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Numerical solution of different population balance models using operational method based on Genocchi polynomials

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Abstract

Genocchi polynomials have exciting properties in the approximation of functions. Their derivative and integral calculations are simpler than other polynomials and, in practice, they give better results with low degrees. For these reasons, in this article, after introducing the important properties of these polynomials, we use them to approximate the solution of different population balance models. In each case, we first discuss the solution method and then do the error analysis. Since we do not have an exact solution, we compare our numerical results with those of other methods. The comparison of the obtained results shows the efficiency of our method. The validity of the presented results is indicated using MATLAB-Simulink.

Keywords. Genocchi polynomials, Genocchi numbers, Population balance models, Error analysis, Numerical results.
2010 Mathematics Subject Classification. 65M70,65M22, 65Gxx.

1. Introduction

In this article, we investigate the numerical solution of different population balance models. These equations include a wide range of ordinary and partial differential equations. Here we focus on the numerical solution of three models of these equations:

Binary equal breakage model. In 1988, Randolph and Larson discovered that various particle processes can be explained using population balance equations [21]. These processes include crystallization from solution, emulsion polymerization, microbial growth, and particle aggregation. These equations are defined in the following way [20]:

$$\frac{dw}{dt} + \left(1 + \varepsilon t^k\right) = 2^{k+1}\varepsilon t^k w\left(2t\right), \quad t \in [0, \alpha]. \tag{1.1}$$

In the Equation (1.1), ε and k are constant and the initial condition is:

$$w\left(0\right) = 1. \tag{1.2}$$

These equations may look simple, but they are complex. The presence of $w\left(2t\right)$ adds to their complexity. So, it is necessary to use numerical methods to approximate the solution.

Binary uniform breakage model. The population balance equation for a continuous, mixed-suspension, mixed-product removal crystallizer with a binary uniform breakage model, can be converted into the conventional integro-differential Equation (1.3)

$$\frac{dw}{dt} + \left(1 + \varepsilon t^k\right) w = 2\varepsilon t \int_t^\infty x^{k-2} w\left(x\right) dx \quad t \in [0, \alpha], \tag{1.3}$$

with the initial condition (1.2).

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Volterra's population model. Researchers have defined this model to describe population growth. The model applies to a closed system. The references for this model are [25, 27]

$$\frac{dw}{dt} - \alpha w + \beta w^2 + \chi w \int_0^{\xi} w(z)\, dz = 0, \quad w(0) = w_0. \tag{1.4}$$

In Equation (1.4), α, β, and χ represent different factors. The factors are the birth rate, crowding factor, and toxicity factor. w_0 represents the initial condition. $w(\xi)$ represents the population at time ξ. The extra integral term represents the accumulated toxicity on the species [2].

Among the numerical methods proposed to solve these equations are:

The Adams-Moultion-Shell method [20] is one approach. Another method is the weighted residual method [26]. There is also the Block-Pulse method [13]. Consider the Shifted Legender method [5] as well. The Wavelet-Galerkin method [6] is another technique. Another option is using rationalized Haar functions [1]. Srivastava et al. introduced and developed a generalized wavelet method with the quasilinearization technique to solve Volterra's population growth model of fractional order. Unlike existing operational matrix methods based on orthogonal functions, they have formulated wavelet operational matrices by general order integration without using block pulse functions [28]. Yuzbasi uses the Bessel collocation method to give solutions for the fractional Volterra's model for the population growth of a species in a closed system [39].

Other methods that have been used to solve population growth model (fractional or ordinary) and similar differential equations are: Gegenbauer wavelet quasi-linearization [9], improved Bessel collocation method [40], a sinc-Gauss-Jacobi collocation method [24], improved Legendre method [41], modified Lagrange polynomial method [3], modified Lyapunov–Razumikhin method (LRM) [37], Razumikhin method [38], and auto-correlation functions of compactly supported wavelets [17].

Operational matrix-based methods (which are used in this paper) have been widely used in solving differential equations, such as orthogonal Polynomials Based Operational Matrices [31, 32], Paraskevopoulos's algorithm with operational matrices of Vieta–Lucas polynomials [33, 34] and a decomposition algorithm coupled with operational matrices [35].

As mentioned above, various methods have been proposed to solve the population balance models, many of which have used orthogonal polynomials. Here we use orthogonal Genocchi polynomials which have not been used before and have much simpler calculations to approximate the solution. The exact solution of Equations (1.1), (1.3), and (1.4) are unknown. Hence, researchers, often compare the results of these methods. As we see in section 2, Genocchi polynomials have very interesting properties and the calculations of their polynomial values, derivatives, and integrals are much simpler than other polynomials such as Chebyshev, Legendre, and Hermite. For this reason, in this article, we solve Eqs. (1.1), (1.3), and (1.4) using Genocchi polynomials. We provide error analysis for each case and compare the numerical results of the introduced method with the other methods.

We arranged the continuation of this article as follows:

In section 2, Genocchi polynomials and numbers and their properties are introduced. It gave the method of approximating an arbitrary function by Genocchi polynomials in section 3. In this section, error bounds for Genocchi polynomials interpolation are also given. In section 4, the methods of solving Eqs. (1.1), (1.3), and (1.4) are given, the error bound is checked and we compare the numerical results with other methods. By interpreting the tables and results of this section, we can see the effectiveness of the introduced method.

2. Genocchi numbers and polynomials

Genocchi numbers $\mathcal{G}_n$, named after Angelo Genocchi, are sequences of integers. These sequences satisfy a specific relation

$$\frac{-2t}{1+e^{-t}} = \sum_{n=0}^{\infty} \mathcal{G}_n \frac{t^n}{n!}.$$

Table 1. The first few Genocchi numbers.

n :	1	2	3	4	5	6	7	8	9	10	11	12
Genocchi number:	1	-1	0	1	0	-3	0	17	0	-155	0	2073

For these numbers we have

$$\mathcal{G}_{2n-1} = 0,$$

$$\mathcal{G}_{2n} = 2\left(1 - 2^{2n}\right) B_{2n} = 2nE_{2n-1}\left(0\right),$$

where $n \in \mathbb{Z}^{>0}$, B_n is a Bernoulli number and $E_n\left(x\right)$ is an Euler polynomial. The first few Genocchi numbers are given in Table 1. The prime Genocchi numbers are only -3 and 17. These occur for $n = 6$ and 8. It can be proven that only these numbers are prime Genocchi numbers [12].
Genocchi polynomials are an important polynomial in the Appell polynomial family. These polynomials are defined as [23].

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n\left(x\right) \frac{t^n}{n!}, \qquad |t| < \pi.$$

The Genocchi polynomial of degree n can be written in the following explicit form

$$\mathcal{G}_n\left(x\right) = \sum_{i=0}^{n} \binom{n}{i} \mathcal{G}_i x^{n-i},$$

and $\mathcal{G}_i$ is a Genocchi number. Now, if we take derivatives from both sides of the above relationship, with respect to x, we get

$$\frac{d}{dx}\mathcal{G}_n\left(x\right) = n\mathcal{G}_{n-1}\left(x\right), \tag{2.1}$$

and as a result

$$\int_{x_1}^{x_2} \mathcal{G}_n\left(x\right) dx = \frac{\mathcal{G}_{n+1}\left(x_2\right) - \mathcal{G}_{n+1}\left(x_1\right)}{n+1}.$$

Below, we give some of the most important properties of Genocchi polynomials.

$$\mathcal{G}_i\left(x+1\right) + \mathcal{G}_i\left(x\right) = 2ix^{i-1} \Rightarrow \mathcal{G}_i\left(1\right) + \mathcal{G}_i\left(0\right) = 0,\ \ i > 1,$$

$$\int_0^1 \mathcal{G}_i\left(x\right) \mathcal{G}_j\left(x\right) dx = \frac{2i(-1)^i! j! \mathcal{G}_{i+j}}{(i+j)!}, \quad i, i \geq 1.$$

Genocchi polynomials and related wavelets have been widely used in solving fractional calculus problems [7, 8, 14]. Loh and Phang, create a new numerical scheme for solving the system of Volterra integro-differential equation with Genocchi polynomials[16]. Some other problems whose solutions have been approximated with high accuracy using these polynomials are: fractional diffusion wave equation and fractional Klein–Gordon equation [15], nonlocal anti-periodic boundary value problem of arbitrary fractional order [18], fractional Abel differential equation [22],variable order fractional optimal control problems [30], Bratu-type equations [11, 29]. In this article, we try to approximate Equations (1.1), (1.3), and (1.4) using Genocchi polynomials, approximate the error and display the results numerically.

3. Function approximation

In this section, how to approximate a function by Genocchi polynomials and the resulting error are investigated. We can use Genocchi polynomials as basic polynomials to approximate function $f\left(x\right) \in L^2\left[0,1\right]$.
For this end, assume

$$\mathbb{G} = Span\left\{\mathcal{G}_1(x), \mathcal{G}_2(x), ..., \mathcal{G}_N(x)\right\}$$

be the space generated by Genocchi polynomials. The approximation of f in this space is

$$f(x) \approx f_N(x) = \sum_{i=1}^{N} \mathtt{f}_i \mathcal{G}_i(x), \tag{3.1}$$

For example, to approximate the $sin(x)$ with $N=4$ we have

$$\begin{aligned}\sin(x) \approx f_4(x) &= \sum_{i=1}^{4} \mathtt{f}_i \mathcal{G}_i(x)\\ &= \mathtt{f}_1 + \mathtt{f}_2(2x-1) + \mathtt{f}_3\left(3x^2-3x\right) + \mathtt{f}_4\left(4x^3-6x^2+1\right)\\ &= \left[\begin{array}{cccc} \mathtt{f}_1 & \mathtt{f}_2 & \mathtt{f}_3 & \mathtt{f}_4 \end{array}\right] \left[\begin{array}{l} 1 \\ 2x-1 \\ 3x^2-3x \\ 4x^3-6x^2+1 \end{array}\right].\end{aligned}$$

Now, with collocation points $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ the following four equations and four unknowns are obtained

$$\begin{array}{lllll} \mathtt{f}_1 & -\mathtt{f}_2 & & +\mathtt{f}_4 & = \sin(0) \\ \mathtt{f}_1 & -\frac{1}{3}\mathtt{f}_2 & -\frac{2}{3}\mathtt{f}_3 & +\frac{13}{27}\mathtt{f}_4 & = \sin\left(\frac{1}{3}\right) \\ \mathtt{f}_1 & +\frac{1}{3}\mathtt{f}_2 & +\frac{2}{3}\mathtt{f}_3 & -\frac{13}{27}\mathtt{f}_4 & = \sin\left(\frac{2}{3}\right) \\ \mathtt{f}_1 & +\mathtt{f}_2 & & -\mathtt{f}_4 & = \sin(1) \end{array}$$

By solving we have

$$\mathtt{f}_1 = 0.2805, \quad \mathtt{f}_2 = 0.4368, \quad \mathtt{f}_3 = -0.3782, \quad \mathtt{f}_4 = -0.1242$$

$$f_4(x) = 0.4968x^3 - 0.3894x^2 + 2.0082x - 0.2805$$

The best approximation is unique in $\mathbb{G}$, then [10]

$$\begin{aligned}&\|f(x) - f_N(x)\| \leq \|f(x) - g(x)\|, \quad \forall g(x) \in \mathbb{G}.\\ &\Rightarrow \quad (f(x) - f_N(x), g(x)) = 0,\end{aligned} \tag{3.2}$$

where $(.\,,\,.)$ is the inner product.
The approximation of the f (3.1), can be written as the following matrix

$$f_N(x) = \mathbf{F}^T \mathbf{G}(x),$$

where

$$\mathbf{F}^T = [\mathtt{f}_1, \mathtt{f}_2, \cdots, \mathtt{f}_N]^T, \quad \mathbf{G}(x) = [\mathcal{G}_1(x), \mathcal{G}_2(x), \cdots, \mathcal{G}_N(x)]^T, \tag{3.3}$$

and according to (3.2), for all $\mathcal{G} \in \mathbb{G}$ we can write

$$\left(f(x) - \mathbf{F}^T \mathbf{G}(x), \mathcal{G}_i(x)\right) = 0.$$

The vector of coefficients in the relation (3.3) (i.e. vector $\mathbf{F}$), can be calculated as follows

$$\mathbf{F} = \mathbf{O}^{-1}(f(x), \mathbf{G}(x)),$$

where

$$\mathbf{O}_{ij} = (\mathbf{G}(x), \mathbf{G}(x))_{ij} = \int_0^1 \mathcal{G}_i(x)\mathcal{G}_j(x)\,dx = \frac{2i(-1)^i !j!\mathcal{G}_{i+j}}{(i+j)!}.$$

According to (2.1), the derivative of function $f(x)$ can be approximated by Genocchi polynomials

$$f'(x) \approx f'_N(x) = \sum_{i=1}^{N} f_i G'_i(x) = \sum_{i=2}^{N} f_i i G_{i-1}(x) = \mathbf{FDG}(x), \tag{3.4}$$

where $\mathbf{D}$ is a operational matrix of derivative and

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & & \cdots & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & \cdots & N & 0 \end{bmatrix}_{N\times N}.$$

Thus, $\mathbf{D}$ is $N \times N$ operational matrix of derivative. Accordingly, the kth derivative of $\mathbf{G}(x)$ can be obtained by

$$\mathbf{G}'(x) = \mathbf{DG}(x) = \mathbf{G}(\mathbf{x})\mathbf{D}^T,$$

$$\mathbf{G}''(x) = \mathbf{DG}'(x) = \mathbf{G}(x)\left(\mathbf{D}^T\right)^{\mathbf{2}},$$

$$\mathbf{G}'''(x) = \mathbf{DG}''(x) = ... = \mathbf{G}(x)\left(\mathbf{D}^T\right)^{\mathbf{3}},$$

$$\vdots$$

$$\mathbf{G}^{(k)}(x) = \mathbf{DG}^{(k-1)}(x) = ... = \mathbf{G}(x)\left(\mathbf{D}^T\right)^{k}.$$

3.1. **Approximation error.**

Theorem 3.1. *Suppose that $f(x) \in C^{N+1}[0,1]$ (N is the order of approximation), and $f_N(x)$ is the best approximation to $f(x)$ out of $\mathbb{G}$, Then a bound for the approximation error can be expressed as,*

$$\|f(x) - f_N(x)\|_2 \leq \frac{\max_{x\in[0,1]}\left|f^{(N+1)}(x)\right|}{(N+1)!\sqrt{2N+3}}.$$

Proof. Suppose $T_N(x)$ is a Taylor expansion for approximate of $f(x)$ about $x = \alpha$, then

$$f(x) \approx T_N(x) = \sum_{j=0}^{N} \frac{f^{(j)}(\alpha)(x-\alpha)^j}{j!},$$

and

$$|f(x) - T_N(x)| \leq \mathrm{M}\frac{|x-\alpha|^{N+1}}{(N+1)!},$$

where M is a maximum value of $\left|f^{(N+1)}\right|$ on $[\alpha, x]$.
Now, since $f_N(x)$ is the best approximation of $f(x)$ and the best approximation is unique, then we can write

$$\|f(x) - f_N(x)\|_2 \leq \|f(x) - T_N(x)\|_2 \leq \left\|\mathrm{M}\tfrac{(x-\alpha)^{N+1}}{(N+1)!}\right\|_2$$

$$= \sqrt{\int_0^1 \left(\mathrm{M}\tfrac{(x-\alpha)^{N+1}}{(N+1)!}\right)^2 dt} = \tfrac{\mathrm{M}}{(N+1)!}\sqrt{\int_0^1 (x-\alpha)^{2N+2} dt}$$

$$= \tfrac{\mathrm{M}}{(N+1)!}\sqrt{\int_0^1 (x-\alpha)^{2N+2} du} = \tfrac{\mathrm{M}}{(N+1)!}\sqrt{\tfrac{(x-\alpha)^{2N+3}}{2N+3}\bigg|_0^1}.$$

If we set the value of α equal to 0, we will have

$$\|f(x) - f_N(x)\|_2 \leq \frac{M}{(N+1)!\sqrt{2N+3}}.$$

□

According to Theorem 3.1, it is clear that

$$\lim_{N\to\infty} \|f(x) - f_N(x)\|_2 = 0.$$

4. Solution of population balance equations by the Genocchi polynomials

In this section, we use the collocation method based on the Genocchi polynomials to solve the population balance equations numerically. To this end, we first approximate $w(t)$, by Genocchi polynomials

$$w(t) \approx \sum_{i=1}^{N} \lambda_i \mathcal{G}_i(t) = \Lambda^T \mathbf{G}(t), \tag{4.1}$$

where

$$\Lambda^T = [\lambda_1, \lambda_2, \cdots, \lambda_N]^T, \quad \mathbf{G}(t) = [\mathcal{G}_1(t), \mathcal{G}_2(t), \cdots, \mathcal{G}_N(t)]^T.$$

Also, using (3.4) to approximate the derivative of $w(x)$, we have

$$w'(t) \approx \sum_{i=0}^{N} \lambda_i \mathcal{G}'_i(t) = \Lambda^T \mathbf{G}'(t) = \Lambda^T \mathbf{D}\mathbf{G}(t). \tag{4.2}$$

4.1. **Binary equal breakage model.** Population balance differential Equation (1.1) can be normalized by changing the independent variable to $t = \alpha x$, then we have

$$\frac{dw(\alpha x)}{d(\alpha x)} + \left(1 + \varepsilon \alpha^k x^k\right) = 2^{k+1} \varepsilon \alpha^k x^k w(2\alpha x), \quad 0 \le x \le 1,$$

$$\frac{1}{\alpha}\frac{dw(\alpha x)}{dx} + \left(1 + \varepsilon \alpha^k x^k\right) = 2^{k+1} \varepsilon \alpha^k x^k w(2\alpha x), \quad 0 \le x \le 1$$

$$\begin{aligned} & a_1 w'(\alpha x) + a_2(x) w(\alpha x) - a_3(x) w(2\alpha x) = 0, \qquad 0 \le x \le 1, \\ & w(0) = 1, \end{aligned} \tag{4.3}$$

where

$$a_1 = \frac{1}{\alpha}, \quad a_2(x) = 1 + \varepsilon \alpha^k x^k, \quad a_3(x) = 2^{k+1} \varepsilon \alpha^k x^k.$$

Therefore, substituting (4.1) and (4.2) in (4.3), we have

$$a_1 \Lambda^T \mathbf{D}\mathbf{G}(\alpha x) + a_2(x) \Lambda^T \mathbf{G}(\alpha x) - a_3(x) \Lambda^T \mathbf{G}(2\alpha x) = 0, \quad 0 < x \le 1. \tag{4.4}$$

From the initial condition, we can write

$$\Lambda^T \mathbf{G}(0) = 1. \tag{4.5}$$

For the approximate $w(x)$, we collocate (4.4) at the below collocation points

$$\begin{aligned} & a_1 \Lambda^T \mathbf{D}\mathbf{G}(\alpha x_i) + a_2(x_i) \Lambda^T \mathbf{G}(\alpha x_i) - a_3(x_i) \Lambda^T \mathbf{G}(2\alpha x_i) = 0, \\ & x_i = \tfrac{i}{N-1}, \quad i = 1, 2, \cdots, N-1. \end{aligned} \tag{4.6}$$

Relation (4.6) along with the initial condition (4.5), contains N equations and N unknowns, which can be solved by using the usual methods of solving algebraic equations.

4.1.1. *Error Analysis.* Assuming that w_N is the approximate solution obtained from Genocchi polynomials for Equation (4.3), we write as follows

$$\mathcal{E}\left(w_N\right) = a_1 {w'}_N + a_2 w_N - a_3 \hat{w}_N.$$

If w is the exact solution of the equation, then $\mathcal{E}\left(w\right)=0$ and can be written as

$$\begin{aligned}\|\mathcal{E}\left(w_N\right)\|_2 &= \|\mathcal{E}\left(w_N\right) - \mathcal{E}\left(w\right)\|_2 \\ &= \|a_1\left({w'}_N - w'\right) + a_2\left(w_N - w\right) - a_3\left(\hat{w}_N - a_3\hat{w}\right)\|_2,\end{aligned}$$

where $\hat{w}_N\left(\alpha x\right) = w_N\left(2\alpha x\right)$ and $\hat{w}\left(\alpha x\right) = w\left(2\alpha x\right)$. If

$$M = \max_{x\in[0,1]}\left\{\left|f^{(N+1)}\left(\alpha x\right)\right|, \left|f^{(N+2)}\left(\alpha x\right)\right|\right\}, \tag{4.7}$$

then

$$\|\mathcal{E}\left(w_N\right)\|_2 \leq a_1\|\left({w'}_N - w'\right)\|_2 + \|\left(a_2 - a_3\right)\left(w_N - w\right)\|_2 \leq \frac{C_1 M}{(N+1)!\sqrt{2N+3}}, \tag{4.8}$$

where

$$a_1 + \|a_2 - a_3\|_2 = \frac{1}{\alpha} + 1 + \left(1 - 2^{k+1}\right)\varepsilon\alpha^k x^k \leq \frac{1}{\alpha} + 1 + \left(1 - 2^{k+1}\right)\varepsilon\alpha^{2k} = C_1. \tag{4.9}$$

4.1.2. *Numerical result.* Due to the existence of the term $w\left(2x\right)$ in Equation (4.3), it is impossible to find the exact solution, therefore, the results obtained from the introduced method are compared with the results of other methods. We solve differential Equation (4.3) for constants $k=4, \varepsilon=\frac{1}{162}$ and $\alpha=6.5$, with different values of N.

($\mathbf{N=3}$):

Here, for small $N=3$, we implement the method step by step:

We write the approximate solution of (4.1) as below

$$w\left(t\right) \approx \begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}\begin{bmatrix}1 \\ 2t-1 \\ 3t^2-3t\end{bmatrix} = \lambda_1 + \lambda_2\left(2t-1\right) + \lambda_3\left(3t^2-3t\right).$$

By replacing in (4.3), we will have

$$(0.1538)\begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}\begin{bmatrix}0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0\end{bmatrix}\begin{bmatrix}1 \\ 13x-1 \\ 126.75x^2-19.5x\end{bmatrix}$$

$$+\left(1+11.0189x^4\right)\begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}\begin{bmatrix}1 \\ 13x-1 \\ 126.75x^2-19.5x\end{bmatrix}$$

$$-352.6049x^4\begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}\begin{bmatrix}1 \\ 26x-1 \\ 507x^2-39x\end{bmatrix} = 0.$$

Now, with collocation points $\frac{1}{2}$ and 1 and initial condition (4.5), the following three equations and three unknowns are obtained

$$\begin{aligned}\lambda_1 \quad -0.6924\lambda_2 \quad -0.4614\lambda_3 &= 1, \\ -20.3491\lambda_1 \quad -254.8583\lambda_2 \quad -2535.9062\lambda_3 &= 0, \\ -340.5860\lambda_1 \quad -8670.5881\lambda_2 \quad -170656.4383\lambda_3 &= 0.\end{aligned}$$

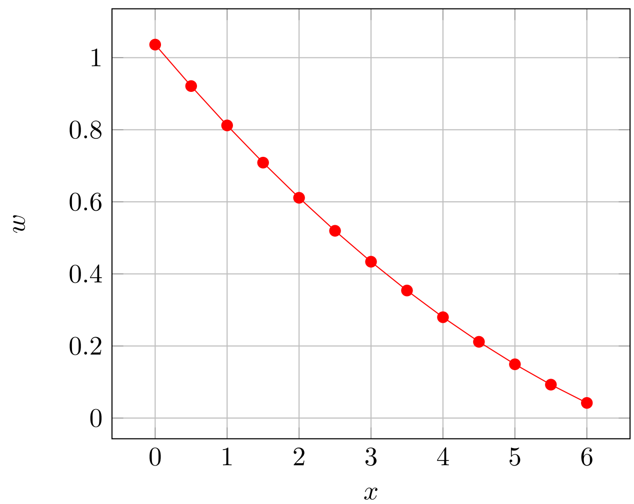

FIGURE 1. Genocchi polynomials solution for (4.3) with $N = 3$.

TABLE 2. Compare the our results ($N = 17$), with the methods presented in previous literature.

x	Genocchi polynomials	Wavelet Galerkin [6]	rattionalized Haar [1]
0	1	1	1
0.5	0.6070423	0.607006	0.6070341
1	0.3744780	0.374462	0.3744773
1.5	0.2417954	0.241759	0.2417950
2	0.1554201	0.155404	0.1554150
2.5	0.0897601	0.089745	0.0897578
3	0.0455771	0.045574	0.0455772
3.5	0.0195103	0.019403	0.0195095
4	0.00639246	0.00639265	0.00639244
4.5	0.00140580	0.00140666	0.00140578
5	0.00017481	0.000175756	0.00017527
5.5	0.0000103	0.0000101114	0.000010058
6	0.0000001042	0.000000207547	0.0000002030

By solving the above equations, the approximate solution is obtained as follows

$$0.9241 - 0.1121\,(2x - 1) + 0.0039\left(3x^2 - 3x\right) = 1.0362 + 0.2359x + 0.0117x^2.$$

The graph of this solution is shown in Figure 1.

($\mathbf{N = 17}$):

Here, the numerical results obtained from the introduced method are compared with the results of the methods presented in previous literature, and the result of this comparison is given in Table 2 ($N = 17$). The presented method is compared with the rationalized Haar method [1] and wavelet-Galerkin method [6]. In Table 3, we have shown the approximate result of the introduced method for different values of N. Figure 2 shows the approximate solution of (4.3) with Genocchi polynomials with $N = 17$.

TABLE 3. Approximate result of the introduced method for different values of N.

x	$N=17$	$N=14$	$N=10$	$N=5$
0.5	0.6070423	0.6070512	0.6070780	0.6071125
1.5	0.2417954	0.2417893	0.2417713	0.2417052
2.5	0.0897601	0.0897504	0.0897112	0.0896958
3.5	0.0195103	0.0195258	0.0194724	0.0194112
4.5	0.00140580	0.00162104	0.00160102	0.00175128
5	0.00017481	0.000172507	0.00017002	0.00018506
5.5	0.0000103	0.000010208	0.000010365	0.000011021
6	0.0000001042	0.0000001085	0.0000001156	0.0000002034

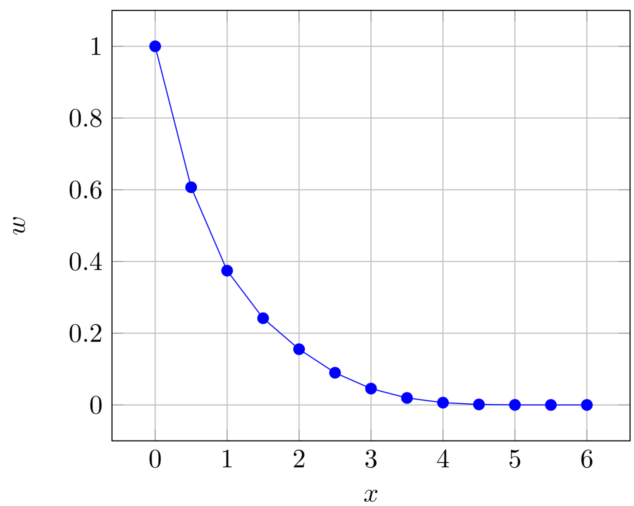

FIGURE 2. Genocchi polynomials solution for (4.3) with $N=17$.

4.2. **Binary uniform breakage model.** By changing variable $t=\alpha x$, the normalized form of Equation (1.3) is obtained as follows

$$a_1 w'(x) + \left(1+\varepsilon\alpha^k x^k\right) w(x) = 2\varepsilon\alpha^k x \int_{\alpha x}^{\infty} \chi^{k-2} w(\chi)\, d\chi, \quad x\in[0,1]. \tag{4.10}$$

Using relations (4.1) and (4.2), the Equation (4.10) will be as follows

$$a_1 \Lambda^T \mathbf{D}\mathbf{G}(x) + \left(1+\varepsilon\alpha^k x^k\right)\Lambda^T \mathbf{G}(x) = 2\varepsilon\alpha^k x \Lambda^T \mathbf{I}(x), \ \ x\in(0,1],$$

$$\Lambda^T \mathbf{G}(0) = 1,$$

where $\mathbf{I}(x)$ is calculated as follows

$$(\mathbf{I}(x))_i = \int_{\alpha x}^{\infty} \chi^{k-2}\mathcal{G}_i(\chi)\, d\chi = \lim_{\eta\to\infty}\left(\int_{\alpha x}^{\eta} \chi^{k-2}\mathcal{G}_i(\chi)\, d\chi\right). \tag{4.11}$$

If we set the value of k equal to 4, we will have:

$$\int_{\alpha x}^{\eta} \chi^2 \mathcal{G}_i(\chi)\, d\chi = (i+1)\,\chi^2 \mathcal{G}_{i+1}(\chi)\Big|_{\alpha x}^{\eta} - 2(i+1)\int_{\alpha x}^{\eta} \chi \mathcal{G}_{i+1}(\chi)\, d\chi$$

$$= (i+1)\,\chi^2 \mathcal{G}_{i+1}(\chi)\Big|_{\alpha x}^{\eta} - 2(i+1)\left[(i+2)\,\chi \mathcal{G}_{i+2}(\chi)\Big|_{\alpha x}^{\eta} - (i+2)\int_{\alpha x}^{\eta} \mathcal{G}_{i+2}(\chi)\,d\chi\right]$$
$$= (i+1)\,\chi^2 \mathcal{G}_{i+1}(\chi)\Big|_{\alpha x}^{\eta} - 2(i+1)\left[(i+2)\,\chi \mathcal{G}_{i+2}(\chi)\Big|_{\alpha x}^{\eta} - (i+2)\left(\frac{\mathcal{G}_{i+3}(\eta) - \mathcal{G}_{i+3}(\alpha x)}{(i+3)}\right)\right].$$

Now, by using the collocation points (4.6), and initial condition, the algebraic equations of N equation and N unknowns are obtained, and by solving it, the approximate solution of w is obtained.

4.2.1. *Error Analysis.* In (4.10), if w_N is the approximate answer obtained from Genocchi polynomials, then

$$a_1 {w'}_N + f w_N - g\int_{\alpha x}^{\infty} \chi^{k-2} w_N(\chi)\,d\chi = \vartheta_N \geq 0,$$

where

$$f(x) = 1 + \varepsilon \alpha^k x^k,$$

$$g(x) = 2\varepsilon \alpha^k x.$$

It is obvious that

$$a_1 w' + f w - g\int_{\alpha x}^{\infty} \chi^{k-2} w(\chi)\,d\chi = 0,$$

where w is a exact solution of (4.10). So that

$$\|\vartheta_N\|_2 \leq a_1\|{w'}_N - w'\|_2 + \|f\|_2\|w_N - w\|_2 - \|g\|_2\left\|\int_{\alpha x}^{\infty} \chi^{k-2}(w_N - w)(\chi)\,d\chi\right\|_2.$$

Now with (4.7), we have

$$\|\vartheta_N\|_2 \leq \left(\frac{1}{\alpha} + A_1\right)\frac{M}{(N+1)!\sqrt{2N+3}} - A_2 I_1,$$

$$\|\vartheta_N\|_2 \leq \left(\frac{1}{\alpha} + A_1\right)\frac{M}{(N+1)!\sqrt{2N+3}} - \frac{MA_2}{(N+1)!\sqrt{2N+3}},$$

$$\|\vartheta_N\|_2 \leq \frac{\rho}{(N+1)!\sqrt{2N+3}},$$

where

$$\rho = \left(\frac{1}{\alpha} + A_1\right)M + MA_2A_3,$$

$$A_1 = 1 + \varepsilon\alpha^{2k},$$

$$A_2 = 1 + \varepsilon\alpha^{k+1},$$

$$A_3 = \textstyle\int_{\alpha x}^{\eta} \|\chi^{k-2}\|_2 d\chi,$$

then

$$\lim_{N\to\infty} \|\vartheta_N\|_2 = 0.$$

TABLE 4. Compare the our results for Equation (1.3) ($N = 18$), with the methods presented in previous literature.

x	Genocchi polynomials	Wavelet Galerkin [6]	Block pulse functions [13]
0	1	1	1
0.5	0.569034	0.589568	0.608522
1	0.360074	0.360626	0.374203
1.5	0.212065	0.220917	0.233135
2	0.134502	0.133500	0.146054
2.5	0.0698754	0.0771257	0.0897002
3	0.0420123	0.0402498	0.0513895
3.5	0.0127603	0.0172186	0.0254139
4	0.0050281	0.00507987	0.00974413
4.5	0.000541123	0.000665560	0.00251911
5	0.000540113	0.0000739025	0.000368119
5.5	0.000000621309	0.0000224699	0.00000452567
6	0.000000892021	0.00000127375	0.000000536893

TABLE 5. Approximate solution using Genocchi polynomials of different orders N.

x	$N = 18$	$N = 15$	$N = 12$	$N = 9$
1	0.360074	0.360095	0.3605804	0.3620892
2	0.134502	0.134612	0.134652	0.134806
3	0.0420123	0.042543	0.0428602	0.4325129
4	0.0050281	0.0050982	0.0051385	0.0055024
5	0.000540113	0.000058289	0.000610286	0.000702586
6	0.000000892021	0.000002058	0.000006282	0.000009852

4.2.2. *Polynomial approximation of the solution for $N = 4$.* By substituting the approximat solution (4.1) in Equation (4.10), and using collocation points (4.6), we will get the following equations

$$\begin{aligned}
&\lambda_1 && -(0.6923)\,\lambda_2 && -(1.1690)\,\lambda_3 && +\lambda_4 && = 1,\\
&-(2.4486e6)\,\lambda_1 && -(3.6485e8)\,\lambda_2 && +(3.3839e52)\,\lambda_3 && -(4.8091e12)\,\lambda_4 && = 0,\\
&-(4.8969e6)\,\lambda_1 && -(7.2969e8)\,\lambda_2 && +(6.7679e52)\,\lambda_3 && -(9.6183e12)\,\lambda_4 && = 0,\\
&-(7.3439e6)\,\lambda_1 && -(1.0945e9)\,\lambda_2 && +(1.0152e53)\,\lambda_3 && -(1.4427e13)\,\lambda_4 && = 0.
\end{aligned}$$

By solving the above equations, the solution is obtained as follows

$$0.8739 - 0.1795\,(2x-1) + 0\left(3x^2 - 3x\right) + 0.0017\left(4x^3 - 6x^2 + 1\right).$$

4.2.3. *Numerical result.* In Table 4, the numerical results obtained for (1.3) using Genocchi polynomials of degree $N = 18$, are compared with the methods presented in [6] and [13]. In reference [6], authors have used the wavelet Galerkin method to approximate the solution. In reference [13], Hwang and Shih have used the block pulse functions to approximate the solution. Here too, we consider the constant values as the values of Equation (1.1), i.e. $k = 4, \varepsilon = \frac{1}{162}$ and $\alpha = 6.5$. Table 5 shows the approximate solution using Genocchi polynomials of different orders N.

4.3. **Volterra's population model.** To solve Equation (1.4), first, we apply scale time and population by introducing the non-dimensional variables [36]

$$t=\frac{\xi\chi}{\beta},\quad y=\frac{w\beta}{\alpha},$$

then we have

$$\mu\frac{dy}{dt}-y+y^2+y\int_0^t y(z)\,dz=0,\quad y(0)=y_0,\tag{4.12}$$

where $\mu=\chi(\alpha\beta)^{-1}$. If $y(0)=y_0$, then the analytical solution of Equation (4.12) can be calculated as follows [36]

$$y(t)=y_0e^{\frac{1}{\mu}\int_0^t\left(1-y(\iota)-\int_0^\iota y(z)dz\right)d\iota}.$$

For simplicity, Volterra's population Equation (4.12) in interval $[0,\alpha]$ convert to interval $[0,1]$

$$\frac{\mu}{\alpha}\frac{dy}{dt}-y+y^2+y\int_0^{\alpha t}y(z)\,dz=0,\quad t\in[0,1],\quad y(0)=y_0.\tag{4.13}$$

By substituting $y(t)\approx\sum\limits_{i=0}^{N}\lambda_i\mathcal{G}_i(t)$ and $y'(t)\approx\sum\limits_{i=0}^{N}\lambda_i\mathcal{G}'_i(t))$ in Equation (4.13), we will have

$$\begin{aligned}
&\tfrac{\mu}{\alpha}\sum_{i=1}^{N}\lambda_i\mathcal{G}'_i(t)-\sum_{i=1}^{N}\lambda_i\mathcal{G}_i(t)+\left(\sum_{i=1}^{N}\lambda_i\mathcal{G}_i(t)\right)^2+\left(\sum_{i=1}^{N}\lambda_i\mathcal{G}_i(t)\right)\left(\int_0^{\alpha t}\sum_{i=1}^{N}\lambda_i\mathcal{G}_i(z)dz\right)\\
&=\sum_{i=1}^{N}\lambda_i\left(\tfrac{\mu}{\alpha}\mathcal{G}'(t)-\mathcal{G}(t)\right)+\left(\sum_{i=1}^{N}\lambda_i\mathcal{G}(t)\right)^2+\left(\sum_{i=1}^{N}\lambda_i\mathcal{G}_i(t)\right)\left(\sum_{i=1}^{N}\lambda_i\int_0^{\alpha t}\mathcal{G}_i(z)\,dz\right)\\
&=\sum_{i=1}^{N}\lambda_i\left(\tfrac{\mu}{\alpha}\mathcal{G}'(t)-\mathcal{G}(t)\right)+\left(\sum_{i=1}^{N}\lambda_i\mathcal{G}(t)\right)^2+\left(\sum_{i=1}^{N}\lambda_i\mathcal{G}_i(t)\right)\left(\sum_{i=1}^{N}\lambda_i\tfrac{\mathcal{G}_{i+1}(\alpha t)-\mathcal{G}_{i+1}(0)}{i+1}\right)=0.
\end{aligned}$$

It can be written briefly

$$\Lambda^T\left(\frac{\mu}{\alpha}\mathbf{DG}(t)-\mathbf{G}(t)\right)+\left(\Lambda^T\mathbf{G}(t)\right)^2+\left(\Lambda^T\mathbf{G}(t)\right)\left(\Lambda^T\mathbf{G}^+(t)\right)=0,$$

where

$$\mathbf{G}^+(t)=\left[\frac{\mathcal{G}_2(\alpha t)-\mathcal{G}_2(0)}{2},\frac{\mathcal{G}_3(\alpha t)-\mathcal{G}_3(0)}{3},\ldots,\frac{\mathcal{G}_{N+1}(\alpha t)-\mathcal{G}_{N+1}(0)}{N+1}\right]^T.$$

If we use the collocation points $t_i=\frac{i}{N-1},\quad i=1,2,\cdots,N-1$ from the above equation then we have

$$\Lambda^T\left(\frac{\mu}{\alpha}\mathbf{DG}(t_i)-\mathbf{G}(t_i)\right)+\left(\Lambda^T\mathbf{G}(t_i)\right)^2+\left(\Lambda^T\mathbf{G}(t_i)\right)\left(\Lambda^T\mathbf{G}^+(t_i)\right)=0.\tag{4.14}$$

Equations (4.14) with the initial condition $\Lambda^T\mathbf{G}(0)=y_0$, gives us N equations to obtain N coefficients $\lambda_i,\ i=1,...,N$, and we can obtain these coefficients by Newton iteration method.

4.3.1. *Error Analysis.*

Theorem 4.1. *If y_N is the approximate solution from (4.14), then*

$$\left\|\frac{\mu}{\alpha}\frac{dy_N}{dt}-y_N+{y_N}^2+y_N\int_0^{\alpha t}y_N(z)\,dz\right\|_2\le\frac{\mathcal{L}_1}{(N+1)!\sqrt{2N+3}}+\frac{\mathcal{L}_2}{\left[(N+1)!\right]^2(2N+3)},$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are constant coefficients.

Proof. We put

$$\mathcal{E}\left(y_N\right)=\frac{\mu}{\alpha}\frac{dy_N}{dt}-y_N+{y_N}^2+y_N\int_0^{\alpha t}y_N\left(z\right)dz.$$

If y is the exact solution of (4.13), then $\mathcal{E}\left(y\right)=0$. Also we can write

$$\begin{aligned}\left\|\mathcal{E}\left(y_N\right)\right\|_2&=\left\|\mathcal{E}\left(y_N\right)-\mathcal{E}\left(y\right)\right\|_2\\&=\left\|\frac{\mu}{\alpha}\frac{d\left(y_N-y\right)}{dt}+\left(y_N-y\right)+\left(y_N-y\right)^2+\left(y_N\int_0^{\alpha t}y_N\left(z\right)dz-y\int_0^{\alpha t}y\left(z\right)dz\right)\right\|_2\\&\leq\frac{\mu}{\alpha}\left\|\frac{d\left(y_N-y\right)}{dt}\right\|_2+\left\|y_N-y\right\|_2+\left\|y_N-y\right\|_2\left\|y_N-y\right\|_2\\&+\left\|\left(y_N\int_0^{\alpha t}y_N\left(z\right)dz-y\int_0^{\alpha t}y\left(z\right)dz\right)\right\|_2.\end{aligned}$$

Now by using Theorem 3.1, we have

$$\left\|y-y_N\right\|_2\leq M_1C\left(N\right),$$

$$\left\|y'-y'_N\right\|_2\leq M_2C\left(N\right),$$

where $M_1=\max_{x\in[0,1]}\left|f^{(N+1)}\left(x\right)\right|$, $M_2=\max_{x\in[0,1]}\left|f^{(N+2)}\left(x\right)\right|$ and $C\left(N\right)=\frac{1}{(N+1)!\sqrt{2N+3}}$. If we let $y-y_N=\mathcal{R}$, then

$$\left\|E\left(y_N\right)\right\|_2\leq\frac{\mu}{\alpha}M_2C\left(N\right)+M_1C\left(N\right)\left[1+M_1C\left(N\right)\right]-\left(\left\|\mathcal{R}\right\|_2\left\|\int_0^{\alpha t}y\left(z\right)dz\right\|_2+\left\|y-\mathcal{R}\right\|_2\left\|\int_0^{\alpha t}\mathcal{R}\left(z\right)dz\right\|_2\right).$$

As we know

$$\begin{aligned}&\left\|\mathcal{R}\right\|_2\leq M_1C\left(N\right),\\&\left\|\int_0^{\alpha t}y\left(z\right)dz\right\|_2\leq\alpha y_{max},\\&\left\|\int_0^{\alpha t}\mathcal{R}\left(z\right)dz\right\|_2\leq\alpha M_1C\left(N\right),\\&\left\|y-\mathcal{R}\right\|_2\leq y_{max}-M_1C\left(N\right).\end{aligned}$$

where [36]

$$y_{\max}=1+\mu\ln\left(\frac{\mu}{1+\mu-y_0}\right).$$

As a result, we will have

$$\left\|E\left(y_N\right)\right\|_2\leq C\left(N\right)\left[\mathcal{L}_1+\mathcal{L}_2C\left(N\right)\right],$$

where

$$\mathcal{L}_1=\tfrac{\mu}{\alpha}M_2+M_1-2M_1y_{max}\alpha,$$

$$\mathcal{L}_2=M_1^2\left(1+\alpha\right).$$

□

TABLE 6. Compare the Approxy_{max} of our method ($N = 15$) with the methods presented in previous literature and Exacty_{max} for Equation (1.4).

μ	Approx y_{max}				Exact y_{max}
	Our method	[2]	[19]	[4]	
0.02	0.92342514	0.923409	0.9038380533	0.9038380646	0.923471721
0.1	0.76975021	0.7697499	0.7651130834	0.7651130842	0.76974144907
0.2	0.65905209	0.6590506	0.6579123080	0.6579123129	0.6590503816
0.5	0.48519101	0.48519018	0.4852823482	0.4852823500	0.4851902914

TABLE 7. Compare the error of our results for Approxy_{max} with the methods presented in previous literature (degree of Genocchi polynomials is 15).

μ	Our method	[2]	[19]	[4]
0.02	$4.65e-5$	$1.81e-5$	$1.72e-2$	$1.95e-2$
0.1	$8.76e-6$	$8.40e-6$	$4.63e-3$	$4.11e-3$
0.2	$1.70e-6$	$2.18e-7$	$1.14e-3$	$1.13e-3$
0.5	$7.18e-7$	$3.88e-7$	$9.56e-5$	$9.50e-5$

TABLE 8. Compare the error of the introduced method for different N.

μ	$N = 15$	$N = 12$	$N = 9$	$N = 5$
		Error		
0.02	$4.65e-5$	$3.93e-4$	$4.62e-4$	$5.75e-3$
0.1	$8.76e-6$	$2.12e-5$	$8.65e-5$	$7.72e-4$
0.2	$1.70e-6$	$3.15e-5$	$1.07e-4$	$8.03e-4$
0.5	$7.18e-7$	$2.46e-6$	$9.05e-6$	$1.02e-4$

4.3.2. *Numerical result.* In the following, we apply our proposed method based on Gnocchi polynomials to approximate the solution of the Volterra population model (4.12).
We have obtained the interpolator polynomial for the case $N = 3$, $y_0 = 0.1$, $\alpha = 5$, and $\mu = 0.1$, as follows (the resulting nonlinear equations have been solved using MATLAB function "fsolve")

$$-0.18x^2 + 0.2668x - 0.6129.$$

We implemented our method for $\mu = 0.02$, 0.1, 0.2 and 0.5, with $y_0 = 0.1$ and $N = 15$. In Table 6, we have compared the maximum value of the population Approxy_{max} which obtain in our method with the methods introduced in articles [2] (auto-correlation functions of compact supported wavelets), [19](combining homotopy perturbation method (HPM) and Pade´ technique) and [4] (modified Adomian decomposition method). The exact value of y_{max} is [36]

$$Exacty_{\max} = 1 + \mu \ln\left(\frac{\mu}{1 + \mu - y_0}\right).$$

Table 7 shows the comparison between errors of methods given in [2, 4, 19], and our proposed Method of Approxy_{max}. Table 8 and Figure 3 also compare the error of the introduced method for different N. Figure 3 shows the error as a function of N for two fixed values of N ($N = 5, 9, 12, 15$) and $\mu = 0.1, 0.5$. In this figure, we have used a logarithmic scale for both axes.

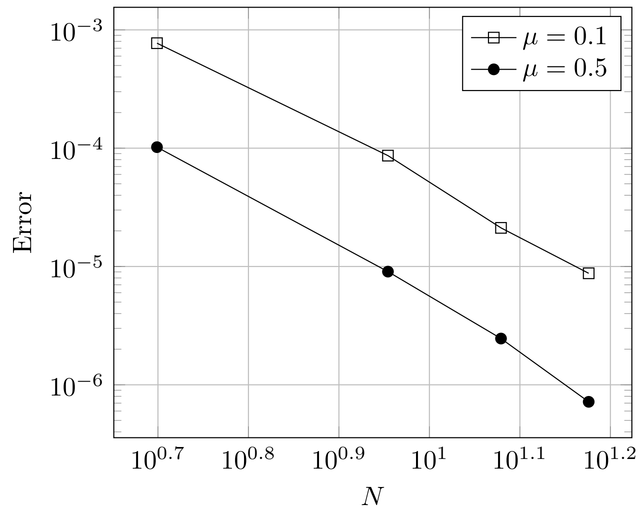

FIGURE 3. The error as a function of N, $\mu = 0.1, 0.5$ for Equation (1.4).

5. Conclusion

In this article, we investigated and numerically solved three models of population balance equations. For this purpose, we used Genocchi polynomials as basic orthogonal polynomials. The interesting properties of these polynomials and their simpler calculations (derivative and integral) are a good justification for using these polynomials. The results obtained for the introduced problems and their comparison with the results of existing methods show the efficiency of these polynomials well.

References

[1] A. Alipanah and M. Dehghan, *Solution of population balance equations via rationalized Haar functions*, Kybernetes, *37* (2008), 1189–1196.

[2] A. Alipanah and M. Zafari, *Collocation method using auto-correlation functions of compact supported wavelets for solving Volterra's population model*, Chaos, Solitons and Fractals, *175* (2023).

[3] I. A. Bhat, L. N. Mishra, N. N., Mishra, C. Tunç, and O. Tunç, *Precision and efficiency of an interpolation approach to weakly singular integral equations*, International Journal of Numerical Methods for Heat and Fluid Flow, *34* (2024), 1479–1499.

[4] J. Biazar and K. Hosseini, *Analytic approximation of Volterras population model*, JAMSI, *13* (2017), 5–17.

[5] R. Y. Chang and M. L. Wang, *Shifted Legendre function approximation of differential equations; application to crystalization processes*, J. Chem. Engng, *8* (1984), 117–25.

[6] M. Q. Chen, C. Hwang, and Y. P. Shih, *A Wavelet-Galerkin method for solving population balance equations*, Computers Chem. Engng, *20* (1996), 131–145.

[7] H. Dehestani, Y. Ordokhani, and M. Razzaghi, *On the applicability of Genocchi wavelet method for different kinds of fractional order differential equations with delay, Numer*, Linear Algebr., *26* (2019).

[8] H. Dehestani, Y. Ordokhani, and M. Razzaghi, *A numerical technique for solving various kinds of fractional partial differential equations via Genocchi hybrid functions*, Revista de la Real Academia de Ciencias Exactas, F´ısicas y Naturales. Serie A. Matem´aticas, *113* (2019), 3297–3321.

[9] A. S. Firdous, M. Irfan, and S. N. Kottakkaran, *Gegenbauer wavelet quasi-linearization method for solving fractional population growth model in a closed system*, Mathematical Methods in the Applied Sciences, *45* (2022), 3605–3623.

[10] B. Fornberg, *A practical guide to pseudospectral methods*, Cambridge, 1999.

[11] F. Ghomanjani, *A numerical method for solving Bratus problem*, Palestine Journal of Mathematics, *11* (2022), 372–377.

[12] S. Hu, M. S. Kim, P. Moree, and M. Sha, *Irregular primes with respect to Genocchi numbers and Artin's primitive root conjecture*, Journal of Number Theory *205* (2019), 50–80.

[13] C. Hwang and Y. P. Shih, *Solutions of population balance equations via block pulse functions*, J. Chem. Engng, *25* (1982), 39–45.

[14] A. Isah and C. Phang, *Genocchi wavelet-like operational matrix and its application for solving nonlinear fractional differential equations*, Open Phys., *14* (2016), 46–472.

[15] A. Kanwal, C. Phang, and U. Iqbal, *Numerical solution of fractional diffusion wave equation and fractional Klein–Gordon equation via two-dimensional Genocchi polynomials with a Ritz–Galerkin method*, Computation, *6* (2018).

[16] J. R. Loh. and C. Phang, *A new numerical scheme for solving system of Volterra integro-differential equation*, Alex. Eng. J., *57* (2018), 1117–1124.

[17] M. Lotfi and A. Alipnah, *Implementation of auto-correlation functions of compactly supported wavelets to population balance differential equation*, The 6th Seminar on Numerical Analysis and Its Applications, Maraghe, Iran (2016).

[18] M. M. Matar, *Existence of solution involving Genocchi numbers for nonlocal anti-periodic boundary value problem of arbitrary fractional order*, Revista de la Real Academia de Ciencias Exactas, F´ısicas y Naturales. Serie A. Matem´aticas, *112* (2018), 945–956.

[19] S. T. Mohyud-Din, A. Yıldırım, and Y. Gülkanat, *Analytical solution of Volterra's population model*, Journal of King Saud University - Science, *22*, (2010), 247–250.

[20] A. D. Randolph, *Effect of crystal breakage on crystal size distribution in a mixed suspension crystallizer.* I and EC Fund. *8* (1969), 58–63.

[21] A. D. Randolph and M. A. Larson, *Theory of Particulate Processes*, 2nd Edn. Academic Press, New York, 1988.

[22] F. Rigi and H. Tajadodi, *Numerical approach of fractional Abel differential equation by genocchi polynomials*, International Journal of Applied and Computational Mathematics, *5* (2019), 1–11.

[23] S. H. Rim, K. H., Park, and E. J. Moon, *On Genocchi numbers and polynomials*, Abstr. Appl. Anal. 2008.

[24] A. Saadatmandi, A. Khani, and M. A. Azizi, *A sinc-Gauss-Jacobi collocation method for solving Volterra's population growth model with fractional order*, Tbilisi Mathematical Journal *11* (2018), 123–137.

[25] F. M. Scudo, *Vito Volterra and theoretical ecology*, Teoretical Population Biology, *2* (1971), 1–23.

[26] P. N. Singh and D. Ramkrishna, *Solution of population balance equations by WRM*, Comput. Chem. Engng, *1* (1977), 23–31.

[27] R. D. Small, *Population growth in a closed system*, SIAM Review, *25* (1983), 93–95.

[28] H. M. Srivastava, F. A. Shah, and M. Irfan, *Generalized wavelet quasilinearization method for solving population growth model of fractional order*, Mathematical Methods in the Applied Sciences, *43* (2020), 8753-8762.

[29] G. Swaminathan, G. Hariharan, V. Selvaganesan, and S. Bharatwaja, *A new spectral collocation method for solving Bratu-type equations using Genocchi polynomials*, Journal of Mathematical Chemistry, (2021).

[30] H. Tajadodi, *Efficient technique for solving variable order fractional optimal control problems*, Alex. Eng. J., *59*, (2020), 5179–5185.

[31] I. Talib and F. Özger, *Orthogonal polynomials based operational matrices with applications to bagley–torvik fractional derivative differential equations*, IntechOpen, (2023).

[32] I. Talib and M. Bohner, *Numerical study of generalized modified caputo fractional differential equations*, International Journal of Computer Mathematics, *100* (2023), 153–176.

[33] I. Talib, A. N. Zulfiqar, Z. Hammouch, and H. Khalil, *Compatibility of the Paraskevopoulos's algorithm with operational matrices of Vieta–Lucas polynomials and applications*, Mathematics and Computers in Simulation, *202* (2022), 442–463.

[34] I. Talib, A. Raza, A. Atangana, and M. B. Riaz, *Numerical study of multi-order fractional differential equations with constant and variable coefficients*, Journal of Taibah University for Science, *16* (2022), 608–620.

[35] I. Talib, N. Alam, D. Baleanu, and D. Zaidi, *decomposition algorithm coupled with operational matrices approach with applications to fractional differential equations*, Thermal Science, *25* (2021), 449–455.
[36] K. G. TeBeest, *Numerical and analytical solutions of Volterras population model*, SIAM Rev, *39* (1997), 484–493.
[37] O. Tunç and C. Tunç, *Solution estimates to Caputo proportional fractional derivative delay integro–differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, *117* (2023).
[38] C. Tunç and O. Tunç, *A note on the qualitative analysis of Volterra integro-differential equations*, Journal of Taibah University for Science, *13* (2019), 490–496.
[39] Ş. Yüzbaşı, *A numerical approximation for Volterra's population growth model with fractional order*, Applied Mathematical Modelling, *37* (2013), 3216–3227.
[40] Ş. Yüzbaşı, *Improved Bessel collocation method for linear Volterra integro-differential equations with piecewise intervals and application of a Volterra population model*, Applied Mathematical Modelling, *40* (2016), 5349–5363.
[41] Ş. Yüzbaşı, M. Sezer, and B. Kemancı, *Numerical solutions of integro-differential equations and application of a population model with an improved Legendre method*, Applied Mathematical Modelling *37* (2013), 2086–2101.