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Dynamical systems in product Łukasiewicz semirings

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Abstract

This paper studies dynamical systems in product Lukasiewicz semirings and we generalize the results of Markechova and Riecan concerning the logical entropy. Also, the notion of logical entropy of a product Lukasiewicz semiring is introduced and it is shown that entropy measure is invariant under isomorphism.

Keywords. Lukasiewicz semiring, Entropy, Dynamical systems, Partition.
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1. Introduction

Shannon entropy in [25] concept of information theory is studied. Kolmogorov and Sinai [11, 26] developed the entropy of a dynamical system. Recall that if two dynamical systems are isomorphic, then they have the same entropy. Recently in the paper [17], in a dynamical system, the notion of logical entropy is studied and it is shown that by replacing the Shannon entropy function with the logical entropy function, the results that are analogous to the case of classical Kolmogorov-Sinai entropy theory are obtained [17, 18]. Here, in a product Lukasiewicz semiring, we introduced logical entropy.

Dynamical systems theory has its origins in the seminal work of Henri Poincar [20] on celestial mechanics. Eventually, dynamical systems theory (or nonlinear dynamics) found broad application beyond celestial mechanics (see, [7]-[16], [24], and [27]). We recall from [19] that there are about entropy of dynamical systems in MV-algebras, especially after its Mundicïi characterization as an interval in a lattice ordered group. Petrovicovï in [21, 22], for the product MV-algebras, a entropy theory of Shannon and Kolmogorov-Sinai type has been provided. Recall that the notion of near semirings, the theory of quantum mechanics, and Lukasiewicz near semirings were introduced in [2] and[5, 6].

In the present paper, we introduce the notion of logical entropy of a product Łukasiewicz semiring dynamical system and prove entropy measure is invariant under isomorphism of product Lukasiewicz semiring dynamical systems.

2. Main results

Recall that a groupoid is a set having one binary operation satisfying only closure and monoid is a set that is closed under an associative binary operation and has an identity element. We have the following definition from [3]. A near semiring is an algebra $(R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that:

1. $(R, +, 0)$ is a commutative monoid,
2. $(R, \cdot, 1)$ is a groupoid with condition $x \cdot 1 = x = 1 \cdot x$,
3. $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$,
4. $x \cdot 0 = 0 \cdot x = 0$,

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for all $x, y, z \in R$.

Also, a near semiring, which is a monoid, is said a semiring if $(R, \cdot, 1)$ satisfies left distributivity: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, for all $x, y, z \in R$. A near semiring R if satisfies in the property $a + a = a$ is called idempotent, for all $a \in R$. It is clear that $(R, +)$ is a semilattice ([3], Remark 1).

By [3], a function α of an idempotent near semiring $(R, +, \cdot,^{\alpha}, 0, 1)$ to $(R, +, \cdot,^{\alpha}, 0, 1)$ is said an involution on R if, for each $a, b \in R$:

1. $a^{\alpha\alpha} = a$,
2. if $a \leq b$ then $b^{\alpha} \leq a^{\alpha}$,

with $\leq$ the induced order.

Definition 2.1. A product Łukasiewicz semiring is an algebraic structure $(R, +, \cdot, ^{\alpha}, \times, 0, 1)$, where $(R, +, \cdot, ^{\alpha}, 0, 1)$ is a Łukasiewicz semiring and $\times$ is an associative and abelian binary operation on R with the following properties:

1. For every $x \in R$, $u \times x = x$;
2. If $((x^{\alpha\alpha} + y^{\alpha}) \cdot y^{\alpha\alpha})^{\alpha} = 0$, then $z \times x + z \times y \leq u$, and $z \times (x + y) = z \times x + z \times y$, for all $x, y, z \in R$.

We recall that n-tuple $X = (x_1, x_2, \ldots, x_n)$ of elements of a product Łukasiewicz semiring $(R, +, \cdot, ^{\alpha}, \times, 0, 1)$ with the property $x_1 + x_2 + \ldots + x_n = 1$ is called a partition of unity 1. There is the following definition of [21]:

Definition 2.2. A dynamical system in a product Łukasiewicz semiring $(R, +, \cdot, ^{\alpha}, \times, 0, 1)$ is a system $(R, \mu, \mathfrak{U})$, where $\mu : R \longrightarrow [0, 1]$ is a state, and $\mathfrak{U} : R \longrightarrow R$ is a function such that $\mathfrak{U}(u) = u$, the following conditions are satisfied:

(i) if $((x^{\alpha\alpha} + y^{\alpha}) \cdot y^{\alpha\alpha})^{\alpha} = 0$, then $((\mathfrak{U}(x))^{\alpha\alpha} + (\mathfrak{U}(y))^{\alpha}) \cdot (\mathfrak{U}(y))^{\alpha\alpha})^{\alpha} = 0$ and $\mathfrak{U}(x + y) = \mathfrak{U}(x) + \mathfrak{U}(y)$;
(ii) $\mathfrak{U}(x \times y) = \mathfrak{U}(x) \times \mathfrak{U}(y)$;
(iii) $\mu(\mathfrak{U}(x)) = \mu(x)$;

for every $x, y \in R$.

Definition 2.3. If $X = (x_1, x_2, \ldots, x_n)$ is a partition of unity, then its entropy is defined by the formula

$$H(X) = \sum_{i=1}^{n} \varphi(\mu(x_i)),$$

where $\varphi(x) = -x \log x$, if $x > 0$, $\varphi(0) = 0$.

Definition 2.4. Suppose $(R, \mu, \mathfrak{U})$ is a dynamical system in a product Łukasiewicz semiring $(R, +, \cdot, ^{\alpha}, \times, 0, 1)$ and θ is a sub-additive generator, where a function $\theta : [0, 1] \longrightarrow [0, \infty)$ is called a sub-additive generator, if the following implication holds

$$c_{ij} \in [0, 1], \; i = 1, \ldots, n, \; j = 1, \ldots, m, \; \sum_{i=1}^{n} c_{ij} = b_j, \; \sum_{j=1}^{m} c_{ij} = a_i, \sum_{i=1}^{n} a_i = 1, \quad \sum_{j=1}^{m} b_j = 1,$$

then

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \theta(c_{ij}) \leq \sum_{i=1}^{n} \theta(a_i) + \sum_{j=1}^{m} \theta(b_j).$$

We can define the θ-entropy of $(R, \mu, \mathfrak{U})$ by the formula

$$H_{\theta}^{\mu}(\mathfrak{U}) = \sup\{H_{\theta}^{\mu}(\mathfrak{U}, X); \; X \text{ is a partition in } (R, +, \cdot, ^{\alpha}, \times, 0, 1),$$

where

$$H_{\theta}^{\mu}(\mathfrak{U}, X) = \lim_{n \longrightarrow \infty} 1/n H_{\theta}^{\mu}(\bigvee_{k=0}^{n-1} \mathfrak{U}^k(X)).$$

Definition 2.5. Suppose $X = (x_1, x_2, \dots, x_n)$ is a partition in a product Lukasiewicz semiring $(R, +, \cdot, {}^\alpha, \times, 0, 1)$, and $\mu : R \longrightarrow [0, 1]$ is a state. By Shannon's formula, we define the entropy of X with respect to μ:

$$H_s^\mu(X) = \Sigma_{i=1}^n s(\mu(x_i)),$$

where $s : [0, 1] \longrightarrow [0, \infty)$ is the Shannon entropy function defined by $s(x) = x \log x$ and the Shannon entropy is a number, for every $x \in [0, 1]$. Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be two partitions in R. Then the conditional entropy of X given Y is defined by

$$H_s^\mu(X/Y) = \Sigma_{i=1}^n \Sigma_{j=1}^m (\mu(x_i \times y_i) \cdot \log \frac{\mu(x_i \times y_i)}{\mu(y_i)}).$$

Recall that from [23] the logical entropy of a dynamical system $(R, \mu, \mathfrak{U})$ in a product Lukasiewicz semiring $(R, +, \cdot, {}^\alpha, \times, 0, 1)$ is defined by

$$H_s^\mu(\mathfrak{U}) = \sup\{H_s^\mu(\mathfrak{U}, x);\ X \ \text{ is a partition in } R\},$$

and if X is a partition in R, then the logical entropy of $\mathfrak{U}$ relative to X is defined by

$$H_s^\mu(\mathfrak{U}, X) = \lim_{n \longrightarrow \infty} H_s^\mu(\bigvee_{k=0}^{n-1} \mathfrak{U}^k(X)).$$

Theorem 2.6. *[18] Let $(R, \mu, \mathfrak{U})$ be a dynamical system in a product Lukasiewicz near semiring $(R, +, \cdot, {}^\alpha, \times, 0, 1)$, and X be a partition in R. For any non-negative integer r, it holds:*

$$H_s^\mu(\mathfrak{U}, X) = H_s^\mu(\mathfrak{U}, \bigvee_{k=0}^{n-1} \mathfrak{U}^k(X)).$$

Definition 2.7. Two product Lukasiewicz near semiring dynamical systems $(R_1, \mu_1, \mathfrak{U}_1)$, $(R_2, \mu_2, \mathfrak{U}_2)$ are isomorphic, if there exists a mapping $\lambda : R_1 \longrightarrow R_2$ with $\lambda(u_1) = u_2$ and satisfying the following conditions:

1. $\lambda(x \times y) = \lambda(x) \times \lambda(y)$;
2. If $((x^{\alpha\alpha} + y^\alpha) \cdot y^{\alpha\alpha})^\alpha = 0$, then $\lambda(x + y) = \lambda(x) + \lambda(y)$;
3. $\mu_2(\lambda(x)) = \mu_1(x)$;
4. $\lambda(\mathfrak{U}_1(x)) = \mathfrak{U}_2(\lambda(x))$;

for every $x, y \in \mathfrak{U}_1$.

Now, we are ready to state the following theorem:

Theorem 2.8. *Let θ be a sub-additive generator, and $(R_1, \mu_1, \mathfrak{U}_1)$, $(R_2, \mu_2, \mathfrak{U}_2)$ be product Łukasiewicz near semiring dynamical systems. If $\mathfrak{U}_1 \simeq \mathfrak{U}_2$, then $H_\theta^{\mu_1}(\mathfrak{U}_1) = H_\theta^{\mu_2}(\mathfrak{U}_2)$.*

Proof. Suppose $\mathfrak{U}_1 \simeq \mathfrak{U}_2$ and λ be an isomorphism between dynamical systems $(R_1, \mu_1, \mathfrak{U}_1)$ and $(R_2, \mu_2, \mathfrak{U}_2)$. Now, assume that $X = (x_1, x_2, \dots, x_n)$ is a partition in a product Łukasiewicz near semiring $(R_1, \times)$ with $x_1 + x_2 + \dots + x_n = u_1$. By definition 2.6, we have $\lambda(x_1 + x_2 + \dots + x_n) = \lambda(x_1) + \lambda(x_2) + \dots + \lambda(x_n) = \lambda(u_1) = u_2$. Therefore, it is shown that $\lambda(X) = (\lambda(x_1), \lambda(x_2), \dots + \lambda(x_n))$ in product Łukasiewicz near semiring dynamical system $(R_2, \mu_2, \mathfrak{U}_2)$ is a partition. Also, by Definition 2.7,

$$H_\theta^{\mu_2}(\lambda(X)) = \sum_{i=1}^n \theta(\mu_2(\lambda(x_i))) = \sum_{i=1}^n \theta(\mu_1(x_i)) = H_\theta^{\mu_1}(X).$$

On the other hand, we have

$$H_\theta^{\mu_2}(\bigvee_{k=0}^{n-1} \mathfrak{U}_2^k(\lambda(X))) = H_\theta^{\mu_2}(\bigvee_{k=0}^{n-1} \lambda(\mathfrak{U}_1^k(X))) = H_\theta^{\mu_2}(\lambda(\bigvee_{k=0}^{n-1} \mathfrak{U}_1^k(X))) = H_\theta^{\mu_1}(\bigvee_{k=0}^{n-1} \mathfrak{U}_1^k(\lambda(X))). \tag{2.1}$$

If $n \longrightarrow \infty$, then we conclude

$$H_\theta^{\mu_2}(\mathfrak{U}_2, \lambda(X)) = \lim_{n\longrightarrow\infty} \frac{1}{n} H_\theta^{\mu_2}(\bigvee_{k=0}^{n-1} \mathfrak{U}_2^k(\lambda(X))) = \lim_{n\longrightarrow\infty} \frac{1}{n} H_\theta^{\mu_1}(\bigvee_{k=0}^{n-1} \mathfrak{U}_1^k(X)) = H_\theta^{\mu_1}(\mathfrak{U}_1, X). \tag{2.2}$$

In fact,

$$H_\theta^{\mu_1}(\mathfrak{U}_1) = \sup\{H_\theta^{\mu_1}(\mathfrak{U}_1, X);$$

X is a partition in $(R_1,\, \mu_1, \mathfrak{U}_1)\} \leq \sup\{H_\theta^{\mu_2}(\mathfrak{U}_2, Y),\, Y$ is a partition in $(R_2,\, \mu_2, \mathfrak{U}_2)\} = H_\theta^{\mu_2}(\mathfrak{U}_2)$.

Now, we prove that $H_\theta^{\mu_2}(\mathfrak{U}_2) \leq H_\theta^{\mu_1}(\mathfrak{U}_1)$. To do this, it is enough to define $\lambda^{-1} : R_2 \longrightarrow R_1$, that

$$\lambda^{-1}(x \times y) = \lambda^{-1}(\lambda(a) \times \lambda(b)) = \lambda^{-1}(\lambda(a \times b)) = a \times b = \lambda^{-1}(x) \times \lambda^{-1}(y),$$

where for every $x,\, y \in R_2$ there exist $a,\, b \in R_1$ such that $\lambda^{-1}(x) = a$ and $\lambda^{-1}(y) = b$. Also,

$$\lambda^{-1}(x + y) = \lambda^{-1}(\lambda(a) + \lambda(b)) = \lambda^{-1}(\lambda(a + b)) = a + b = \lambda^{-1}(x) + \lambda^{-1}(y),$$

which $x + y \leq u_1$ and $a + b \leq u_2$. In addition, we have

$$\lambda^{-1}(\mathfrak{U}_2(x)) = \lambda^{-1}(\mathfrak{U}_2(\lambda(a))) = \lambda^{-1}(\lambda(\mathfrak{U}_1(a))) = \mathfrak{U}_1(a) = \mathfrak{U}_1(\lambda^{-1}(x)),$$

where $x \in R_2$. Therefore, with the same argument for $H_\theta^{\mu_1}(\mathfrak{U}_1) \leq H_\theta^{\mu_2}(\mathfrak{U}_2)$, we obtain

$$H_\theta^{\mu_2}(\mathfrak{U}_2) \leq H_\theta^{\mu_1}(\mathfrak{U}_1).$$

□

Entropy of a Łukasiewicz near semiring dynamical system $(R,\, \mu, \mathfrak{U})$ is defined by the

$$\varphi(\mathfrak{U}) = \sup\{\varphi(\mathfrak{U}, X) \,|\, X \text{ is a partition of unity}\}, \text{ where } \varphi(\mathfrak{U}, X) = \lim_{n\longrightarrow\infty} H_n(\mathfrak{U}, X).$$

Definition 2.9. Two Łukasiewicz semiring dynamical systems $(R_1,\, \mu_1, \mathfrak{U}_1)$, $(R_2,\, \mu_2, \mathfrak{U}_2)$ are equivalent, if there exists a function $\theta : R_1 \longrightarrow R_2$ satisfying the following conditions:

(i) θ is a bijection;
(ii) if $a, b, c \in R_1$, $a = b+c$, then $\theta(a) = \theta(b)+\theta(c)$,
(iii) $\theta(u_1) = u_2$;
(iv) $\mu_2(\theta(a)) = \mu_1(a)$ for any $a \in R_1$;
(v) $\mathfrak{U}_2(\theta(a)) = \theta(\mathfrak{U}_1(a))$ for any $a \in R_1$.

Corollary 2.10. *If $(R_1,\, \mu_1, \mathfrak{U}_1)$, $(R_2,\, \mu_2, \mathfrak{U}_2)$ are equivalent, then $\varphi(\mathfrak{U}_1) = \varphi(\mathfrak{U}_2)$.*

Proof. We know that if X is any partition of u_1, then $\varphi(x)$ is a partition of u_2, and $H(X) = H(\varphi(X))$. Suppose that δ is an arbitrary positive number and we get a common refinement $\mathcal{C}$ of $X, \ldots, \mathfrak{U}^{n-1}(X)$ such that $H_n + \delta > H(\mathcal{C})$. Clearly that $H(\mathcal{C}) \geq H_n(\theta(X))$. Since $H_n(X) + \delta \geq H_n(\theta(X))$ holds for any $\delta > 0$, we have $H_n(X) \geq H_n(\theta(X))$. Also we have

$$\varphi_1(\mathfrak{U}_1, X) = \lim_{n\longrightarrow\infty} \frac{1}{n} H_n(X) \geq \lim_{n\longrightarrow\infty} \frac{1}{n} H_n(\theta(X)) = \varphi_2(\mathfrak{U}_2, \theta(X)),$$

and also

$$\varphi_1(\mathfrak{U}_1) = \sup\{\varphi_1(\mathfrak{U}_1, X), X\} \geq \varphi_2(\mathfrak{U}_2, \theta(X)).$$

Suppose B is any partition of u_2. Then $X = \theta^{-1}(B)$ is a partition of u_1 and $\theta(X) = B$. Therefore $\varphi_1(\mathfrak{U}_1) \geq \varphi_2(\mathfrak{U}_2, \theta(X)) = \varphi_2(\mathfrak{U}_2, B)$ for any B, hence $\varphi_1(\mathfrak{U}_1) \geq \varphi_2(\mathfrak{U}_2)$.

□

3. Conclusions

The study of the concept of entropy is very important in contemporary sciences. Entropy has been applied in information theory, physics, computer sciences, statistics, chemistry, biology, sociology, general systems theory and many other fields. In the paper, by using the concept of logical entropy of a partition in a product Łukasiewicz near semiring, we introduced the notion of logical entropy of a product Łukasiewicz near semiring dynamical systems. Also, we have defined a general type of entropy of a product MV-algebra dynamical system. Specifically, we introduced the notion of logical entropy of a product Łukasiewicz semiring and prove that entropy measure is invariant under isomorphism of product Łukasiewicz semiring dynamical systems.

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