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On the dynamics of newly generated analytical solutions and conserved vectors of a generalized 3D KP-BBM equation

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Abstract

This paper examines a high-dimensional non-linear partial differential equation called the generalized Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation exists in three dimensions. The Lie symmetry analysis of the equation is carried out step-by-step. As a result, we found symmetries from which various group-invariant solutions arise, leading to numerous solutions of interest that satisfy the KP-BBM equation. Secured solutions of interest include hyperbolic functions and elliptic functions, with the latter being the more general of the two solutions. Additionally, a significant number of algebraic solutions with arbitrary functions are also obtained. Furthermore, the dynamics of the solutions are further explored diagrammatically using computer software. In the concluding section, various conservation laws of the underlying model are derived via the multiplier method and the Noether theorem.

Keywords. A generalized three-dimensional KP-BBM equation, Lie symmetry analysis, Group-invariant and analytical solutions, Conservation laws.

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1. Introduction

The world around us is inherently non-linear. Meanwhile, it is evident that non-linear partial differential equations (NLPDEs) are extensively utilized as models in delineating various complex and non-complex physical phenomena. One of the basic bottlenecks for these models has to do with the way their traveling wave solutions can be achieved. Thus, the interest and attention given to find traveling wave solutions of NLPDEs is quite increasing and has now turned out to be a hot topic for scientists and other various researchers. Lately, many researchers who have a keen interest in the non-linear physical phenomena delve into examining exact solutions of NLPDEs due to their relevance in analyzing the outcome of any given model. Therefore, it is essential that the research into closed-form solutions to NLPDEs serves a very crucial purpose in observing certain physical circumstances. Furthermore, the diversity of solutions of NLPDEs holds an essential position in a variety of areas of sciences inclusive of optical fibers, chemical physics, geochemistry, biology, hydrodynamics, chemical kinetics, meteorology, heat flow, plasma physics, together with electromagnetic theory. Given the aforementioned and for emphasis, having realized that significant scientists have contemplated non-linear science as the most outstanding borderline for fundamental cognition of nature, we present some pertinent models that include a 3D generalized non-linear potential Yu-Toda-Sasa-Fukuyama equation in physics alongside pngineering, recently investigated by the authors in [3]. Moreover, the authors in [7] examined another generalized NLPDEs called advection-diffusion equation with power law non-linearity in fluid mechanics.

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This generalized equation characterizes buoyancy-propelled plume movement embedded in a medium that is bent on nature. Further to that, a generalized structure of Korteweg-de Vries-Zakharov-Kuznetsov model in the paper [27] was investigated. The dilution of warm isentropic fluid alongside cold static framework species together with hot isothermal, applicable in fluid dynamics was recounted via the use of the model. Besides, an investigation in [17] was carried out on the modified as well as the generalized Zakharov-Kuznetsov model, delineating the ion-acoustic meandering solitary waves resident in a magneto-plasma and possessive of electron-positron-ion observable in the autochthonous universe. This model was utilized in representing dust-magneto-acoustic, ion-acoustic, together with dust-ion-acoustic waves in the laboratory dusty plasmas. Additionally, the vector bright solitons, alongside their various interaction attribute related to the coupled Fokas-Lenells system [55] were studied in the given reference. The femto-second optical pulses embedded in a double-refractive optical fiber, modeled into an NLPDEs, were further investigated. Furthermore, the Boussinesq-Burgers-type system recounting shallow water waves and also emerging near ocean beaches and lakes was given attention in the paper [20]. We can continue with the list, but we mention a few. See more of the applications of NLPDEs in the references [2, 4–6, 8–11, 16, 18, 34, 44, 47].

Now, having established the fact that no general technique in achieving various exact traveling wave results of NLPDEs have been found, mathematicians and physicists have come up with some sound, effective, and efficient techniques lately so that the seemingly nagging problem could be nipped in the bud. Some of these techniques include: bifurcation technique [54], Painlevé expansion [50], homotopy perturbation technique [14], tanh-coth approach [49], extended homoclinic test approach [15], Cole-Hopf transformation technique [39], Adomian decomposition approach [46], Bäcklund transformation [22], Lie symmetry analysis [36, 38], F-expansion technique [56], rational expansion technique [53], multiple exp-function method [26], extended simplest equation approach [29], the Kudryashov's technique [30], Hirota technique [25], Darboux transformation [32], tanh-function technique [48], the $\left(\frac{G'}{G}\right)$-expansion technique [43], sine-Gordon equation expansion technique [12], generalized unified technique [37], exponential function technique [23], and so on and so forth.

In an avenue where the transversal imbalance can be safely disregarded, the Benjamin-Bona-Mahony model (BBMahM) has been demonstrated as a well-founded conjecture for the unidirectional propagation of long waves possessing small amplitude. The BBMahM equation reads

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

where $u = u(t,x)$ is considered in a specified class of real non-periodic functions. The Korteweg-de Vries (KdV) equation, along with the BBMahM equation, both have the same basis as a conjecture of waves possessing small but finite amplitude and moderate length. It is argued that the BBMahM equation, which circumvents several problematic aspects of the KdV equation, is in many ways the preferred model.

In 2005, Wazwaz divulged two structures of BBMahM that were orchestrated in the sense of Kadomtsev-Petviashvili. These models are commonly referred to as the Kadomtsev-Petviashvili Benjamin-Bona-Mahony (KP-BBM) equation, which reads [45].

$$\left(\psi_t + \psi_x - p(u^2)_x - q\psi_{txx}\right)_x + s\psi_{yy} = 0, \tag{1.1}$$

and

$$(\psi_t + \psi_x - p(u^n)_x - q(u)^n_{txx})_x + s\psi_{yy} = 0, \tag{1.2}$$

with constants $p, q, s, n \neq 0$. The given models (1.1) and (1.2) were examined in [45] where Wazwaz obtained solutions: periodic, solitons, and compactons, together with solitary patterns via the sin-cos and tan-hyperbolic techniques. Besides, the finite difference technique of Crank-Nicholson was utilized in securing approximate results [33]. Abdou in [1] invoked the extended mapping approach to fetch periodic solutions of (1.1) and (1.2). Furthermore, in [40, 41], the theory of dynamical systems via bifurcation techniques was employed in attaining various traveling waves of the models. Hoque investigated three types of rogue wave solutions by selecting dissimilar test functions [24]. Recently, consideration was given to a 3-D KP-BBM model expressed as:

$$u_{tx} + au_{xx} + b\left(uu_x\right)_x - cu_{txxx} + du_{yy} + eu_{zz} = 0, \tag{1.3}$$

with parameters:

$$a = L_x^2,\ b = \Delta/D,\ c = (DL_x/L)^2,\ d = L_y^2,\ e = L_z^2,\ \text{and}\ L = \sqrt{L_x^2 + L_y^2 + L_z^2},$$

where L_z, L_y, and L_x connote wavelengths z, y along with x path, separately. Moreover, $u(t, x, y, z)$ portrays function of involved wave amplitude, possessing t temporal coordinate alongside propagation distance labeled z, y, and x. Meanwhile, Δ stands for the wave amplitude alongside the depth of water D [52].

This equation models fluid movement on an offshore structure [42]. The Kadomtsev-Petviashvili model, whose emergence in diverse modeling scenarios has been found for non-linear dispersive waves propagating primarily in the x-axis direction, with fragile dispersive effects running in a parallel direction to the y-axis. Additionally, the movement navigates normally to the basic direction of propagation, referred to as the Kadomtsev-Petviashvili-Benjamin-Bona-Mahony model (KP-BBM). Tian [13, 52] obtained breather wave results as well as lump wave outcomes using the Hirota technique in accordance with symbolic computation. Despite the above, Liu further employed the Hirota approach to derive the first, second, and third-order rogue wave solutions to (1.3) based on a symbolic computation approach that does not require a Hirota bilinear structure [31]. Furthermore, in [19], Guo calculated auto-Bäcklund as well as non-auto-Bäcklund transformations alongside some soliton results for a generalized system of long-wave-containing-dispersion model and a two-dimensional system of the Boussinesq-Burgers model.

This research paper presents a generalized 3-D KP-BBM model to be investigated [51].

$$u_{tx} + \alpha u_y u_{xx} + \beta u_x u_{xy} - b u_{txxx} + a u_{xx} + c u_{yy} + d u_{zz} = 0, \tag{1.4}$$

with non-zero constants α, β, a, b, c, and d. Xie and Li [51] investigated a generalized 3-D KP-BBM model (1.4), in which they selected $u_x u_y$ as the non-linear convection term which can be used to connote more dispersive effects. This ensures the equation possesses more meaningful and functional characteristics than the case of two dimensions. Moreover, having learned that multiple-order rogue wave solutions comprising breathers with symmetric peaks, as well as capable of scattering into various kinds of breathers, were yet to be examined for (1.4), the authors in [51] took it up. Therefore, they searched for such a rogue wave that satisfies Equation (1.4). In addition, the authors engaged in some variable transformations to reduce the equation and then constructed its bilinear structure derived by potential transformation. Subsequently, they secured the first, second, third, and fourth-order rogue waves through symbolic computation. Furthermore, the scattering behaviors of the results are investigated. Comparing their research outcome to Hoque's [24] of the two-dimensional case, the soliton gained consists of independent breathers, obviously different from the multi-lump soliton (that is from first to third order) constructed by Hoque.

However, in this research, we employ the Lie symmetry method to identify all of the related Lie point symmetries to the model under consideration. Thus, our investigation in the paper is organized as follows: Section 1 presents the introduction and literature review of the model under study, while section 2 reveals the stepwise procedure in obtaining the symmetries of KP-BBM (1.4) and then uses them to carry out symmetry reductions. Furthermore, we obtain some non-linear differential equations (NODE), which are later solved via direct integration alongside the simplest technique to produce their analytical solutions. Additionally, in section 3, we compute the conservation laws of the understudy model using the two methods, namely Noether's theorem and the multiplier method. Thereafter, concluding remarks are provided.

2. Exact solutions of (1.4)

This section provides the closed-form solutions of (1.4) by applying Lie group theory.

2.1. **Lie point symmetries of KP-BBM model (1.4).**

Determination of Lie point symmetries related to KP-BBM (1.4) is first ensured. Thus, the group of symmetries is computed for the model via the vector field

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \psi \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial z}, \tag{2.1}$$

with functions τ, ξ, η, ψ, and ϕ dependent on t, x, y, z along-side u. Therefore, X as given in (2.1) is referred to as a point symmetry of model (1.4), if we have

$$X^{[4]}\Xi\Big|_{\Xi=0} = 0, \tag{2.2}$$

with

$$\Xi \equiv u_{tx} + \alpha u_y u_{xx} + a u_{xx} + \beta u_x u_{xy} + c u_{yy} - b u_{txxx} + d u_{zz}.$$

In this instance, $X^{[4]}$ portrays the fourth extension of X. This can be calculated by invoking the general relation

$$X^{[4]}\Xi = X + \eta^x \partial/\partial u_x + \eta^y \partial/\partial u_y + \eta^{tx} \partial/\partial u_{tx} + \eta^{xx} \partial/\partial u_{xx} + \eta^{yy} \partial/\partial u_{yy} + \eta^{zz} \partial/\partial u_{zz} + \eta^{txxx} \partial/\partial u_{txxx}. \tag{2.3}$$

where the ζs are defined as

$$\begin{aligned}
&\eta^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\psi) - u_z D_x(\phi),\\
&\eta^y = D_y(\eta) - u_t D_y(\tau) - u_x D_y(\xi) - u_y D_y(\psi) - u_z D_y(\phi),\\
&\eta^{xy} = D_y(\eta^x) - u_{tx} D_y(\tau) - u_{xx} D_y(\xi) - u_{xy} D_y(\psi) - u_{yz} D_y(\phi),\\
&\eta^{tx} = D_x(\eta^t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi) - u_{ty} D_x(\psi) - u_{tz} D_t(\phi),\\
&\eta^{xx} = D_x(\eta^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\psi) - u_{xz} D_x(\phi),\\
&\eta^{yy} = D_y(\eta^y) - u_{ty} D_y(\tau) - u_{xy} D_y(\xi) - u_{yy} D_y(\psi) - u_{yz} D_y(\phi),\\
&\eta^{zz} = D_z(\eta^z) - u_{tz} D_z(\tau) - u_{xz} D_z(\xi) - u_{yz} D_z(\psi) - u_{zz} D_z(\phi),\\
&\eta^{txxx} = D_x(\eta^{txx}) - u_{ttxx} D_x(\tau) - u_{txxx} D_x(\xi) - u_{txxy} D_x(\psi) - u_{txxz} D_x(\phi),
\end{aligned}$$

with operators D_z, D_y, D_t, and D_x connoting the complete derivatives written as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots,\\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{ty} \frac{\partial}{\partial u_t} + \cdots,\\
D_z &= \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{tz} \frac{\partial}{\partial u_t} + u_{xz} \frac{\partial}{\partial u_x} + \cdots.
\end{aligned} \tag{2.4}$$

Substituting the respective ζ's and decomposing the result on various involved-derivatives of u, furnish the determining partial differential equations (PDEs)

$$\tau_z = 0,\ \tau_x = 0,\ \tau_y = 0,\ \tau_u = 0, \tag{2.5}$$

$$\xi_z = 0,\ \xi_x = 0,\ \xi_t = 0,\ \xi_y = 0,\ \xi_u = 0, \tag{2.6}$$

$$\phi_y = 0,\ \phi_t = 0,\ \phi_x = 0,\ \phi_u = 0, \tag{2.7}$$

$$\psi_z = 0,\ \psi_t = 0,\ \psi_t = 0,\ \psi_u = 0, \tag{2.8}$$

$$\phi_{zz} = 0, \tag{2.9}$$

$$\psi_y = \phi_z, \tag{2.10}$$

$$\tau_t = 2\phi_z, \tag{2.11}$$

$$\eta_u = -\phi_z, \tag{2.12}$$

$$\eta_x = 0, \tag{2.13}$$

$$\eta_{zz} = 0, \tag{2.14}$$

$$\alpha \eta_y + 2a\, \phi_z = 0. \tag{2.15}$$

On solving the determining equations given in (2.5)–(2.15), the infinitesimals of (1.4) are calculated as

$$\begin{aligned}\tau =&2c_2 t + c_5\ \xi = c_1,\ \phi = c_2 z + c_3,\ \psi = c_2\, y + c_4,\\ \eta =& - c_2\, u - \frac{2}{\alpha} a\, c_2\, y + F(t) z + H(t).\end{aligned}$$

We invoke the vector field (2.1) to determine Lie point symmetries of Equation (1.4)

$$\begin{aligned}&X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial z},\ X_5 = 2\alpha t \frac{\partial}{\partial t} + \alpha y \frac{\partial}{\partial y} + \alpha z \frac{\partial}{\partial z} - (2a\, y + \alpha u)\frac{\partial}{\partial u},\\ &X_H = H(t)\frac{\partial}{\partial u},\ X_F = zF(t)\frac{\partial}{\partial u}.\end{aligned} \tag{2.16}$$

2.2. **Symmetry reductions and solutions of** (1.4)**.** We utilize the secured Lie point symmetries to gain various exact solutions of (1.4) through symmetry reduction process.

2.2.1. *Reductions using* $G_1 = X_1 + X_H$. We first take into account the symmetry $G_1 = X_1 + X_H$. We solve its Lagrangian system and obtain group invariant calculated as

$$u(t, x, y, z) = Q(\xi, \eta, \zeta) + \int H(t) dt,\ \text{where}\ \xi = x,\ \eta = y,\ \zeta = z.$$

Inserting the value of u in (1.4), we achieve a NLPDE computed as

$$aQ_{\xi\xi} + cQ_{\eta\eta} + \alpha Q_\eta Q_{\xi\xi} + \beta Q_\xi Q_{\xi\eta} + dQ_{\zeta\zeta} = 0. \tag{2.17}$$

We secure a solution of KP-BBM (1.4) via (2.17) as

$$u(t, x, y, z) = A_3 + A_4 (A_0 x + A_1 y + A_2 z + A_3) + \int H(t) dt, \tag{2.18}$$

where $A_j = j = 0, \ldots, 4$ are arbitrary constants. The symmetries of (2.17) are

$$\begin{aligned}&J_1 = \frac{\partial}{\partial \xi},\ J_2 = \frac{\partial}{\partial \eta},\ J_3 = \frac{\partial}{\partial \zeta},\ J_4 = \zeta \frac{\partial}{\partial Q},\ J_5 = \frac{\partial}{\partial Q},\\ &J_6 = \xi \frac{\partial}{\partial \xi} + \left(\frac{2a}{\alpha}\eta + 2Q\right)\frac{\partial}{\partial Q},\ J_7 = \eta \frac{\partial}{\partial \eta} + \zeta \frac{\partial}{\partial \zeta} - \left(\frac{2a}{\alpha}\eta + Q\right)\frac{\partial}{\partial Q}.\end{aligned}$$

From symmetry generator J_1, we have the invariant $Q(\xi, \eta, \zeta) = E(r, s)$, $r = \eta$ and $s = \zeta$. Using the results, we reduce (2.17) to $cE_{rr} + dE_{ss} = 0$ and it solves to give

$$u(t, x, y, z) = f_1\left(z - \sqrt{\frac{-d}{c}}\, y\right) + f_2\left(z + \sqrt{\frac{-d}{c}}\, y\right) + \int H(t) dt. \tag{2.19}$$

Repeating the same process for J_2, we have invariant $Q(\xi, \eta, \zeta) = E(r, s)$, $r = \xi$ and $s = \zeta$ that yields a similar equation $aE_{rr} + dE_{ss} = 0$ which when solved gives

$$u(t, x, y, z) = f_1\left(z - \sqrt{\frac{-d}{a}}\, x\right) + f_2\left(z + \sqrt{\frac{-d}{a}}\, x\right) + \int H(t) dt. \tag{2.20}$$

Next, we engage J_3 and secure $Q(\xi, \eta, \zeta) = E(r, s)$, $r = \xi$ and $s = \eta$. Eventually, we transform Equation (2.17) further to NLPDE

$$aE_{rr} + cE_{ss} + \alpha E_s E_{rr} + \beta E_r E_{rs} = 0. \tag{2.21}$$

The equation then occasions a solution of KP-BBM (1.4) as

$$u(t, x, y, z) = C_2 + C_4 (C_0 x + C_1 y + C_2) + \int H(t) dt, \tag{2.22}$$

with arbitrary constants $C_k, k = 0, 1, 2, 3$. Lie symmetries admitted by (2.21) are

$$L_1 = \frac{\partial}{\partial s},\ L_2 = \frac{\partial}{\partial r},\ L_3 = \frac{\partial}{\partial E},\ L_4 = s\frac{\partial}{\partial s} - \left(\frac{2a}{\alpha}s + E\right)\frac{\partial}{\partial E},$$
$$L_5 = r\frac{\partial}{\partial r} + 2\left(\frac{a}{\alpha}s + E\right)\frac{\partial}{\partial E}.$$

We first take a linear combination of $L_1 - L_3$, whose characteristic equations solve to secure $E(r, s) = W(p) + r$, $p = s - r$. Inserting the result into (2.21), we have

$$aW''(p) + cW''(p) - \beta W''(p) + \alpha W'(p)W''(p) + \beta W'(p)W''(p) = 0. \tag{2.23}$$

Solving the nonlinear ordinary differential equation (NORDE) (2.23), we get

$$u_1 = A_0 + x - \frac{1}{\alpha+\beta}(a + c - \beta)(y - x) + \int H(t)dt, \tag{2.24}$$
$$u_2 = B_0 + x + \frac{1}{\alpha+\beta}\left\{\beta - a - c + \sqrt{\beta^2 + a^2 + c^2 - 2\alpha B_2 + \Omega_0}\right\}(y - x) + \int H(t)dt, \tag{2.25}$$
$$u_3 = C_0 + x - \frac{1}{\alpha+\beta}\left\{a - \beta + c + \sqrt{\beta^2 + a^2 + c^2 - 2\alpha B_2 + \Omega_0}\right\}(y - x) + \int H(t)dt, \tag{2.26}$$

which are the solutions of (1.4), with $\Omega_0 = 2ac - 2\beta B_2 - 2a\beta - 2\beta c$ as well as A_0, B_0, and C_0 arbitrary constants. Now, we make do with L_4 and so we have invariant $E(r, s) = \frac{1}{\alpha s}(\alpha W(p) + r) - as^2$, where $p = r$. Putting this in (2.21) purveys NORDE

$$2cW(p) - \beta W'(p)^2 - \alpha W(p)W''(p) = 0, \tag{2.27}$$

which solves to give the integral relation which we present as

$$\int_0^{W(p)} \pm\frac{\alpha + 2\beta}{\sqrt{(\alpha + 2\beta)\left\{4cw + \exp\left(\frac{1}{\alpha}\left[\ln\left(w^{-2\beta}C_0\right) + 2i\pi\right]\right)\right\}}}dw - p - C_1 = 0, \tag{2.28}$$

where constants C_0 and C_1 are arbitrary. Lastly under J_3, we consider L_5 which furnishes $E(r, s) = \frac{1}{\alpha}\left(\alpha r^2 W(p) - as\right)$, $p = s$. Thus we have a transformed NORDE

$$cW''(p) + 2\alpha W(p)W'(p) + 4\beta W(p)W'(p) = 0, \tag{2.29}$$

which solves to produce a solution of Equation (1.4) as

$$u(t, x, y, z) = x^2\left\{\frac{\sqrt{c(\alpha + 2\beta)C_0}}{(\alpha + 2\beta)}\tanh\left(\frac{y + C_1}{c}\sqrt{c(\alpha + 2\beta)C_0}\right)\right\} - \frac{a}{\alpha}y + \int H(t)dt, \tag{2.30}$$

where constants C_0 and C_1 are arbitrary. On invoking $J = J_1 + J_2 + \cdots + J_5$ and adopting the usual procedure, we have a more condensed NLPDE furnished as

$$\begin{aligned}&aE_{rr} - \beta E_{rr} + cE_{rr} + \alpha E_r E_{rr} + \beta E_r E_{rr} + 2\alpha E_r E_{rs} + \beta E_r E_{rs} + \alpha E_r E_{ss} + dE_{ss}\\ &+ \beta E_s E_{rr} - \beta s E_{rr} + \beta E_s E_{rs} - \beta s E_{rs} - \alpha E_r + 2aE_{rs} - \beta E_{rs} + aE_{ss} - a = 0.\end{aligned} \tag{2.31}$$

Equation (2.31) produces the four Lie symmetries which we present as

$$I_1 = \frac{\partial}{\partial r},\ I_2 = \frac{\partial}{\partial E},\ I_3 = \frac{\partial}{\partial s} + s\frac{\partial}{\partial E},\ I_4 = \left(\frac{2}{3}r + \frac{1}{3}s\right)\frac{\partial}{\partial r}$$
$$+ s\frac{\partial}{\partial s} + \frac{1}{3\alpha\beta}\left(\beta\left((s^2 - s + 4E)\alpha + 2(r - s)(a + d)\right) - 2\alpha ds\right)\frac{\partial}{\partial E}.$$

We linearly combine Lie generators I_1, I_2 and I_3. So we secure invariant

$$E(r, s) = -\frac{1}{2}r^2 + rs + r + W(p),\ \text{with}\ p = s - r. \tag{2.32}$$

On using (2.32) in (2.31), we achieve an ordinary differential equation (ORDE) $cW''(p) + dW''(p) - c = 0$. Solving the equation gives a solution of (1.4) as

$$u(t,x,y,z) = \frac{1}{2(c+d)}\Big\{d\left((2(z-x) - 2C_1 + 2)(y-x) - (y-x)^2 + 2C_1(z-x)\right.$$
$$\left.+2C_2\right) - 2c\left[(C_1-1)(y-x) - \frac{1}{2}(z-x)^2 - C_1(z-x) - C_2\right]\Big\}$$
$$-\frac{1}{2}x^2 + xz + x + \int H(t)dt. \tag{2.33}$$

Now, we engage I_4 and consequently achieve the invariant

$$Q(\xi,\eta,\zeta) = \frac{1}{\alpha\beta}\Big\{2\alpha\beta s^{4/3}W(p) + \alpha\beta s^2 - 2a\beta r + 2a\beta s + 2\alpha\beta s$$
$$+ 4\alpha ds - 2\beta dr + 2\beta ds\Big\}, \text{ where } p = \frac{r-s}{s^{2/3}}. \tag{2.34}$$

Hence, using the achieved relation (2.34) in Equation (2.31), we gain

$$9cW''(p) + 9dW''(p) + 4\alpha p^2W'(p)W''(p) + 4\beta p^2W'(p)W''(p) - 6\alpha pW'(p)^2$$
$$-4\beta pW'(p)^2 - 8\beta pW(p)W''(p) + 4\alpha pW(p)W'(p) + 8\beta W(p)W'(p) + 9d = 0. \tag{2.35}$$

It is now the turn of J_6 whose invariant gives $Q(\xi,\eta,\zeta) = \frac{1}{\alpha}\left(\alpha\xi^2E(r,s) - a\eta\right)$, $r=\eta$ and $s=\zeta$. On lodging the new function in (2.17) one achieves the reduced NLPDE

$$cE_{rr} + dE_{ss} + 2\alpha EE_r + 4\beta EE_r = 0. \tag{2.36}$$

In this case, we secure a solution of (1.4) in this regard as

$$u(t,x,y,z) = x^2\left\{\frac{cB_1^2 + dB_2^2}{B_1(\alpha+2\beta)}\tanh\left(B_1y + B_2z + B_0\right)\right\} - \frac{a}{\alpha}y + \int H(t)dt, \tag{2.37}$$

where constants B_0 and B_1 are arbitrary. Besides, (2.36) admits Lie symmetries

$$L_1 = \frac{\partial}{\partial r},\ L_2 = \frac{\partial}{\partial s},\ L_3 = s\frac{\partial}{\partial s} + r\frac{\partial}{\partial r} - E\frac{\partial}{\partial E}.$$

Exploring $L = c_0L_1 + c_1L_2$ furnishes $E(r,s) = W(p)$, where $p = s - (c_1/c_0)\,r$. Putting the new function into (2.36) we secure the second order NORDE

$$2\alpha c_0c_1W(p)W'(p) + 4\beta c_0c_1W(p)W'(p) - cc_1^2W''(p) - c_0^2dW''(p) = 0. \tag{2.38}$$

Consequently, solution to KP-BBM (1.4) in this instance purveys

$$u(t,x,y,z) = x^2\left\{\frac{\Delta_0}{c_0c_1(\alpha+2\beta)}\tan\left(\frac{\Delta_0}{(cc_1^2+dc_0^2)}\left[A_0 + z - (c_1/c_0)\,y\right]\right)\right\} - \frac{a}{\alpha}y + \int H(t)dt, \tag{2.39}$$

with $\Delta_0 = \sqrt{c_0c_1A_1(\alpha+2\beta)(cc_1^2+dc_0^2)}$ and arbitrary constants A_0 and A_1. In the case of L_3, we have invariant $E(r,s) = \frac{1}{r}W(p)$, where $p = s/r$. Substituting the new relation in (2.36), we compute a reduced form of Equation (1.4) as

$$4cpW'(p) + 2cW(p) + dW''(p) - 2\alpha pW(p)W'(p) - 4\beta pW(p)W'(p) + cp^2W''(p) - 2\alpha W'(p)^2 - 4\beta W'(p)^2 = 0. \tag{2.40}$$

Finally, in the case of J_7, we have $Q(\xi,\eta,\zeta) = \frac{1}{\alpha\eta}\left(\alpha E(r,s) - a\eta^2\right)$, $r=\eta$ and $s=\zeta/\eta$. Putting the current function in (2.36), one achieves the NLPDE

$$2cE + dE_{ss} - \beta E_r^2 - \alpha sE_sE_{rr} - \beta sE_rE_{rs} + 4csE_s + cs^2E_{ss} - \alpha EE_{rr} = 0. \tag{2.41}$$

In this instance, we achieve the solution to KP-BBM (1.4) as

$$u(t,x,y,z) = \frac{1}{y}\left\{f_1(x)f_2\left(\frac{z}{y}\right)\right\} - \frac{a}{\alpha}y + \int H(t)dt. \tag{2.42}$$

Additionally, one secures two symmetries of (2.41) as $I_1 = \partial_r$ and $I_2 = r\partial_r + 2E\partial_E$. Combining the two symmetry generators linearly gives the invariant $E(r,s) = (1+r)^2 W(p)$, where $p = s$. Using the relation reduces (2.41) to NORDE

$$\begin{aligned} &4\beta W'(p)^2 + 2\alpha p W(p)W'(p) - 4cpW'(p) - dW''(p) + 4\beta p W(p)W'(p) \\ &- 2cW(p) - cp^2W''(p) + 2\alpha W'(p)^2 = 0. \end{aligned} \tag{2.43}$$

2.2.2. *Reductions of* (1.4) *using* $G_2 = X_2 + X_F$. Now, we explore $G_2 = X_2 + X_F$ in order to reduce (1.4) and so G_2 gives invariant

$$u(t,x,y,z) = xzF(t) + Q(\xi,\eta,\zeta), \text{ where } \xi = t,\ \eta = y,\ \zeta = z.$$

Inserting the new relation in KP-BBM (1.4) gives the partial differential equation (PADE)

$$cQ_{\eta\eta} + dQ_{\zeta\zeta} + \zeta F(t) = 0. \tag{2.44}$$

Consequently, we obtain two possible solutions of KP-BBM (1.4) as

$$u(t,x,y,z) = f_1(t) + f_2(y) + f_3(z) + xzF(t), \tag{2.45a}$$

$$u(t,x,y,z) = \frac{1}{cdA_0}\left\{cd\,H_1\left(\frac{1}{2}y - \frac{cz}{2\sqrt{-cd}}\right) + cd\,H_2\left(z + \frac{\sqrt{-cd}}{c}y\right) - \frac{1}{8}A_0\left(cz^2 + dy^2\right)z\right\}F'(t) + xzF(t),$$

where constant A_0 as well as functions $f_1(t)$, $f_2(t)$, $f_3(z)$, $H_1(t)$, and $H_2(t)$ are arbitrary with the functions depending on their various arguments.

2.2.3. *Reductions of* (1.4) *using* $G_3 = X_3 + X_H$. In this part of the reduction task, we compute the invariant of $G_3 = X_3 + X_H$ as

$$u(t,x,y,z) = yH(t) + Q(\xi,\eta,\zeta), \text{ where } \xi = t,\ \eta = x,\ \zeta = z.$$

Inserting the new value of $u(t,x,y,z)$ in (1.4) furnishes equation

$$Q_{\xi\eta} + \alpha H(t)Q_{\eta\eta} - bQ_{\xi\eta\eta\eta} + aQ_{\eta\eta} + dQ_{\zeta\zeta} = 0. \tag{2.46}$$

Eventually, one succeeds in obtaining two solutions of KP-BBM (1.4) here as

$$u(t,x,y,z) = f_1(t) + f_2(z) - \frac{1}{2}x\left(2\alpha c_1\int H(t)dt + 2ac_1t + 2dc_2t - c_1x - 2A_1 - 2A_2\right), \tag{2.47a}$$

$$u(t,x,y,z) = H(t)y + f_3(t) + \frac{1}{2}c_3x^2 + (B_1 - ac_3t - dc_4t)\,x + \frac{1}{2}c_4z^2 + B_1z + B_2(x+1) - \alpha c_3x\int H(t)dt,$$

where constants $c_j, j = 1,2,3,4$ and $A_i, B_i, i = 1,2$ as well as functions $f_k, k = 1,2,3$, depending on their arguments are all arbitrary. Now in order to have more interesting solutions of (1.4), we take a special case of (2.46) with $H(t) = 1$. Therefore, we obtain four symmetries of model (1.4) which are explicated as

$$J_1 = \frac{\partial}{\partial\xi},\ J_2 = \frac{\partial}{\partial\eta},\ J_3 = \frac{\partial}{\partial\zeta},\ J_4 = Q\frac{\partial}{\partial Q}.$$

We take a look at J_1 and so we achieve invariant $Q(\xi,\eta,\zeta) = E(r,s)$, $r = \eta$, and $s = \zeta$. Plugging the new function in (2.46) gives $(a+\alpha)\,E_{rr} + dE_{ss} = 0$, yielding

$$u(t,x,y,z) = y + f_1\left(z - \sqrt{\frac{-d}{a+\alpha}}\,x\right) + f_2\left(z + \sqrt{\frac{-d}{a+\alpha}}\,x\right), \tag{2.48}$$

with arbitrary functions f_1 and f_2. In the case of J_2, $Q(\xi,\eta,\zeta) = E(r,s)$, $r = \xi$, and $s = \zeta$ which reduces (1.4) to $E_{ss} = 0$. Thus, we have a solution of KP-BBM (1.4) as

$$u(t,x,y,z) = y + zf_3(t) + f_4(t), \tag{2.49}$$

where $f_3(t)$ and $f_4(t)$ are arbitrary functions of t. Next, we look at J_3. Hence, we have $Q(\xi,\eta,\zeta) = E(r,s)$, $r = \xi$, and $s = \eta$. Invoking the new function in (2.46) yields the NLPDE expressed as

$$E_{rs} + \alpha E_{ss} - bE_{rsss} + aE_{ss} = 0. \tag{2.50}$$

We secure a solution of KP-BBM model (1.4) in this instance as

$$\begin{aligned} u(t,x,y,z) = \; & y + f_5(t) + \frac{1}{\Delta_1}\Bigg\{ C_0C_1 \exp\left(c_1 t + \left[\frac{1}{2bc_1}\sqrt{4bc_1^2 + \alpha^2 + 2a\alpha + \alpha^2} \; + \frac{a}{2bc_1} + \frac{\alpha}{2bc_1}\right] x \right) \\ & + C_0C_2 \exp\left(c_1 t - \left[\frac{1}{2bc_1}\sqrt{4bc_1^2 + \alpha^2 + 2a\alpha + \alpha^2} \; + \frac{a}{2bc_1} + \frac{\alpha}{2bc_1}\right] x \right) \Bigg\}, \end{aligned} \tag{2.51}$$

with $\Delta_1 = \frac{a}{2bc_1} + \frac{\alpha}{2bc_1} + \frac{1}{2bc_1}\sqrt{4bc_1^2 + \alpha^2 + 2a\alpha + \alpha^2}$ and arbitrary constants c_1, C_0, C_1, C_2 as well as function $f_5(t)$ which is depending on t. We consider symmetry $L = c_0\partial_r + c_1\partial_s$ of Equation (2.50) which yields invariant $E(r,s) = W(p)$, where $p = s - (c_1/c_0)\,r$. Inserting the new function into (2.50) we secure the second order NORDE

$$\alpha c_0 W''(p) + ac_0 W''(p) + bc_1 W''''(p) - c_1 W''(p) = 0. \tag{2.52}$$

Solving the obtained Equation (2.52), we achieve a solution of (1.4) as

$$\begin{aligned} u(t,x,y,z) = \; & A_0 \sin\left(\sqrt{\frac{ac_0 + \alpha c_0 - c_1}{bc_1}} \left[x - \left(\frac{c_1}{c_0}\right) t \right] \right) + A_1 \cos\left(\sqrt{\frac{ac_0 + \alpha c_0 - c_1}{bc_1}} \right. \\ & \times \left. \left[x - \left(\frac{c_1}{c_0}\right) t \right] \right) + A_2 \left[x - \left(\frac{c_1}{c_0}\right) t \right] + y + A_3, \end{aligned} \tag{2.53}$$

where constants $A_j, j = 0,1,2,3$ are arbitrary. We contemplate linear combinations of $J_1, \ldots, J_4$ as $J = a_0J_1 + a_1J_2 + a_2J_3 + J_4$, thereby furnishing the invariant

$$Q(\xi,\eta,\zeta) = \exp\left(\frac{1}{a_0}\xi\right) E(r,s), \;\; r = \eta - \left(\frac{a_1}{a_0}\right)\xi, \;\text{ and } s = \zeta - \left(\frac{a_2}{a_0}\right)\xi. \tag{2.54}$$

Exchanging the function with that obtainable in (2.46) produces

$$E_r + a_0 dE_{ss} + aa_0E_{rr} + \alpha a_0 E_{rr} - a_1E_{rr} - a_2E_{rs} - bE_{rrr} + a_1bE_{rrrr} + a_2bE_{rrrs} = 0. \tag{2.55}$$

Application of Lie theoretic approach to (2.55) purveys four Lie symmetries

$$I_1 = \frac{\partial}{\partial r}, \;\; I_2 = \frac{\partial}{\partial s}, \;\; I_3 = \frac{\partial}{\partial E}, \;\; I_4 = E\frac{\partial}{\partial E}.$$

From symmetry generators I_1, I_2, and I_3, we obtain $E(r,s) = \exp\left(\frac{1}{c_0}r\right)W(p)$, where $p = s - \left(\frac{c_1}{c_0}\right) r$ and the obtained result further reduces (2.55) to equation

$$\begin{aligned} & aa_0c_0^2W(p) + \alpha a_0c_0^2W(p) - a_1c_0^2c_1^2W''(p) + a_2c_0^3c_1W''(p) - 4a_1bc_1^3W'''(p) \\ & + 6ba_1c_1^2W''(p) - 3bc_0c_1^2W''(p) + 2a_1c_0^2c_1W'(p) - 4ba_1c_1W'(p) + ba_2c_0W'(p) \\ & + 3bc_0c_1W'(p) + da_0c_0^4W''(p) + ba_1c_1^4W''''(p) - a_2c_0^3W'(p) - c_0^3c_1W'(p) \\ & - a_1c_0^2W(p) + a_1bW(p) - bc_0W(p) + c_0^3W(p) + aa_0c_0^2c_1^2W''(p) + \alpha a_0c_0^2c_1^2W''(p) \\ & - a_2bc_0c_1^3W''''(p) - 2aa_0c_0^2c_1W'(p) - 2\alpha a_0c_0^2c_1W'(p) + 3a_2bc_0c_1^2W'''(p) \\ & + bc_0c_1^3W'''(p) - 3a_2bc_0c_1W''(p) = 0. \end{aligned} \tag{2.56}$$

Finally, we consider $I = \nu_0I_1 + \nu_1I_2 + I_3 + I_4$, and so we have invariant $E(r,s) = \exp\left(\frac{1}{\nu_0}r\right)W(p) - 1$, where $p = s - \left(\frac{\nu_1}{\nu_0}\right) r$. Using the relation reduces Equation (2.55) to NORDE

$$\begin{aligned} & aa_0\nu_0^2W(p) + \alpha a_0\nu_0^2W(p) - a_1\nu_0^2\nu_1^2W''(p) + a_2\nu_0^3\nu_1W''(p) - 4a_1b\nu_1^3W'''(p) \\ & + 6ba_1\nu_1^2W''(p) - 3b\nu_0\nu_1^2W''(p) + 2a_1\nu_0^2\nu_1W'(p) - 4ba_1\nu_1W'(p) + ba_2\nu_0W'(p) \end{aligned}$$

$$
\begin{aligned}
&+3b\nu_0\nu_1W'(p)+da_0\nu_0^4W''(p)+ba_1\nu_1^4W''''(p)-a_2\nu_0^3W'(p)-\nu_0^3\nu_1W'(p)\\
&-a_1\nu_0^2W(p)+a_1bW(p)-b\nu_0W(p)+\nu_0^3W(p)+aa_0\nu_0^2\nu_1^2W''(p)+\alpha a_0\nu_0^2\nu_1^2W''(p)\\
&-a_2b\nu_0\nu_1^3W''''(p)-2aa_0\nu_0^2\nu_1W'(p)-2\alpha a_0\nu_0^2\nu_1W'(p)+3a_2b\nu_0\nu_1^2W'''(p)\\
&+b\nu_0\nu_1^3W'''(p)-3a_2b\nu_0\nu_1W''(p)=0. \qquad (2.57)
\end{aligned}
$$

2.2.4. *Reductions of* (1.4) *using* $G_4=X_4+X_F$. Now, we focus on symmetry generator $G_4=X_4+X_F$. We then solve the related characteristic equation and get the mathematical relation

$$u(t,x,y,z)=Q(\xi,\eta,\zeta)+\frac{1}{2}z^2F(t),\ \text{where } \xi=t,\ \eta=x,\ \zeta=y.$$

Substituting the value of u here in (1.4), we achieve a NLPDE calculated as

$$Q_{\xi\eta}+aQ_{\eta\eta}+cQ_{\zeta\zeta}+\alpha Q_\zeta Q_{\eta\eta}+\beta Q_\eta Q_{\eta\zeta}-bQ_{\xi\eta\eta\eta}+dF(t)=0. \qquad (2.58)$$

Thus, we obtain two solutions of KP-BBM (1.4) via (2.58) as

$$u(t,x,y,z)=\ f_1(x)-\frac{c}{\alpha c_2}\exp\left(-\frac{\alpha c_2}{c}y\right)f_2(t)-\frac{d}{\alpha c_2}yF(t)-\frac{a}{\alpha}y+\frac{1}{2}z^2F(t)+f_3(t), \qquad (2.59a)$$

$$
\begin{aligned}
u(t,x,y,z)=\ &\frac{1}{2\alpha c_2}\left\{\left(\alpha c_2z^2-2dy\right)F(t)-2c_2\left[ay-\alpha f_4(t)-\frac{1}{2}\alpha c_2x^2\ -\alpha\left(C_1x+C_2\right)\right]\right.\\
&\left.-\ 2c\exp\left(-\frac{\alpha c_2}{c}y\right)f_5(t)\right\}, \qquad (2.59b)
\end{aligned}
$$

with constant C_1,C_2, and c_2 as well as $f_1(x)$ and $f_j(t), j=2,\dots,5$, arbitrary functions depending on x and t respectively. Moreover, in order to obtain more results of interest for (1.4) using G_4, we adopt the same assumption as earlier done in the case of G_3. As a result, we have three symmetry generators given as

$$J_1=\frac{\partial}{\partial\xi}+F_1(\xi)\frac{\partial}{\partial Q},\ J_2=\frac{\partial}{\partial\eta}+F_2(\xi)\frac{\partial}{\partial Q},\ J_3=\frac{\partial}{\partial\zeta}+F_3(\xi)\frac{\partial}{\partial Q}.$$

We take $F_i(\xi)=0,i,1,2,3$ and so, following the usual steps we obtain for J_1, invariant $Q(\xi,\eta,\zeta)=E(r,s)$, $r=\eta$ and $s=\zeta$, which reduces NLPDE (2.58) to

$$aE_{rr}+cE_{ss}+\alpha E_sE_{rr}+\beta E_rE_{rs}+d=0. \qquad (2.60)$$

Consequently, we secure a solution of (1.4) in this situation as

$$u(t,x,y,z)=\frac{1}{2}z^2+f_1(x)+f_2(y). \qquad (2.61)$$

Besides, we compute the symmetries admitted by (2.60) as

$$L_1=\frac{\partial}{\partial r},\ L_2=\frac{\partial}{\partial s},\ L_3=\frac{\partial}{\partial E},\ \text{and } L_4=\frac{3}{2}r\frac{\partial}{\partial r}+s\frac{\partial}{\partial s}+\left(\frac{a}{\alpha}s+2E\right)\frac{\partial}{\partial E}.$$

We take a look at $L=e_1L_1+e_2L_2+L_3$ and so we achieve invariant $E(r,s)=\frac{1}{e_1}\left(e_1W(p)+r\right)$, where $p=s-\left(\frac{e_2}{e_1}\right)r$. Substituting the function in (2.60) purveys

$$ae_2^2W''(p)+ce_1^2W''(p)-\beta e_2W''(p)+\alpha e_2^2W'(p)W''(p)+\beta e_2^2W'(p)W''(p)+de_1^2=0. \qquad (2.62)$$

Integration of Equation (2.62) yields the relation

$$\int\mp\frac{1}{e_2^2\left(\alpha+\beta\right)}\left\{\beta e_2-ae_2^2-ce_2^2+\sqrt{\beta^2e_2^2-2\beta ce_1^2e_2-2\beta de_1^2e_2^2p+\Omega_1}\right\}dp+A_0, \qquad (2.63)$$

where both A_0 and A_1 are integration constants with $\Omega_1=c^2e_1^4-2a\beta e_2^3+2ace_1^2e_2^2+a^2e_2^4-2\alpha de_1^2e_2^2p-2A_1\beta de_1^2e_2^2-2A_1\alpha de_1^2e_2^2$. Now, we explore L_4 and as we usual do, following a certain known procedure, we obtain $E(r,s)=\frac{1}{\alpha}\left(\alpha r^{4/3}W(p)-as\right)$, where $p=\frac{s}{r^{2/3}}$. Invoking the new result in (2.60) gives us the second order NORDE

$$9cW''(p)+8\beta W(p)W'(p)+4\alpha W(p)W'(p)-4\beta pW'(p)^2-6\alpha pW'(p)^2$$

$$- 8\beta pW(p)W''(p) + 4\beta p^2W'(p)W''(p) + 4\alpha p^2W'(p)W''(p) + 9d = 0. \tag{2.64}$$

Next, we do J_2 so $Q(\xi,\eta,\zeta) = E(r,s)$, $r = \xi$ and $s = \zeta$, thus reducing NLPDE (2.58) to $cE_{ss} + d = 0$. We solve the equation and get a solution of (1.4) as

$$u(t,x,y,z) = \frac{1}{2}z^2 - \frac{d}{2c}y^2 + f_1(t)y + f_2(t). \tag{2.65}$$

For J_3, we have $Q(\xi,\eta,\zeta) = E(r,s)$, $r = \xi$ and $s = \eta$, thus reducing (1.4) to

$$E_{rs} - bE_{rsss} + aE_{ss} + d = 0. \tag{2.66}$$

We solve the linear partial differential equation (LNPADE) (2.66) and get

$$\begin{aligned} u(t,x,y,z) = {}& \frac{1}{2}z^2 + f(t) + \frac{1}{\Delta_2}\left\{B_0B_1 \exp\left(c_1t + \left[\frac{a}{2bc_1} + \frac{1}{2bc_1}\sqrt{a^2+4bc_1^2}\right]x\right)\right. \\ &\left. + B_0B_2 \exp\left(c_1t + \left[\frac{a}{2bc_1} - \frac{1}{2bc_1}\sqrt{a^2+4bc_1^2}\right]x\right)\right\} - \frac{d}{2a}x^2 - \frac{dB_1}{aB_0}x, \end{aligned} \tag{2.67}$$

with $\Delta_2 = \frac{a}{2bc_1} + \frac{1}{2bc_1}\sqrt{a^2+4bc_1^2}$, arbitrary constants c_1, B_0, B_1, B_2 and function $f(t)$ which is depending on t. Further, (2.66) admits the symmetries calculated as

$$I_1 = \frac{\partial}{\partial r},\ I_2 = \frac{\partial}{\partial s},\ \text{and } I_3 = \left(\frac{d}{2a}s^2 + E\right)\frac{\partial}{\partial E}.$$

Firstly, we do $I = e_0I_1 + e_1I_2$ so we get $E(r,s) = W(p)$, where $p = s - \left(\frac{e_1}{e_0}\right)r$. Substituting the expression of $E(r,s)$ in (2.66), we have the fourth order ORDE

$$ae_0W''(p) - e_1W''(p) + be_1W''''(p) + de_0 = 0, \tag{2.68}$$

which solves to yield a solution of KP-BBM (1.4) as

$$\begin{aligned} u(t,x,y,z) = {}& \frac{1}{2}z^2 - \frac{1}{ae_0 - e_1}\left\{C_0be_1 \sin\left(\sqrt{\frac{ae_0-e_1}{be_1}}\left[x - \left(\frac{e_1}{e_0}\right)t\right]\right)\right. \\ &\left. + C_1be_1\cos\left(\sqrt{\frac{ae_0-e_1}{be_1}}\left[x - \left(\frac{e_1}{e_0}\right)t\right]\right) + \frac{de_0}{2}C_2\left[x - \left(\frac{e_1}{e_0}\right)t\right]\right\} \\ & + C_3\left[x - \left(\frac{e_1}{e_0}\right)t\right] + C_4, \end{aligned} \tag{2.69}$$

where constants $C_j, j = 0,1,2,3,4$ are arbitrary. Now, we combine the three symmetries and obtain an invariant $E(r,s) = \frac{1}{2a}\left(2ae^rW(p) - ds^2 - 2ds - 2d\right)$, where $p = s - r$. Invoking the new function in (2.66), gives the linear ORDE (LORDE)

$$W'(p) - W''(p) + aW''(p) - bW'''(p) + bW''''(p) = 0. \tag{2.70}$$

Now, we consider $J = J_1 + J_2 + J_3$ so we have invariant $Q(\xi,\eta,\zeta) = E(r,s)$, $r = \eta - \left(\frac{a_1}{a_0}\right)\xi$ and $s = \zeta - \left(\frac{a_2}{a_0}\right)\xi$, which transforms NLPDE (2.58) to

$$a_0cE_{ss} - a_1E_{rr} - a_2E_{rs} + aa_0E_{rr} + \alpha a_0E_sE_{rr} + \beta a_0E_rE_{rs} + a_1bE_{rrrr} + a_2bE_{rrrs} + a_0d = 0. \tag{2.71}$$

Therefore, we calculate the translation symmetries admitted by (2.71) as

$$L_1 = \frac{\partial}{\partial r},\ L_2 = \frac{\partial}{\partial s},\ \text{and } L_3 = \frac{\partial}{\partial E}.$$

We consider $L = e_0L_1 + e_1L_2 + L_3$ so we achieve invariant $E(r,s) = \frac{1}{e_0}\left(e_0W(p) + r\right)$, where $p = s - \left(\frac{e_1}{e_0}\right)r$. Putting the function in (2.71) gives

$$a_2e_0^3e_1W''(p) - a_1e_0^2e_1^2W''(p) - a_0e_0^2e_1\beta W''(p) + \alpha a_0e_0^2e_1^2W'(p)W''(p)$$

$$+ aa_0e_0^2e_1^2W''(p) + ca_0e_0^4W''(p) + a_1be_1^4W''''(p) - a_2be_0e_1^3W''''(p)$$
$$+ \beta a_0e_0^2e_1^2W'(p)W''(p) + a_0e_0^4d = 0. \tag{2.72}$$

2.2.5. *Reductions of* (1.4) *using* $G_5 = X_5$. Finally, we explore $G_5 = X_5$ in order to reduce (1.4) so we have invariant

$$u(t,x,y,z) = \frac{1}{\alpha\sqrt{t}}\left(\alpha\, Q(\xi,\eta,\zeta) - ay\sqrt{t}\right), \text{ where } \xi = x,\ \eta = \frac{y}{\sqrt{t}},\ \zeta = \frac{z}{\sqrt{t}}.$$

Inserting the new relation in KP-BBM (1.4) gives the PADE

$$2cQ_{\eta\eta} + 2dQ_{\zeta\zeta} - \zeta Q_{\xi\zeta} - Q_\xi + 2\alpha Q_\eta Q_{\xi\xi} + 2\beta Q_\xi Q_{\xi\eta} - \eta Q_{\xi\eta} + bQ_{\xi\xi\xi} + b\zeta Q_{\xi\xi\xi\zeta} + b\eta Q_{\xi\xi\xi\eta} = 0. \tag{2.73}$$

Thus, Lie symmetry analysis of NLPDE (2.73) yields three symmetries

$$J_1 = \frac{\partial}{\partial\xi},\ J_2 = \frac{\partial}{\partial\eta},\ J_3 = \frac{\partial}{\partial Q}.$$

We examine J_1 and get invariant $Q(\xi,\eta,\zeta) = E(r,s)$, $r = \eta$ and $s = \zeta$, which reduces (1.4) to $2cE_{rr} + 2dE_{ss} = 0$. Solving the PADE yields a solution of (1.4) as

$$u(t,x,y,z) = \frac{1}{\alpha\sqrt{t}}\left\{\alpha\left[f_1\left(\frac{z}{\sqrt{t}} - i\sqrt{\frac{d}{ct}}y\right) + f_2\left(\frac{z}{\sqrt{t}} + \sqrt{\frac{-d}{ct}}y\right)\right] - ay\sqrt{t}\right\}, \tag{2.74}$$

where f_1 and f_2 are arbitrary functions. We study linear combination of the translation symmetries and get $Q(\xi,\eta,\zeta) = E(r,s) + \xi\zeta + \xi$, $r = \eta$ and $s = \zeta$. On applying the function in (1.4), we have $2cE_{rr} + 2dE_{ss} - 2s - 1 = 0$, which solves to give the solution of (1.4)

$$\begin{aligned} u(t,x,y,z) = \frac{1}{\sqrt{t}}&\left\{f_1\left(\frac{1}{2}\frac{y}{\sqrt{t}} - \frac{c}{2\sqrt{-cd}}\frac{z}{\sqrt{t}}\right) + f_2\left(\frac{z}{\sqrt{t}} + \frac{1}{c}\sqrt{-cd}\,\frac{y}{\sqrt{t}}\right) - \frac{1}{8cd}\sqrt{-cd}\right. \\ &\times\left(\frac{z}{\sqrt{t}} + \frac{1}{c}\sqrt{-cd}\,\frac{y}{\sqrt{t}}\right)^2\left(\frac{1}{2}\frac{y}{\sqrt{t}} - \frac{c}{2\sqrt{-cd}}\frac{z}{\sqrt{t}}\right) - \frac{1}{4c}\left(\frac{z}{\sqrt{t}} + \frac{1}{c}\sqrt{-cd}\,\frac{y}{\sqrt{t}}\right) \\ &\times\left(\frac{1}{2}\frac{y}{\sqrt{t}} - \frac{c}{2\sqrt{-cd}}\frac{z}{\sqrt{t}}\right)^2 - \frac{1}{4cd}\sqrt{-cd}\left(\frac{z}{\sqrt{t}} + \frac{1}{c}\sqrt{-cd}\,\frac{y}{\sqrt{t}}\right) \\ &\times\left.\left(\frac{1}{2}\frac{y}{\sqrt{t}} - \frac{c}{2\sqrt{-cd}}\frac{z}{\sqrt{t}}\right)\right\} - \frac{a}{\alpha}y + x\left(\frac{z}{\sqrt{t}} + 1\right), \end{aligned} \tag{2.75}$$

where f_1 and f_2 are arbitrary functions of their respective arguments. Further exploration gives no results of importance.

Case 1. We contemplate the translation symmetries X_1, X_2, X_3 together with X_4 combination, achieved as

$$X = X_1 + X_2 + X_3 + \mu X_4,$$

where μ is a constant. From the linear combination we get the invariants $q = y - t$, $k = z - \mu t\, p = x - t$, which yield group invariant result viz; $G(p,q,k) = u(t,x,y,z)$. This further transforms (1.4) to the NLPDE

$$aG_{pp} - \mu G_{kp} - G_{pq} + \beta G_pG_{pq} + b\mu G_{pppk} + \alpha G_qG_{pp} + bG_{pppq} + bG_{pppp} - G_{pp} + cG_{qq} + dG_{kk} = 0. \tag{2.76}$$

Equation (2.76) have five symmetries namely,

$$\Gamma_1 = \frac{\partial}{\partial p}, \Gamma_2 = \frac{\partial}{\partial q}, \Gamma_3 = \frac{\partial}{\partial k}, \Gamma_4 = \frac{\partial}{\partial G}, \Gamma_5 = k\frac{\partial}{\partial G}.$$

Similarly, characteristic equations from combination of translation symmetries yield,

$$\Gamma = \Gamma_1 + \Gamma_2 + \gamma\Gamma_3,$$

with $\gamma \neq 0$ a constant. This provides the invariant solutions: $g = k - \gamma p\, f = q - p$, and $U(f,g) = G(p,q,k)$. Equation (2.76) reduces to

$$2\alpha\gamma U_fU_{fg} + \alpha\gamma U_fU_{gg} + \beta\gamma U_fU_{fg} + \beta\gamma U_gU_{ff} + \beta\gamma^2U_gU_{fg} - 3b\mu\gamma U_{ffgg} \tag{2.77}$$

$$
\begin{aligned}
&- 3b\mu\gamma^2 U_{fggg} - b\mu\gamma^3 U_{gggg} - \gamma^2 U_{gg} + \mu U_{fg} + aU_{ff} - \gamma U_{fg} + cU_{ff} + dU_{gg}\\
&+ b\gamma^4 U_{gggg} + a\gamma^2 U_{gg} + \beta U_f U_{ff} + 2a\gamma U_{fg} + \mu\gamma U_{gg} + \alpha U_f U_{ff} + 3b\gamma^2 U_{ffgg}\\
&+ b\gamma U_{fffg} + 3b\gamma^3 U_{fggg} - b\mu U_{fffg} = 0.
\end{aligned}
$$

Equation (2.77) gives three symmetries, namely

$$R_1 = \frac{\partial}{\partial f},\ R_2 = \frac{\partial}{\partial g},\ R_3 = \frac{\partial}{\partial U}.$$

Furthermore, combination of $R_1 + \theta R_2$, where θ is a constant gives invariant solution $g - \theta f = r$, and $H(r) = U(f, g)$.

Finally, invoking $H(r) = U(f, g)$ into (2.77), transforms the NLNPADE to an NODE

$$
\begin{aligned}
&2\alpha\gamma\theta^2 H'H'' - \alpha\gamma^2\theta H'H'' + 2\beta\gamma\theta^2 H'H'' - \beta\gamma^2\theta H'H'' - 3b\mu\gamma\theta^2 H''''\\
&+ 3b\mu\gamma^2\theta H'''' - b\mu\gamma^3 H'''' - \gamma^2 H'' - \mu\theta H'' + a\,\theta^2 H'' + \gamma\theta H'' + c\,\theta^2 H''\\
&+ dH'' + b\gamma^4 H'''' + a\gamma^2 H'' - \beta\theta^3 H'H'' - 2a\gamma\theta H'' + \mu\gamma H'' - \alpha\theta^3 H'H''\\
&+ 3b\gamma^2\theta^2 H'''' - b\gamma\theta^3 H'''' - 3b\gamma^3\theta H'''' + b\mu\theta^3 H'''' = 0.
\end{aligned}
\tag{2.78}
$$

Equation (2.78) simplifies into

$$
\begin{aligned}
&(2\alpha\gamma\theta^2 - \alpha\gamma^2\theta + 2\beta\gamma\theta^2 - \beta\gamma^2\theta - \beta\theta^3 - \alpha\theta^3)H'H''\\
&+ (a\gamma^2 + \mu\gamma - 2a\gamma\theta + a\,\theta^2 + d - \gamma^2 - \mu\theta + \gamma\theta + c\,\theta^2)H''\\
&+ (b\gamma^4 + 3b\mu\gamma^2\theta - b\mu\gamma^3 - 3b\mu\gamma\theta^2 + 3b\gamma^2\theta^2 - b\gamma\theta^3 - 3b\gamma^3\theta + b\mu\theta^3)H'''' = 0.
\end{aligned}
\tag{2.79}
$$

Hence, we then have

$$\mathcal{R}H'''' + \mathcal{P}H'H'' + \mathcal{Q}H'' = 0, \tag{2.80}$$

in which

$$
\begin{aligned}
&\mathcal{P} = 2\alpha\gamma\theta^2 - \alpha\gamma^2\theta + 2\beta\gamma\theta^2 - \beta\gamma^2\theta - \beta\theta^3 - \alpha\theta^3,\\
&\mathcal{Q} = a\gamma^2 - 2a\gamma\theta + \mu\gamma + d - \gamma^2 - \mu\theta + a\,\theta^2 + \gamma\theta + c\,\theta^2,\\
&\mathcal{R} = b\gamma^4 + 3b\mu\gamma^2\theta - b\mu\gamma^3 - 3b\mu\gamma\theta^2 + 3b\gamma^2\theta^2 - b\gamma\theta^3 - 3b\gamma^3\theta + b\mu\theta^3.
\end{aligned}
$$

2.3. **Solution of KP-BBM (1.4) using direct integration.** This section seeks to reveal the approach to securing the general solution of the KP-BBM Equations (1.4) by examining NODE (2.80). Integrating (2.80) regarding r yields

$$\frac{\mathcal{P}}{2}H'^2 + \mathcal{Q}H' + \mathcal{R}H''' + K_0 = 0, \tag{2.81}$$

with K_0 an arbitrary integration constant. Suppose $\Phi(r) = H'(r)$, then (2.81) becomes

$$\frac{\mathcal{P}}{2}\Phi^2 + \mathcal{Q}\Phi + \mathcal{R}\Phi'' + K_0 = 0. \tag{2.82}$$

To perform the integration of (2.82), we multiply it first by $\Phi'(r)$ to get

$$\frac{\mathcal{P}}{2}\Phi^2\Phi' + \mathcal{Q}\Phi\Phi' + \mathcal{R}\Phi''\Phi' + K_0\Phi' = 0. \tag{2.83}$$

Integrating (2.83) with respect to r gives

$$\frac{\mathcal{R}}{2}\Phi'^2 + \frac{\mathcal{P}}{6}\Phi^3 + \frac{\mathcal{Q}}{2}\Phi^2 + K_0\Phi + K_1 = 0, \tag{2.84}$$

where K_1 is an arbitrary constant. Then,

$$\Phi'^2 + \frac{\mathcal{P}}{3\mathcal{R}}\Phi^3 + \frac{\mathcal{Q}}{\mathcal{R}}\Phi^2 + \frac{2}{\mathcal{R}}K_0\Phi + \frac{2}{\mathcal{R}}K_1 = 0. \tag{2.85}$$

Suppose that m_1, m_2 and m_3 are real roots $(m_1 > m_2 > m_3)$ of a cubic equation

$$\Phi^3 + \frac{3\mathcal{Q}}{\mathcal{P}}\Phi^2 + \frac{6}{\mathcal{P}}K_0\Phi + \frac{6}{\mathcal{P}}K_1 = 0. \tag{2.86}$$

and satisfies the conditions

$$m_1m_2m_3 = -\frac{6}{\mathcal{P}}K_1,\ m_1m_2 + m_1m_3 + m_2m_3 = \frac{6}{\mathcal{P}}K_0,\ m_1 + m_2 + m_3 = -\frac{3\mathcal{Q}}{\mathcal{P}}.$$

Then equation (2.85) is written as

$$\Phi'^2 = -\frac{\mathcal{P}}{3\mathcal{R}}(\Phi - m_1)(\Phi - m_2)(\Phi - m_3),$$

and has the solution

$$\Phi(r) = m_2 + (m_1 - m_2)\mathrm{cn}^2\left\{\sqrt{\frac{\mathcal{P}(m_1 - m_3)}{12\mathcal{R}}}\,(r - r_0)\,\middle|K^2\right\},\ K^2 = \frac{m_1 - m_2}{m_1 - m_3}, \tag{2.87}$$

with r_0 a constant as well as cn denoting the Jacobi cosine function. Thus by retro-grading to the fundamental variables, one gains the result of KP-BBM (1.4) as [28]

$$u(t,x,y,z) = \mathcal{Q}_0\left[\mathrm{EllipticE}\left\{\mathrm{sn}(\mathcal{Q}_1(r - r_0), K^2), K^2\right\}\right] + \left\{m_2 - (m_1 - m_2)\frac{1 - K^4}{K^4}\right\} \times (r - r_0) + k_1, \tag{2.88}$$

where

$$\mathcal{Q}_0 = \sqrt{\frac{12\mathcal{R}(m_1 - m_2)^2}{(m_1 - m_3)\mathcal{P}K^8}},\ \mathcal{Q}_1 = \sqrt{\frac{\mathcal{P}(m_1 - m_2)}{12\mathcal{R}}},$$

where we have an integration constant k_1, and variable $r = (\gamma - \mu)t + (\theta - \gamma)x - \theta y + z$, while EllipticE[r,k] depicts the incomplete elliptic integral [21, 28].

2.4. **Analytic solutions of (1.4) via simplest equation approach.** The simplest equation approach is engaged in this subsection in solving the NODE (2.80).

2.4.1. *Solutions of (1.4) via the Riccati equation as the simplest equation.* Balancing procedure furnishes term $M = 1$. Then solutions of (2.80) purvey the form,

$$H(r) = a_0 + a_1Y(r).$$

Putting this secured value of $H(r)$ into (2.80), also invoking Riccati equation, secures the system calculated in terms of a_0, a_1:

$$\begin{aligned}
&\mathcal{A}{a_1}^2l^3 + 12\,\mathcal{C}a_1l^4 = 0,\\
&\mathcal{A}{a_1}^2l^2m + 12\,\mathcal{C}a_1l^3m = 0,\\
&\mathcal{A}{a_1}^2mn^2 + 8\,\mathcal{C}a_1lmn^2 + \mathcal{C}a_1m^3n + \mathcal{B}a_1mn = 0,\\
&6\,\mathcal{A}{a_1}^2lmn + \mathcal{A}{a_1}^2m^3 + 60\,\mathcal{C}a_1l^2mn + 15\,\mathcal{C}a_1lm^3 + 3\,\mathcal{B}a_1lm = 0,\\
&2\,\mathcal{A}{a_1}^2l^2n + 2\,\mathcal{A}{a_1}^2lm^2 + 20\,\mathcal{C}a_1l^3n + 25\,\mathcal{C}a_1l^2m^2 + \mathcal{B}a_1l^2 = 0,\\
&2\,\mathcal{A}{a_1}^2ln^2 + 2\,\mathcal{A}{a_1}^2m^2n + 16\,\mathcal{C}a_1l^2n^2 + 22\,\mathcal{C}a_1lm^2n + \mathcal{C}a_1m^4 + 2\,\mathcal{B}a_1ln + \mathcal{B}a_1m^2 = 0.
\end{aligned}$$

On the engagement of Maple, solution to the gained-system furnishes

$$a_0 = a_0,\ a_1 = -12\,\frac{\mathcal{C}l}{\mathcal{A}},\ \mathcal{B} = 4\,\mathcal{C}(ln - m^2). \tag{2.89}$$

One establishes the fact that the outcome associated to model (1.4) via Ricatti equation given, can be expressed as

$$u(t,x,y,z) = a_0 + a_1\left\{-\frac{m}{2l} - \frac{\omega}{2l}\tanh\left[-\frac{1}{2}\omega(r + D)\right]\right\}, \tag{2.90}$$

and

$$u(t,x,y,z) = a_0 + a_1 \left\{ -\frac{m}{2l} - \frac{\omega}{2l} \tanh\left(\frac{\omega\, r}{2}\right) + \frac{\operatorname{sech}\left(\frac{\omega \mathrm{r}}{2}\right)}{D\cosh\left(\frac{\omega r}{2}\right) - \frac{2\,l}{\omega}\sinh\left(\frac{\omega r}{2}\right)} \right\}, \tag{2.91}$$

with $\omega = \sqrt{m^2 - 4ln}$ and D is an integration constant.

2.4.2. *Solutions of* (1.4) *via the Bernoulli equation as the simplest equation.* This case yields balancing term $M = 1$ and just as earlier presented gives the result of (2.80) in the structure

$$H(r) = a_0 + a_1 Y(r).$$

On inserting this expression of $H(r)$ into (2.80) and applying Bernoulli equation earlier-given furnishes the system in terms of a_0, a_1 presented as

$$\begin{aligned}
&\mathcal{A}{a_1}^2 m^3 + 12\,\mathcal{C} a_1 m^4 = 0,\\
&\mathcal{C} a_1 l^4 + \mathcal{B} a_1 l^2 = 0,\\
&\mathcal{A}{a_1}^2 l m^2 + 12\,\mathcal{C} a_1 l m^3 = 0,\\
&2\,\mathcal{A}{a_1}^2 l^2 m + 25\,\mathcal{C} a_1 l^2 m^2 + \mathcal{B} a_1 m^2 = 0,\\
&\mathcal{A}{a_1}^2 l^3 + 15\,\mathcal{C} a_1 l^3 m + 3\,\mathcal{B} a_1 l m = 0,
\end{aligned}$$

thus resulting via the use of Maple to values

$$a_0 = a_0,\ a_1 = -\frac{12\mathcal{C}m}{\mathcal{A}},\ \mathcal{B} = -Cl^2, \tag{2.92}$$

Hence, solutions of the KP-BBM (1.4) are

$$u(t,x,y,z) = a_0 + \,a_1 l \left\{ \frac{\cosh\left[l(r+D)\right] + \sinh\left[l(r+D)\right]}{1 - m\cosh\left[l(r+D)\right] - m\sinh\left[l(r+D)\right]} \right\}, \tag{2.93}$$

with D taken as an arbitrary integration constant and $r = (\gamma-\mu)t + (\theta-\gamma)x - \theta y + z$.

3. Graphical depictions of some of the solutions

Here, we present the graphical descriptions of some of the obtained solutions in the previous section. The findings comprise various solutions of interest ranging from trigonometry, and hyperbolic, to Jacobi elliptic functions. Additionally, a number of algebraic solutions consisting of arbitrary functions were achieved. These arbitrary functions can assume any possible mathematical functions with the result satisfying (1.4). Therefore, using computer software, we present some graphical displays of solitary waves in the form of three-dimensional (3D), two-dimensional (2D), and density plots.

In the first place, we consider algebraic solution (2.19) and let $f_1(x,z) = \operatorname{sech}(x,z)$, $f_2(x,z) = \sin(x,z)$, $H(t) = -\sin(t)$ where parameters $a = 10$, $d = -5$, $t = 0.02$, $y = 0$ and $-9 < x, z < 9$. Using the listed data, Figure 1 is plotted. Furthermore, we explore the solution with the same parametric values but allowing a decrease in time to $t = 0.03$ and $t = 0.05$ but expanding the given interval by two and three units. Thus, we accordingly achieve Figures 2 and 3. We observe that the Figures reveal a steady decrease in the number of periods which forms the background, with a corresponding increase in time t. Next, we contemplate hyperbolic-algebraic function solution (2.30) furnishing Figure 4 with unalike parameter values $\alpha = 100$, $\beta = 1$, $a = 10$, $c = 0.1$, $C_0 = 200$, $C_1 = 700$ where $x = 0.1$, $z = 0$ and $-9 < t, y < 9$. In addition using the same constant values and setting $z = 0$ with $x = 0.4$ and $-13 < t, y < 13$, we plot Figure 5. Now, we shift attention to trigonometric solution (2.53). In exhibiting Figure 6, we utilize dissimilar constant values $\alpha = 10$, $c_0 = 1$, $c_1 = 1$, $a = 10$, $b = 1$, $A_0 = 0.02$, $A_1 = 0.7$, $A_2 = 0.3$, $A_3 = 0$, where $y = 10$, $z = 0$ and $-6 < t, x < 6$. In Figure 7, we adopt parametric values $\alpha = 10$, $c_0 = 1$, $c_1 = 1$, $a = 10$, $b = 4$, $A_0 = 0.02$, $A_1 = 0.9$, $A_2 = 0.06$, $A_3 = 0.06$, where $y = 10$, $z = 0$ and $-6 < t, x < 6$. In addition, we use $\alpha = 10$, $c_0 = 1$, $c_1 = 1$, $a = 10$, $b = 10$, $A_0 = 0.02$, $A_1 = 0.9$, $A_2 = 0.06$, $A_3 = 10$, with $y = 10$ as well as $z = 0$ such that $-6 < t, x < 6$, to plot Figure 8. It is observed that at fixed values of y and z the periodicity changes. Hence, we can infer that the movement of the wave particles can still occur based on other parameters in the solution beside that of the variables.

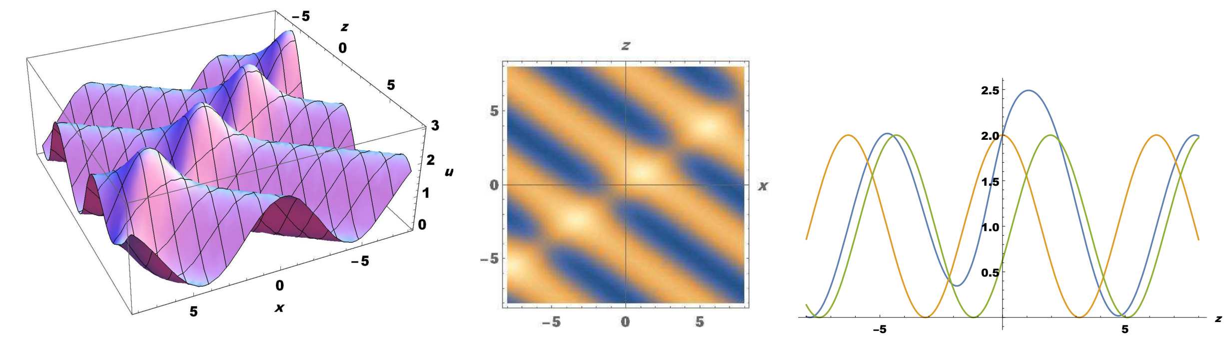

Figure 1. Wave profile exhibiting algebraic solution (2.19) at $t = 0.02$ and $y = 0$.

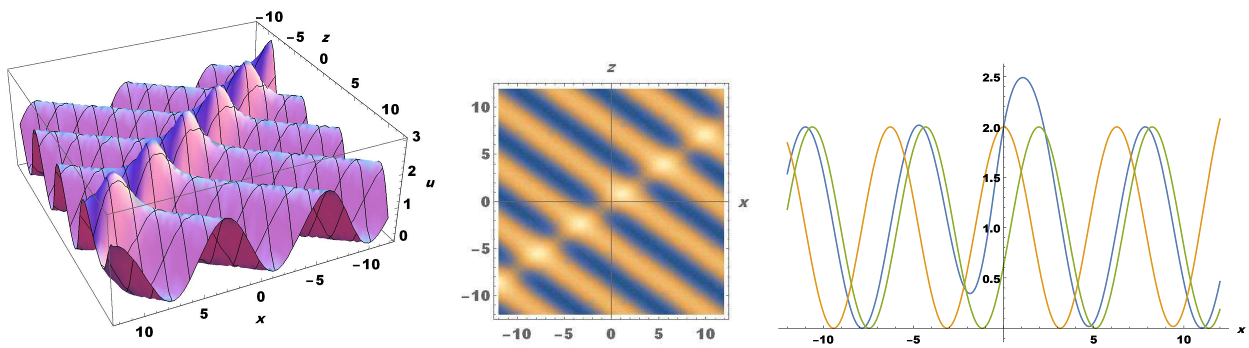

Figure 2. Wave profile exhibiting algebraic solution (2.19) at $t = 0.03$ and $y = 0$.

We contemplate solution (2.59) which gives the plot in Figures 9 and 10 where we let $f_1(x) = \cosh(x)$, $f_2(t) = \sin(t)$, $f_3(t) = \cos(t)$, $F(t) = \mathrm{Si}(t)$. The involved parameters are allocated values for the Figures respectively as $a = 100$, $\alpha = 0.3$, $c = 0.6$, $c_2 = 0.001$, $d = 100$ where $y = 2$, $z = 1$ using the interval $-11 < t, x < 11$ and $a = 1$, $\alpha = 0.3$, $c = 0.4$, $c_2 = 0.001$, $d = 1000$ where $y = 0.2$, $z = 10$ with the interval $-14 < t, x < 14$. Next, we focus on the wave depictions of periodic solution (2.69). In Figure 11, we assign $e_0 = 1$, $e_1 = 0.1$, $a = 10$, $b = 100$, $A_0 = 0.02$, $A_1 = 0.7$, $A_2 = 0.3$, $A_3 = A_4 = 0$, where $y = 0$, $z = 0.02$ and $-9 < t, x < 9$. In the case of Figure 12, we allocate parameter values as $e_0 = 1$, $e_1 = 0.1$, $a = 10$, $b = 100$, $A_0 = 0.2$, $A_1 = 0.9$, $A_2 = 0.1$, $A_3 = A_4 = 0$, where $y = 0.1$, $z = 0.01$, using the interval $-11 < t, x < 11$. Additionally, to plot Figure 13, we engage the previous values with the change in constants $A_0 = 10$, $A_1 = 1.4$, $A_2 = 0.1$, $A_3 = 0.02$, $A_4 = 0.2$, with $y = 0.3$, $z = 0.2$ and $-16 < t, x < 16$.

Figure 14 exhibits solution (2.88) for the parameter values $\theta = 0.6, \mu = 0.2, \gamma = -4, t = -14, k_1 = 1, m_1 = 100, m_2 = 50.05, m_3 = -60, \mathcal{R} = 1, \mathcal{P} = 0.287$ and $r_0 = 0$. Figure 15 portrays solution (2.90) for parameters $\gamma = 0.2, \mu = 0.3, \theta = 1.9, t = 6, \mathcal{C} = 1.9, \mathcal{A} = 2, D = 0.9, m = 2, a_0 = 10, \omega = 2, z = 1$. Figure 16 demonstrates the solution (2.91) for the values $\theta = 1.9, \mu = 0.3, \gamma = 0.2, \mathcal{C} = 1.9, \mathcal{A} = 2, D = 0.9, m = 2, a_0 = 10, \omega = 2, t = 6, z = -1$. Figure 17 depicts solution (2.93) for the values $\theta = 1.9, \mu = 0.3, \gamma = 0.2, \mathcal{C} = 1.9, \mathcal{A} = 2, D = 0.8, m = 2, a_0 = 10, t = 6, z = 1$.

4. Conserved vectors of (1.4)

We give conserved vectors of the KP-BBM (1.4) by using two variant methods. They are the multiplier approach [36] and Noether's theorem [35].

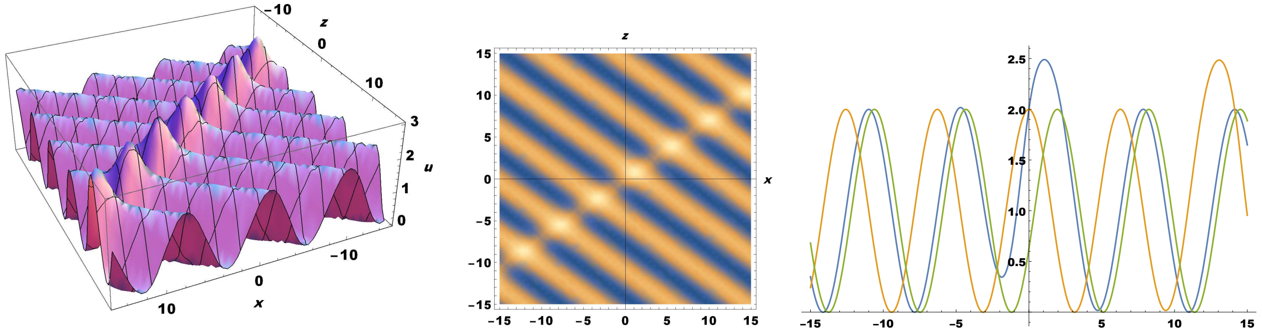

Figure 3. Wave profile exhibiting algebraic solution (2.19) at $t = 0.05$ and $y = 0$.

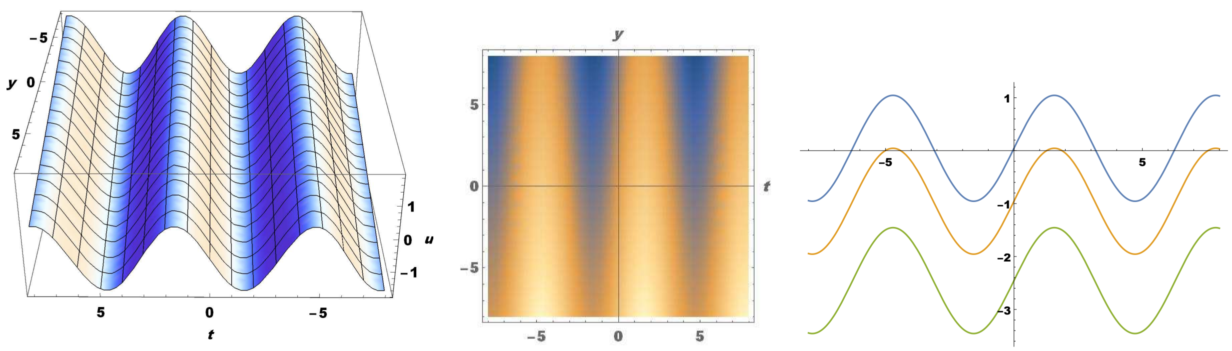

Figure 4. Graphical display of hyperbolic-algebraic solution (2.30) at $x = 0.1$ and $z = 0$.

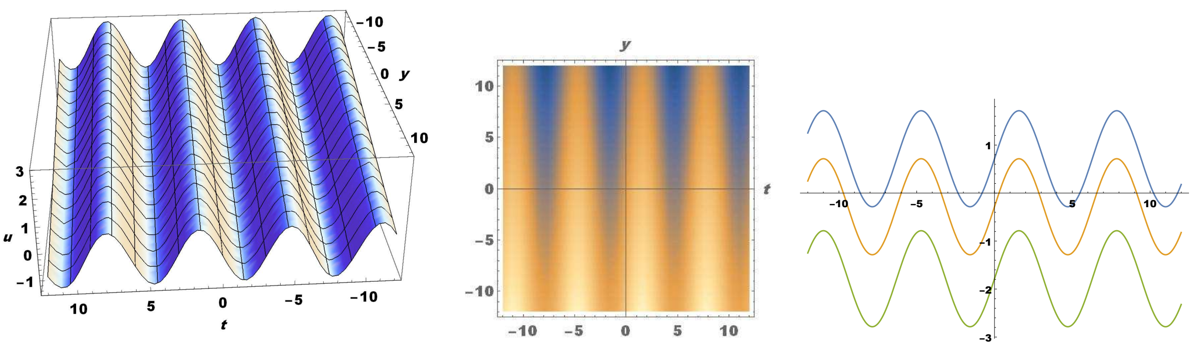

Figure 5. Graphical display of hyperbolic-algebraic solution (2.30) at $x = 0.4$ and $z = 0$.

4.1. **Conservation laws via the multiplier method.** We seek multipliers $\mathcal{M}$ that depend on t, x, y and u only. The determining relation is expressed, viz;

$$\frac{\delta}{\delta u}\left\{\mathcal{M}\left(\alpha u_y u_{xx} + \beta u_y u_{xy} - b u_{txxx} + a u_{xx} + c u_{yy} + d u_{zz} + u_{tx}\right)\right\} = 0, \tag{4.1}$$

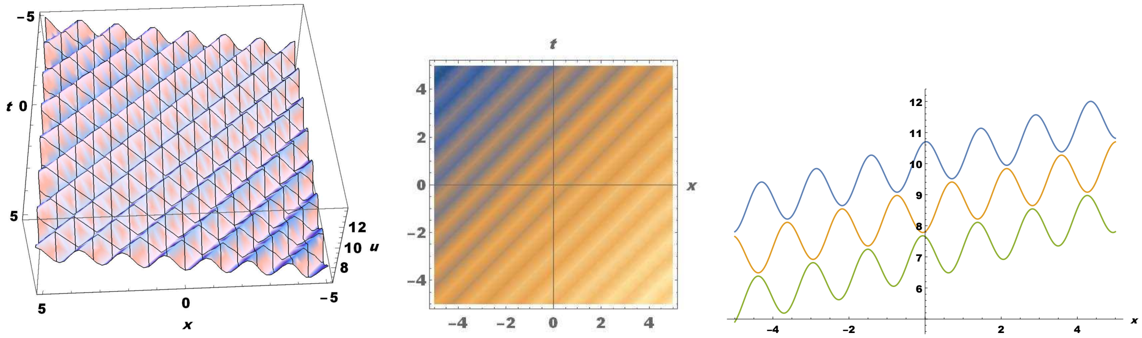

Figure 6. Graphical exhibition of solitary wave solution (2.53) at $y = 10$ and $z = 0$.

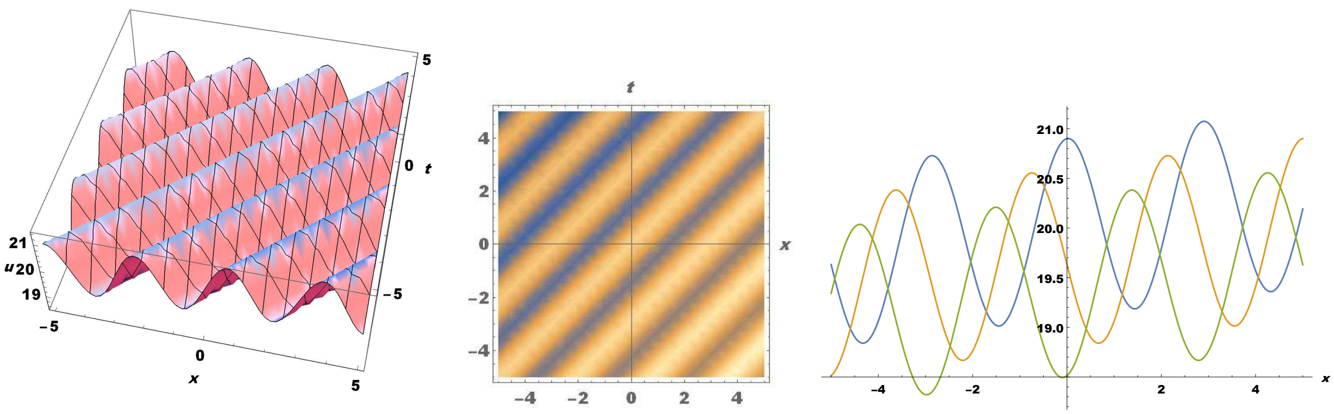

Figure 7. Graphical exhibition of solitary wave solution (2.53) at $y = 10$ and $z = 0$.

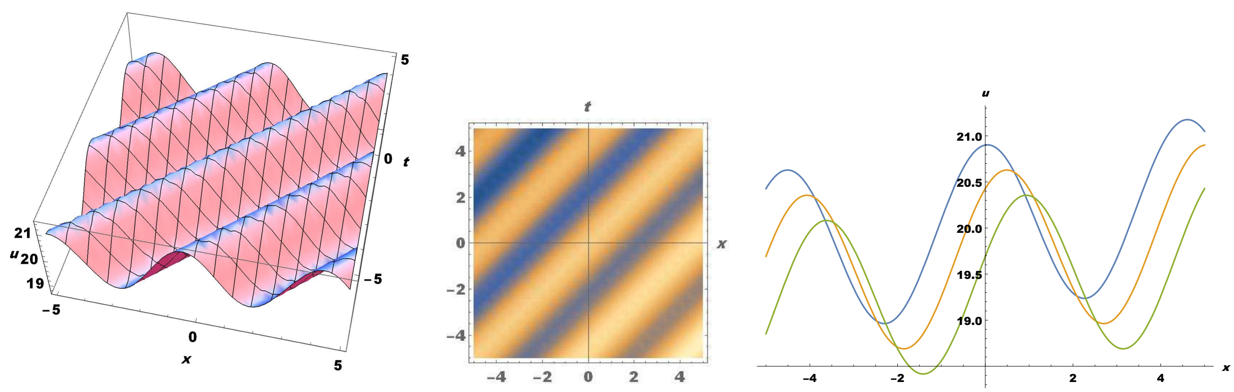

Figure 8. Graphical exhibition of solitary wave solution (2.53) at $y = 10$ and $z = 0$.

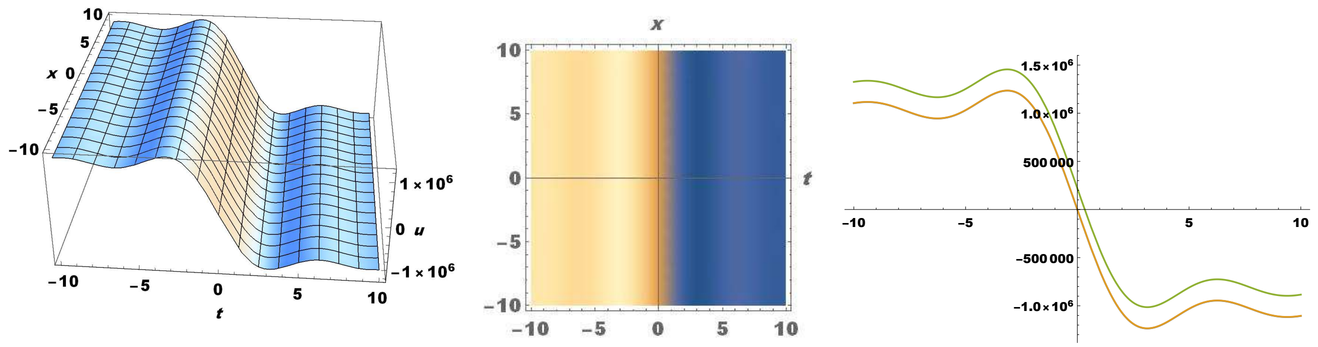

Figure 9. Graphical depiction of alg-exponential function solution (2.59) at $y = 2$ and $z = 1$.

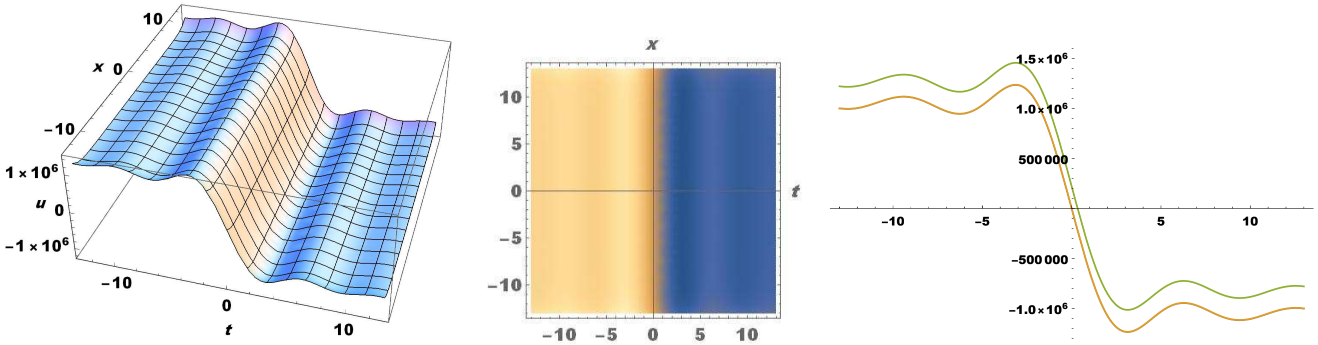

Figure 10. Graphical depiction of alg-exponential function solution (2.59) at $y = 0.2$ and $z = 10$.

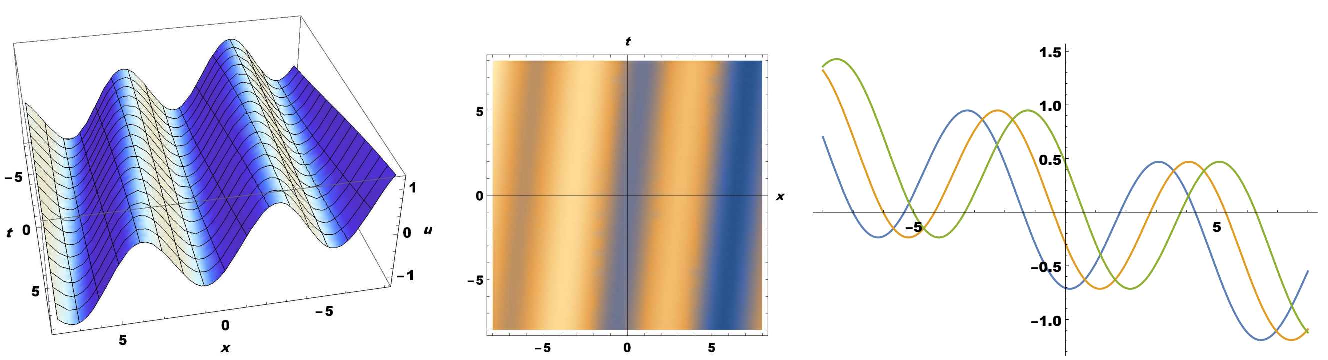

Figure 11. Graphical depiction of solitary wave solution (2.69) at $x = 0$.

with Euler operator $\delta/\delta u$, given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_y \frac{\partial}{\partial u_y} - D_x \frac{\partial}{\partial u_x} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_y^2 \frac{\partial}{\partial u_{yy}}$$

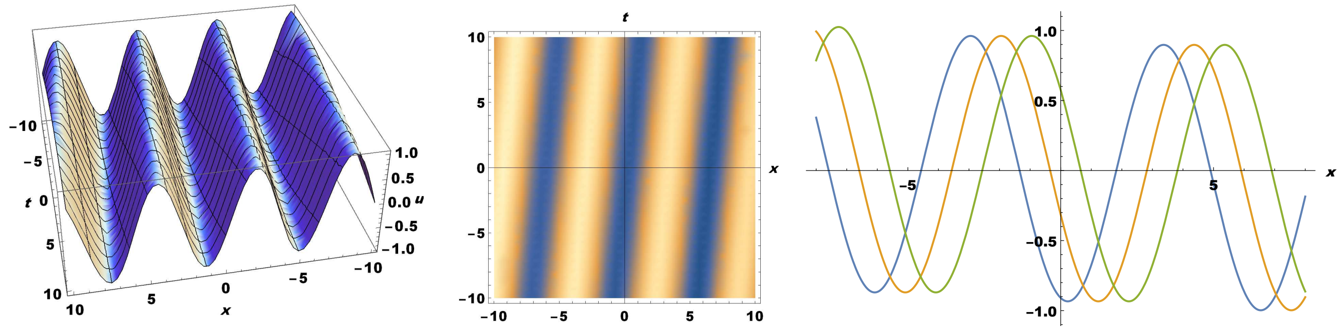

Figure 12. Graphical depiction of solitary wave solution (2.69) at $x = 0$.

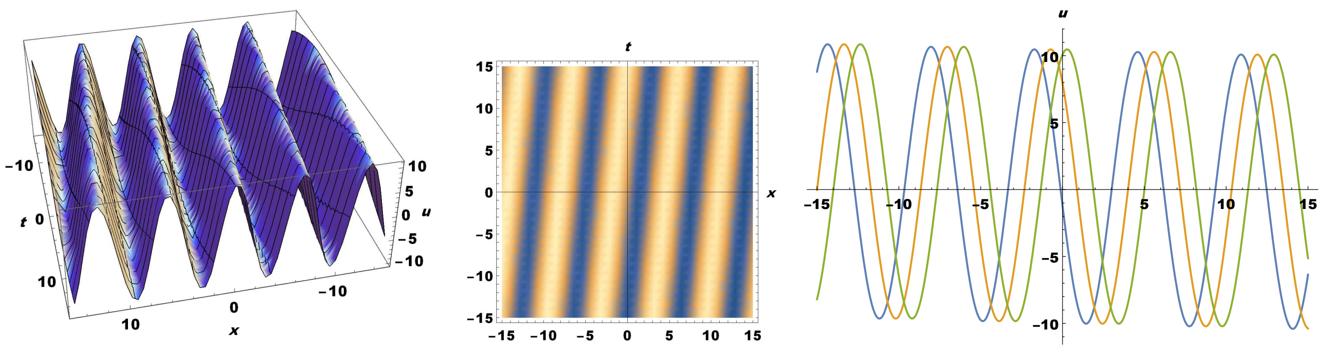

Figure 13. Graphical depiction of solitary wave solution (2.69) at $x = 0$.

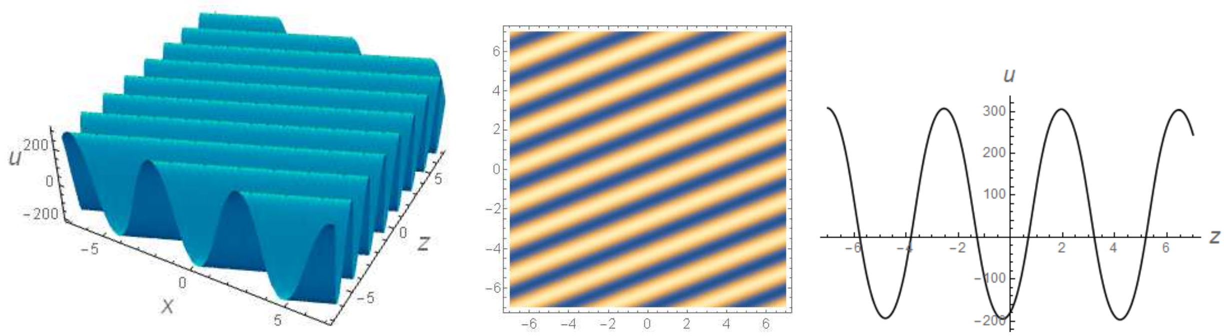

Figure 14. Three and two dimensional together-with density plots for (2.88).

$$+ D_z^2 \frac{\partial}{\partial u_{zz}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_t D_x^3 \frac{\partial}{\partial u_{txxx}} \tag{4.2}$$

with total derivative operators (D_t, D_x, D_y, D_z) as explicated in (2.4). Simplification of (4.1) and decomposing on the derivatives of u furnishes

$$\mathcal{M}_x = 0, \ \mathcal{M}_y = 0, \ \mathcal{M}_{zz} = 0, \ \mathcal{M}_u = 0,$$

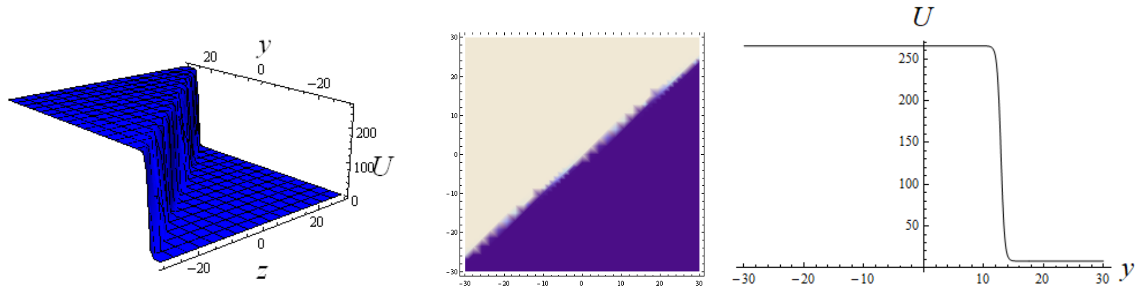

Figure 15. Three and two dimensional together-with density plots of (2.90).

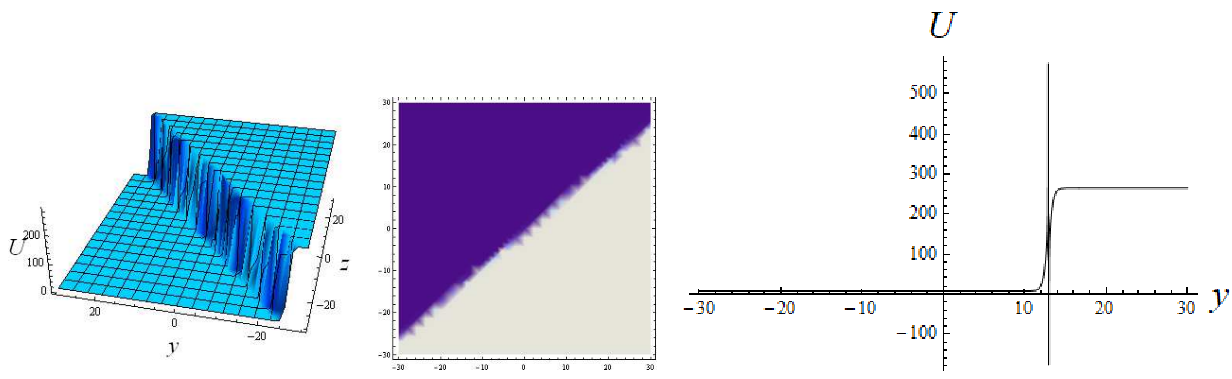

Figure 16. Three and two dimensional together-with density plots of (2.91).

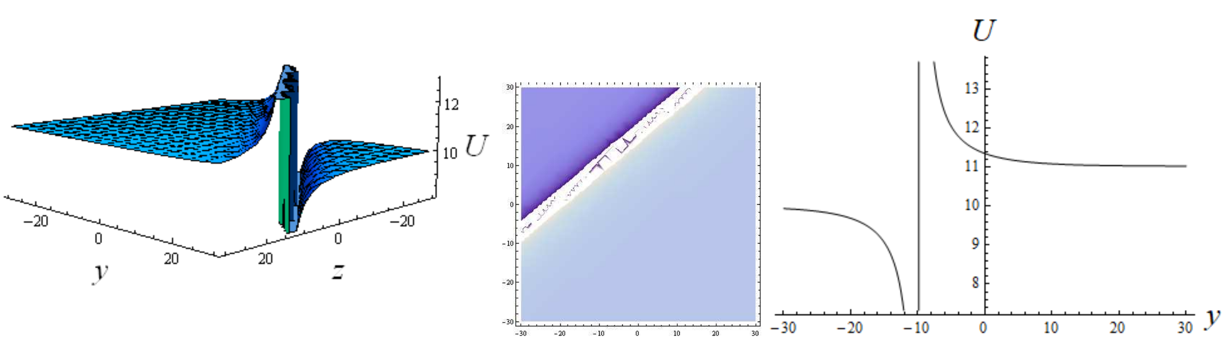

Figure 17. Three and two dimensional together-with density plots of (2.93).

whose solution is

$$\mathcal{M} = A(t)z + B(t),$$

with A and B being functions dependent on t. The conserved currents of model KP-BBM (1.4) are achieved by invoking divergence identity

$$D_tT^t + D_yT^y + D_xT^x + D_zT^z = \mathcal{M}\left[\frac{\delta}{\delta u}\left\{\mathcal{M}\left(\beta u_y u_{xy} + \alpha u_y u_{xx} - bu_{txxx} + cu_{yy} + au_{xx} + u_{tx} + du_{zz}\right)\right\}\right], \tag{4.3}$$

with spatial fluxes T^x, T^y , T^z, and T^t being conserved density. Invoking (4.3), one achieves conserved vectors related to the two secured multipliers in the subsequent manner:

Case 1. For $\mathcal{M}_1 = A(t)z$, the related components of its conserved vector are

$$\begin{aligned}
T^t &= \frac{1}{2} z u_x A(t) - \frac{1}{4} b z u_{xxx} A(t),\\
T^x &= \frac{1}{2}\alpha z A(t) u_x u_y - \frac{1}{2}\alpha z A(t) u u_{xy} - \frac{1}{4}\beta z A(t) u u_{yy} + \frac{1}{4}\beta z A(t) u_y^2\\
&\quad + \frac{1}{4} b z F'(t) u_{xx} - \frac{3}{4} b z A(t) u_{txx} + a z A(t) u_x - \frac{1}{2} z F'(t) u + \frac{1}{2} z A(t) u_t,\\
T^y &= \frac{1}{2}\alpha z A(t) u u_{xx} + c\, z A(t) u_y + \frac{1}{4}\beta z A(t) u_x u_y + \frac{1}{4}\beta z A(t) u u_{xy},\\
T^z &= d z A(t) u_z - d A(t) u.
\end{aligned}$$

Case 2. For $\mathcal{M}_2 = B(t)$, associated components of its conserved vector are

$$\begin{aligned}
T^t &= \frac{1}{2} u_x B(t) - \frac{1}{4} b u_{xxx} B(t),\\
T^x &= \frac{1}{2}\alpha B(t) u_x u_y - \frac{1}{2}\alpha B(t) u u_{xy} - \frac{1}{4}\beta B(t) u u_{yy} + \frac{1}{4}\beta B(t) u_y^2 + \frac{1}{4} b G'(t) u_{xx}\\
&\quad - \frac{3}{4} b B(t) u_{txx} + a B(t) u_x - \frac{1}{2} G'(t) u + \frac{1}{2} B(t) u_t,\\
T^y &= \frac{1}{2}\alpha B(t) u u_{xx} + c\, B(t) u_y + \frac{1}{4}\beta B(t) u_x u_y + \frac{1}{4}\beta B(t) u u_{xy},\\
T^z &= d B(t) u_z.
\end{aligned}$$

4.2. **Conservation laws via the Noether's theorem.** The fourth-order KP-BBM (1.4) doesn't own a Langrangian. However, we consider a condition $\beta = 2\alpha$ and Equation (1.4) becomes

$$u_{tx} + \alpha u_y u_{xx} + 2\alpha u_x u_{xy} - b u_{txxx} + c u_{yy} + a u_{xx} + d u_{zz} = 0. \tag{4.4}$$

We already gave Euler operator $\delta/\delta u$ in (4.2), (see also [3, 27, 36]). Now, $\delta\mathcal{L}/\delta u = 0$ for

$$\mathcal{L} = -\frac{1}{2} a u_x^2 - \frac{1}{2} b u_{xx} u_{tx} - \frac{1}{2} c u_y^2 - \frac{1}{2} d u_z^2 - \frac{1}{2} u_t u_x - \frac{1}{2}\alpha u_x^2 u_y, \tag{4.5}$$

therefore $\mathcal{L}$ is the Lagrangian for (4.4).

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial u}, \tag{4.6}$$

with the coefficients τ, ξ, ϕ and ψ being function of the variables t, x, y, z and u. Thus, Lie-Bäcklund operator given as X and explicated in (4.6) is a Noether operator interrelated to $\mathcal{L}$ provided

$$\mathcal{L}\left[D_t(\tau) + D_y(\phi) + D_x(\xi) + D_z(\psi)\right] + X^{[2]}(\mathcal{L}) = D_x(B^2) + D_t(B^1) + D_y(B^3) + D_z(B^4), \tag{4.7}$$

is satisfied where B^1, B^2, B^3, and B^4 depending on (t, x, y, z) are gauge functions. Expanding (4.7) and disintegrating the outcome on u derivatives, yields a system of linear PDEs, whose solutions gives

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\ \ B^1 = 0,\ \ B^2 = 0,\ \ B^3 = 0,\ \ B^4 = 0,\\
X_2 &= \frac{\partial}{\partial x},\ \ B^1 = 0,\ \ B^2 = 0,\ \ B^3 = 0,\ \ B^4 = 0,\\
X_3 &= \frac{\partial}{\partial y},\ \ B^1 = 0,\ \ B^2 = 0,\ \ B^3 = 0,\ \ B^4 = 0,\\
X_4 &= \frac{\partial}{\partial z},\ \ B^1 = 0,\ \ B^2 = 0,\ \ B^3 = 0,\ \ B^4 = 0,
\end{aligned}$$

$$
\begin{aligned}
X_5 &= 2\alpha t\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial y} + \alpha z\frac{\partial}{\partial z} - (\alpha u + 2ay)\frac{\partial}{\partial u}, \;\; B^1 = 0, \;\; B^2 = 0, B^3 = 2acu, \;\; B^4 = 0,\\
X_G &= G(t)\frac{\partial}{\partial u}, \;\; B^1 = 0, \;\; B^2 = -\frac{1}{2}vG'(t), \;\; B^3 = 0, \;\; B^4 = 0,\\
X_F &= zF(t)\frac{\partial}{\partial u}, \;\; B^1 = 0, \;\; B^2 = -\frac{1}{2}vzF'(t), \;\; B^3 = 0, \;\; B^4 = -duF(t).
\end{aligned}
$$

In relation to the above Noether symmetries, one gains the following conserved vectors:

$$
\begin{aligned}
T_1^t &= -\frac{1}{4}bu_t u_{xxx} - \frac{1}{2}au_x^2 - \frac{1}{4}bu_{xx}u_{tx} - \frac{1}{2}cu_y^2 - \frac{1}{2}\alpha u_x^2 u_y - \frac{1}{2}du_z^2,\\
T_1^x &= au_t u_x - \frac{3}{4}bu_t u_{,\,txx} + \frac{1}{2}bu_{tx}^2 + \frac{1}{4}bu_{tt}u_{xx} + \alpha u_t u_x u_y + \frac{u_t^2}{2},\\
T_1^y &= cu_t u_y + \frac{1}{2}\alpha u_t u_x^2,\\
T_1^z &= du_t u_z;
\end{aligned}
$$

$$
\begin{aligned}
T_2^t &= \frac{1}{4}bu_{xx}^2 - \frac{1}{4}bu_{xxx}u_x + \frac{u_x^2}{2},\\
T_2^x &= \frac{1}{2}au_x^2 + \frac{1}{4}bu_{xx}u_{tx} - \frac{3}{4}bu_x u_{txx} - \frac{1}{2}cu_y^2 - \frac{1}{2}du_z^2 + \frac{1}{2}\alpha u_x^2 u_y,\\
T_2^y &= cu_x u_y + \frac{1}{2}\alpha u_x^3,\\
T_2^z &= du_x u_z;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t &= \frac{1}{4}bu_{xx}u_{xy} - \frac{1}{4}bu_{xxx}u_y + \frac{1}{2}u_x u_y,\\
T_3^x &= au_x u_y - \frac{3}{4}bu_y u_{txx} + \frac{1}{4}bu_{xx}u_{ty} + \frac{1}{2}bu_{tx}u_{xy} + \frac{1}{2}u_t u_y + \alpha u_x u_y^2,\\
T_3^y &= \frac{1}{2}cu_y^2 - \frac{1}{2}au_x^2 - \frac{1}{2}bu_{xx}u_{tx} - \frac{1}{2}du_z^2 - \frac{1}{2}u_t u_x,\\
T_3^z &= du_y u_z;
\end{aligned}
$$

$$
\begin{aligned}
T_4^t &= \frac{1}{4}bu_{xx}u_{xz} - \frac{1}{4}bu_{xxx}u_z + \frac{1}{2}u_x u_z,\\
T_4^x &= au_x u_z + \frac{1}{4}bu_{xx}u_{tz} + \frac{1}{2}bu_{tx}u_{xz} - \frac{3}{4}bu_z u_{txx} + \frac{1}{2}u_t u_z + \alpha u_x u_y u_z,\\
T_4^y &= cu_y u_z + \frac{1}{2}\alpha u_x^2 u_z,\\
T_4^z &= \frac{1}{2}du_z^2 - \frac{1}{2}au_x^2 - \frac{1}{2}bu_{xx}u_{tx} - \frac{1}{2}cu_y^2 - \frac{1}{2}u_t u_x - \frac{1}{2}\alpha u_x^2 u_y;
\end{aligned}
$$

$$
\begin{aligned}
T_5^t &= \frac{1}{2}z\alpha u_x u_z - dt\alpha u_z^2 - \frac{1}{4}bz\alpha u_{xxx}u_z - ct\alpha u_y^2 - at\alpha u_x^2 - \frac{1}{2}bt\alpha u_{xx}u_{tx}\\
&\quad + ayu_x - t\alpha^2 u_y u_x^2 + \frac{1}{2}\alpha u u_x + \frac{1}{2}y\alpha u_y u_x + \frac{1}{4}b\alpha u_x u_{xx} + \frac{1}{4}bz\alpha u_{xz}u_{xx}\\
&\quad + \frac{1}{4}by\alpha u_{xy}u_{xx} - \frac{1}{2}abyu_{xxx} - \frac{1}{4}b\alpha u u_{xxx} - \frac{1}{4}by\alpha u_y u_{xxx} - \frac{1}{2}bt\alpha u_{xxx}u_t,\\
T_5^x &= 2yu_x a^2 + \alpha u u_x a + z\alpha u_z u_x a + 3y\alpha u_y u_x a + yu_t a + 2t\alpha u_x u_t a
\end{aligned}
$$

$$
\begin{aligned}
&+ t\alpha u_t^2 - \frac{3}{2}byu_{txx}a + bt\alpha u_{tx}^2 + y\alpha^2 u_y^2 u_x + \alpha^2 uu_y u_x + z\alpha^2 u_z u_y u_x \\
&+ \frac{1}{2}\alpha uu_t + \frac{1}{2}z\alpha u_z u_t + \frac{1}{2}y\alpha u_y u_t + 2t\alpha^2 u_y u_x u_t + \frac{3}{4}b\alpha u_{xx}u_t + \frac{1}{4}bz\alpha u_{xx}u_{tz} \\
&+ \frac{1}{4}by\alpha u_{xx}u_{ty} + \frac{1}{2}b\alpha u_x u_{tx} + \frac{1}{2}bz\alpha u_{xz}u_{tx} + \frac{1}{2}by\alpha u_{xy}u_{tx} - \frac{3}{4}b\alpha uu_{txx} \\
&+ \frac{1}{2}bt\alpha u_{xx}u_{tt} - \frac{3}{4}bz\alpha u_z u_{txx} - \frac{3}{4}by\alpha u_y u_{txx} - \frac{3}{2}bt\alpha u_t u_{txx},
\end{aligned}
$$

$$
\begin{aligned}
T_5^y =& \frac{1}{2}z\alpha^2 u_x^2 u_z - \frac{1}{2}dy\alpha u_z^2 + cz\alpha u_y u_z + \frac{1}{2}cy\alpha u_y^2 + \frac{1}{2}ay\alpha u_x^2 + \frac{1}{2}\alpha^2 uu_x^2 \\
&+ 2acyu_y + c\alpha uu_y + t\alpha^2 u_x^2 u_t + 2ct\alpha u_y u_t - \frac{1}{2}y\alpha u_x u_t - \frac{1}{2}by\alpha u_{xx}u_{tx}, \\
T_5^z =& \frac{1}{2}dz\alpha u_z^2 + 2adyu_z + d\alpha uu_z + dy\alpha u_y u_z - \frac{1}{2}cz\alpha u_y^2 \\
&+ 2dt\alpha u_t u_z - \frac{1}{2}az\alpha u_x^2 - \frac{1}{2}z\alpha^2 u_y u_x^2 - \frac{1}{2}z\alpha u_x u_t - \frac{1}{2}bz\alpha u_{xx}u_{tx};
\end{aligned}
$$

$$
\begin{aligned}
T_G^t =& \frac{1}{4}bG(t)u_{xxx} - \frac{1}{2}G(t)u_x, \\
T_G^x =& \frac{3}{4}bG(t)u_{txx} - aG(t)u_x - \alpha G(t)u_x u_y - \frac{1}{2}G(t)u_t - \frac{1}{4}bG'(t)u_{xx}, \\
T_G^y =& -cG(t)u_y - \frac{1}{2}\alpha G(t)u_x^2, \\
T_G^z =& -dG(t)u_z;
\end{aligned}
$$

$$
\begin{aligned}
T_F^t =& -\frac{1}{2}zF(t)u_x + \frac{1}{4}bzF(t)u_{xxx}, \\
T_F^x =& -azF(t)u_x + \frac{3}{4}bzF(t)u_{txx} - \alpha zF(t)u_x u_y - \frac{1}{4}bzF'(t)u_{xx} - \frac{1}{2}zF(t)u_t, \\
T_F^y =& -czF(t)u_y - \frac{1}{2}\alpha zF(t)u_x^2, \\
T_F^z =& -dzF(t)u_z.
\end{aligned}
$$

5. Conclusion

This research work examines the 3D KP-BBM Equation (1.4), which recently appeared in [51] and has numerous applications. Firstly, Lie point symmetries of the equation were computed and then these were used to transform (1.4) to nonlinear ordinary differential equations. Direct integration of one of the ordinary differential equations provided us with an exact solution in terms of the incomplete elliptic integral. We further engage in a detailed reduction process to transform the KP-BBM equation as much as possible into solvable differential equations from which diverse solutions of interest were achieved. These solutions include trigonometric, hyperbolic, algebraic-trigonometric, algebraic-exponential, algebraic-hyperbolic, and complex-algebraic functions. Furthermore, the simplest equation approach was utilized via the Riccati and Bernoulli equations, and solutions to (1.4) were secured. The solutions obtained were presented graphically with an adequate choice of parametric values. Finally, the multiplier method was utilized to generate conserved vectors for (1.4). Moreover, Noether's theorem was invoked, and conserved vectors were obtained for $\beta = 2\alpha$.

References

[1] M. A. Abdou, *Exact periodic wave solutions to some nonlinear evolution equations*, Int. J. Nonlinear Sci., *6* (2008), 145–153.

[2] M. J. Ablowitz and P. A. Clarkson, *Nonlinear Evolution Equations and Inverse Scattering*, Solitons, Cambridge University Press, Cambridge, UK, 1991.

[3] O. D. Adeyemo, C. M. Khalique, Y. S. Gasimov, and F. Villecco, *Variational and non-variational approaches with Lie algebra of a generalized (3+1)-dimensional nonlinear potential Yu-Toda-Sasa-Fukuyama equation in Engineering and Physics*, Alex. Eng. J., *63* (2023), 17–43.

[4] O. D. Adeyemo, L. Zhang, and C. M. Khalique, *Bifurcation theory, Lie group-invariant solutions of subalgebras and conservation laws of a generalized (2+1)-dimensional BK equation Type II in plasma physics and fluid mechanics*, Mathematics, *10* (2022), 2391.

[5] O. D. Adeyemo, L. Zhang, and C. M. Khalique, *Optimal solutions of Lie subalgebra, dynamical system, travelling wave solutions and conserved currents of (3+1)-dimensional generalized Zakharov–Kuznetsov equation type I*, Eur. Phys. J. Plus, *137* (2022), 954.

[6] O. D. Adeyemo and C. M. Khalique, *An optimal system of Lie subalgebras and group-invariant solutions with conserved currents of a (3+1)-D fifth-order nonlinear model with applications in electrical electronics, chemical engineering and pharmacy*, J. Nonlinear Math. Phys., *30* (2023), 843–916.

[7] O. D. Adeyemo, T. Motsepa, and C. M. Khalique, *A study of the generalized nonlinear advection-diffusion equation arising in engineering sciences*, Alex. Eng. J., *61* (2022), 185–194.

[8] O. D. Adeyemo, *Applications of cnoidal and snoidal wave solutions via an optimal system of subalgebras for a generalized extended (2+1)-D quantum Zakharov-Kuznetsov equation with power-law nonlinearity in oceanography and ocean engineering*, J. Ocean Eng. Sci., *9* (2024), 126–153.

[9] A. S. Alhasanat, *On the minimal speed determinacy of traveling waves to a three-species competition model*, Adv. Math. Models Appl., *8* (2023), 34–44.

[10] U. Al Khawajaa, H. Eleuchb, and H. Bahloulid, *Analytical analysis of soliton propagation in microcavity wires*, Results Phys., *12* (2019), 471–474.

[11] B. Babajanov and F. Abdikarimov, Expansion method for the loaded modified Zakharov-Kuznetsov equation, Adv. Math. Models Appl., *7* (2022), 168–177.

[12] Y. Chen and Z Yan, *New exact solutions of (2+1)-dimensional Gardner equation via the new sine-Gordon equation expansion method*, Chaos Solitons Fract., *26* (2005), 399–406.

[13] S. S. Chen, B. Tian, J. Chai, X. Y. Wu, and Z. Du, *Lax pair, Lax pair binary Darboux transformations and dark-soliton interaction of a fifth-order defocusing nonlinear Schrödinger equation for the attosecond pulses in the optical fiber communication*, Wave Random Complex, *30* (2020), 389–402.

[14] C. Chun and R. Sakthivel, *Homotopy perturbation technique for solving two point boundary value problems-comparison with other methods*, Comput. Phys. Commun., *181* (2010), 1021–1024.

[15] M. T. Darvishi and M. Najafi, *A modification of extended homoclinic test approach to solve the (3+1)-dimensional potential-YTSF equation*, Chin. Phys. Lett., *28* (2011), 040202.

[16] S.T. Demiray and S. Duman, *MTEM to the (2+1)-dimensional ZK equation and Chafee-Infante equation*, Adv. Math. Models Appl., *6* (2021), 63–70.

[17] X. X. Du, B. Tian, Q. X. Qu, Y. Q. Yuan, and X. H. Zhao, *Lie group analysis, solitons, self-adjointness and conservation laws of the modified Zakharov-Kuznetsov equation in an electron-positron-ion magnetoplasma*, Chaos Solitons Fract., *134* (2020), 109709.

[18] A.S. Farajov, *On a solvability of the nonlinear inverse boundary value problem for the Boussinesq equation*, Adv. Math. Models Appl., *7* (2022), 241–248.

[19] X. Y. Gao, Y. J. Guo, and W. R. Shan, *Shallow water in an open sea or a wide channel: Auto- and non-auto-Bäcklund transformations with solitons for a generalized (2+1)-dimensional dispersive long-wave system*, Chaos Solit. Fractals, *138* (2020), 109950.

[20] X. Y. Gao, Y. J. Guo, and W. R. Shan, *Water-wave symbolic computation for the Earth, Enceladus and Titan: The higher-order Boussinesq-Burgers system, auto-and non-auto-Bäcklund transformations*, Appl. Math. Lett.,

104 (2020), 106170.
[21] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 7th Edition, Academic Press, New York, 2007.
[22] C. H. Gu, *Soliton Theory and Its Application*, Zhejiang Science and Technology Press, Zhejiang, China, 1990.
[23] J. H. He and X. H. Wu, *Exp-function method for nonlinear wave equations*, Chaos Solitons Fract., *30* (2006), 700–708.
[24] M. F. Hoque, H. O. Roshid, and F. S. Alshammari, *Higher-order rogue wave solutions of the Kadomtsev Petviashvili-Benjanim Bona Mahony (KP-BBM) model via the Hirota bilinear approach*, Phys. Scr., *95* (2020), 115215.
[25] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, UK, 2004.
[26] W. X. Ma, T. Huang, and Y. Zhang, *A multiple exp-function method for nonlinear differential equations and its application*, Phys. Scr., *82* (2010), 065003.
[27] C. M. Khalique and O. D. Adeyemo, *A study of (3+1)-dimensional generalized Korteweg-de Vries-Zakharov-Kuznetsov equation via Lie symmetry approach*, Results Phys., *18* (2020), 103197.
[28] C. M. Khalique and O. D. Adeyemo, *Closed-form solutions and conserved vectors of a generalized (3+1)-dimensional breaking soliton equation of engineering and nonlinear science*, Mathematics, *8* (2020), 1692.
[29] N. A. Kudryashov and N. B. Loguinova, *Extended simplest equation method for nonlinear differential equations*, Appl. Math. Comput., *205* (2008), 396-402.
[30] N. A. Kudryashov, *Simplest equation method to look for exact solutions of nonlinear differential equations*, Chaos Solitons Fract., *24* (2005), 1217–1231
[31] S. Liu, *Multiple rogue wave solutions for the (3+1)-dimensional generalized Kadomtsev-Petviashvili Benjamin-Bona-Mahony equation*, Chinese. J. Phys., *68* (2020), 961–970.
[32] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, New York, USA, 1991.
[33] A. Mekki and M. M. Ali, *Numerical simulation of Kadomtsev-Petviashvili-Benjamin-Bona-Mahony equations using finite difference method*, Appl. Math. Comput., *219* (2013), 11214–11222.
[34] I. E. Mhlanga and C. M. Khalique, *A study of a generalized Benney–Luke equation with time-dependent coefficients*, Nonlinear Dyn., *90* (2017), 1535–1544.
[35] E. Noether, *Invariante variations probleme, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen*, Math-phys. Klasse, *2* (1918), 235–257.
[36] P. J. Olver, *Applications of Lie Groups to Differential Equations*, second ed., Springer-Verlag, Berlin, Germany, 1993.
[37] M. S. Osman, *One-soliton shaping and inelastic collision between double solitons in the fifth-order variable-coefficient Sawada-Kotera equation*, Nonlinear Dynam., *96* (2019), 1491–1496.
[38] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, USA, 1982.
[39] A. H. Salas and C. A. Gomez, *Application of the Cole-Hopf transformation for finding exact solutions to several forms of the seventh-order KdV equation*, Math. Probl. Eng., 2010.
[40] M. Song, C. X. Yang, B. and G. Zhang, *Exact solitary wave solutions of the Kadomtsev-Petviashvili-Benjamin-Bona-Mahony equation*, Appl. Math. Comput., *217* (2010), 1334–1339.
[41] S. Q. Tang, X. L. Huang, and W. T. Huang, *Bifurcations of travelling wave solutions for the generalized KP-BBM equation*, Appl. Math. Comput., *216* (2010), 2881–2890.
[42] K. U. H. Tariq and A. R. Seadawy, *Soliton solutions of (3+1)-dimensional Korteweg-de Vries Benjamin-Bona-Mahony, Kadomtsev-Petviashvili Benjamin-Bona-Mahony and modified Korteweg de Vries-Zakharov-Kuznetsov equations and their applications in water waves*, J. King. Saud. Univ. Sci., *31* (2019), 8–13.
[43] M. Wang, X. Li, and J. Zhang, *The* $(G'/G)-$ *expansion method and travelling wave solutions for linear evolution equations in mathematical physics*, Phys. Lett. A, *24* (2005), 1257–1268.
[44] A. M. Wazwaz, *Exact soliton and kink solutions for new (3+1)-dimensional nonlinear modified equations of wave propagation*, Open Eng., *7* (2017), 169–174.
[45] A. M. Wazwaz, *Exact solutions of compact and noncompact structures for the KP-BBM equation*, Nonlinear. Dyn., *169* (2005), 700–712.

[46] A. M. Wazwaz, *Partial Differential Equations*, CRC Press, Boca Raton, Florida, USA, 2002.
[47] A. M. Wazwaz, *The tanh and sine-cosine method for compact and noncompact solutions of nonlinear Klein Gordon equation*, Appl. Math. Comput., *167* (2005) 1179–1195.
[48] A. M. Wazwaz, *The tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations*, Appl. Math. Comput., *169* (2005), 321–338.
[49] A. M. Wazwaz, *Traveling wave solution to (2+1)-dimensional nonlinear evolution equations*, J. Nat. Sci. Math., *1* (2007), 1–13.
[50] J. Weiss, M. Tabor, and G. Carnevale, *The Painlévé property and a partial differential equations with an essential singularity*, Phys. Lett. A, *109* (1985), 205–208.
[51] Y. Xie and L. Li. *Multiple-order breathers for a generalized (3+1)-dimensional Kadomtsev-Petviashvili Benjamin-Bona-Mahony equation near the offshore structure*, Math. Comput. Simul., *193* (2022), 19–31.
[52] Y. Yin, B. Tian, X.Y. Wu, H.M. Yin, Ch. R. Zhang, *Lump waves and breather waves for a (3+1)-dimensional generalized Kadomtsev-Petviashvili Benjamin-Bona-Mahony equation for an offshore structure*, Mod. Phys. Lett. B., *32* (2018), 1850031.
[53] X. Zeng and D. S. Wang, *A generalized extended rational expansion method and its application to (1+1)-dimensional dispersive long wave equation*, Appl. Math. Comput., *212* (2009), 296–304.
[54] L. Zhang and C. M. Khalique, *Classification and bifurcation of a class of second-order ODEs and its application to nonlinear PDEs*, Discrete Contin. Dyn. Syst. - S., *11* (2018), 777–790.
[55] C. R. Zhang, B. Tian, Q. X. Qu, L. Liu, and H. Y. Tian, *Vector bright solitons and their interactions of the couple Fokas–Lenells system in a birefringent optical fiber*, Z. Angew. Math. Phys., *71* (2020), 1–19.
[56] Y. Zhou, M. Wang, and Y. Wang, *Periodic wave solutions to a coupled KdV equations with variable coefficients*, Phys. Lett. A, *308* (2003), 31–36.