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Convergence analysis for piecewise Lagrange interpolation method of fractal fractional model of tumor-immune interaction with two different kernels

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Abstract

Ahmad et al. (see [1]) presented a piecewise Lagrange interpolation method for solving tumor-immune interaction models with fractal fractional operators using a power law and exponential kernel. We suggest a convergence analysis for this method and we obtain the order of convergence. Of course, there are some mistakes in this numerical method that were corrected. Furthermore, Numerical illustrations are demonstrated to show the effectiveness of the corrected numerical method.

Keywords. Tumor-immune interaction, Fractal fractional model, Piecewise Lagrange interpolation, Convergence order.
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1. Introduction

One of the most interesting topics in applied mathematics is the explanation of phenomena in terms of models that are as close as possible to the observed reality. In this sense, the fractional calculus (see e.g. [5]), has been playing a leading role in the current scientific scencere and many efforts have been made to find the numerical and analytical solutions to the fractional differential and integral equations; for example see [6, 8, 11–13].

In the field of mathematical biology, scientists build mathematical models to study specific phenomena, and extract biological knowledge from it. In this case as well, fractional calculus offers a more precise depiction of the biological phenomena. Since cancer is one of the most prevalent diseases that can ultimately result in the loss of a person's life, various models have been suggested for it. One of them is an ordinary differential equation model of the tumor-immune interaction that introduced by Castiglione and Piccoli [3] (to appear). Dendritic cells, which are the most efficient in presenting antigen in vertebrate immune systems ([7]) play a crucial role in presenting tumor associated antigen. Dendritic cells are introduced externally and ignite the immune response against themselves and, as a side effect, also against the tumor cells. In fact, the clone expansion of cytotoxic T cells is able to recognize the tumor-associated-antigen loaded by Dendritic cells, also favors tumor killing since cancer cells naturally display the same tumor-associated antigens on their cell surface [4]. Authors in [1], considered a fractal fractional model of this tumor-immune interaction in Caputo and Caputo-Fabrizio sense [2] as

$$\begin{cases} {}^{\mathcal{FFP}}\mathcal{D}_{0,t}^{\theta;\omega} H(t) = \alpha^* + \beta^* DH\left(1-\frac{H}{\sigma^*}\right) - \gamma^* H, \\ {}^{\mathcal{FFP}}\mathcal{D}_{0,t}^{\theta;\omega} C(t) = \alpha' + \beta' I(M+D)C\left(1-\frac{C}{\sigma'}\right) - \gamma' C, \\ {}^{\mathcal{FFP}}\mathcal{D}_{0,t}^{\theta;\omega} M(t) = \tilde{\beta} M\left(1-\frac{M}{\tilde{\sigma}}\right) - d^* MC, \\ {}^{\mathcal{FFP}}\mathcal{D}_{0,t}^{\theta;\omega} D(t) = -d' DC, \\ {}^{\mathcal{FFP}}\mathcal{D}_{0,t}^{\theta;\omega} I(t) = \beta_0 DH - e_0 IC - \gamma_0 I. \end{cases} \tag{1.1}$$

with initial conditions

$$H(0) = H_0, \qquad C(0) = C_0, \qquad M(0) = M_0, \qquad D(0) = D_0, \qquad I(0) = I_0,$$

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Table 1. Description and values of parameters.

Parameters	Description	Values
α^*	CD4+ T birth rate	10^{-4}
β^*	CD4+ T proliferation rate	10^{-1}
γ^*	CD4+ T death rate	0.005
σ^*	Carrying capacity of CD4+ T	1
α'	CD8+ T birth rate	10^{-4}
β'	CD8+ T proliferation rate	10^{-2}
γ'	CD8+ T death rate	0.005
σ'	Carrying capacity of CD8+ T	1
$\widetilde{\beta}$	1/2 satur const of tumor	0.02
d^*	Killing by CD8+ of tumor	0.1
$\widetilde{\sigma}$	Carrying capacity of tumor	1
d'	CD8+ T killing of DC	0.1
β_0	IL-2 production by CD4+ T	10^{-2}
γ_0	IL-2 degradation rate	10^{-2}
e_0	IL-2 uptake by CD8+ T	10^{-7}

where H represents CD4 + T cells, C denotes CD8 + T cells, M represents cancer cells, D denotes dendritic cells, and I represents IL-2. The specific values and a description of the model parameters can be found in the Table 1 [4]. They (Ahmad et al.) suggested a numerical scheme for both power law (Caputo operator) and exponential decay kernels (Caputo-Fabrizio oprator) based on Lagrangian piecewise interpolation. However, there are some mistakes the derivation of the methods; for example, the authors obtained the following linear system for exponential decay type kernel (Eq. 27 of [1])

$$
\begin{aligned}
H^{b+1} &= H^0 + \frac{\omega t_b^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_1\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\omega t_{b-1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_1\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right)\\
&\quad + \frac{\theta\omega}{\mu(\theta)}\frac{3}{2}(\Delta t)t_b^{\omega-1}\mathcal{H}_1\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\theta\omega}{\mu(\theta)}\frac{\Delta t}{2}t_{b-1}^{\omega-1}\mathcal{H}_1\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right),\\
C^{b+1} &= C^0 + \frac{\omega t_b^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_2\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\omega t_{b-1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_2\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right)\\
&\quad + \frac{\theta\omega}{\mu(\theta)}\frac{3}{2}(\Delta t)t_b^{\omega-1}\mathcal{H}_2\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\theta\omega}{\mu(\theta)}\frac{\Delta t}{2}t_{b-1}^{\omega-1}\mathcal{H}_2\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right),\\
M^{b+1} &= M^0 + \frac{\omega t_b^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_3\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\omega t_{b-1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_3\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right)\\
&\quad + \frac{\theta\omega}{\mu(\theta)}\frac{3}{2}(\Delta t)t_b^{\omega-1}\mathcal{H}_3\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\theta\omega}{\mu(\theta)}\frac{\Delta t}{2}t_{b-1}^{\omega-1}\mathcal{H}_3\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right),\\
D^{b+1} &= D^0 + \frac{\omega t_b^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_4\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\omega t_{b-1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_4\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right)\\
&\quad + \frac{\theta\omega}{\mu(\theta)}\frac{3}{2}(\Delta t)t_b^{\omega-1}\mathcal{H}_4\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\theta\omega}{\mu(\theta)}\frac{\Delta t}{2}t_{b-1}^{\omega-1}\mathcal{H}_4\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right),\\
I^{b+1} &= I^0 + \frac{\omega t_b^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_5\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\omega t_{b-1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_5\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right)\\
&\quad + \frac{\theta\omega}{\mu(\theta)}\frac{3}{2}(\Delta t)t_b^{\omega-1}\mathcal{H}_5\left(t_b, H^b, C^b, M^b, D^b, I^b\right) - \frac{\theta\omega}{\mu(\theta)}\frac{\Delta t}{2}t_{b-1}^{\omega-1}\mathcal{H}_5\left(t_{b-1}, H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right).
\end{aligned}
$$

However, we cannot use this relation to get the unknown values because we will need two initial values $\left(H^b, C^b, M^b, D^b, I^b\right)$ and $\left(H^{b-1}, C^{b-1}, M^{b-1}, D^{b-1}, I^{b-1}\right)$ whereas we only have one initial value; on the other hand the model has a singularity in zero for $\omega < 1$ and $t_b^{\omega-1}$ (or $t_{b-1}^{\omega-1}$) is undefined, then the second (or third) term on the right side of the above equations cannot be calculated. Similar contradictions also exist for the power-law kernel (see Eq. (22) of [1]).

Here, we present the corrected numerical method in section 2 and its convergence analysis in section 3. In section 4, we simulate the numerical results for the fractal and fractional orders used in [1] and the final section contains some concluding remarks and future works.

2. Description of the numerical method

2.1. **Power law kernel.** Consider the following system of the proposed model [1]:

$$\mathcal{J}(t)=\mathcal{J}(0)+\frac{\omega}{\Gamma(\theta)}\int_0^t \psi^{\omega-1}(t-\psi)^{\theta-1}\Lambda(\psi,\mathcal{J}(\psi))d\psi, \qquad t\in[0,T], \tag{2.1}$$

where

$$\Lambda(t,\mathcal{J}(t))=\begin{cases}\mathcal{H}_1(t,H,C,M,D,I)=\alpha^*+\beta^*DH\left(1-\frac{H}{\sigma^*}\right)-\gamma^*H,\\ \mathcal{H}_2(t,H,C,M,D,I)=\alpha'+\beta'I(M+D)C\left(1-\frac{C}{\sigma'}\right)-\gamma'C,\\ \mathcal{H}_3(t,H,C,M,D,I)=\tilde{\beta}M\left(1-\frac{M}{\tilde{\sigma}}\right)-d^*MC,\\ \mathcal{H}_4(t,H,C,M,D,I)=-d'DC,\\ \mathcal{H}_5(t,H,C,M,D,I)=\beta_0DH-e_0IC-\gamma_0I,\end{cases}$$

$$\mathcal{J}(t)=\begin{cases}H(t),\\ C(t),\\ M(t),\\ D(t),\\ I(t),\end{cases}\qquad \mathcal{J}(0)=\begin{cases}H(0),\\ C(0),\\ M(0),\\ D(0),\\ I(0).\end{cases}$$

We assume that the nonlinear continuous function $\Lambda(t,\mathcal{J}(t))$ satisfies the Lipschitz condition

$$|\Lambda(s,\mathcal{J}(s))-\Lambda(s,\overline{\mathcal{J}}(s))|\le\mathcal{L}_\Lambda|\mathcal{J}(s)-\overline{\mathcal{J}}(s)|, \qquad \forall\mathcal{J},\overline{\mathcal{J}}\in C[0,T]\times C[0,T]\times C[0,T]\times C[0,T]\times C[0,T].$$

Now, consider two cases; $\omega=1$ and $0<\omega<1$:
Case I ($\omega=1$): Let $\Pi:=\{t_i:0=t_0<t_1<...<t_M=T\}$ be a uniform partition of the interval $[0,T]$ with step size $h=\frac{T}{M}$. Collocating Eq. (2.1) at the points $t=t_{b+1}$ $(b=0,1,...,M-1)$ yields

$$\begin{aligned}
H(t_{b+1})&=H_0+\frac{\omega}{\Gamma(\theta)}\sum_{s=0}^{b}\int_{t_s}^{t_{s+1}}\psi^{\omega-1}\left(t_{b+1}-\psi\right)^{\theta-1}\mathcal{H}_1(\psi,H,C,M,D,I)d\psi,\\
C(t_{b+1})&=C_0+\frac{\omega}{\Gamma(\theta)}\sum_{s=0}^{b}\int_{t_s}^{t_{s+1}}\psi^{\omega-1}\left(t_{b+1}-\psi\right)^{\theta-1}\mathcal{H}_2(\psi,H,C,M,D,I)d\psi,\\
M(t_{b+1})&=M_0+\frac{\omega}{\Gamma(\theta)}\sum_{s=0}^{b}\int_{t_s}^{t_{s+1}}\psi^{\omega-1}\left(t_{b+1}-\psi\right)^{\theta-1}\mathcal{H}_3(\psi,H,C,M,D,I)d\psi,\\
D(t_{b+1})&=D_0+\frac{\omega}{\Gamma(\theta)}\sum_{s=0}^{b}\int_{t_s}^{t_{s+1}}\psi^{\omega-1}\left(t_{b+1}-\psi\right)^{\theta-1}\mathcal{H}_4(\psi,H,C,M,D,I)d\psi,\\
I(t_{b+1})&=I_0+\frac{\omega}{\Gamma(\theta)}\sum_{s=0}^{b}\int_{t_s}^{t_{s+1}}\psi^{\omega-1}\left(t_{b+1}-\psi\right)^{\theta-1}\mathcal{H}_5(\psi,H,C,M,D,I)d\psi.
\end{aligned}\tag{2.2}$$

For brevity, we introduce the notations $H_{b+1}\approx H(t_{b+1})$, $C_{b+1}\approx C(t_{b+1})$,... , $I_{b+1}\approx I(t_{b+1})$. Now, the Lagrange interpolation in the finite interval $[t_s,t_{s+1}]$ allows us to approximate the kernel $\psi^{\omega-1}\mathcal{H}_i(\psi,H,C,M,D,I)$, $(i=1,2,...,5)$ as

$$\begin{aligned}\psi^{\omega-1}\mathcal{H}_i(\psi,H,C,M,D,I)\simeq{}&\frac{\psi-t_{s+1}}{t_s-t_{s+1}}t_s^{\omega-1}\mathcal{H}_i\left(t_s,H_s,C_s,M_s,D_s,I_s\right)\\&+\frac{\psi-t_s}{t_{s+1}-t_s}t_{s+1}^{\omega-1}\mathcal{H}_i\left(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1}\right).\end{aligned}$$

Therefore, the system (2.2) becomes

$$\begin{aligned}
H_{b+1} &= H(0) + \frac{\omega}{h\Gamma(\theta)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_1\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \int_{t_s}^{t_{s+1}} (t_{s+1}-\psi)(t_{b+1}-\psi)^{\theta-1} d\psi \\
&+ t_{s+1}^{\omega-1} \mathcal{H}_1\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \int_{t_s}^{t_{s+1}} (\psi - t_s)(t_{b+1}-\psi)^{\theta-1} d\psi \Big], \\
C_{b+1} &= C(0) + \frac{\omega}{h\Gamma(\theta)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_2\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \int_{t_s}^{t_{s+1}} (t_{s+1}-\psi)(t_{b+1}-\psi)^{\theta-1} d\psi \\
&+ t_{s+1}^{\omega-1} \mathcal{H}_2\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \int_{t_s}^{t_{s+1}} (\psi - t_s)(t_{b+1}-\psi)^{\theta-1} d\psi \Big], \\
M_{b+1} &= M(0) + \frac{\omega}{h\Gamma(\theta)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_3\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \int_{t_s}^{t_{s+1}} (t_{s+1}-\psi)(t_{b+1}-\psi)^{\theta-1} d\psi \\
&+ t_{s+1}^{\omega-1} \mathcal{H}_3\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \int_{t_s}^{t_{s+1}} (\psi - t_s)(t_{b+1}-\psi)^{\theta-1} d\psi \Big], \\
D_{b+1} &= D(0) + \frac{\omega}{h\Gamma(\theta)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_4\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \int_{t_s}^{t_{s+1}} (t_{s+1}-\psi)(t_{b+1}-\psi)^{\theta-1} d\psi \\
&+ t_{s+1}^{\omega-1} \mathcal{H}_4\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \int_{t_s}^{t_{s+1}} (\psi - t_s)(t_{b+1}-\psi)^{\theta-1} d\psi \Big], \\
I_{b+1} &= I(0) + \frac{\omega}{h\Gamma(\theta)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_5\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \int_{t_s}^{t_{s+1}} (t_{s+1}-\psi)(t_{b+1}-\psi)^{\theta-1} d\psi \\
&+ t_{s+1}^{\omega-1} \mathcal{H}_5\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \int_{t_s}^{t_{s+1}} (\psi - t_s)(t_{b+1}-\psi)^{\theta-1} d\psi \Big].
\end{aligned}$$

Calculating the integrals leads to

$$\begin{aligned}
H_{b+1} &= H_0 + \frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_1\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \Big((b-s)^{\theta+1} + (b-s+1)^\theta (\theta - b + s) \Big) \\
&- t_{s+1}^{\omega-1} \mathcal{H}_1\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \Big((b-s)^\theta (\theta + b - s + 1) - (b+1-s)^{\theta+1} \Big) \Big], \\
C_{b+1} &= C_0 + \frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_2\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \Big((b-s)^{\theta+1} + (b-s+1)^\theta (\theta - b + s) \Big) \\
&- t_{s+1}^{\omega-1} \mathcal{H}_2\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \Big((b-s)^\theta (\theta + b - s + 1) - (b+1-s)^{\theta+1} \Big) \Big], \\
M_{b+1} &= M_0 + \frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_3\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \Big((b-s)^{\theta+1} + (b-s+1)^\theta (\theta - b + s) \Big) \\
&- t_{s+1}^{\omega-1} \mathcal{H}_3\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \Big((b-s)^\theta (\theta + b - s + 1) - (b+1-s)^{\theta+1} \Big) \Big], \\
D_{b+1} &= D_0 + \frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_4\left(t_s, H_s, C_s, M_s, D_s, I_s\right) \Big((b-s)^{\theta+1} + (b-s+1)^\theta (\theta - b + s) \Big) \\
&- t_{s+1}^{\omega-1} \mathcal{H}_4\left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \Big((b-s)^\theta (\theta + b - s + 1) - (b+1-s)^{\theta+1} \Big) \Big],
\end{aligned} \tag{2.3}$$

$$I_{b+1} = I_0 + \frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=0}^{b} \Big[t_s^{\omega-1} \mathcal{H}_5 \left(t_s, H_s, C_s, M_s, D_s, I_s\right) \Big((b-s)^{\theta+1} + (b-s+1)^\theta (\theta - b + s) \Big)$$
$$- t_{s+1}^{\omega-1} \mathcal{H}_5 \left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \Big((b-s)^\theta (\theta + b - s + 1) - (b+1-s)^{\theta+1} \Big) \Big]. \quad (2.4)$$

Thus by knowing H_0, C_0, M_0, D_0, I_0 we can obtain the remaining values of the unknown functions at the mesh points by solving nonlinear system (2.3) for $b = 0, 1, ..., M-1$. These nonlinear systems may be readily solved by Newton's iterative method.

Case II ($\omega < 1$): In this case, the considered model has another singularity in $t_0 = 0$, and the introduced method in previous section is not applicable, therefore a small change is required in the scheme as follows

$$\begin{aligned} H(t_{b+1}) &= H_0 + \frac{\omega}{\Gamma(\theta)} \int_0^{t_1} \psi^{\omega-1} \left(t_{b+1} - \psi\right)^{\theta-1} \mathcal{H}_1(\psi, H, C, M, D, I) d\psi \\ &+ \frac{\omega}{\Gamma(\theta)} \sum_{s=1}^{b} \int_{t_s}^{t_{s+1}} \psi^{\omega-1} (t_{b+1} - \psi)^{\theta-1} \mathcal{H}_1(\psi, H, C, M, D, I) d\psi \\ &\approx H_0 + \frac{\omega}{\Gamma(\theta)} \Big[\mathcal{H}_1(t_0, H_0, C_0, M_0, D_0, I_0) \int_0^{t_1} \frac{\psi - t_1}{-h} \psi^{\omega-1} (t_{b+1} - \psi)^{\theta-1} d\psi \\ &+ \mathcal{H}_1(t_1, H_1, C_1, M_1, D_1, I_1) \int_0^{t_1} \frac{\psi - t_0}{h} \psi^{\omega-1} (t_{b+1} - \psi)^{\theta-1} d\psi \Big] \\ &+ \frac{\omega}{h\Gamma(\theta)} \sum_{s=1}^{b} \Big[t_s^{\omega-1} \mathcal{H}_1(t_s, H_s, C_s, M_s, D_s, I_s) \int_{t_s}^{t_{s+1}} (t_{s+1} - \psi)(t_{b+1} - \psi)^{\theta-1} d\psi \\ &+ t_{s+1}^{\omega-1} \mathcal{H}_1(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}) \int_{t_s}^{t_{s+1}} (\psi - t_s)(t_{b+1} - \psi)^{\theta-1} d\psi \Big]. \end{aligned} \quad (2.5)$$

In fact, in the first interval ($[0, t_1]$), $\mathcal{H}_i(\psi, H, C, M, D, I)$ is approximated by using the Lagrange interpolation in the interval $[0, t_1]$ and the same process as the previous section is used for the other intervals.

The first and second integrals in Eq. (2.5) can not be calculated accurately and must be approximated by using quadrature rules. Using a similar process for other equations leads to

$$\begin{aligned} H_{b+1} &= H_0 + \frac{\omega}{h\Gamma(\theta)} \left[\alpha_1 \mathcal{H}_1(t_1, H_1, C_1, M_1, D_1, I_1) - \alpha_2 \mathcal{H}_1(t_0, H_0, C_0, M_0, D_0, I_0) \right] \\ &+ \frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=1}^{b} \Big[t_s^{\omega-1} \mathcal{H}_1 \left(t_s, H_s, C_s, M_s, D_s, I_s\right) \Big((b-s)^{\theta+1} + (b-s+1)^\theta (\theta - b + s) \Big) \\ &- t_{s+1}^{\omega-1} \mathcal{H}_1 \left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \Big((b-s)^\theta (\theta + b - s + 1) - (b+1-s)^{\theta+1} \Big) \Big], \\ C_{b+1} &= C_0 + \frac{\omega}{h\Gamma(\theta)} \left[\alpha_1 \mathcal{H}_2(t_1, H_1, C_1, M_1, D_1, I_1) - \alpha_2 \mathcal{H}_2(t_0, H_0, C_0, M_0, D_0, I_0) \right] \\ &+ \frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=1}^{b} \Big[t_s^{\omega-1} \mathcal{H}_2 \left(t_s, H_s, C_s, M_s, D_s, I_s\right) \Big((b-s)^{\theta+1} + (b-s+1)^\theta (\theta - b + s) \Big) \\ &- t_{s+1}^{\omega-1} \mathcal{H}_2 \left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \Big((b-s)^\theta (\theta + b - s + 1) - (b+1-s)^{\theta+1} \Big) \Big], \\ M_{b+1} &= M_0 + \frac{\omega}{h\Gamma(\theta)} \left[\alpha_1 \mathcal{H}_3(t_1, H_1, C_1, M_1, D_1, I_1) - \alpha_2 \mathcal{H}_3(t_0, H_0, C_0, M_0, D_0, I_0) \right] \\ &+ \frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=1}^{b} \Big[t_s^{\omega-1} \mathcal{H}_3 \left(t_s, H_s, C_s, M_s, D_s, I_s\right) \Big((b-s)^{\theta+1} + (b-s+1)^\theta (\theta - b + s) \Big) \\ &- t_{s+1}^{\omega-1} \mathcal{H}_3 \left(t_{s+1}, H_{s+1}, C_{s+1}, M_{s+1}, D_{s+1}, I_{s+1}\right) \Big((b-s)^\theta (\theta + b - s + 1) - (b+1-s)^{\theta+1} \Big) \Big], \\ D_{b+1} &= D_0 + \frac{\omega}{h\Gamma(\theta)} \left[\alpha_1 \mathcal{H}_4(t_1, H_1, C_1, M_1, D_1, I_1) - \alpha_2 \mathcal{H}_4(t_0, H_0, C_0, M_0, D_0, I_0) \right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{\omega h^\theta}{\Gamma(\theta+2)}\sum_{s=1}^{b}\Big[t_s^{\omega-1}\mathcal{H}_4\left(t_s,H_s,C_s,M_s,D_s,I_s\right)\Big((b-s)^{\theta+1}+(b-s+1)^\theta(\theta-b+s)\Big)\\
&-t_{s+1}^{\omega-1}\mathcal{H}_4\left(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1}\right)\Big((b-s)^\theta(\theta+b-s+1)-(b+1-s)^{\theta+1}\Big)\Big],\\
I_{b+1}&=I_0+\frac{\omega}{h\Gamma(\theta)}\left[\alpha_1\mathcal{H}_5(t_1,H_1,C_1,M_1,D_1,I_1)-\alpha_2\mathcal{H}_5(t_0,H_0,C_0,M_0,D_0,I_0)\right]\\
&+\frac{\omega h^\theta}{\Gamma(\theta+2)}\sum_{s=1}^{b}\Big[t_s^{\omega-1}\mathcal{H}_5\left(t_s,H_s,C_s,M_s,D_s,I_s\right)\Big((b-s)^{\theta+1}+(b-s+1)^\theta(\theta-b+s)\Big)\\
&-t_{s+1}^{\omega-1}\mathcal{H}_5\left(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1}\right)\Big((b-s)^\theta(\theta+b-s+1)-(b+1-s)^{\theta+1}\Big)\Big],
\end{aligned}
\tag{2.6}
$$

where

$$
\alpha_1\approx\int_0^{t_1}\psi^\omega(t_{b+1}-\psi)^{\theta-1}d\psi,\qquad \alpha_2\approx\int_0^{t_1}(\psi^\omega-h\psi^{\omega-1})(t_{b+1}-\psi)^{\theta-1}d\psi.
$$

Again, the unknown values at the mesh points are obtained by solving this nonlinear system for $b=0,1,...,M-1$.

2.2. **Exponential decay type kernel.** Consider the tumor-immune model in the Caputo-Fabrizio fractional operator sense as [1]

$$
\begin{aligned}
H(t)&=H(0)+\frac{\omega t^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_1(t,H,C,M,D,I)+\frac{\theta\omega}{\mu(\theta)}\int_0^t\psi^{\omega-1}\mathcal{H}_1(\psi,H,C,M,D,I)d\psi,\\
C(t)&=C(0)+\frac{\omega t^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_2(t,H,C,M,D,I)+\frac{\theta\omega}{\mu(\theta)}\int_0^t\psi^{\omega-1}\mathcal{H}_2(\psi,H,C,M,D,I)d\psi,\\
M(t)&=M(0)+\frac{\omega t^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_3(t,H,C,M,D,I)+\frac{\theta\omega}{\mu(\theta)}\int_0^t\psi^{\omega-1}\mathcal{H}_3(\psi,H,C,M,D,I)d\psi,\\
D(t)&=D(0)+\frac{\omega t^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_4(t,H,C,M,D,I)+\frac{\theta\omega}{\mu(\theta)}\int_0^t\psi^{\omega-1}\mathcal{H}_4(\psi,H,C,M,D,I)d\psi,\\
I(t)&=I(0)+\frac{\omega t^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_5(t,H,C,M,D,I)+\frac{\theta\omega}{\mu(\theta)}\int_0^t\psi^{\omega-1}\mathcal{H}_5(\psi,H,C,M,D,I)d\psi,
\end{aligned}
\tag{2.7}
$$

by collocating Eq. (2.7) on the mesh grid points Π_M, we obtain

$$
\begin{aligned}
H(t_{b+1})&=H(0)+\frac{\omega t_{b+1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_1(t_{b+1},H_{b+1},C_{b+1},M_{b+1},D_{b+1},I_{b+1})\\
&+\frac{\theta\omega}{\mu(\theta)}\int_0^{t_1}\psi^{\omega-1}\mathcal{H}_1(\psi,H,C,M,D,I)d\psi+\sum_{s=1}^{b}\int_{t_s}^{t_{s+1}}\psi^{\omega-1}\mathcal{H}_1(\psi,H,C,M,D,I)d\psi.
\end{aligned}
\tag{2.8}
$$

Note that in this model, we have a singularity in zero; therefore applying piecewise Lagrange interpolation yields

$$
\begin{aligned}
H_{b+1}&=H(0)+\frac{\omega t_{b+1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_1(t_{b+1},H_{b+1},C_{b+1},M_{b+1},D_{b+1},I_{b+1})\\
&+\frac{\theta\omega}{h\mu(\theta)}\Big[\mathcal{H}_1(t_0,H_0,C_0,M_0,D_0,I_0)\int_0^{t_1}(h\psi^{\omega-1}-\psi^\omega)d\psi\\
&+\mathcal{H}_1(t_1,H_1,C_1,M_1,D_1,I_1)\int_0^{t_1}\psi^\omega d\psi\\
&+\sum_{s=1}^{b}\Big(\mathcal{H}_1(t_s,H_s,C_s,M_s,D_s,I_s)t_s{}^{\omega-1}\int_{t_s}^{t_{s+1}}(t_{s+1}-\psi)d\psi
\end{aligned}
$$

$$+\ \mathcal{H}_1(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1}){t_{s+1}}^{\omega-1}\int_{t_s}^{t_{s+1}}(\psi-t_s)d\psi\Big)\Big], \tag{2.9}$$

or, equivalently

$$\begin{aligned} H_{b+1} &= H_0+\frac{\omega t_{b+1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_1(t_{b+1},H_{b+1},C_{b+1},M_{b+1},D_{b+1},I_{b+1}) \\ &+\frac{\theta}{\mu(\theta)(\omega+1)}h^{\omega}\left[\mathcal{H}_1(t_0,H_0,C_0,M_0,D_0,I_0)+\mathcal{H}_1(t_1,H_1,C_1,M_1,D_1,I_1)\right] \\ &+\frac{\theta\omega h}{2\mu(\theta)}\sum_{s=1}^{b}\left[{t_s}^{\omega-1}\mathcal{H}_1(t_s,H_s,C_s,M_s,D_s,I_s)+{t_{s+1}}^{\omega-1}\mathcal{H}_1(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1})\right]. \end{aligned} \tag{2.10}$$

Using a similar process for the other equations gives a nonlinear algebraic equations system as follows, which can be solved by using Newton's iterative method.

$$\begin{aligned} H_{b+1} &= H_0+\frac{\omega t_{b+1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_1(t_{b+1},H_{b+1},C_{b+1},M_{b+1},D_{b+1},I_{b+1}) \\ &+\frac{\theta}{\mu(\theta)(\omega+1)}h^{\omega}\left[\mathcal{H}_1(t_0,H_0,C_0,M_0,D_0,I_0)+\mathcal{H}_1(t_1,H_1,C_1,M_1,D_1,I_1)\right] \\ &+\frac{\theta\omega h}{2\mu(\theta)}\sum_{s=1}^{b}\left[{t_s}^{\omega-1}\mathcal{H}_1(t_s,H_s,C_s,M_s,D_s,I_s)+{t_{s+1}}^{\omega-1}\mathcal{H}_1(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1})\right], \\ C_{b+1} &= C_0+\frac{\omega t_{b+1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_2(t_{b+1},H_{b+1},C_{b+1},M_{b+1},D_{b+1},I_{b+1}) \\ &+\frac{\theta}{\mu(\theta)(\omega+1)}h^{\omega}\left[\mathcal{H}_2(t_0,H_0,C_0,M_0,D_0,I_0)+\mathcal{H}_2(t_1,H_1,C_1,M_1,D_1,I_1)\right] \\ &+\frac{\theta\omega h}{2\mu(\theta)}\sum_{s=1}^{b}\left[{t_s}^{\omega-1}\mathcal{H}_2(t_s,H_s,C_s,M_s,D_s,I_s)+{t_{s+1}}^{\omega-1}\mathcal{H}_2(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1})\right], \\ M_{b+1} &= M_0+\frac{\omega t_{b+1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_3(t_{b+1},H_{b+1},C_{b+1},M_{b+1},D_{b+1},I_{b+1}) \\ &+\frac{\theta}{\mu(\theta)(\omega+1)}h^{\omega}\left[\mathcal{H}_3(t_0,H_0,C_0,M_0,D_0,I_0)+\mathcal{H}_3(t_1,H_1,C_1,M_1,D_1,I_1)\right] \\ &+\frac{\theta\omega h}{2\mu(\theta)}\sum_{s=1}^{b}\left[{t_s}^{\omega-1}\mathcal{H}_3(t_s,H_s,C_s,M_s,D_s,I_s)+{t_{s+1}}^{\omega-1}\mathcal{H}_3(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1})\right], \\ D_{b+1} &= D_0+\frac{\omega t_{b+1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_4(t_{b+1},H_{b+1},C_{b+1},M_{b+1},D_{b+1},I_{b+1}) \\ &+\frac{\theta}{\mu(\theta)(\omega+1)}h^{\omega}\left[\mathcal{H}_4(t_0,H_0,C_0,M_0,D_0,I_0)+\mathcal{H}_4(t_1,H_1,C_1,M_1,D_1,I_1)\right] \\ &+\frac{\theta\omega h}{2\mu(\theta)}\sum_{s=1}^{b}\left[{t_s}^{\omega-1}\mathcal{H}_4(t_s,H_s,C_s,M_s,D_s,I_s)+{t_{s+1}}^{\omega-1}\mathcal{H}_4(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1})\right], \\ I_{b+1} &= I_0+\frac{\omega t_{b+1}^{\omega-1}(1-\theta)}{\mu(\theta)}\mathcal{H}_5(t_{b+1},H_{b+1},C_{b+1},M_{b+1},D_{b+1},I_{b+1}) \\ &+\frac{\theta}{\mu(\theta)(\omega+1)}h^{\omega}\left[\mathcal{H}_5(t_0,H_0,C_0,M_0,D_0,I_0)+\mathcal{H}_5(t_1,H_1,C_1,M_1,D_1,I_1)\right] \\ &+\frac{\theta\omega h}{2\mu(\theta)}\sum_{s=1}^{b}\left[{t_s}^{\omega-1}\mathcal{H}_5(t_s,H_s,C_s,M_s,D_s,I_s)+{t_{s+1}}^{\omega-1}\mathcal{H}_5(t_{s+1},H_{s+1},C_{s+1},M_{s+1},D_{s+1},I_{s+1})\right], \quad b=0,1,...,M-1. \end{aligned} \tag{2.11}$$

It is well known that the guesses for Newton's method can be crucial; then, we choose H_b, C_b, ..., I_b as initial guesses for solving the nonlinear systems.

3. Convergence analysis

In this section, we state the error analysis for the introduced numerical method under the Caputo fractal fractional operator; a similar argument can be used for the Caputo-Fabrizio operator.
The following lemma will be used in our theoretical analysis.

Lemma 3.1. *(Discrete Gronwall lemma)[9]. Assume that $\{k_j\}_{j\geq 0}$ is a non-negative sequence, and the sequence $\{\epsilon_n\}$ satisfies $\epsilon_0 \leq \rho_0$ and*

$$\epsilon_n \leq \rho_0 + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} k_j \epsilon_j, \qquad n \geq 1,$$

with $\rho_0 \geq 0, q_j \geq 0$. Then

$$\epsilon_n \leq (\rho_0 + \sum_{j=0}^{n-1} q_j) \exp(\sum_{j=0}^{n-1} k_j), \qquad n \geq 1.$$

Theorem 3.2. *Let $\mathcal{J}(t)$ be the exact solution of the Equation (2.1) and $\mathcal{J}_{b+1}$ be the numerical solution (2.3) (in the point t_{b+1}) and assume that $\Lambda(\psi, \mathcal{J}(\psi))$ satisfies Lipshitz condition. Then the proposed method is convergent and for sufficiently smooth Λ, its order of convergence is at least $2+\theta$.*

Proof. Let $\mathbf{e}_{b+1} := \mathcal{J}(t_{b+1}) - \mathcal{J}_{b+1}$; then by subtracting (2.3) from (2.2) we have

$$\mathbf{e}_{b+1} = \mathcal{J}(t_{b+1}) - \mathcal{J}_{b+1} = \frac{\omega}{\Gamma(\theta)} \sum_{s=0}^{b} \int_{t_s}^{t_{s+1}} \psi^{\omega-1}(t_{b+1}-\psi)^{\theta-1}\Lambda(\psi, \mathcal{J}(\psi))d\psi$$
$$-\frac{\omega h^\theta}{\Gamma(\theta+2)} \sum_{s=0}^{b} \Big[c_s t_s^{\omega-1}\Lambda(t_s, \mathcal{J}_s) - c_{s+1} t_{s+1}^{\omega-1}\Lambda(t_{s+1}, \mathcal{J}_{s+1}) \Big],$$

where c_i for $i = 0, 1, ..., b+1$ are constant. By adding and diminishing the necessary terms and using Lipschitz condition, we obtain

$$|\mathbf{e}_{b+1}| \leq \frac{\omega}{\Gamma(\theta)} \sum_{s=0}^{b} \Big[| \int_{t_s}^{t_{s+1}} \psi^{\omega-1}(t_{b+1}-\psi)^{\theta-1}\Lambda(\psi, \mathcal{J}(\psi))d\psi$$
$$-\frac{h^\theta}{\theta(\theta+1)} c_s t_s^{\omega-1}\Lambda(t_s, \mathcal{J}(t_s)) - \frac{h^\theta}{\theta(\theta+1)} c_{s+1} t_{s+1}^{\omega-1}\Lambda(t_{s+1}, \mathcal{J}(t_{s+1}))|$$
$$+ \frac{h^\theta}{\theta(\theta+1)} |c_s| t_s^{\omega-1}\mathcal{L}_\Lambda |\mathcal{J}(t_s) - \mathcal{J}_s| + \frac{h^\theta}{\theta(\theta+1)} |c_{s+1}| t_{s+1}^{\omega-1}\mathcal{L}_\Lambda |\mathcal{J}(t_{s+1}) - \mathcal{J}_{s+1}| \Big],$$

where $\mathcal{L}_\Lambda$ is Lipschitz constant.

In the following, we can put the integrals $\int_{t_s}^{t_{s+1}} \frac{(\psi-t_{s+1})}{-h}(t_{b+1}-\psi)^{\theta-1}d\psi$ and $\int_{t_s}^{t_{s+1}} \frac{(\psi-t_s)}{h}(t_{b+1}-\psi)^{\theta-1}d\psi$ instate of constants c_s and c_{s+1}, respectively. Therefore we can write

$$\begin{aligned}
|\mathbf{e}_{b+1}| &\leq \frac{\omega}{\Gamma(\theta)} \sum_{s=0}^{b} \Big[| \int_{t_s}^{t_{s+1}} (t_{b+1}-\psi)^{\theta-1} \Big(\psi^{\omega-1}\Lambda(\psi, \mathcal{J}(\psi)) - t_s^{\omega-1}\Lambda(t_s, \mathcal{J}(t_s)) \frac{(\psi - t_{s+1})}{-h} \\
&- t_{s+1}^{\omega-1}\Lambda(t_{s+1}, \mathcal{J}(t_{s+1})) \frac{(\psi - t_s)}{h} \Big) d\psi | \\
&+ \frac{h^\theta}{\theta(\theta+1)} \left(|c_s| t_s^{\omega-1}\mathcal{L}_\Lambda |\mathbf{e}_s| + |c_{s+1}| t_{s+1}^{\omega-1}\mathcal{L}_\Lambda |\mathbf{e_{s+1}}| \right) \Big] \\
&\leq \frac{\omega}{\Gamma(\theta)} \sum_{s=0}^{b} \Big[| \int_{t_s}^{t_{s+1}} (t_{b+1}-\psi)^{\theta-1} I_s(\psi) d\psi | + \frac{h^\theta}{\theta(\theta+1)} \left(|c_s| t_s^{\omega-1}\mathcal{L}_\Lambda |\mathbf{e}_s| + |c_{s+1}| t_{s+1}^{\omega-1}\mathcal{L}_\Lambda |\mathbf{e_{s+1}}| \right) \Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \frac{\omega}{\Gamma(\theta)}||I||_\infty \sum_{s=0}^{b} |\int_{t_s}^{t_{s+1}} (t_{b+1}-\psi)^{\theta-1} d\psi| \\
&+ \quad \frac{\omega h^\theta}{\theta(\theta+1)\Gamma(\theta)} \sum_{s=0}^{b} \left(|c_s|(sh)^{\omega-1}\mathcal{L}_\Lambda|\mathbf{e}_s| + |c_{s+1}|((s+1)h)^{\omega-1}\mathcal{L}_\Lambda|\mathbf{e_{s+1}}|\right), \qquad (3.1)
\end{aligned}
$$

where I_s is the Lagrange interpolation error. Calculate the integral leads to

$$
\begin{aligned}
|\mathbf{e}_{b+1}| &\leq \frac{\omega h^\theta}{\theta\Gamma(\theta)}||I||_\infty \sum_{s=0}^{b}((b-s)^\theta - (b+1-s)^\theta) \\
&+ \frac{\omega h^{\theta+\omega-1}}{\Gamma(\theta+2)}\mathcal{L}_\Lambda \sum_{s=0}^{b} \left(|c_s|s^{\omega-1}|\mathbf{e}_s| + |c_{s+1}|(s+1)^{\omega-1}|\mathbf{e_{s+1}}|\right). \qquad (3.2)
\end{aligned}
$$

The inequality (3.2) can be simplified as follows

$$
\begin{aligned}
&|\mathbf{e}_{b+1}| \leq C_1 h^\theta ||I||_\infty + C_2 h^{\theta+\omega-1} \sum_{s=0}^{b} |\mathbf{e}_s| + C_3 h^{\theta+\omega-1}|\mathbf{e}_{b+1}|, \\
&|\mathbf{e}_{b+1}| \leq \frac{C_1 h^\theta ||I||_\infty}{1-C_3 h^{\theta+\omega-1}} + \frac{C_2 h^{\theta+\omega-1}}{1-C_3 h^{\theta+\omega-1}} \sum_{s=0}^{b} |\mathbf{e}_s|. \qquad (3.3)
\end{aligned}
$$

with constants C_1, C_2 and C_3. Finally, using the Gronwall inequality Lemma 3.1, yields

$$
|\mathbf{e}_{b+1}| \leq \frac{C_1 h^\theta ||I||_\infty}{1-C_3 h^{\theta+\omega-1}} \exp(\sum_{s=0}^{b} \frac{C_2 h^{\theta+\omega-1}}{1-C_3 h^{\theta+\omega-1}}).
$$

Thus $|\mathbf{e}_{b+1}| \to 0$ as $h \to 0$; since $||I||_\infty$ is of order h^2 (see [10]), the convergence order will be $O(h^{2+\theta})$. □

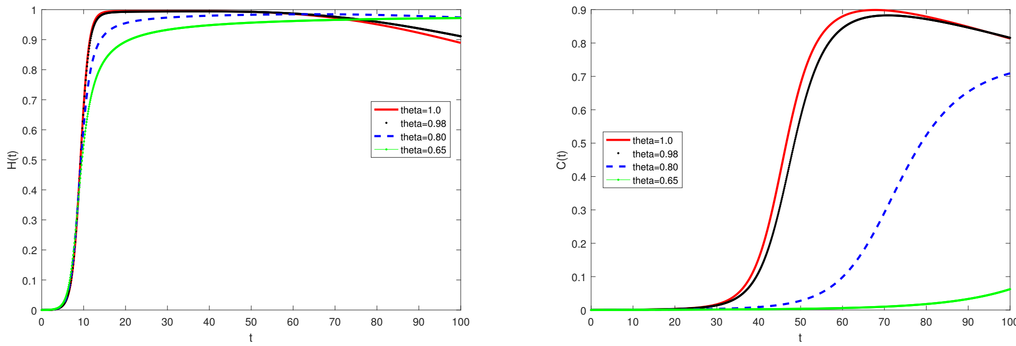
4. Numerical simulations

In order to illustrate the presented results in the previous sections, similar to [1], we consider the initial conditions $H(0) = 0$, $C(0) = 0$, $M(0) = 1$, $D(0) = 10$, $I(0) = 0$ and the parametric values presented in Table 1. Tumor-immune interaction for the Caputo operator by using introduced algorithms in subsection 2.1 which are shown in Figures 1-6 for values of fractional and fractal order similar to [1]. Also, Figures 7-12, show the results for the Caputo-Fabrizio model. We observe that the obtained results are near to expected results in [1]. The novel Caputo- Fabrizio operator produces excellent dynamics of the model. Stability occurs more quickly in the Caputo-Fabrizio case than in the Caputo operator, for example, see the $H(t)$ dynamic in Figure 1 and Figure 7. In Figure 1, $CD4+T$ cells rise to their maximal value 1, while in Figure 7, in addition to the maximal value of 1, we have stability at $t = 20$. Similarly, we can compare the other unknowns of the model for the Caputo and Caputo-Fabrizio case. This leads us to conclude than the Caputo-Fabrizio operator performs better than the Caputo operator.

We have performed all of the numerical computations using software Matlab.

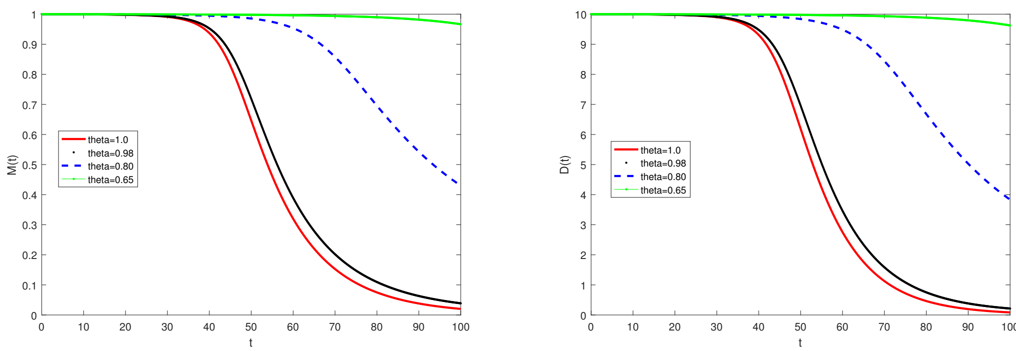
5. Conclusion

In this article, we corrected the introduced numerical methods in [1] for the tumor-immune interaction model and we obtained an error analysis, displaying a convergence order of $O(h^{2+\theta})$. Of course, the model is considered in a large interval ($T = 100$) and it will be more effective to use methods with a higher order of convergence or useful methods for large intervals. Numerical results were given to illustrate the theoretical results.



(a) CD4+T cells ($H(t)$). (b) CD8+T cells ($C(t)$).

FIGURE 1. Dynamic of CD4+T cells ($H(t)$) and CD8+T cells ($C(t)$) for Caputo operator with different values of $\theta, \omega = 1$ and $h = 0.1$.



(a) cancer cells ($M(t)$). (b) dendritic cells ($D(t)$).

FIGURE 2. Dynamic of cancer cells ($M(t)$) and dendritic cells ($D(t)$) for Caputo operator with different values of $\theta, \omega = 1$ and $h = 0.1$.

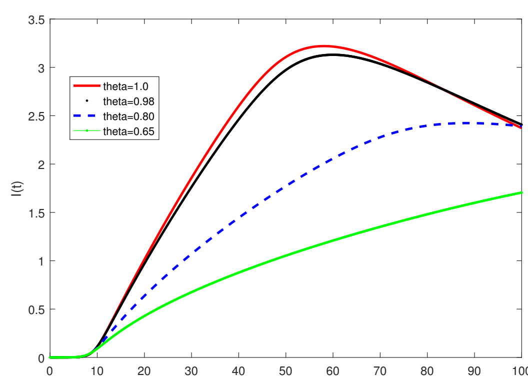
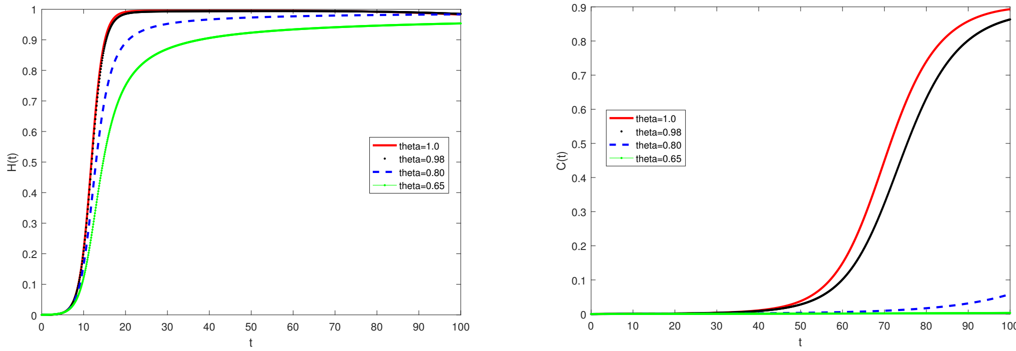
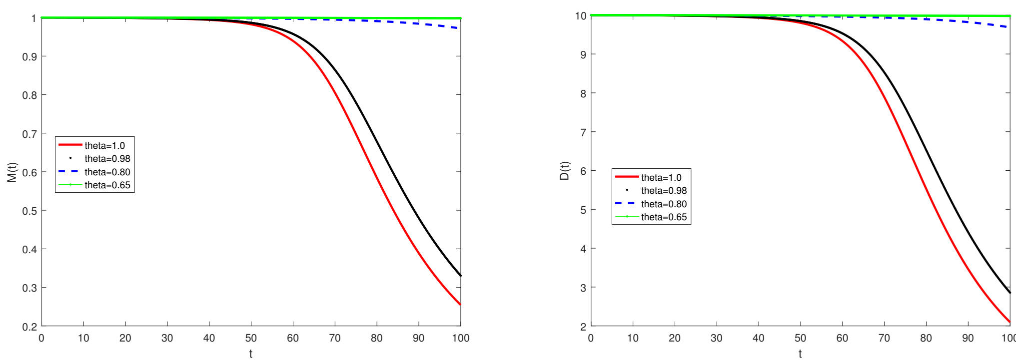


FIGURE 3. Dynamic of IL-2 ($I(t)$) for Caputo operator with different values of $\theta, \omega = 1$ and $h = 0.1$.



(a) CD4+T cells ($H(t)$). (b) CD8+T cells ($C(t)$).

FIGURE 4. Dynamic of CD4+T cells ($H(t)$) and CD8+T cells ($C(t)$) for Caputo operator with different values of $\theta, \omega = 0.9$ and $h = 0.1$.



(a) cancer cells ($M(t)$). (b) dendritic cells ($D(t)$).

FIGURE 5. Dynamic of cancer cells ($M(t)$) and dendritic cells ($D(t)$) for Caputo operator with different values of $\theta, \omega = 0.9$ and $h = 0.1$.

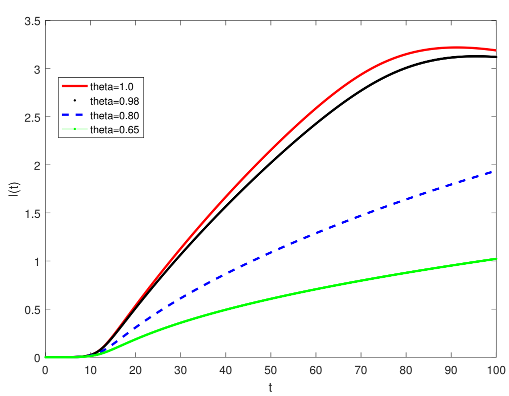
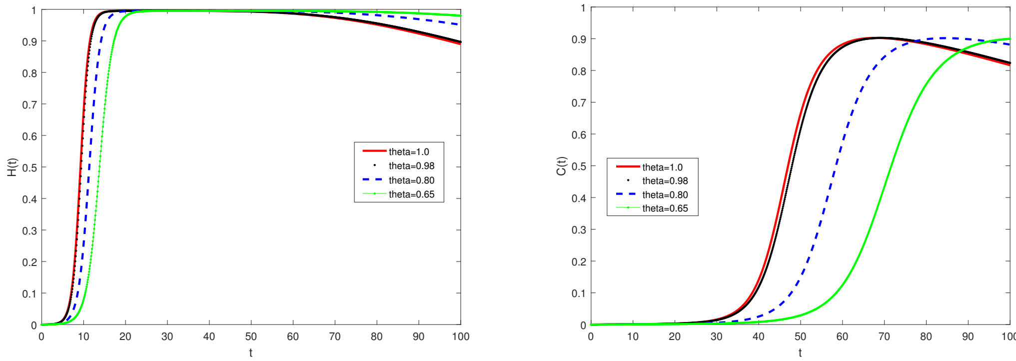
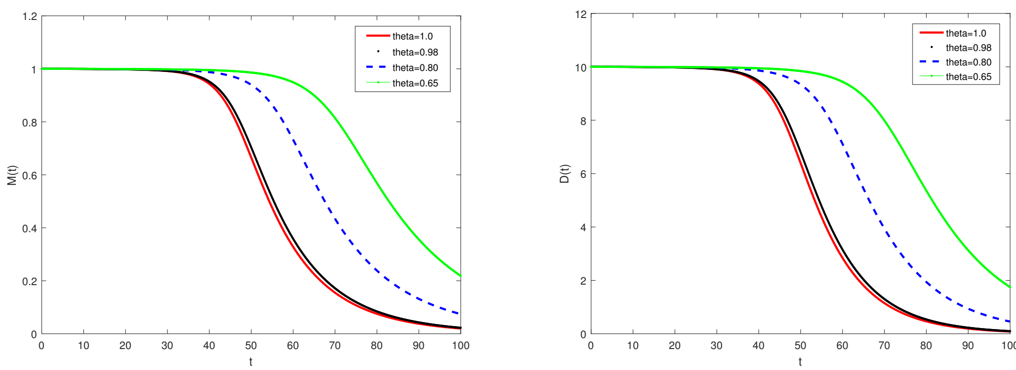


FIGURE 6. Dynamic of IL-2 ($I(t)$) for Caputo operator with different values of $\theta, \omega = 0.9$ and $h = 0.1$.



(a) CD4+T cells ($H(t)$). (b) CD8+T cells ($C(t)$).

FIGURE 7. Dynamic of CD4+T cells ($H(t)$) and CD8+T cells ($C(t)$) for Caputo-Fabrizio operator with different values of $\theta, \omega = 1$ and $h = 0.1$.



(a) cancer cells ($M(t)$). (b) dendritic cells ($D(t)$).

FIGURE 8. Dynamic of cancer cells ($M(t)$) and dendritic cells ($D(t)$) for Caputo-Fabrizio operator with different values of $\theta, \omega = 1$ and $h = 0.1$.

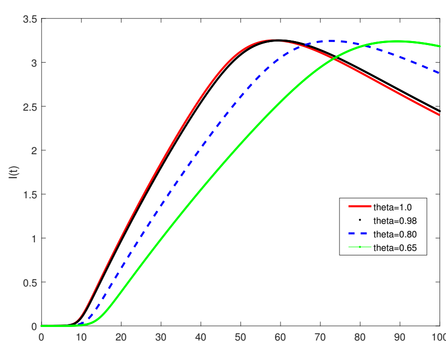
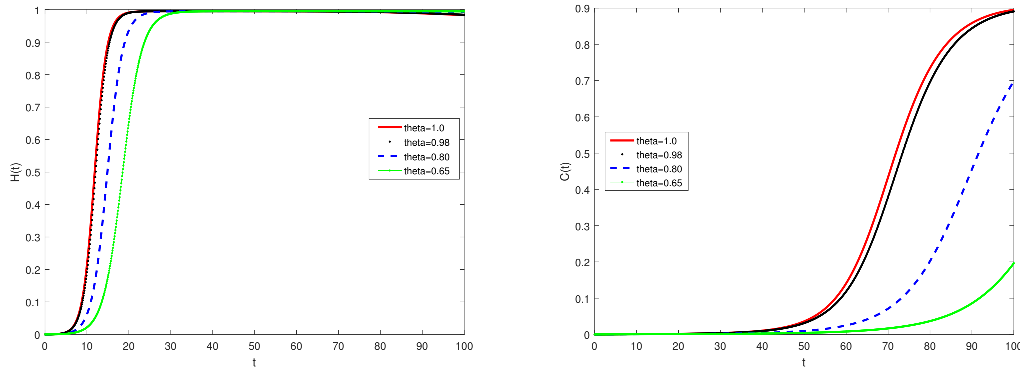
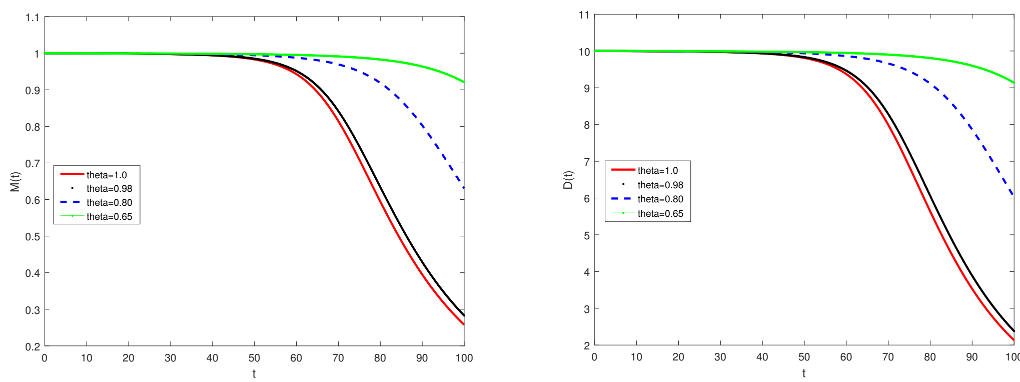


FIGURE 9. Dynamic of IL-2 ($I(t)$) for Caputo-Fabrizio operator with different values of $\theta, \omega = 1$ and $h = 0.1$.



(a) CD4+T cells ($H(t)$). (b) CD8+T cells ($C(t)$).

FIGURE 10. Dynamic of CD4+T cells ($H(t)$) and CD8+T cells ($C(t)$) for Caputo-Fabrizio operator with different values of $\theta, \omega = 0.9$ and $h = 0.1$.



(a) cancer cells ($M(t)$). (b) dendritic cells ($D(t)$).

FIGURE 11. Dynamic of cancer cells ($M(t)$) and dendritic cells ($D(t)$) for Caputo-Fabrizio operator with different values of $\theta, \omega = 0.9$ and $h = 0.1$.

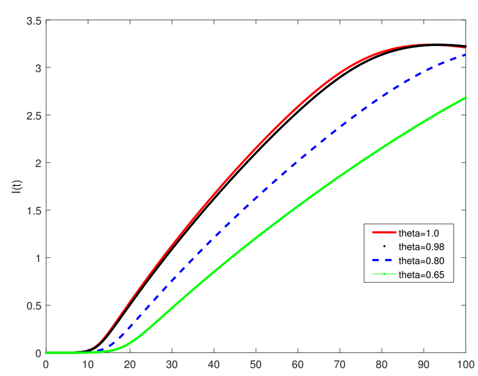


FIGURE 12. Dynamic of IL-2 ($I(t)$) for Caputo-Fabrizio operator with different values of $\theta, \omega = 0.9$ and $h = 0.1$.

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