqrt{\Delta}}}(\rho - x_0), \tag{3.7}$$

and for $y_0 > 0,\ \rho > x_0$, we have

$$P(\rho) = x_0 - e^{\frac{\mu\pi}{\sqrt{\Delta}}}(P_1(\rho) - x_0)f(T_3) - \left(\frac{e^{\frac{\mu\pi}{\sqrt{\Delta}}}(P_1(\rho) - x_0)(a - d) + 2by_0}{\sqrt{\Delta}}\right)g(T_3), \tag{3.8}$$

where $P_1(\rho) = x_0 + (\rho - x_0)f(T_1) + \left(\frac{(a - d)(\rho - x_0) - 2by_0}{\sqrt{\Delta}}\right)g(T_1)$, and T_1, T_3 are the smallest positive roots of

$$g(T_3) = \frac{e^{\frac{-\mu\pi}{\sqrt{\Delta}}}y_0\sqrt{\Delta}}{2c(P_1(\rho) - x_0)},\ \tan\left(\frac{\sqrt{\Delta}}{2}T_1\right) = \left(\frac{y_0\sqrt{\Delta}}{2c(\rho - x_0) + (a - d)y_0}\right).$$

Proof. Eigenvalues of the coefficient matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of the system (3.1) are $\frac{\mu \pm i\sqrt{\Delta}}{2}$.

Let $v = v_1 + iv_2$ be an eigenvector of the matrix A associated to the eigenvalue $\frac{\mu + i\sqrt{\Delta}}{2}$, where

$$v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}, v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \in \mathbb{R}^2.$$

Then,

$$\left(A - \frac{\mu + i\sqrt{\Delta}}{2}I\right)(v_1 + iv_2) = 0$$

implies that

$$\left(A - \frac{\mu}{2}I\right)v_1 + \frac{\sqrt{\Delta}}{2}v_2 = 0, \text{ and } \left(A - \frac{\mu}{2}I\right)v_2 - \frac{\sqrt{\Delta}}{2}v_1 = 0.$$

Hence,

$$\frac{2}{\sqrt{\Delta}}\left(A - \frac{\mu}{2}I\right) = [-v_2, v_1][v_1, v_2]^{-1}$$

$$\Rightarrow \frac{2}{\sqrt{\Delta}}\begin{bmatrix} \frac{a-d}{2} & b \\ c & -\frac{a-d}{2} \end{bmatrix} = [-v_2, v_1][v_1, v_2]^{-1}.$$

Let

$$B = \frac{1}{ad - bc}\begin{bmatrix} -d\alpha + b\beta \\ c\alpha - a\beta \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Then the general solution of (3.1) is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = (f(t)[v_1, v_2] + g(t)[-v_2, v_1])\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + B. \tag{3.9}$$

Let ρ be any real number. Orbit of the differential system starting at $(\rho, 0)$ satisfy the initial conditions $x(0) = \rho$ and $y(0) = 0$, so that from (3.9), we get,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [v_1, v_2]^{-1}\begin{bmatrix} \rho - x_0 \\ -y_0 \end{bmatrix}. \tag{3.10}$$

Hence, a solution of (3.1) satisfying the initial conditions $x(0) = \rho$ and $y(0) = 0$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left(f(t)I + g(t)[-v_2, v_1][v_1, v_2]^{-1}\right) \begin{bmatrix} \rho - x_0 \\ -y_0 \end{bmatrix} + B. \tag{3.11}$$

If P is the Poincare half-return map that maps the point $(\rho, 0)$ to the point $(P(\rho), 0)$, satisfy the equations

$$\begin{bmatrix} P(\rho) - x_0 \\ -y_0 \end{bmatrix} = \left(f(T)I + g(T)[-v_2, v_1][v_1, v_2]^{-1}\right) \begin{bmatrix} \rho - x_0 \\ -y_0 \end{bmatrix}, \tag{3.12}$$

where T is the least time required for the orbit to reach the point $(P(\rho), 0)$ from the point $(\rho, 0)$. From (3.12), we get,

$$P(\rho) - x_0 = (\rho - x_0)f(T) + \left(\frac{(a-d)(\rho - x_0) - 2by_0}{\sqrt{\Delta}}\right) g(T), \text{ and}$$

$$-y_0 = -y_0 f(T) + \left(\frac{2c(\rho - x_0) + (a-d)y_0}{\sqrt{\Delta}}\right) g(T).$$

Assume that for $y_0 \geq 0, \rho > x_0$, $c > 0$ and $b < 0$. If $y_0 = 0$, then $g(T) = e^{\frac{\mu}{2}T} \sin\left(\sqrt{\Delta}\frac{T}{2}\right) = 0$ and hence, $T = \frac{2\pi}{\sqrt{\Delta}}$ and $f(T) = -e^{\frac{\mu\pi}{\sqrt{\Delta}}}$.
Therefore,

$$P(\rho) = x_0 - e^{\frac{\mu\pi}{\sqrt{\Delta}}}(\rho - x_0) = -e^{\frac{\mu\pi}{\sqrt{\Delta}}}\left(\rho - \frac{b\beta - d\alpha}{ad - bc}\right) + \frac{b\beta - d\alpha}{ad - bc}. \tag{3.13}$$

Next, to obtain an expression of the half-return map for the system (3.1) that sends the point $(\rho, 0), \rho > x_0$, to the point $(P(\rho), 0)$, when the solutions are oriented in an anticlockwise direction and its equilibrium point (x_0, y_0) is in the lower half-plane, i.e., (x_0, y_0) is virtual for the system (3.1). If the solutions of the system (3.1) are oriented in the clockwise direction, then again reversing the orientation using the change of the time variable, t to $-t$, we get the same expression of the half-return map. Using the translation $x = u + x_0, y = v + y_0$, the Equations (3.1) get transformed to

$$(\dot{u}, \dot{v}) = (au + bv, cu + dv). \tag{2.1*}$$

System (2.1*) has an equilibrium point at $(u, v) = (0, 0)$, and its solution is given by $u(t) = x(t) - x_0, v(t) = y(t) - y_0$, where $(x(t), y(t))$ is a solution of the system (3.1). Similar to the case, when $y_0 = 0$ in Lemma 3.2, expression for the half-return map of (2.1*) will be

$$P(\rho) = -e^{\frac{\mu\pi}{\sqrt{\Delta}}}\rho, \text{ for } u = \rho > 0.$$

Hence, for (3.1),

$$P(\rho) = x_0 - e^{\frac{\mu\pi}{\sqrt{\Delta}}}(\rho - x_0), \text{ for } x = \rho > x_0.$$

.

Now assume that $y_0 > 0$. Let T_1, T_2 and T_3 be the times of flights of the trajectories from the initial point $(\rho, 0)$ to $(P_1(\rho), y_0)$, from $(P_1(\rho), y_0)$ to $(P_2(\rho), y_0)$, and from $(P_2(\rho), y_0)$ to the point $(P(\rho), 0)$, respectively (see Figure 2(b).

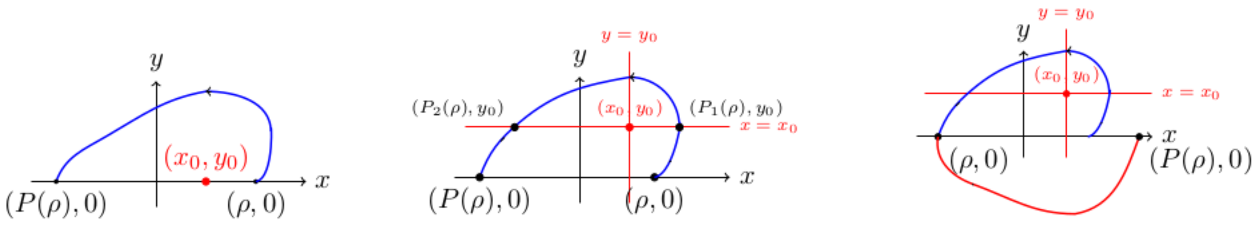
First, to find an expression for $P_1(\rho)$, note that $(P_1(\rho), y_0)$ and $(\rho, 0)$ correspond to $(t = T_1, x = P_1(\rho), y = y_0)$ and $(t = 0,\ x = \rho, y = 0)$, respectively. Applying these conditions to the general solution given by the Equation (3.12), we get

$$P_1(\rho) - x_0 = (\rho - x_0)f(T_1) + \left(\frac{(a-d)(\rho - x_0) - 2by_0}{\sqrt{\Delta}}\right) g(T_1), \tag{3.14}$$

$$0 = -y_0 f(T_1) + \left(\frac{2c(\rho - x_0) + (a-d)y_0}{\sqrt{\Delta}}\right) g(T_1). \tag{3.15}$$

From (3.15), we get

$$\tan(T_1\sqrt{\Delta}/2) = \frac{g(T_1)}{f(T_1)} = \frac{y_0\sqrt{\Delta}}{2c(\rho - x_0) + (a-d)y_0}.$$



(a) $y_0 = 0, \rho > x_0$. (b) $y_0 > 0, \rho > x_0$. (c) $y_0 > 0, \rho < x_0$.

FIGURE 2. Half-return maps for focus.

Hence,

$$T_1 = \frac{2}{\sqrt{\Delta}} \tan^{-1}\left(\frac{y_0\sqrt{\Delta}}{2c(\rho - x_0) + (a-d)y_0}\right). \tag{3.16}$$

Since T_1 is the time of flight, it is the smallest positive root of the Equation (3.16).

Next, to find an expression for P_2, note that the orbit (a solution) starts at $(P_1(\rho), y_0)$, and ends at $(P_2(\rho), y_0)$ in time T_2, which corresponds to the conditions $(x(0) = P_1(\rho),\ y(0) = y_0)$ and $(x(0) = P_2(\rho),\ y(0) = y_0)$, respectively. On similar lines, as in the case when $y_0 = 0$, from (3.12), we obtain

$$P_2(\rho) = x_0 - e^{\frac{\mu\pi}{\sqrt{\Delta}}}(P_1(\rho) - x_0). \tag{3.17}$$

Finally, to obtain an expression for $P(\rho)$, applying initial conditions $x = P_2(\rho),\ y = y_0$ to the Equation (3.12), we can find the equation of the trajectory starting at $(P_2(\rho), y_0)$ as,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left(fI + g[-v_2, v_1][v_1, v_2]^{-1}\right)\begin{bmatrix} P_2(\rho) - x_0 \\ 0 \end{bmatrix} + B.$$

Since $x(T_3) = P(\rho)$ and $y(T_3) = 0$, we get,

$$\begin{bmatrix} P(\rho) \\ 0 \end{bmatrix} = \left(f(T_3)I + g(T_3)[-v_2, v_1][v_1, v_2]^{-1}\right)\begin{bmatrix} P_2(\rho) - x_0 \\ 0 \end{bmatrix} + B. \tag{3.18}$$

Thus,

$$P(\rho) - x_0 = (P_2(\rho) - x_0)f(T_3) + \left(\frac{(a-d)(P_2(\rho) - x_0) - 2by_0}{\sqrt{\Delta}}\right)g(T_3),$$

$$-y_0 = \left(\frac{2c(P_2(\rho) - x_0)}{\sqrt{\Delta}}\right)g(T_3).$$

Hence, from the Equation (3.17), we get,

$$P(\rho) = x_0 - e^{\frac{\mu\pi}{\sqrt{\Delta}}}(P_1(\rho) - x_0)f(T_3) - \left(\frac{e^{\frac{\mu\pi}{\sqrt{\Delta}}}(P_1(\rho) - x_0)(a-d) + 2by_0}{\sqrt{\Delta}}\right)g(T_3), \text{ and}$$

$$g(T_3) = \frac{e^{-\frac{\mu\pi}{\sqrt{\Delta}}}y_0\sqrt{\Delta}}{2c(P_1(\rho) - x_0)}.$$

Since T_3 is the time of flight, it is the smallest positive root of the above equation. □

If $\mu > 0, \Delta > 0$ and $y_0 > 0$, then the system (3.1) has a real focus at (x_0, y_0) in the upper half-plane. Then the expression for the half-return map P for the system (3.1) involves the time of flight T explicitly. In this case, it is not easy to find an expression of the half-return map P, which sends points $(\rho, 0)$ to $(P(\rho), 0)$ that depends only on ρ, and parameters of the system. In such cases, we can find the first few coefficients in powers of $\rho - x_0$ in Taylor's series of $P(\rho)$ about point $\rho = x_0$. In the following lemma, we obtain the initial two terms in Taylor's series of the half-return map of the system (3.1), with a focus at (x_0, y_0) in an upper half part of the plane. In the proof, there is a remarkable use of Taylor's series expansion of the inverse of an analytic function, as in [14], [1].

Lemma 3.3. *In system* (3.1)*, assume that* $c > 0, \mu = a + d \neq 0,\ \delta = ad - bc,\ \Delta = 4\delta - \mu^2 > 0$*, and* $y_0 \neq 0$*. Suppose that* P *is a half-return map for the system* (3.1)*, which map the point* $(\rho, 0)$ *to the point* $(P(\rho), 0)$*. Then the linear approximation of* $P(\rho) - x_0$ *is given by*

$$P(\rho) - x_0 = L(\rho - x_0) + M,$$

where $L = (1 + aA + (ax_0 - by_0)B)\left(\dfrac{b{y_0}^2}{c}e^{-\frac{\mu\pi}{\sqrt{\Delta}}} - e^{\frac{\mu\pi}{\sqrt{\Delta}}}\right)$,

$$M = -\frac{1}{4}\frac{b^2y_0^2(A + Bx_0)^2\sqrt{\Delta}\mu}{c} - \left(-\frac{b{y_0}^2}{c}e^{-\frac{\mu\pi}{\sqrt{\Delta}}} - e^{\frac{\mu\pi}{\sqrt{\Delta}}}\right)by_0(A + Bx_0) - \frac{1}{2}\frac{(a-d)\,y_0}{c},$$

$$A = \frac{2}{\sqrt{\Delta}}\tan^{-1}\left(\frac{y_0\sqrt{\Delta}}{-2cx_0 + (a-d)y_0}\right), \ \textit{and}\ B = \frac{y_0}{b{y_0}^2 + x_0\ (a-d)\,y_0 - c{x_0}^2}.$$

Proof. Using the linear approximations of $f(T_1)$ and $g(T_1)$ from expression of $P_1(\rho) - x_0$ obtained in Lemma 3.2, we get

$$\begin{aligned} P_1(\rho) - x_0 =& (\rho - x_0)\left(1 + \frac{1}{2}\mu A + \frac{1}{2}\mu Bx_0 + \frac{1}{2}\mu B(\rho - x_0)\right) \\ &+ \left(\frac{(a-d)}{2}(\rho - x_0) - by_0\right)(A + Bx_0 + B(\rho - x_0)), \end{aligned}$$

where

$$\begin{aligned} A =& \frac{2}{\sqrt{\Delta}}\tan^{-1}\left(\frac{y_0\sqrt{\Delta}}{-2cx_0 + (a-d)y_0}\right) = \frac{2}{\sqrt{\Delta}}\tan^{-1}\left(\frac{y_0\sqrt{\Delta}}{2\beta - \mu y_0}\right), \text{ and} \\ B =& \frac{y_0}{b{y_0}^2 + x_0\ (a-d)\,y_0 - c{x_0}^2} = \frac{y_0}{\beta x_0 - \alpha y_0}. \end{aligned}$$

Hence, the linear approximation of $P_1(\rho) - x_0$ is given by

$$\begin{aligned} P_1(\rho) - x_0 &= -by_0(Bx_0 + A) + (1 + aA + (ax_0 - by_0)B)(\rho - x_0) \\ &= C + D(\rho - x_0), \end{aligned} \tag{3.19}$$

where $C = -by_0(Bx_0 + A)$ and $D = aA + (ax_0 - by_0)B$.

Also, using the linear approximations of $f(T_3)$, $P_1(\rho) - x_0$ and $g(T_3)$ in the expression of $P(\rho) - x_0$ from Lemma 3.2, we get

$$\begin{aligned} P(\rho) - x_0 =& -e^{\frac{\mu\pi}{\sqrt{\Delta}}}(C + D(\rho - x_0))\left(1 + \frac{\mu T_3}{2}\right) \\ &- \frac{y_0}{2c(C + D(\rho - x_0))}\left((C + D(\rho - x_0))(a-d) + 2by_0e^{\frac{-\mu\pi}{\sqrt{\Delta}}}\right) \\ =& -e^{\frac{\mu\pi}{\sqrt{\Delta}}}(C + D(\rho - x_0))\left(1 + \frac{\mu T_3}{2}\right) - \frac{a-d}{2c}y_0 - \frac{b}{c}y_0^2\frac{e^{-\frac{\mu\pi}{\sqrt{\Delta}}}}{C + D(\rho - x_0)}. \end{aligned} \tag{3.20}$$

From $g(T_3)$ in Lemma 3.2, linear approximation of T_3 using Taylor's series of holomorphic functions, for instance, see [14], is given by

$$
\begin{aligned}
T_3 &= g^{-1}\left(e^{\frac{-\mu\pi}{\sqrt{\Delta}}}\frac{y_0\sqrt{\Delta}}{2c(C+D(\rho-x_0))}\right)\\
&= g^{-1}\left(e^{\frac{-\mu\pi}{\sqrt{\Delta}}}\frac{y_0\sqrt{\Delta}}{2cC}\left(1-\frac{D}{C}(\rho-x_0)\right)+o((\rho-x_0)^2)\right)\\
&= e^{\frac{-\mu\pi}{\sqrt{\Delta}}}\frac{y_0\sqrt{\Delta}}{2c}\left(C-D(\rho-x_0)\right).
\end{aligned}
\tag{3.21}
$$

From (3.20) and (3.21), expression for $P(\rho)-x_0$ becomes

$$
\begin{aligned}
P(\rho)-x_0 = &-\left(C+D(\rho-x_0)\right)\left(e^{\frac{\mu\pi}{\sqrt{\Delta}}}+\frac{\mu y_0\sqrt{\Delta}\left(C-D(\rho-x_0)\right)}{4c}\right)\\
&-\frac{a-d}{2c}y_0-\frac{by_0^2}{c}e^{-\frac{\mu\pi}{\sqrt{\Delta}}}(C-D(\rho-x_0)).
\end{aligned}
\tag{3.22}
$$

Hence, the linear approximation for $P(\rho)-x_0$ is given by

$$
\begin{aligned}
P(\rho)-x_0 =& D\left(\frac{by_0{}^2}{c}e^{-\frac{\mu\,\pi}{\sqrt{\Delta}}}-e^{\frac{\mu\,\pi}{\sqrt{\Delta}}}\right)(\rho-x_0)\\
&-\frac{1}{4}\frac{C^2\sqrt{\Delta}\mu}{c}+\left(-\frac{by_0{}^2}{c}e^{-\frac{\mu\,\pi}{\sqrt{\Delta}}}-e^{\frac{\mu\,\pi}{\sqrt{\Delta}}}\right)C-\frac{1}{2}\frac{(a-d)\,y_0}{c}\\
=& L(\rho-x_0)+M,\\
\text{where } L =& (1+aA+(ax_0-by_0)B)\left(\frac{by_0{}^2}{c}e^{-\frac{\mu\,\pi}{\sqrt{\Delta}}}-e^{\frac{\mu\,\pi}{\sqrt{\Delta}}}\right),\ \text{and}\\
M =& -\frac{1}{4}\frac{b^2y_0^2(A+Bx_0)^2\sqrt{\Delta}\mu}{c}-\left(-\frac{by_0{}^2}{c}e^{-\frac{\mu\,\pi}{\sqrt{\Delta}}}-e^{\frac{\mu\,\pi}{\sqrt{\Delta}}}\right)by_0(A+Bx_0)\\
&-\frac{1}{2}\frac{(a-d)\,y_0}{c}.
\end{aligned}
$$

□

In the following lemma, we find expressions of full Poincaré return maps for the PWL system (2.2) of FF type, which is a composition of the half-return maps obtained in Lemma 3.1 and Lemma (3.2).

Lemma 3.4. *In system (2.2), let* $\mu^\pm = a^\pm + d^\pm$, $\delta^\pm = a^\pm d^\pm - b^\pm c^\pm \neq 0$, $\Delta^\pm = 4\delta^\pm - (\mu^\pm)^2 > 0, x_0^\pm = \dfrac{b^\pm\beta^\pm - d^\pm\alpha^\pm}{a^\pm d^\pm - b^\pm c^\pm}$ *and* $y_0^\pm = \dfrac{c^\pm\alpha^\pm - a^\pm\beta^\pm}{a^\pm d^\pm - b^\pm c^\pm}$.
Then, for $c^\pm > 0$ *and* $\rho \in \Sigma_r^c, P^+(\rho) \in \Sigma_l^c$, *we have the following:*

(1) *If* $\mu^+ = \mu^- = 0$, *then the full return map of the system (2.2) is given by*

$$
P(\rho) = \rho + 2\left(x_0^- - x_0^+ - \frac{a^-}{c^-}y_0^- + \frac{a^+}{c^+}y_0^+\right).
\tag{3.23}
$$

(2) *If* $\mu^+\mu^- \neq 0$ *and* $y_0^+ = y_0^- = 0$, *then the full return map of the system (2.2) is given by*

$$
P(\rho) = e^{\frac{\mu^+\pi}{\sqrt{\Delta^+}}+\frac{\mu^-\pi}{\sqrt{\Delta^-}}}\rho - e^{\frac{\mu^-\pi}{\sqrt{\Delta^-}}}\left(e^{\frac{\mu^+\pi}{\sqrt{\Delta^+}}}+1\right)x_0^+ + \left(e^{\frac{\mu^-\pi}{\sqrt{\Delta^-}}}+1\right)x_0^-.
\tag{3.24}
$$

(3) If $\mu^+ \neq 0,\ \mu^- = 0$ and $y_0^+ = 0$, then the full return map of the system (2.2) is given by

$$P(\rho) = e^{\frac{\mu^+\pi}{\sqrt{\Delta^+}}}\rho + 2\left(x_0^- - \frac{a^-}{c^-}y_0^-\right) - \left(1 + e^{\frac{\mu^+\pi}{\sqrt{\Delta^+}}}\right)x_0^+. \tag{3.25}$$

(4) If $\mu^+ = 0,\ \mu^- \neq 0$ and $y_0^- = 0$, then full return map of the system (2.2) is given by

$$P(\rho) = e^{\frac{\mu^-\pi}{\sqrt{\Delta^-}}}\rho + \left(1 + e^{\frac{\mu^-\pi}{\sqrt{\Delta^-}}}\right)x_0^- - 2e^{\frac{\mu^-\pi}{\sqrt{\Delta^-}}}\left(x_0^+ - \frac{a^+}{c^+}y_0^+\right). \tag{3.26}$$

Proof. The expression of Poincare maps in Equations (3.23), (3.24), (3.25), and (3.26) follows from the composition of expression of half-return map obtained in Lemma 3.1 and Lemma 3.2. □

Now we discuss the domain on which the full return map is well-defined. The crossing region of the PWL system (2.2) is

$$\Sigma^c = \left\{(x,0) \in \Sigma : (c^+x + \beta^+)\cdot(c^-x + \beta^-) > 0\right\}.$$

For simplicity, let us assume that the solutions of the system (2.2) are oriented in an anticlockwise direction. Therefore, $c^\pm > 0, b^\mp < 0$. Also, assume that subsystems of the system (2.2) have real equilibrium points $(x_0^\pm, y_0^\pm)$. Therefore, $y_0^+ > 0$ and $y_0^- < 0$. In this case, the crossing region of the system (2.2) is given by

$$\Sigma^c = \left\{(x,0) \in \Sigma : x > x_0^+ + \frac{d^+}{c^+}y_0^+, x > x_0^- + \frac{d^-}{c^-}y_0^-\right\}$$
$$\cup\left\{(x,0) \in \Sigma : x < x_0^+ + \frac{d^+}{c^+}y_0^+, x < x_0^- + \frac{d^-}{c^-}y_0^-\right\}.$$

Let us denote

$$\Sigma_r^c = \left\{(x,0) \in \Sigma : x > x_0^+ + \frac{d^+}{c^+}y_0^+ \text{ and } x > x_0^- + \frac{d^-}{c^-}y_0^-\right\},$$
$$\Sigma_l^c = \left\{(x,0) \in \Sigma : x < x_0^+ + \frac{d^+}{c^+}y_0^+ \text{ and } x < x_0^- + \frac{d^-}{c^-}y_0^-\right\}.$$

Then,

$$\Sigma^c = \Sigma_r^c \cup \Sigma_l^c.$$

We have the following cases:

Case(i): $x_0^- + \frac{d^-}{c^-}y_0^- < x_0^+ + \frac{d^+}{c^+}y_0^+$,

Case(ii): $x_0^+ + \frac{d^+}{c^+}y_0^+ < x_0^- + \frac{d^-}{c^-}y_0^-$.

In Case (i), we have

$$\Sigma_r^c = \left\{(x,0) \in \Sigma\colon x_0^+ + \frac{d^+}{c^+}y_0^+ < x\right\},\ \Sigma_l^c = \left\{(x,0) \in \Sigma\colon x < x_0^- + \frac{d^-}{c^-}y_0^-\right\},$$
$$\Sigma^s = \Sigma\setminus\Sigma^c = \left\{(x,0) \in \Sigma\colon x_0^- + \frac{d^-}{c^-}y_0^- < x < x_0^+ + \frac{d^+}{c^+}y_0^+\right\}.$$

If the upper half orbit Γ^+ of the system (2.2) begins at a point $(\rho,0)$ in the region Σ_r^c, and reaches at the point $(P^+(\rho),0)$ in the region Σ_l^c, and the lower half orbit Γ^- which starts at $(P^+(\rho),0)$, reaches at the point $(P(\rho) = P^-(P^+(\rho)),0)$ in Σ_r^c, then the full return map $P = P^- \circ P^+$ is well defined, and we have a full orbit of the system (2.2), which is of crossing type. Therefore, if $\rho \in \Sigma_r^c$, i.e., $x_0^+ + \frac{d^+}{c^+}y_0^+ < \rho$ and $P^+(\rho) \in \Sigma_l^c$, i.e., $P^+(\rho) = x_0^+ - e^{\frac{\mu^+\pi}{\sqrt{\Delta^+}}}(\rho - x_0^+) < x_0^- + \frac{d^-}{c^-}y_0^-$, then the full return map $P = P^- \circ P^+$ is well defined, and the full orbits of the system (2.2) are of crossing type. Thus, for $\rho > \max\left\{x_0^+ + \frac{d^+}{c^+}y_0^+,\ x_0^+ - e^{\frac{\mu^+\pi}{\sqrt{\Delta^+}}}\frac{d^+}{c^+}y_0^+\right\}$, $P(\rho)$ is well defined.

Similarly, in case(ii), the full return map $P(\rho)$ is defined for each

$\rho > \max\left\{ x_0^- + \frac{d^-}{c^-}y_0^-,\ x_0^- - e^{\frac{\mu^-\pi}{\sqrt{\Delta^-}}}\frac{d^-}{c^-}y_0^- \right\}$. Further, if the subsystems of the system (2.2) have foci at $(x_0^\pm, y_0^\pm = 0)$, then $P(\rho)$ is defined for $\rho > \max\{x_0^+, x_0^-\}$.

Now, we will obtain the conditions under which the system (2.2) has a sliding limit cycle. From the expression of P^+, observe that $\rho \in \Sigma_r^c$, $P^+(\rho) \in \Sigma_l^s$ if and only if

$$x_0^+ + \frac{d^+}{c^+}y_0^+ < \rho < e^{-\frac{\mu^+\pi}{\sqrt{\Delta^+}}}\left(x_0^+ - x_0^- - \frac{d^-}{c^-}y_0^-\right) + x_0^+.$$

Remark 3.5. If $c^+ - c^- \neq 0$ or $\beta^+ - \beta^- \neq 0$, then the sliding vector field of the system (2.2) is given by

$$\begin{aligned} Z^s(x,0) &= \frac{Z_1^+(x,0)Z_2^-(x,0) - Z_1^-(x,0)Z_2^+(x,0)}{Z_2^+(x,0) - Z_2^-(x,0)} \\ &= \frac{(a^+x+\alpha^+)(c^-x+\beta^-) - (a^-x+\alpha^-)(c^+x+\beta^+)}{c^+x+\beta^+ - c^-x - \beta^-} \\ &= \frac{(a^+c^- - a^-c^+)x^2 + (a^+\beta^- - a^-\beta^+ + \alpha^+c^- - c^+\alpha^-)x + (\alpha^+\beta^- - \alpha^-\beta^+)}{(c^+ - c^-)x + (\beta^+ - \beta^-)}. \end{aligned}$$

Let $D = a^+\beta^- - a^-\beta^+ + \alpha^+c^- - c^- - c^+\alpha^-)^2 - 4(a^+c^- - a^-c^+)(\alpha^+\beta^- - \alpha^-\beta^+)$. Observe that the sliding vector field Z^s has two equilibrium points when $D > 0$, one equilibrium point when $D = 0$, and no equilibrium point when $D < 0$.

Using the full return map, we will give the conditions for the period annulus of the system (2.2) when it is FF type with $y_0^\pm = 0$.

Proposition 3.6. *In the system (2.2), let* $\mu^\pm = a^\pm + d^\pm$, $\delta^\pm = a^\pm d^\pm - b^\pm c^\pm \neq 0$, $\Delta^\pm = 4\delta^\pm - (\mu^\pm)^2 > 0, x_0^\pm = \dfrac{b^\pm\beta^\pm - d^\pm\alpha^\pm}{a^\pm d^\pm - b^\pm c^\pm}$ *and* $y_0^\pm = \dfrac{c^\pm\alpha^\pm - a^\pm\beta^\pm}{a^\pm d^\pm - b^\pm c^\pm}$.
Assume that $\mu^\pm \neq 0$ *whenever* $y_0^\pm = 0$. *Then for the piecewise linear system (2.2), a period annulus occurs in the following cases only:*

(1) $\mu^+ = \mu^- = 0$ *and* $c_0^+x_0^+ - a^+y_0^+ - c^-x_0^- + a^-y_0^- = 0$.
(2) $\mu^+\mu^- \neq 0$, $y_0^+ = y_0^- = 0$, $\frac{\mu^+}{\sqrt{\Delta^+}} = -\frac{\mu^-}{\sqrt{\Delta^-}}$ *and* $x_0^+ = x_0^-$.

Proof. The orbits starting at points $(\rho, 0)$, with $(\rho, 0)$ in the interval I in Σ_r^c, are periodic orbits if and only if $P(\rho) = \rho$. Hence, from the equation, $P(\rho) = \rho$ for all ρ in the interval $I \subseteq \Sigma_r^c$, we get required conditions. □

4. Normal forms of the types focus-focus and saddle-saddle

In accordance with Proposition 3.6, a period annulus exists around the equilibrium point $(x_0^+ = x_0^-, 0)$ of the PWL system (2.2) in canonical form if it satisfies either
(1) $\mu^+ = \mu- = 0$, i.e., both Z^+ and Z^- have centers at (x_0^+, y_0^+) and (x_0^-, y_0^-), respectively, or
(2) $\mu^+\mu^- \neq 0$ and Z^+ and Z^- both have foci at $(x_0^+ = x_0^-, 0)$.

Using the above information, we show that the normal form of a piecewise smooth differential system (2.1), if it has a period annulus around the origin, are nothing but PWL systems in some neighbourhood of the origin.
Consider the following hypotheses about the system (2.1):

AI: The system (2.1) has a center at the origin, i.e., a period annulus around the origin.

AII: Both Z^+ and Z^- have focus only at the origin when both Z^+ and Z^- are extended independently on $\mathbb{R}^2$.

AIII: Z^+ has a saddle only at (x_0^+, y_0^+) and Z^- has a saddle only at (x_0^-, y_0^-).

Assuming hypotheses (AI) and (AII), we claim that the PWL system (2.1) is Σ-conjugate to the PWL system given by

$$Z_0(x,y) = \begin{cases} (x-y, x+y), & \text{if } y \in \Sigma^+ = \{(x,y) \in \mathbb{R}^2 \colon y > 0\} \\ (-x-y, x-y), & \text{if } y \in \Sigma^- = \{(x,y) \in \mathbb{R}^2 \colon y < 0\} \end{cases}. \tag{4.1}$$

On $\Sigma = \{(x,0) \colon (x,0) \in \mathbb{R}^2\}$, the system (4.1) is defined by the Filippov convex combination.

Proposition 4.1. *Suppose that the system (2.1) satisfies the hypotheses (AI) and (AII). Then there exist open subsets U and V in $\mathbb{R}^2$ containing the origin such that the systems (2.1) on U and (4.1) on V are Σ-conjugate.*

Proof. Proof is similar to the proof of Theorem 2 in [15]. □

If hypotheses (AI) and (AIII) hold, then it can be seen that the PWS system (2.1) is Σ-conjugate to PWL system

$$Z_0(x,y) = \begin{cases} (y-1, x) & \text{if } y \in \Sigma^+ = \{(x,y) \in \mathbb{R}^2 \colon y > 0\} \\ (y+1, x) & \text{if } y \in \Sigma^- = \{(x,y) \in \mathbb{R}^2 \colon y < 0\} \end{cases}. \tag{4.2}$$

On $\Sigma = \{(x,0) \colon (x,0) \in \mathbb{R}^2\}$, the system (4.2) is defined by the Filippov convex combination.

Proposition 4.2. *Suppose that the hypotheses (AI) and (AIII) hold for the system (2.1). Then there exist open subsets U and V of $\mathbb{R}^2$ containing the origin such that the system (2.1) on U and the system (4.2) on V are Σ-conjugate to each other.*

Proof. The proof is similar to the proof of Theorem 2 in [15]. □

5. Bifurcation of limit cycles from period annulus

The zeros of the Melnikov function of order one are used to find the number of limit cycles bifurcated from the period annulus of a differential system. We find an upper bound for the number of limit cycles that bifurcates from the period annulus of the systems (4.1) and (4.2) due to perturbation.

5.1. Focus-Focus.

Consider a perturbation of the system (4.1) in a class of polynomial functions of degree n as,

$$Z_\varepsilon(x,y) = \begin{cases} (x-y+\varepsilon f^+(x,y), x+y+\varepsilon g^+(x,y)), & x \geq 0, \\ (-x-y+\varepsilon f^-(x,y), x-y+\varepsilon g^-(x,y)), & x < 0, \end{cases} \tag{5.1}$$

where

$$f^\pm(x,y) = \sum_{i+j=0}^{n} a^\pm_{i,j} x^i y^j, \; g^\pm(x,y) = \sum_{i+j=0}^{n} b^\pm_{ij} x^i y^j.$$

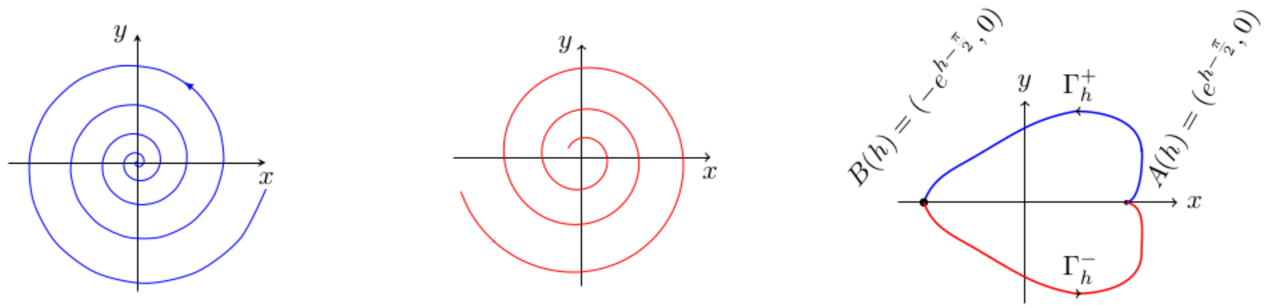
System (5.1) at $\varepsilon = 0$ is integrable, and the first integral is

$$H(x,y) = \begin{cases} H^+(x,y) = \frac{1}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{x}{y}\right) & \text{if } y > 0, \\ \pi/2 & \text{if } y = 0, \\ H^-(x,y) = \frac{1}{2}\ln(x^2+y^2) - \tan^{-1}\left(\frac{x}{y}\right) & \text{if } y < 0 \end{cases}$$

with the integrating factors $\xi(x,y)^\pm = \dfrac{1}{x^2+y^2}$.

Note that the system (5.1) at $\varepsilon = 0$ satisfies the following hypotheses:

BI: There is an interval $I = (-\infty, \infty)$ and two points $A(h) = (a(h), 0) = (e^{h-\pi/2}, 0)$ and $B(h) = (b(h), 0) = (-e^{h-\pi/2}, 0)$ such that
$H^+(A(h)) = H^+(B(h)) = h$, and $H^-(A(h)) = H^-(B(h)) = h, \; a(h) \neq b(h), h \in I$.



(a) Orbit of $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}x-y\\ x+y\end{pmatrix}$. (b) Orbit of $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-x-y\\ x-y\end{pmatrix}$. (c) Closed orbit of (5.1) at $\varepsilon = 0$.

FIGURE 3. Trajectory of Focus-Focus type.

BII: The system (5.1) at $\varepsilon = 0$ has an orbital arc

$$\Gamma_h^+ : H^+(x,y) = h, \;\; h \in (-\infty, \infty)$$

from $A(h)$ to $B(h)$ in an upper half plane and has an orbital arc

$$\Gamma_h^- : H^-(x,y) = h, \;\; h \in (-\infty, \infty)$$

from $B(h)$ to $A(h)$ in a lower half plane.

Under the above hypotheses (BI) and (BII), system (5.1) at $\varepsilon = 0$ has a family of closed orbits

$$\Gamma_h = \Gamma_h^+ + \Gamma_h^-, \quad h \in (-\infty, \infty)$$

and each of Γ_h is piecewise smooth. Observe that

$$\lim_{(x,y)\to(h,0^+)} H^+(x,y) = \lim_{(x,y)\to(h,0^-)} H^-(x,y) = \begin{cases} \ln(h) + \frac{\pi}{2} & \text{if } h > 0 \\ \ln(-h) - \frac{\pi}{2} & \text{if } h < 0. \end{cases},$$

and $\lim_{(x,y)\to(h,0^+)} H_x^+(x,y) = 1/h = \lim_{(x,y)\to(h,0^-)} H_x^-(x,y)$.

Therefore,

$$\frac{H_x^+(A(h))}{H_x^-(A(h))} = 1 = \frac{H_x^-(B(h))}{H_x^+(B(h))}.$$

Proposition 5.1. *The system (5.1) has at most n limit cycles. Moreover, there exist polynomials $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$ such that (5.1) has exactly n limit cycles.*

Proof. From Theorem 1.1 in [23], the Melnikov function of first order, $M(h)$, for the system (5.1), is given by

$$\begin{aligned}
M(h) &= \int_{\Gamma_h^+} \xi^+(g^+(x,y)dx - f^+(x,y)dy) + \int_{\Gamma_h^-} \xi^-(g^-(x,y)dx - f^-(x,y)dy \\
&= \int_{\Gamma_h^+} \xi^+(g^+(x,y)dx - f^+(x,y)dy) + \int_{\Gamma_h^-} \xi^-(g^-(x,y)dx - f^-(x,y)dy \\
&= \int_{\Gamma_h^+} \frac{1}{x^2+y^2}\left(\left(\sum_{i+j=0}^{n} b_{ij}^+ x^i y^j\right) dx - \left(\sum_{i+j=0}^{n} a_{ij}^+ x^i y^j\right) dy\right) \\
&\quad + \int_{\Gamma_h^-} \frac{1}{x^2+y^2}\left(\left(\sum_{i+j=0}^{n} b_{ij}^- x^i y^j\right) dx - \left(\sum_{i+j=0}^{n} a_{ij}^- x^i y^j\right) dy\right) \\
&= \sum_{i+j=0}^{n}\left(\int_{\Gamma_h^+} \frac{1}{x^2+y^2}\left(b_{ij}^+ x^i y^j dx - a_{ij}^+ x^i y^j dy\right) + \int_{\Gamma_h^-} \frac{1}{x^2+y^2}\left(b_{ij}^- x^i y^j dx - a_{ij}^- x^i y^j dy\right)\right). \qquad (5.2)
\end{aligned}$$

To find the line integrals over Γ_h^+ and Γ_h^-, we write Γ_h^+ and Γ_h^- in polar coordinates, $x = r\cos\theta, y = r\sin\theta$. Then

$$\Gamma_h^+ : re^{\theta} = e^h,\ 0 < \theta < \pi.$$

Thus, in parametric form

$$\Gamma_h : x(\theta) = e^{-\theta+h}\cos(\theta), y(\theta) = e^{-\theta+h}\sin(\theta),\ 0 < \theta < \pi.$$

Similarly,

$$\Gamma_h^- : x(\theta) = e^{\theta+h}\cos(\theta), y(\theta) = e^{\theta+h}\sin(\theta),\ \pi < \theta < 2\pi.$$

From (5.2), the first-order Melnikov function becomes,

$$\begin{aligned}
M(h) &= \sum_{i+j=0}^{n} e^{h(i+j-1)} \int_0^{\pi} e^{-\theta(i+j+3)} \cos^i\theta \sin^j\theta \left(b_{ij}^+(-\cos\theta - \sin\theta) - a_{ij}^+(-\sin\theta + \cos\theta)\right) d\theta \\
&+ \sum_{i+j=0}^{n} e^{h(i+j-1)} \int_{\pi}^{2\pi} e^{-\theta(i+j+3)} \cos^i\theta \sin^j\theta \left(b_{ij}^-(-\cos\theta - \sin\theta) - a_{ij}^-(-\sin\theta + \cos\theta)\right) d\theta \\
&= \sum_{i+j=0}^{n} \alpha_{ij} e^{h(i+j-1)}, \qquad (5.3)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{ij} &= \int_0^{\pi} e^{-\theta(i+j+3)} \cos^i\theta \sin^j\theta \left(b_{ij}^+(-\cos\theta - \sin\theta) - a_{ij}^+(-\sin\theta + \cos\theta)\right) d\theta \\
&+ \int_{\pi}^{2\pi} e^{-\theta(i+j+3)} \cos^i\theta \sin^j\theta \left(b_{ij}^-(-\cos\theta - \sin\theta) - a_{ij}^-(-\sin\theta + \cos\theta)\right) d\theta.
\end{aligned}$$

We can write (5.3) as

$$e^h M(h) = \sum_{i+j=0}^{n} \alpha_{ij} e^{h(i+j)} \text{ for } h \neq 0, \qquad (5.4)$$

Put $e^h = \tilde{h} > 0$ in (5.4), we get

$$\tilde{h} M(\ln(\tilde{h})) = \sum_{i+j=0}^{n} \alpha_{ij} \tilde{h}^{i+j} \text{ for } \tilde{h} > 0. \qquad (5.5)$$

The right-hand side of Equation (5.5) is a n degree polynomial in $\tilde{h}$, and hence it has at most n positive zeros. Therefore, $M(h)$ has at most n positive zeros. Thus, the system (5.1) has at most n limit cycles.

Further, the right-hand side of the Equation (5.5) is a polynomial of degree n in $\tilde{h}$, and the coefficient of $\tilde{h}^k$ is $u_k = \sum_{i+j=k} \alpha_{ij}$, it is clear that $\dfrac{\partial(u_0, u_1, ..., u_n)}{\partial(\alpha_{00}, \alpha_{01}, ..., \alpha_{nn})}$ has a full row rank. Also, note that α_{ij}'s are linear combinations of a_{ij}'s and b_{ij}'s. Therefore, there exist parameters α_{ij}, and hence parameters a_{ij}, b_{ij} such that $M(h)$ has exactly n positive zeros. Hence, there exist polynomials $f^{\pm}(x, y)$ and $g^{\pm}(x, y)$ such that the system (5.1) has exactly n limit cycles. □

Now we present an example of a piecewise smooth system of the type focus-focus with polynomial perturbation. Here, we discuss the polynomial perturbation of degree $n = 5$.

Example 5.2. Consider the system (5.1) with $n = 5$, where

$$
\begin{array}{llll}
a^+_{0,0} = -600, & b^+_{0,0} = a^-_{0,0} = b^-_{0,0} = 0, & a^+_{1,0} = -\frac{40}{7}\frac{1}{1-e^{-4\pi}}\frac{274}{2}, & b^+_{1,0} = a^-_{1,0} = b^-_{1,0} = 0,\\
a^+_{0,1} = -\frac{40}{1}\frac{1}{1+e^{-4\pi}}\frac{274}{2}, & b^+_{0,1} = a^-_{0,1} = b^-_{0,1} = 0, & a^+_{2,0} = -\frac{221}{33}\frac{1}{1+e^{-5\pi}}\frac{225}{3}, & b^+_{2,0} = a^-_{2,0} = b^-_{2,0} = 0,\\
a^+_{1,1} = -\frac{442}{9}\frac{1}{1+e^{-5\pi}}\frac{225}{3}, & b^+_{1,1} = a^-_{1,1} = b^-_{1,1} = 0, & a^+_{0,2} = -221\frac{1}{1+e^{-5\pi}}\frac{225}{3}, & b^+_{0,2} = a^-_{0,2} = b^-_{0,2} = 0,\\
a^+_{3,0} = \frac{208}{27}\frac{1}{1-e^{-6\pi}}\frac{85}{4}, & b^+_{3,0} = a^-_{3,0} = b^-_{3,0} = 0, & a^+_{1,2} = \frac{3120}{11}\frac{1}{1-e^{-6\pi}}\frac{85}{4}, & b^+_{1,2} = a^-_{1,2} = b^-_{1,2} = 0,\\
a^+_{2,1} = \frac{3120}{49}\frac{1}{1-e^{-6\pi}}\frac{85}{4}, & b^+_{2,1} = a^-_{2,1} = b^-_{2,1} = 0, & a^+_{0,3} = 1040\frac{1}{1-e^{-6\pi}}\frac{85}{4}, & b^+_{0,3} = a^-_{0,3} = b^-_{0,3} = 0,\\
a^+_{4,0} = -\frac{26825}{3077}\frac{1}{1+e^{-7\pi}}\frac{15}{5}, & b^+_{4,0} = a^-_{4,0} = b^-_{4,0} = 0, & a^+_{3,1} = -\frac{26825}{332}\frac{1}{1+e^{-7\pi}}\frac{15}{5}, & b^+_{3,1} = a^-_{3,1} = b^-_{3,1} = 0,\\
a^+_{1,3} = -\frac{53650}{39}\frac{1}{1+e^{-7\pi}}\frac{15}{5}, & b^+_{1,3} = a^-_{1,3} = b^-_{1,3} = 0, & a^+_{2,2} = -\frac{26825}{39}\frac{1}{1+e^{-7\pi}}\frac{15}{5}, & b^+_{2,2} = a^-_{2,2} = b^-_{2,2} = 0,\\
a^+_{0,4} = -\frac{26825}{36}\frac{1}{1+e^{-7\pi}}\frac{15}{5}, & b^+_{0,4} = a^-_{0,4} = b^-_{0,4} = 0, & a^+_{5,0} = \frac{54400}{5889}\frac{1}{1-e^{-8\pi}}\frac{1}{5}, & b^+_{5,0} = a^-_{5,0} = b^-_{5,0} = 0,\\
a^+_{4,1} = \frac{54400}{543}\frac{1}{1-e^{-8\pi}}\frac{1}{5}, & b^+_{4,1} = a^-_{4,1} = b^-_{4,1} = 0, & a^+_{3,2} = \frac{54400}{101}\frac{1}{1-e^{-8\pi}}\frac{1}{5}, & b^+_{3,2} = a^-_{3,2} = b^-_{3,2} = 0,\\
a^+_{2,3} = \frac{54400}{17}\frac{1}{1-e^{-8\pi}}\frac{1}{5}, & b^+_{2,3} = a^-_{2,3} = b^-_{2,3} = 0, & a^+_{1,4} = \frac{54400}{9}\frac{1}{1-e^{-8\pi}}\frac{1}{5}, & b^+_{1,4} = a^-_{1,4} = b^-_{1,4} = 0,\\
a^+_{0,5} = \frac{54400}{3}\frac{1}{1-e^{-8\pi}}\frac{1}{5}, & b^+_{0,5} = a^-_{0,5} = b^-_{0,5} = 0. & &
\end{array}
\tag{5.6}
$$

Then the Melnikov function of (5.1) is given by

$$M(h) = \sum_{i+j=1}^{n} \alpha_{ij} e^{h(i+j-1)}, \tag{5.7}$$

where

$$\alpha_{0,0} = -\frac{2}{5}\left(e^{-3\pi}+1\right)b^{+}_{0,0} + \frac{1}{5}\left(e^{-3\pi}+1\right)a^{+}_{0,0} + \frac{2}{5}\left(e^{-3\pi}+e^{-6\pi}\right)b^{-}_{0,0} + \frac{1}{5}\left(e^{-3\pi}+e^{-6\pi}\right)a^{-}_{0,0},$$
$$\alpha_{1,0} = -11\left(e^{-4\pi}+1\right)b^{+}_{1,0} + 7\left(e^{-4\pi}+1\right)a^{+}_{1,0} - 11\left(e^{-4\pi}+e^{-8\pi}\right)b^{-}_{1,0} - 7\left(e^{-4\pi}+e^{-8\pi}\right)a^{-}_{1,0},$$
$$\alpha_{0,1} = -3\left(e^{-4\pi}+1\right)b^{+}_{0,1} + \left(e^{-4\pi}+1\right)a^{+}_{0,1} - 3\left(e^{-4\pi}+e^{-8\pi}\right)b^{-}_{0,1} - \left(e^{-4\pi}+e^{-8\pi}\right)a^{-}_{0,1},$$
$$\alpha_{2,0} = -47\left(e^{-5\pi}+1\right)b^{+}_{2,0} + 33\left(e^{-5\pi}+1\right)a^{+}_{2,0} + 47\left(e^{-5\pi}+e^{-10\pi}\right)b^{-}_{2,0} + 33\left(e^{-5\pi}+e^{-10\pi}\right)a^{-}_{2,0},$$
$$\alpha_{1,1} = -19\left(e^{-5\pi}+1\right)b^{+}_{1,1} + 9\left(e^{-5\pi}+1\right)a^{+}_{1,1} + 19\left(e^{-5\pi}+e^{-10\pi}\right)b^{-}_{1,1} + 9\left(e^{-5\pi}+e^{-10\pi}\right)a^{-}_{1,1},$$
$$\alpha_{0,2} = -4\left(e^{-5\pi}+1\right)b^{+}_{0,2} + \left(e^{-5\pi}+1\right)a^{+}_{0,2} + 4\left(e^{-5\pi}+e^{-10\pi}\right)b^{-}_{0,2} + \left(e^{-5\pi}+e^{-10\pi}\right)a^{-}_{0,2},$$
$$\alpha_{3,0} = \frac{181}{1040}\left(e^{-6\pi}-1\right)b^{+}_{3,0} + \frac{27}{208}\left(e^{-6\pi}-1\right)a^{+}_{3,0} - \frac{181}{1040}\left(e^{-6\pi}-e^{-12\pi}\right)b^{-}_{3,0} + \frac{27}{208}\left(e^{-6\pi}-e^{-12\pi}\right)a^{-}_{3,0},$$
$$\alpha_{1,2} = 29\left(e^{-6\pi}-1\right)b^{+}_{1,2} - 11\left(e^{-6\pi}-1\right)a^{+}_{1,2} - 29\left(e^{-6\pi}-e^{-12\pi}\right)b^{-}_{1,2} + 11\left(e^{-6\pi}-e^{-12\pi}\right)a^{-}_{1,2},$$

$$\alpha_{2,1} = 89\left(e^{-6\pi}-1\right)b^{+}_{2,1} + 49\left(e^{-6\pi}-1\right)a^{+}_{2,1} - 89\left(e^{-6\pi}-e^{-12\pi}\right)b^{-}_{2,1} - 49\left(e^{-6\pi}-e^{-12\pi}\right)a^{-}_{2,1},$$
$$\alpha_{0,3} = -\frac{1}{208}\left(e^{-6\pi}-1\right)b^{+}_{0,3} + \frac{1}{1040}\left(e^{-6\pi}-1\right)a^{+}_{0,3} - \frac{1}{208}\left(e^{-6\pi}+e^{-12\pi}\right)b^{-}_{0,3} - \frac{1}{1040}\left(e^{-6\pi}+e^{-12\pi}\right)a^{-}_{0,3},$$
$$\alpha_{4,0} = \frac{e^{-14\pi}}{26825}\left(e^{7\pi}+1\right)\left(3077\,a^{+}_{4,0}e^{7\pi} - 3958\,b^{+}_{4,0}e^{7\pi} + 3077\,a^{-}_{4,0} + 3958\,b^{-}_{4,0}\right),$$
$$\alpha_{3,1} = \frac{e^{-14\pi}}{26825}\left(e^{7\pi}+1\right)\left(332\,a^{+}_{3,1}e^{7\pi} - 549\,b^{+}_{3,1}e^{7\pi} + 332\,a^{-}_{3,1} + 549\,b^{-}_{3,1}\right),$$
$$\alpha_{2,2} = \frac{e^{-14\pi}}{26825}\left(e^{7\pi}+1\right)\left(68\,a^{+}_{2,2}e^{7\pi} - 149\,b^{+}_{2,2}e^{7\pi} + 68\,a^{-}_{2,2} + 149\,b^{-}_{2,2}\right),$$
$$\alpha_{1,3} = \frac{e^{-14\pi}}{53650}\left(3\,e^{7\pi}+3\right)\left(13\,a^{+}_{1,3}e^{7\pi} - 41\,b^{+}_{1,3}e^{7\pi} + 13\,a^{-}_{1,3} + 41\,b^{-}_{1,3}\right),$$
$$\alpha_{0,4} = \frac{e^{-14\pi}}{26825}\left(6\,e^{7\pi}+6\right)\left(a^{+}_{0,4}e^{7\pi} - 6\,b^{+}_{0,4}e^{7\pi} + c_{0,4} + 6\,b^{-}_{0,4}\right),$$
$$\alpha_{5,0} = \frac{e^{-16\pi}}{54400}\left(e^{8\pi}-1\right)\left(5589\,a^{+}_{5,0}e^{8\pi} - 6973\,b^{+}_{5,0}e^{8\pi} - 5589\,a^{-}_{5,0} - 6973\,b^{-}_{5,0}\right),$$
$$\alpha_{4,1} = \frac{e^{-16\pi}}{54400}\left(e^{8\pi}-1\right)\left(543\,a^{+}_{4,1}e^{8\pi} - 841\,b^{+}_{4,1}e^{8\pi} - 543\,a^{-}_{4,1} - 841\,b^{-}_{4,1}\right),$$
$$\alpha_{3,2} = \frac{e^{-16\pi}}{54400}\left(e^{8\pi}-1\right)\left(101\,a^{+}_{3,2}e^{8\pi} - 197\,b^{+}_{3,2}e^{8\pi} - 101\,a^{-}_{3,2} - 197\,b^{-}_{3,2}\right),$$
$$\alpha_{2,3} = \frac{e^{-16\pi}}{54400}\left(3\,e^{8\pi}-3\right)\left(9\,a^{+}_{2,3}e^{8\pi} - 23\,b^{+}_{2,3}e^{8\pi} - 9\,a^{-}_{2,3} - 23\,b^{-}_{2,3}\right),$$
$$\alpha_{1,4} = \frac{e^{-16\pi}}{54400}\left(3\,e^{8\pi}-3\right)\left(3\,a^{+}_{1,4}e^{8\pi} - 11\,b^{+}_{1,4}e^{8\pi} - 3\,a^{-}_{1,4} - 11\,b^{-}_{1,4}\right),$$
$$\alpha_{0,5} = \frac{e^{-16\pi}}{54400}\left(3\,e^{8\pi}-3\right)\left(a^{+}_{0,5}e^{8\pi} - 7\,b^{+}_{0,5}e^{8\pi} - c_{0,5} - 7\,b^{-}_{0,5}\right).$$

After simplification, Equation (5.7) reduces to

$$\begin{aligned} M(h) &= h^5 - 15h^4 + 85h^3 - 225h^2 + 274h - 120 \\ &= (h-1)(h-2)(h-3)(h-4)(h-5). \end{aligned} \tag{5.8}$$

Hence, the corresponding system has five limit cycles.

Thus, for $n = 5$, there is a systems with polynomial perturbation of degree five that has exactly five limit cycles.

Similarly, for any positive integer n, there exist polynomial functions $f^{\pm}, g^{\pm}$ of degree n such that the system (5.1) has exactly n limit cycles.

5.2. **Saddle-Saddle.** Now consider the piecewise system of the type saddle-saddle;

$$(\dot{x},\dot{y}) = \begin{cases} (y-1, x) & y>0 \\ (y+1, x) & y<0, \end{cases} \tag{5.9}$$

and its perturbation in the class of polynomial function of degree n,

$$(\dot{x},\dot{y}) = \begin{cases} (y-1+\varepsilon f^{+}(x,y), x+\varepsilon g^{+}(x,y)) & y>0 \\ (y+1+\varepsilon f^{-}(x,y), x+\varepsilon g^{-}(x,y)) & y<0, \end{cases} \tag{5.10}$$

where $f^{\pm}(x,y) = \sum_{i+j=0}^{n} a_{ij}^{\pm} x^i y^j, g^{\pm}(x,y) = \sum_{i+j=0}^{n} b_{ij}^{\pm} x^i y^j$.

System (5.9) is Hamiltonion with integral,

$$H(x,y) = \begin{cases} H^{+}(x,y) = \frac{y^2}{2} - \frac{x^2}{2} - y, & y>0 \\ H^{-}(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + y, & y<0, \end{cases}$$

and it satisfies the following hypotheses;

CI: For $h \in (0,1)$, we have

$$H^{+}(A(h)) = H^{+}(B(h)) = -\frac{h}{2} \text{ and } H^{-}(A(h)) = H^{-}(B(h)) = -\frac{h}{2},$$

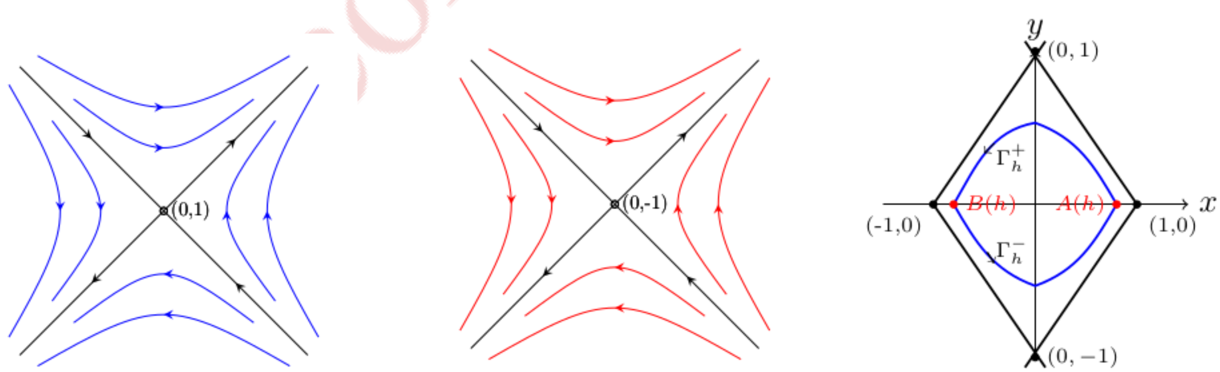
where $A(h) = (\sqrt{h}, 0)$ and $B(h) = (-\sqrt{h}, 0)$.

CII: The system (5.9) has an orbit arc Γ_h^{+} from $A(h)$ to $B(h)$ which is defined by $H^{+}(x,y) = -h/2, y \geq 0$ and an orbit Γ_h^{-} from $B(h)$ to $A(h)$ is defined by $H^{-}(x,y) = -h/2, y<0$.

CIII: System (5.9) has a family of periodic orbits,

$$\Gamma_h = \Gamma_h^{+} \cup \Gamma_h^{-} = \{(x,y) : H^{+}(x,y) = -\frac{h}{2}, y \geq 0\} \cup \{(x,y) : H^{-}(x,y) = -\frac{h}{2}, y<0\}$$

for $h \in (0,1)$.



(a) Orbits of $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} y-1 \\ x \end{pmatrix}$. (b) Orbits of $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} y+1 \\ x \end{pmatrix}$. (c) Period annulus of (5.9).

FIGURE 4. Saddle-saddle.

Lemma 5.3. *The first-order Melnikov function for the system* (5.10) *can be expressed in the form*

$$M(h) = p(h)\mathcal{I}_0(h) + q(h)\mathcal{I}_1(h) + r(h)\mathcal{J}_0(h) + s(h)\mathcal{J}_1(h),$$

where $\mathcal{I}_0(h) = I_{0,0}(h), \mathcal{I}_1(h) = I_{0,1}(h), \mathcal{J}_0(h) = J_{0,0}(h), \mathcal{J}_0(h) = J_{0,1}(h)$,
$I_{ij}(h) = \displaystyle\int_{\Gamma_h^+} x^i y^j dx, J_{ij}(h) = \int_{\Gamma_h^-} x^i y^j dx, i, j \in \mathbb{N}$, *and* $p(h), q(h), r(h), s(h)$ *are polynomials in* h *with* $\deg p(h) \leq [\frac{n}{2}], \deg q(h) \leq [\frac{n-1}{2}], \deg r(h) \leq [\frac{n}{2}]$ *and* $\deg s(h) \leq [\frac{n-1}{2}]$.

Proof. The first-order Melnikov function for the system (5.10) is given by

$$M(h) = \frac{H_y^+(A(h))}{H_y^-(A(h))} \left[\frac{H_y^-(B(h))}{H_y^+(B(h))} \int_{\Gamma_h^+} (g^+(x,y)dx - f^+(x,y)dy) + \int_{\Gamma_h^-} (g^-(x,y)dx - f^-(x,y)dy) \right]. \tag{5.11}$$

Observe here that,

$$\frac{H_y^+(A(h))}{H_y^-(A(h))} \cdot \frac{H_y^-(B(h))}{H_y^+(B(h))} = 1.$$

Hence, for $h \in (0,1)$, the Melnikov function (5.11) becomes

$$\begin{aligned} M(h) &= \int_{\Gamma^+{}_h} (g^+(x,y)dx - f^+(x,y)dy) + \int_{\Gamma_h^-} (g^-(x,y)dx - f^-(x,y)dy) \\ &= \int_{\Gamma_h^+} \sum_{i+j=0}^{n} b_{ij}^+ x^i y^j dx - \int_{\Gamma_h^+} \sum_{i+j=0}^{n} a_{ij}^+ x^i y^j dy \\ &\quad + \int_{\Gamma_h^-} \sum_{i+j=0}^{n} b_{ij}^- x^i y^j dx - \int_{\Gamma_h^-} \sum_{i+j=0}^{n} a_{ij}^- x^i y^j dy \\ &= \sum_{i+j=0}^{n} b_{ij}^+ \int_{\Gamma_h^+} x^i y^j dx - \sum_{i+j=0}^{n} a_{ij}^+ \int_{\Gamma_h^+} x^i y^j dy \\ &\quad + \sum_{i+j=0}^{n} b_{ij}^- \int_{\Gamma_h^-} x^i y^j dx - \sum_{i+j=0}^{n} a_{ij}^- \int_{\Gamma_h^-} x^i y^j dy. \end{aligned} \tag{5.12}$$

We have, by Green's formula,

$$\int_{\Gamma_h^+} x^i y^j dy = \int_{\widehat{AB} \cup \overrightarrow{BA}} x^i y^j dy - \int_{\overrightarrow{BA}} x^i y^j dy = \int_{\widehat{AB} \cup \overrightarrow{BA}} x^i y^j dy = i \iint_{R^+} x^{i-1} y^j dx dy, \tag{5.13}$$

where $\widehat{AB}$ is a path along the trajectory Γ_h^+ traversed from A to B, and R^+ is the region bounded by Γ_h^+ and the segment BA.

Similarly,

$$\int_{\Gamma_h^+} x^{i-1} y^{j+1} dx = -(j+1) \iint_{R^+} x^{i-1} y^j dx dy. \tag{5.14}$$

From (5.13) and (5.14), we get,

$$\int_{\Gamma_h^+} x^i y^j dy = -\frac{i}{j+1} \int_{\Gamma_h^+} x^{i-1} y^{j+1} dx. \tag{5.15}$$

Similarly, we can obtain,

$$\int_{\Gamma_h^-} x^i y^j dy = -\frac{i}{j+1}\int_{\Gamma_h^-} x^{i-1}y^{j+1}dx. \tag{5.16}$$

Substituting (5.15) and (5.16) in (5.12) gives,

$$\begin{aligned} M(h) = & \sum_{i+j=0}^{n} b_{ij}^{+}\int_{\Gamma_h^+} x^i y^j dx + \sum_{i+j=0, i\geq 1}^{n} \frac{i}{j+1} a_{ij}^{+}\int_{\Gamma_h^+} x^{i-1}y^{j+1}dx \\ & + \sum_{i+j=0}^{n} b_{ij}^{-}\int_{\Gamma_h^-} x^i y^j dx + \sum_{i+j=0, i\geq 1}^{n} \frac{i}{j+1} a_{ij}^{-}\int_{\Gamma_h^-} x^{i-1}y^{j+1}dx \\ = & \sum_{i+j=0}^{n} b_{ij}^{+} I_{ij} + \sum_{i+j=0, i\geq 1}^{n} \frac{i}{j+1} a_{ij}^{+} I_{i-1,j+1} + \sum_{i+j=0}^{n} b_{ij}^{-} J_{ij} + \sum_{i+j=0, i\geq 1}^{n} \frac{i}{j+1} a_{ij}^{-} J_{i-1,j+1}, \end{aligned} \tag{5.17}$$

where

$$I_{ij}(h) = \int_{\Gamma_h^+} x^i y^j dx, J_{ij}(h) = \int_{\Gamma_h^-} x^i y^j dx, i, j \in \mathbb{N}.$$

Now differentiating $H^+(x,y) = \frac{y^2}{2} - \frac{x^2}{2} - y$ with respect to x, we get

$$y\frac{\partial y}{\partial x} - \frac{\partial y}{\partial x} - x = 0.$$

Multiplying this equation by $x^{i-1}y^j dx$ and integrating along Γ_h^+, we get

$$I_{i,j}(h) = -\frac{i-1}{j+2} I_{i-2,j+2}(h) + \frac{i-1}{j+1} I_{i-2,j+1}(h). \tag{5.18}$$

Now multiplying $H^+(x,y) = \frac{y^2}{2} - \frac{x^2}{2} - y = -\frac{h}{2}$ by $x^i y^{j-1} dx$ and integrate over Γ_h^+ gives

$$\int_{\Gamma_h^+} \frac{x^i y^{j+1}}{2} dx - \int_{\Gamma_h^+} \frac{x^{i+2} y^{j-1}}{2} dx - \int_{\Gamma_h^+} x^i y^j = -\frac{h}{2}\int_{\Gamma_h^+} x^i y^{j-1} dx.$$

That is,

$$I_{i,j} = \frac{1}{2}(I_{i,j+1} - I_{i+2,j-1} + hI_{i,j-1}). \tag{5.19}$$

Substituting the expression for $I_{i-2,j+1}$ from (5.19) into (5.18), we get

$$I_{i,j} = -\frac{j(i-1)}{(2j+i+1)(j+2)} I_{i-2,j+2} + \frac{i-1}{2j+i+1} hI_{i-2,j}. \tag{5.20}$$

Also, substituting expression of $I_{i+2,j-1}$ obtained from (5.18) into (5.19), we get

$$I_{i,j} = \frac{j(i+j+2)}{(i+2j+1)(j+1)} I_{i,j+1} + \frac{j}{i+2j+1} hI_{i,j-1}. \tag{5.21}$$

Replacing j by $j-1$ in (5.21), we get

$$I_{i,j} = \frac{j(i+2j-1)}{(j-1)(i+j+1)} I_{i,j-1} - \frac{j}{i+j+1} hI_{i,j-2}. \tag{5.22}$$

Substitute expression of $I_{i-2,j+2}$ from (5.22) in (5.18), we get

$$I_{i,j} = -\frac{i-1}{j+1}\left[\frac{j^2+5j+i+5}{(j+1)(i+j+1)} I_{i-2,j+1} - \frac{j+2}{i+j+1} hI_{i-2,j}\right]. \tag{5.23}$$

Note that, since $\Gamma^+{}_h$ and $\Gamma^-{}_h$ are symmetric about the y-axis,

$$I_{2i+1,j}(h) = J_{2i+1,j} = 0 \text{ for all } i,j \in \mathbb{N}\cup\{0\}.$$

Therefore, we need to consider only $I_{2i,j}(h)$ and $J_{2i,j}(h)$ for $h \in (0,1),\ i,j \in \mathbb{N}\cup\{0\}$. Observe that equation (5.22) holds true for all $0 \le i \le n$ and $2 \le j \le n$, and equation (5.23) holds true for all $2 \le i \le n$ and $0 \le j \le n$.

We shall use induction on $k = i + j$. From equations (5.22) and (5.23), for $i + j = k = 2, 3$, we have

$$\begin{cases} I_{2,0}(h) = \frac{1}{3} h I_{0,0}(h), \\ I_{0,2}(h) = 2I_{0,1} - \frac{2}{3} h I_{0,0}, \\ I_{2,1}(h) = \frac{1}{12} h I_{0,0} + \frac{1}{4}(h-1) I_{0,1}, \\ I_{0,3}(h) = -\frac{5}{4} h I_{0,0} + \frac{3}{8}(10-2h) I_{0,1}. \end{cases} \tag{5.24}$$

Assume that the result holds for all $i + j \le k - 1, k \ge 3$.

For $i+j = k$ odd, take the pairs $(i,j) = (k-1,1), (k-3,3), (k-5,5), \cdots, (2,k-2)$ in the equation (5.23), and $(0,k)$ in the equation (5.22), we get

$$\begin{cases} I_{k-1,1}(h) = -\frac{k-2}{2}\left[\frac{3(k+4)}{2(k+1)} - 1\right] I_{k-3,2} - \frac{3}{k+1} h I_{k-3,1}, \\ I_{k-3,3}(h) = -\frac{k-4}{4}\left[\frac{k+26}{6(k+1)} I_{k-5,4} - \frac{5}{k+1} h I_{k-5,3}\right], \\ \qquad \vdots \\ I_{2,k-2}(h) = -\frac{1}{(k-1)}\left[\frac{(k-2)(k+3)+7}{(k-1)(k+1)} I_{0,k-1} - \frac{k}{k+1} h I_{0,k-2}\right], \\ I_{0,k}(h) = \frac{k(2k-1)}{(k-1)(k+1)} I_{0,k-1} - \frac{k}{k+1} h I_{0,k-2}. \end{cases} \tag{5.25}$$

Now, if k is even, then for $i + j = k, k \ge 2$, from (5.22) and (5.23), we obtain,

$$\begin{cases} I_{k,0}(h) = -\frac{(k-1)(k+5)}{k+1} I_{k-2,1} + \frac{2(k-1)}{k+1} h I_{k-2,0} \\ I_{k-2,2}(h) = 2I_{k-2,1} - \frac{2}{k+1} h I_{k-2,0} \\ I_{k-4,4}(h) = \frac{4(k+3)}{3(k+1)} I_{k-4,3} - \frac{4}{k+1} h I_{k-4,2} \\ \qquad \vdots \\ I_{2,k-2}(h) = \frac{(k-2)(2k-1)}{(k-3)(k+1)} I_{2,k-3} - \frac{k-2}{k+1} h I_{2,k-4} \\ I_{0,k}(h) = \frac{k(2k-1)}{(k-1)(k+1)} I_{0,k-1} - \frac{k}{k+1} h I_{0,k-2}. \end{cases} \tag{5.26}$$

From Equations (5.24), it is clear that for $i + j = k = 2, 3$, we have

$I_{i,j} = p(h)\mathfrak{I}_0(h) + q(h)\mathfrak{I}_1(h)$, where $\deg p(h) \le [\frac{k}{2}]$ and $\deg q(h) \le [\frac{k-1}{2}]$.

From the recurrence in (5.25) and (5.26), it is clear that for $i + j = k \ge 3$, we have

$$I_{i,j} = p(h)\mathfrak{I}_0(h) + q(h)\mathfrak{I}_1(h),$$

where $\deg p(h) \le [\frac{k}{2}]$ and $\deg q(h) \le [\frac{k-1}{2}]$.

Thus, for $2i+j=k, 2\leq k\leq n$, we have

$$\begin{aligned}I_{2i,j}(h) =&p_{k-1}(h)I_{0,0}(h)+q_{k-1}(h)I_{0,1}(h)+h[p_{k-2}(h)I_{0,0}(h)+q_{k-2}(h)I_{0,1}(h)]\\ =&p_k(h)\mathfrak{I}_0+q_k(h)\mathfrak{I}_1(h),\end{aligned} \tag{5.27}$$

where $p_{k-1}, p_{k-2}, q_{k-1}$, and q_{k-2} represent the polynomials in h that satisfy,
$\deg p_{k-1}(h) \leq [\frac{k-1}{2}], \deg p_{k-2}(h) \leq [\frac{k-2}{2}], \deg q_{k-1}(h) \leq [\frac{k-2}{2}]$ and $\deg q_{k-2}(h) \leq [\frac{k-3}{2}]$, respectively. Note that, $\max\{[\frac{k-1}{2}],[\frac{k-2}{2}]\} \leq [\frac{k}{2}]$, we have $\deg p_k(h)\leq [\frac{k}{2}]$ and
$\deg q_k(h)\leq[\frac{k-1}{2}]$.
Similarly, it can be shown that for $i+j=k, 0\leq k\leq n$,

$$J_{2i,j}(h)=r_k(h)\mathcal{J}_0(h)+s_k(h)\mathcal{J}_1(h), \tag{5.28}$$

where $r_k(h)$ and $s_k(h)$ are polynomials with $\deg r_k(h)\leq[\frac{k}{2}]$ and $\deg s_k(h)\leq[\frac{k-1}{2}]$. □

Proposition 5.4. *The number of limit cycles of the system* (5.10) *is less than or equal to* $\left[\frac{n}{2}\right]+\left[\frac{n-1}{2}\right]+3$. *Moreover, there exist polynomials* $f^{\pm}$ *and* $g^{\pm}$ *such that the perturbed system* (5.10) *has a number of limit cycles equal to* $\left[\frac{n}{2}\right]+\left[\frac{n-1}{2}\right]+3$.

Proof. We have

$$\mathfrak{I}_0=I_{0,0}=\int_{\Gamma_h^+}x^0y^0dx=\int_{\sqrt{h}}^{-\sqrt{h}}dx=-2\sqrt{h} \tag{5.29}$$

and

$$\begin{aligned}\mathfrak{I}_1(h)=I_{0,1}&=\int_{\Gamma_h^+}x^0y^1dx=\int_{\sqrt{h}}^{-\sqrt{h}}ydx=\int_{\sqrt{h}}^{-\sqrt{h}}(1-\sqrt{x^2+1-h})dx\\ &=-3\sqrt{h}-\frac{1-h}{2}\ln\left(\frac{1-\sqrt{h}}{\sqrt{h}+1}\right).\end{aligned} \tag{5.30}$$

Similarly,

$$\mathcal{J}_0=J_{0,0}=\int_{\Gamma_h^-}x^0y^0dx=\int_{-\sqrt{h}}^{\sqrt{h}}dx=2\sqrt{h} \tag{5.31}$$

and

$$\begin{aligned}\mathcal{J}_1=J_{0,1}&=\int_{\Gamma_h^-}x^0y^1dx=\int_{-\sqrt{h}}^{\sqrt{h}}ydx=\int_{-\sqrt{h}}^{\sqrt{h}}(-1+\sqrt{x^2+1-h})dx\\ &=3\sqrt{h}+\frac{1-h}{2}\ln\left(\frac{1-\sqrt{h}}{\sqrt{h}+1}\right).\end{aligned} \tag{5.32}$$

Therefore, from Lemma 5.3 and Equations (5.29)-(5.32), the first order Melnikov function becomes

$$\begin{aligned}M(h)&=p(h)\mathfrak{I}_0(h)+q(h)\mathfrak{I}_1(h)+r(h)\mathcal{J}_0(h)+s(h)\mathcal{J}_1(h)\\ &=p(h)(-2\sqrt{h})+q(h)\left(-3\sqrt{h}-\frac{1-h}{2}\ln\left(\frac{1-\sqrt{h}}{\sqrt{h}+1}\right)\right)\\ &+r(h)(-3\sqrt{h})+s(h)\left(3\sqrt{h}+\frac{1-h}{2}\ln\left(\frac{1-\sqrt{h}}{\sqrt{h}+1}\right)\right)\\ &=u(h)\sqrt{h}+v(h)\ln\left(\frac{1-\sqrt{h}}{\sqrt{h}+1}\right),\end{aligned} \tag{5.33}$$

where $u(h) = (p(h) - 3q(h) - 3r(h) + 3s(h)), v(h) = (q(h) + s(h))\frac{1-h}{2}$.

It is clear that, $\deg(u(h)) \le [\frac{n}{2}]$, $\deg(v(h)) \le [\frac{n-1}{2}] + 1$.

Let $\#f(t)$ denote the number of zeros of a polynomial function f. Then,

$$\#M(h) \le \#\left(\frac{M(h)}{v(h)}\right) + \#(v(h)). \tag{5.34}$$

Differentiating $\dfrac{M(h)}{v(h)}$ with respect to h, we get

$$\frac{d}{dh}\left(\frac{M(h)}{v(h)}\right) = \frac{(1-h)u(h)v(h) + h(1-h)(v(h)u'(h) - v'(h)u(h)) - 2(v(h))^2}{\sqrt{h}(1-h)(v(h))^2}. \tag{5.35}$$

Since the numerator of the Equation (5.35) is a polynomial of degree less than or equal to n, we can say that $\#\left(\frac{M(h)}{v(h)}\right) \le 2[\frac{n-1}{2}] + 2$. Hence, from the Equation (5.34), we get that

$$\#(M(h)) \le \#\left(\frac{M(h)}{v(h)}\right) + \#(v(h)) \le 3\left[\frac{n-1}{2}\right] + 3. \tag{5.36}$$

Observe that,

$$\sqrt{h}M(h) \in \operatorname{span}\left\{h, h^2, \cdots, h^k, h^{k+1}, \sqrt{h}\ln\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right), \cdots, \sqrt{h}h^{j+1}\ln\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right)\right\},$$

where $k = \left[\dfrac{n}{2}\right], j = \left[\dfrac{n-1}{2}\right]$.

Consider the linear combination

$$\sum_{i=1}^{k+1} a_i h^i + \sum_{i=0}^{j+1} b_i h^i \sqrt{h}\ln\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right) = 0. \tag{5.37}$$

Expand the left-hand side of (5.37) into Taylor's series about 0 and equate the coefficient of each $h^i, i = 0, 1, 2, \cdots$ to 0, we get infinite number equations in a_i's and b_i's. Solving the first $k + j + 4$ equations explicitly, we get $a_i = b_i = 0$ for $i = 1, 2, \ldots k + 1$. Hence,

$$\mathcal{B} = \left\{h, h^2, \cdots, h^k, h^{k+1}, \sqrt{h}\ln\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right), \cdots, \sqrt{h}h^{j+1}\ln\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right)\right\},$$

is linearly independent set of C^∞ functions on $(0, 1)$. Therefore, the Wronskian of any ordered subset of $\mathcal{B}$ is nonzero. Hence, by Theorem 1.3 in [2], there exists a function in span $\mathcal{B}$ that has exactly $\left[\frac{n}{2}\right] + \left[\frac{n-1}{2}\right] + 3$ zeros. Therefore, there exists a perturbed system (4.2) such that the Melnikov function $M(h)$ in (5.33) has exactly $\left[\frac{n}{2}\right] + \left[\frac{n-1}{2}\right] + 3$ zeros. Thus, Melnikov function $M(h)$ has at most $\left[\frac{n}{2}\right] + \left[\frac{n-1}{2}\right] + 3$ zeros. □

Now we present an example of a piecewise smooth differential system of the type saddle-saddle when perturbed in the class of polynomial functions of degree $n = 2$. The method followed here can be used to obtain a perturbation in the class of polynomial functions of degree n such that the perturbed systems has exactly $\left[\frac{n}{2}\right] + \left[\frac{n-1}{2}\right] + 3$ limit cycles.

Example 5.5. Consider the system (5.10), with $f^\pm, g^\pm$ being polynomial functions of degree $n = 2$. Let

$$r = 2\sqrt{h}, \text{ and } s = 3\sqrt{h} + \frac{1-h}{2}\ln\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right).$$

Then the Melnikov function is given by,

$$M(h) = \frac{1}{120}(A_1 h + A_2)s + \frac{2}{15}r\left(B_1 h^2 + B_2 h + B_3\right), \tag{5.38}$$

where

$$
\begin{aligned}
A_1 =& 30\, a^{+}_{1,2} - 30\, a^{+}_{2,1} - 60\, a^{+}_{2,2} - 40\, a^{-}_{1,2} + 40\, b^{-}_{2,1} - 120\, b^{-}_{2,2}, && (5.39)\\
A_2 =& 300\, a^{-}_{1,2} + 60\, b^{-}_{2,0} - 120\, b^{-}_{2,1} + 225\, b^{-}_{2,2} - 180\, a^{-}_{1,1} - 360\, b^{-}_{0,2} - 120\, a^{+}_{1,1} - 150\, a^{+}_{1,2}\\
&- 240\, a^{+}_{0,2} + 30\, a^{+}_{2,1} + 60\, a^{+}_{2,2} - 120\, a^{+}_{1,0} - 120\, a^{+}_{0,1} + 120\, a^{-}_{1,0} + 120\, b^{-}_{0,1}, && (5.40)\\
B_1 =& a^{+}_{2,2} - \frac{15}{8} b^{-}_{2,2}, && (5.41)\\
B_2 =& \frac{-15}{2} b^{-}_{0,2} + \frac{15}{4} b^{-}_{2,0} + \frac{5}{2} a^{+}_{1,1} + \frac{25}{8} a^{+}_{1,2} - \frac{5}{8} a^{+}_{2,1} - \frac{5}{4} a^{+}_{2,2} - \frac{15}{4} a^{-}_{1,1} + \frac{25}{4} a^{-}_{1,2}\\
&- \frac{5}{2} b^{-}_{2,1} + \frac{75}{16} b^{-}_{2,2} + 5\, a^{+}_{0,2} - \frac{5}{2} a^{+}_{2,0}, \text{ and} && (5.42)\\
B_3 =& -15/2\, a^{+}_{0,0} + 15/2\, b^{-}_{0,0}. && (5.43)
\end{aligned}
$$

Now we choose the numbers A_1, A_2, B_1, B_2 and B_3 so that,

$$
\begin{aligned}
M(h) =& \frac{1}{120}\left(A_1 h + A_2\right) s + \frac{2}{15} r \left(B_1 h^2 + B_2 h + B_3\right)\\
=& \left(1000\sqrt{h} - 1\right)\left(4000\sqrt{h} - 1\right)\Bigg[8.2944\, h^{3/2} - 2.0736\, \ln\left(-\frac{-1+\sqrt{h}}{1+\sqrt{h}}\right) h\\
&+ 16.4416\sqrt{h} + 2.0736\, \ln\left(-\frac{-1+\sqrt{h}}{1+\sqrt{h}}\right) - 24.48\, h\Bigg]. \qquad (5.44)
\end{aligned}
$$

Therefore,

$$A_1 = 2.952638889, A_2 = 0.6035542051 \times 10^{-3}, B_1 = 1, B_2 = -\frac{1}{800}, \text{ and } B_3 = \frac{1}{4000000}.$$

Note that for these values of A_1, A_2, B_1, B_2, and B_3, the Equations (5.39)-(5.43) represent a system of five linear equations with 27 unknowns that has nontrivial solutions. Thus, there exist $a^{\pm}_{i,j}, b^{\pm}_{i,j}, 0 \le i, j \le 2$, not all zero, that satisfy the Equations (5.39)-(5.43).

The roots of Melnikov function (5.44) are

$$h_1 = 0.000001000000000, h_2 = 0.00000006250000000, h_3 = 0.9507439781, \text{ and } h_4 = 0.6845346498.$$

Hence, for $n = 2$, there exist polynomial functions $f^{\pm}$ and $g^{\pm}$ such that the system (5.10) has exactly $\left[\frac{n}{2}\right] + \left[\frac{n-1}{2}\right] + 3 = 4$ limit cycles.

In general, for any positive integer n, there are polynomial functions $f^{\pm}, g^{\pm}$ of degree n such that the system (5.10) has exactly $\left[\frac{n}{2}\right] + \left[\frac{n-1}{2}\right] + 3$ limit cycles.

6. Discussion and conclusions

In this paper, Poincaré half-return maps for center and focus are obtained in a simpler form. Several possibilities for the equilibrium point (x_0, y_0), such as $x_0 > 0, x_0 < 0, y_0 = 0, y_0 > 0$ etc., are discussed. Using the half-return maps, the full-return map for a piecewise linear system of focus-focus type is established, and conditions for the period annulus are developed. This idea can be extended to establish the Poincaré maps for a general class of piecewise differential systems to identify the number and location of limit cycles.

Normal forms of piecewise smooth differential systems of focus-focus and saddle-saddle types are obtained, and the limit cycle bifurcation of these systems when perturbed in polynomial functions is discussed. The Chebyshev criterion, Picard-Fuchs equations, and Taylor's series for the inverse function are used in the analysis of limit cycles. The upper bound for the number of limit cycles established in this paper can be further improved. This technique can be utilized to obtain the finer bounds on the number of bifurcated limit cycles from the period annulus for a general class of planar systems.

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