Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 13, No. 1, 2025, pp. 294-306
DOI:10.22034/cmde.2024.59327.2522

Hopf bifurcation and Turing instability in a cross-diffusion prey-predator system with group defense behavior

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Abstract

This paper is concerned with a cross-diffusion prey-predator system in which the prey species is equipped with the group defense ability under the Neumann boundary conditions. The tendency of the predator to pursue the prey is expressed in the cross-diffusion coefficient, which can be positive, zero, or negative. We first select the environmental protection of the prey population as a bifurcation parameter. Next, we discuss the Turing instability and the Hopf bifurcation analysis on the proposed cross-diffusion system. We show that the system without cross-diffusion is stable at the constant positive stationary solution but it becomes unstable when the cross-diffusion appears in the system. Furthermore, the stability of bifurcating periodic solutions and the direction of Hopf bifurcation are examined.

Keywords. Turing instability, Prey-predator model, Hopf bifurcation, Cross-diffusion.
2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

The prey-predator system usually comes from the phenomenon of pursuit and evasion of species. The prey moves away from the predator to expand the distance from the predator. In population dynamics, one of the strategies used to predator-prey interaction is group defense of species. This strategy is based on when the number of the prey is large enough, the prey to defend gathers together, instead of escape [22]. The group defense behavior of species in the predator-prey system has to be modeled by a nonmonotonic functional response, that has several properties described in [17]. For more information about the group defense mechanism, see [3, 15]. As examples of functional responses describing the group defense mechanism we can mention Holling-type IV [12], simplified Monod-Haldane [17], Ivlev-type, and Ivlev-like [7]. Also, the functional response

$$R(u) = \frac{mu}{c + u^p}, \quad p > 1, \tag{1.1}$$

discussed in [10], describes the group defense behavior in prey species.

In [11], Patra et al. introduced the prey-predator system with the functional response (1.1) as follows

$$\begin{cases} \frac{du}{dt} = ru(1 - \frac{u}{k}) - \dfrac{muv}{u^p + c}, \\ \frac{dv}{dt} = (d - \dfrac{e}{u + a})v^2, \end{cases} \tag{1.2}$$

where u and v denote the population densities of the prey and the predator. The parameter k represents the maximum population size of the prey that the environment can maintain indefinitely and r represents the growth rate of the prey population in the scarcity of any limitation. d is the reproduction rate of the predator, m is the maximum consumption rate of prey by the predator, c is the environmental protection of the prey, a is the residual loss of the predator species due to severe scarcity of the prey species and e shows the maximum mortality rate of the predator

Received: 23 November 2023 ; Accepted: 27 March 2024.
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population. All the mentioned parameters are considered as positive constants. Patra et al. studied the stability of stationary points and the Hopf bifurcation for system (1.2) in the presence of delay. Also, the effect of delay on the logistic growth of the prey for the same model was discussed [10].

In this paper, we study the cross-diffusion prey-predator system,

$$\begin{cases} u_t - \mu\Delta u = ru(1-\frac{u}{k}) - \dfrac{muv}{u^p+c}, & (x,t)\in\Omega\times(0,\infty),\\ v_t - \Delta[(\alpha+\beta u)v] = (d-\dfrac{e}{u+a})v^2, & (x,t)\in\Omega\times(0,\infty),\\ u(x,0)=u_0(x)\geq 0, v(x,0)=v_0(x)\geq 0, & x\in\Omega,\\ \partial_\nu u = \partial_\nu v = 0, & (x,t)\in\partial\Omega\times(0,\infty), \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and ν is the outward unit normal vector on $\partial\Omega$. The diffusion coefficient μ represents the diffusive rate of the prey and α represents the diffusive rate of the predator. The cross-diffusion coefficient β demonstrates the tendency of the predator to pursue the prey. Also, μ and α are considered as positive constants and $\beta\in\mathbb{R}$. The positive cross-diffusion coefficient in predator species indicates that the predator species tend to move in the direction of the lower density of the prey species. The negative cross-diffusion coefficient in predator species means the predator species tend to move in the direction of the higher density of the prey species.

In the modeling of several chemical, biological, and physical phenomena, the diffusion, and cross-diffusion terms have crucial effect. Alan Turing in [16] focused on the effect of diffusion on the stability of a reaction-diffusion system. He derived that the diffusion can lead an instability of the system. In [13], the authors generally analyzed the effect of diffusion and cross-diffusion on the stability in reaction-diffusion systems. In [1] the authors found the Turing instability using the local stability analysis and they analyzed Turing patterns caused by cross-diffusion for a reaction-diffusion prey-predator system. Furthermore, there has been some activity on Turing patterns driven by diffusion or cross-diffusion, see [4, 5, 9, 14, 18–20, 23]. In some cases, many researchers have paid more attention to the effect of diffusion and cross-diffusion on the occurrence of Hopf bifurcation for the prey-predator model. For instance, in [8], the authors studied the stability of bifurcation solutions and the direction Hopf bifurcation for a predator-prey model with memory-based diffusion. The existence, direction, and stability of the Hopf bifurcation were studied in a predator-prey model with general group defense for prey [22]. The authors carried out the effect of the delay and diffusion on the Hopf bifurcation. In [2], the existence of two intersecting curves of positive steady-state solutions to system (1.3) has been studied.

As far as we know there is no result about the Hopf bifurcation and Turing instability for the cross-diffusion system (1.3). Our aim is to study the two mentioned concepts for the system (1.3). We obtain the sufficient conditions for the existence of Turing instability in the system (1.3). That is in the absence of cross-diffusion the coexistence equilibrium point of system (1.3) is stable and it becomes unstable when the cross-diffusion appears in the system. We also investigate the direction and stability of periodic solutions of the Hopf bifurcation via the center manifold theorem and the normal form theory. We show that the bifurcating periodic solutions through the Hopf bifurcation are stable.

The structure of the rest of the paper is as follows: In section 2, we discuss the Turing instability in the system (1.3). In section 3, the direction of the Hopf bifurcation and stability of periodic solutions are investigated. In section 4, we express some numerical examples to confirm the theoretical results.

2. Turing instability

We consider the instability of coexistence stationary solution driven by the cross-diffusion for system (1.3). The constant stationary solutions of system (1.3) are given by

$$U_* = (u_*, v_*) = \left(\frac{e-ad}{d}, \frac{r}{mk}(k+\frac{ad-e}{d})((\frac{e-ad}{d})^p+c)\right), \quad (0,0), \quad (k,0). \tag{2.1}$$

The components of U_* are positive when

$$0 < \frac{e}{d} - a < k, \tag{2.2}$$

that is $u_* < k$. Set

$$X := \{(\psi, \varphi) \in \mathbb{H}^2(\Omega) \times \mathbb{H}^2(\Omega) : \partial_\nu \psi = \partial_\nu \varphi = 0\}. \tag{2.3}$$

The linearization of (1.3) at U_* is given by

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} := L \begin{pmatrix} u \\ v \end{pmatrix} = (D\Delta + G) \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.4}$$

where

$$L := \begin{pmatrix} \mu\Delta - \frac{ru_*}{k} + \frac{pu_*^p r(1 - \frac{u_*}{k})}{u_*^p + c} & -\frac{mu_*}{u_*^p + c} \\ \beta v_* \Delta + \frac{d^2 v_*^2}{e} & (\alpha + \beta u_*)\Delta \end{pmatrix}. \tag{2.5}$$

and

$$D := \begin{pmatrix} \mu & 0 \\ \beta v_* & \alpha + \beta u_* \end{pmatrix}, \; G = \begin{pmatrix} G_1 & G_2 \\ G_3 & 0 \end{pmatrix} := \begin{pmatrix} \frac{-ru_*}{k} + \frac{pu_*^p r(1 - \frac{u_*}{k})}{u_*^p + c} & -\frac{mu_*}{u_*^p + c} \\ \frac{d^2 v_*^2}{e} & 0 \end{pmatrix}. \tag{2.6}$$

Consider c as the bifurcation parameter and denote the eigenvalues of $-\Delta$ in Ω under Neumann boundary conditions by

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_n \le .... \tag{2.7}$$

Also, we represent the corresponding normalized eigenfunction of λ_n by φ_n for $n = 0, 1, 2, ...$ Then the eigenvalues of L are obtained from the equation

$$\lambda^2 - T_n(c)\lambda + D_n(c) = 0, \tag{2.8}$$

for $n \in \mathbb{N} \cup \{0\}$, related to the matrix $L_n = -\lambda_n D + G$, where

$$T_0(c) = -\frac{ru_*}{k} + \frac{pu_*^p r(1 - \frac{u_*}{k})}{u_*^p + c}, \tag{2.9}$$

$$D_0(c) = \frac{d^2 m u_* v_*^2}{e(u_*^p + c)}, \tag{2.10}$$

$$T_n(c) = -(\beta u_* + \alpha + \mu)\lambda_n + T_0(c) = -(\beta u_* + \alpha + \mu)\lambda_n - \frac{ru_*}{k} + \frac{pu_*^p r(1 - \frac{u_*}{k})}{u_*^p + c}, \tag{2.11}$$

$$\begin{aligned} D_n(c) &= \mu(\beta u_* + \alpha)\lambda_n^2 - \left((\beta u_* + \alpha) T_0(c) + \frac{\beta m u_* v_*}{u_*^p + c} \right) \lambda_n + D_0(c) \\ &= \mu(\alpha + \beta u_*)\lambda_n^2 - \left((\alpha + \beta u_*)\left(\frac{-ru_*}{k} + \frac{pu_*^p r(1 - \frac{u_*}{k})}{u_*^p + c}\right) + \frac{\beta m u_* v_*}{u_*^p + c} \right) \lambda_n + \frac{d^2 m u_* v_*^2}{e(u_*^p + c)}. \end{aligned} \tag{2.12}$$

The roots of Equation (2.8) are given by

$$\lambda = \frac{T_n(c)}{2} \pm i \frac{\sqrt{4D_n(c) - (T_n(c))^2}}{2} =: \alpha_n(c) \pm i\beta_n(c). \tag{2.13}$$

Under the condition

$$(\frac{p+1}{p}) u_* < k, \tag{2.14}$$

the equation $T_0(c) = 0$ has the unique positive solution

$$c_* := u_*^{p-1}(kp - (p+1)u_*). \tag{2.15}$$

We can rewrite $D_n(c)$ as

$$D_n(c) := \beta E_n + F_n, \tag{2.16}$$

where

$$\begin{aligned} E_n &= \mu u_* \lambda_n^2 - u_* \lambda_n T_0(c) - \frac{m u_* v_* \lambda_n}{u_*^p + c} \\ &= \mu u_* \lambda_n^2 - u_* \lambda_n T_0(c) - u_* \lambda_n r(1 - \frac{u_*}{k}) \\ &= \mu u_* \lambda_n^2 - u_* \lambda_n (\frac{-r u_*}{k} + \frac{p u_*^p r(1 - \frac{u_*}{k})}{u_*^p + c}) - u_* \lambda_n r(1 - \frac{u_*}{k}), \end{aligned} \tag{2.17}$$

$$\begin{aligned} F_n &= \mu \alpha \lambda_n^2 - \alpha \lambda_n T_0(c) + D_0(c) \\ &= \mu \alpha \lambda_n^2 - \alpha \lambda_n (\frac{-r u_*}{k} + \frac{p u_*^p r(1 - \frac{u_*}{k})}{u_*^p + c}) + \frac{d^2 m u_* v_*^2}{e(u_*^p + c)}. \end{aligned} \tag{2.18}$$

Theorem 2.1. *Assume (2.2) is satisfied. If $c < c_*$, then the equilibrium point U_* is unstable.*

Proof. Since $T_0(c)$ is deceasing, for $c < c_*$, $0 = T_0(c_*) < T_0(c)$. Also $D_0(c) > 0$ for all $c \geq 0$. Hence Equation (2.8) has a solution λ with $Re(\lambda) > 0$. Then the stationary solution U_* is unstable for $c < c_*$. □

In the next theorem, we investigate the Turing instability in system (1.3).

Theorem 2.2. *Assume $c > c_*$ and let (2.14) be satisfied.*

- (i) *If $\beta = 0$ (system (1.3) without cross-diffusion), then the system is locally asymptotically stable at U_*.*
- (ii) *Let $\beta > 0$.*
  - (1) *If*

$$\mu \lambda_1 > r(1 - \frac{u_*}{k}), \tag{2.19}$$

then system (1.3) is locally asymptotically stable at U_*.

  - (2) *If*

$$\mu \lambda_1 < r(1 - \frac{2u_*}{k}), \tag{2.20}$$

then system (1.3) is unstable at U_* for $\beta > \frac{-F_1}{E_1}$.

- (iii) *Let $\beta < 0$.*
  - (1) *If*

$$\alpha + \beta u_* > 0, \tag{2.21}$$

then system (1.3) is locally asymptotically stable at U_*.

  - (2) *If*

$$\beta < \frac{-F_1}{E_1} < 0, \tag{2.22}$$

then system (1.3) is unstable at U_*.

Proof. Firstly note that for $c > c_*$, $T_0(c) < 0$ and $D_0(c) > 0$. So, for $c > c_*$

$$F_1 = \mu\alpha\lambda_1^2 - \alpha\lambda_1 T_0(c) + D_0(c) > 0.$$

Now we proceed the proof.
(i). Let $\beta = 0$. From (2.11) and (2.12), for $n \in \mathbb{N}$, we get

$$T_n(c) = -(\mu+\alpha)\lambda_n + T_0(c) < -(\mu+\alpha)\lambda_n < 0,$$
$$D_n(c) = \mu\alpha\lambda_n^2 - \alpha T_0(c)\lambda_n + D_0(c) > 0.$$

Then for $n \in \mathbb{N} \cup \{0\}$, $T_n(c) < 0$ and $D_n(c) > 0$. So according to Equation (2.8), all eigenvalues of the operator L have negative real parts. Therefore, system (1.3) is locally stable at U_*.
(ii). Let $\beta > 0$. We have $T_n(c) < 0$ for $c > c_*$. By (2.17), we have

$$\begin{aligned} E_n =&\mu u_*\lambda_n^2 - u_*\lambda_n T_0(c) - u_*\lambda_n r(1-\frac{u_*}{k}) \\ =&u_*\lambda_n(\mu\lambda_n - r(1-\frac{u_*}{k}) - T_0(c)). \end{aligned}$$

With the condition (2.19) and $\lambda_{n+1} > \lambda_n$ for $n \in \mathbb{N} \cup \{0\}$, we obtain $E_n > 0$. Since $F_n > 0$ and by (2.16), then $D_n > 0$. So the case (1) is satisfied. From (2.20), and (2.17) for $n = 1$, we obtain

$$\begin{aligned} E_1 &= u_*\lambda_1\left(\mu\lambda_1 - r(1-\frac{u_*}{k}) + \frac{ru_*}{k} - \frac{pu_*^p r(1-\frac{u_*}{k})}{u_*^p + c}\right) \\ &= u_*\lambda_1\left(\mu\lambda_1 - r(1-\frac{2u_*}{k}) - \frac{pu_*^p r(1-\frac{u_*}{k})}{u_*^p + c}\right) < 0. \end{aligned}$$

Also we have $D_1(c) = \beta E_1 + F_1 < 0$ for $\beta > \dfrac{-F_1}{E_1}$. Then Equation (2.8) has a solution with positive real part. Hence U_* is unstable for system (1.3).
(iii). Let $\beta < 0$. We know $T_0(c) < 0$ and $D_0(c) > 0$ for $c > c_*$. Using conditions (2.21), (2.11), and (2.12), we have

$$T_n(c) = -(\beta u_* + \alpha + \mu)\lambda_n + T_0(c) < 0,$$
$$D_n(c) = \mu(\beta u_* + \alpha)\lambda_n^2 - \left((\beta u_* + \alpha)T_0(c) + \frac{\beta m u_* v_*}{u_*^p + c}\right)\lambda_n + D_0(c) > 0,$$

for $n \in \mathbb{N}$. So the system (1.3) is locally stable at U_*.

By (2.22), we get $D_1(c) = \beta E_1 + F_1 < 0$. Then Equation (2.8) has a solution with a positive real part. Hence U_* is unstable. □

3. Hopf bifurcation analysis

We investigate the Hopf bifurcation for system (1.3) at the equilibrium point U_*. Also, we study the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions.

Theorem 3.1. *Let (2.14) be satisfied. For $c = c_*$ we have*

(i) if $\beta > 0$, and (2.19) is satisfied, then system (1.3) under the condition $k \neq 3u_*$ has a Hopf bifurcation at U_*;
(ii) let $\beta < 0$ and (2.21) be satisfied. Then system (1.3) under the condition $k \neq 3u_*$ has a Hopf bifurcation at U_*.

Proof. From (2.15), we get $T_0(c_*) = 0$ and $D_0(c_*) > 0$. Let $\beta > 0$. For $n = 1, 2, ...$, we have

$$T_{n+1}(c_*) = -(\mu + \alpha + \beta u_*)\lambda_{n+1} < -(\mu + \alpha + \beta u_*)\lambda_n = T_n(c_*) < 0,$$

using (2.7) and (2.19) for $n = 0, 1, 2, ...$, we obtain

$$D_{n+1}(c_*) = \mu(\alpha + \beta u_*)\lambda_{n+1}^2 - \beta u_* r(1-\frac{u_*}{k})\lambda_{n+1} + D_0(c_*)$$

$$= \mu\alpha\lambda_{n+1}^2 + \beta u_* \lambda_{n+1}(\mu\lambda_{n+1} - r(1 - \frac{u_*}{k})) + D_0(c_*)$$
$$> D_n(c_*) > 0.$$

Let $\beta < 0$. Using condition (2.21) , we have

$$T_{n+1}(c_*) = -(\mu + \alpha + \beta u_*)\lambda_{n+1} < -(\mu + \alpha + \beta u_*)\lambda_n = T_n(c_*) < 0,$$

and

$$\begin{aligned} D_{n+1}(c_*) &= \mu(\alpha + \beta u_*)\lambda_{n+1}^2 - \beta u_* r(1 - \frac{u_*}{k})\lambda_{n+1} + D_0(c_*) \\ &> \mu(\alpha + \beta u_*)\lambda_n^2 - \beta u_* r(1 - \frac{u_*}{k})\lambda_n + D_0(c_*) \\ &= D_n(c_*) > 0. \end{aligned}$$

Then in these cases, L has a pair imaginary eigenvalues at $c = c_*$. On the other hand, since $k \neq 3u_*$,

$$\alpha_0^{'}(c_*) = \frac{r(k - 3u_*)}{2k(u_*^p + c_*)} \neq 0. \tag{3.1}$$

where $\alpha_0(c)$ is defined by (2.13). Hence the cases (i) and (ii) follow from the Hopf theorem [6]. □

In the sequel, we investigate the Hopf bifurcation for the system (1.3) on $\Omega = (0, l\pi)$ for $l \in \mathbb{R}^+$. System (1.3) on $\Omega = (0, l\pi)$ has the following form

$$\begin{cases} u_t = \mu u_{xx} + ru(1 - \frac{u}{k}) - \dfrac{muv}{u^p + c}, & x \in (0, l\pi),\ t > 0, \\ v_t = \big((\alpha + \beta u)v\big)_{xx} + (d - \dfrac{e}{u + a})v^2, & x \in (0, l\pi),\ t > 0, \\ u_x(0,t) = u_x(l\pi, t) = 0, & t > 0, \\ v_x(0,t) = v_x(l\pi, t) = 0, & t > 0. \end{cases} \tag{3.2}$$

It is remarked that for $n = 0, 1, 2, ...$, the eigenvalue λ_n and the corresponding eigenfunction φ_n of the operator $u \longrightarrow -u_{xx}$ on $(0, l\pi)$ with zero Neumann boundary conditions are expressed by

$$\lambda_0 = 0,\ \varphi_0 = \sqrt{\frac{1}{l\pi}},$$
$$\lambda_n = \frac{n^2}{l^2},\ \varphi_n(x) = \sqrt{\frac{2}{l\pi}}\cos(\frac{nx}{l}), \quad n = 1, 2, ... \ .$$

Let

$$X = \{(\varphi, \psi) \in \mathbb{H}^2(0, l\pi) \times \mathbb{H}^2(0, l\pi) : \varphi_x(0,t) = \varphi_x(l\pi, t) = 0,\ \psi_x(0,t) = \psi_x(l\pi, t) = 0\}.$$

For the sake of convenience, in system (3.2) we translate the equilibrium point U_* to the origin. Then system (3.2) is rewritten as,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = D\begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + G\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u,v,c) \\ f_2(u,v,c) \end{pmatrix} =: LU + F(U,c), \tag{3.3}$$

where D and G are defined by (2.6) and

$$\begin{aligned} f_1(u,v,c) = {} & (\frac{-r}{k} + \frac{mv_* p u_*^{p-1}((u_*^p + c)(p+1) - 2pu_*^p)}{2(u_*^p + c)^3})u^2 - (\frac{m(u_*^p + c - pu_*^p)}{(u_*^p + c)^2})uv \\ & + (\frac{mpu_*^{p-1}((u_*^p + c)(p+1) - 2pu_*^p)}{2(u_*^p + c)^3})u^2 v + (\frac{mv_* u_*^{p-2}(p^3 - p)}{6(u_*^p + c)^2} - \frac{mv_* u_*^{2p-2}p^3}{(u_*^p + c)^3} \\ & + \frac{mv_* u_*^{3p-2}p^3}{(u_*^p + c)^4})u^3 + O(|u|^4, |u|^3|v|), \end{aligned} \tag{3.4}$$

$$f_2(u,v,c) = \beta(uv)_{xx} - (\frac{d^3 v_*^2}{e^2})u^2 + (\frac{2d^2 v_*}{e})uv - (\frac{2d^3 v_*}{e^2})u^2 v + (\frac{d^2}{e})uv^2 + (\frac{d^4 v_*^2}{e^3})u^3$$

$$+ O(|u|^4, |v||u|^3, |v|^2|u|^2). \tag{3.5}$$

In this paper we denote the standard inner product in $\mathbb{L}^2(0, l\pi) \times \mathbb{L}^2(0, l\pi)$ by $< g, h >= \int_0^{l\pi} \bar{g}^T h dx$. Now define the operator L^* on X by

$$L^* \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mu\Delta - \dfrac{ru_*}{k} + \dfrac{pu_*^p r(1 - \dfrac{u_*}{k})}{u_*^p + c} & \beta v_* \Delta + \dfrac{d^2 v_*^2}{e} \\ -\dfrac{mu_*}{u_*^p + c} & (\alpha + \beta u_*)\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.6}$$

Using Green's formulas and the zero Neumann boundary condition, for $W, U \in X$, we have

$$\begin{aligned}
< W, LU > &=< W, D\Delta(U) + G(U) > \qquad (3.7)\\
&= \int_0^{l\pi} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \cdot \left[\begin{pmatrix} \mu & 0 \\ \beta v_* & \alpha + \beta u_* \end{pmatrix} \Delta \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} G_1 u + G_2 v \\ G_3 u \end{pmatrix} \right] dx \\
&= \int_0^{l\pi} \left[w_1 \Big(\mu\Delta u + G_1 u + G_2 v \Big) + w_2 \Big(\beta v_* \Delta u + (\alpha + \beta u_*)\Delta v + G_3 u \Big) \right] dx \\
&= \int_0^{l\pi} \Big[\mu u \Delta w_1 + u G_1 w_1 + v G_2 w_1 + u \beta v_* \Delta w_2 + v(\alpha + \beta u_*)\Delta w_2 + u G_3 w_2 \Big] dx \\
&= \int_0^{l\pi} \left[\Big(\mu\Delta w_1 + G_1 w_1 + \beta v_* \Delta w_2 + G_3 w_2 \Big) u + \Big(G_2 w_1 + (\alpha + \beta u_*)\Delta w_2 \Big) v \right] dx \\
&= \int_0^{l\pi} \left[\begin{pmatrix} \mu & \beta v_* \\ 0 & \alpha + \beta u_* \end{pmatrix} \Delta \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} G_1 & G_3 \\ G_2 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] \cdot \begin{pmatrix} u \\ v \end{pmatrix} dx \\
&=< L^* W, U > . \qquad (3.8)
\end{aligned}$$

where D and G are defined by (2.6). Then L^* is the adjoint of L. Now put

$$q := \begin{pmatrix} \dfrac{e\beta_0 i}{d^2 v_*^2} \\ 1 \end{pmatrix}, \ q^* := \frac{1}{2l\pi} \begin{pmatrix} \dfrac{d^2 v_*^2 i}{e\beta_0} \\ 1 \end{pmatrix}, \tag{3.9}$$

where i is the imaginary unit and $\beta_0 := \beta_0(c_*) = \sqrt{D_0} = \sqrt{\dfrac{mu_* d^2 v_*^2}{e(u_*^p + c_*)}}$. Hence

$$L(c_*)q = \begin{pmatrix} \mu\Delta & -\dfrac{mu_*}{u_*^p + c_*} \\ \beta v_* \Delta + \dfrac{d^2 v_*^2}{e} & (\alpha + \beta u_*)\Delta \end{pmatrix} \begin{pmatrix} \dfrac{e\beta_0 i}{d^2 v_*^2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\dfrac{mu_*}{u_*^p + c_*} \\ \dfrac{d^2 v_*^2}{e} (\dfrac{e\beta_0 i}{d^2 v_*^2}) \end{pmatrix}. \tag{3.10}$$

Since

$$\frac{mu_*}{u_*^p + c_*} = \frac{e\beta_0^2(c_*)}{d^2 v_*^2}, \tag{3.11}$$

from (3.10) we have

$$L(c_*)q = \begin{pmatrix} -\dfrac{e\beta_0^2}{d^2 v_*^2} \\ i\beta_0 \end{pmatrix} = i\beta_0 q. \tag{3.12}$$

Again using (3.11), we obtain

$$L^*(c_*)q^* = \frac{1}{2l\pi} \begin{pmatrix} \mu\Delta & \beta v_* \Delta + \dfrac{d^2 v_*^2}{e} \\ -\dfrac{mu_*}{u_*^p + c_*} & (\alpha + \beta u_*)\Delta \end{pmatrix} \begin{pmatrix} \dfrac{d^2 v_*^2 i}{e\beta_0} \\ 1 \end{pmatrix} = \frac{1}{2l\pi} \begin{pmatrix} \dfrac{d^2 v_*^2}{e} \\ \dfrac{-mu_* d^2 v_*^2 i}{e\beta_0(u_*^p + c_*)} \end{pmatrix}$$

$$= \frac{1}{2l\pi}\begin{pmatrix}\frac{d^2v_*^2}{e}\\ -i\beta_0\end{pmatrix} = -i\beta_0 q^*. \tag{3.13}$$

Also, we have $< q^*, q >= 1$, $< q^*, \bar{q} >= 0$. According to [6], $X = X^s \oplus X^c$ where

$$X^c = \{\bar{z}\bar{q} + zq : z \in \mathbb{C}\}\,,\; X^s = \{\omega \in X :< q^*, \omega >= 0\}\,. \tag{3.14}$$

Therefore, for each $U = (u, v)^T \in X$, in can be written in the form

$$\begin{pmatrix}u\\ v\end{pmatrix} = zq + \bar{z}\bar{q} + \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}. \tag{3.15}$$

where $zq + \bar{z}\bar{q} \in X^c$ and $\omega^T = (\omega_1, \omega_2) \in X^s$. Hence

$$z =< q^*, U >,\; \omega = U- < q^*, U > q- < \bar{q^*}, U > \bar{q}. \tag{3.16}$$

Therefore (3.3) in (z, ω) coordinates becomes,

$$\begin{cases}\dfrac{dz}{dt} = i\beta_0 z + < q^*, f^\vee >,\\ \dfrac{d\omega}{dt} = L\omega + H(z, \bar{z}, \omega),\end{cases} \tag{3.17}$$

where $f^\vee = (f_1, f_2)^T$ defined by (3.4), (3.5) at $c = c_*$ and

$$H(z, \bar{z}, \omega) := f^\vee - < q^*, f^\vee > q- < \bar{q^*}, f^\vee > \bar{q}. \tag{3.18}$$

Then, we get

$$H(z, \bar{z}, \omega) = \begin{pmatrix}f_1\\ f_2\end{pmatrix} - (\frac{-d^2v_*^2 i}{2e\beta_0}f_1 + \frac{f_2}{2})\begin{pmatrix}\frac{e\beta_0 i}{d^2v_*^2}\\ 1\end{pmatrix} - (\frac{d^2v_*^2 i}{2e\beta_0}f_1 + \frac{f_2}{2})\begin{pmatrix}\frac{-e\beta_0 i}{d^2v_*^2}\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}f_1\\ f_2\end{pmatrix} - \begin{pmatrix}\frac{f_1}{2} + \frac{e\beta_0 i}{2d^2v_*^2}f_2 + \frac{f_1}{2} - \frac{e\beta_0 i}{2d^2v_*^2}f_2\\ -\frac{d^2v_*^2 i}{2e\beta_0}f_1 + \frac{f_2}{2} + \frac{d^2v_*^2 i}{2e\beta_0}f_1 + \frac{f_2}{2}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

So the center manifold of system (3.17) can be expressed by

$$\omega = (\frac{\omega_{20}}{2})z^2 + \omega_{11}z\bar{z} + (\frac{\omega_{02}}{2})\bar{z}^2 + O(|\, z\,|^3), \tag{3.19}$$

such that $(2i\beta_0 - L)\omega_{20} = 0$, $L\omega_{11} = 0$ and $(2i\beta_0 + L)\omega_{02} = 0$. Hence we obtain $\omega_{20} = \omega_{02} = \omega_{11} = 0$. Based on [6], we can rewrite $f^\vee$ as follows,

$$f^\vee(U) = \frac{1}{2}B(U, U) + \frac{1}{6}C(U, U, U) + O(|U|^4), \tag{3.20}$$

where

$$B(U, V) = \begin{pmatrix}B_1(U, V)\\ B_2(U, V)\end{pmatrix},\; C(U, V, W) = \begin{pmatrix}C_1(U, V, W)\\ C_2(U, V, W)\end{pmatrix}, \tag{3.21}$$

with the following components

$$B_i(U, V) = \sum_{j,k=1}^{2}\frac{\partial^2 f_i(\gamma, c_*)}{\partial\gamma_j\partial\gamma_k}u_j v_k\,|_{\gamma=0},\; i = 1, 2\;\;, \tag{3.22}$$

$$C_i(U, V, W) = \sum_{j,k,l=1}^{2}\frac{\partial^3 f_i(\gamma, c_*)}{\partial\gamma_j\partial\gamma_k\partial\gamma_l}u_j v_k\omega_l\,|_{\gamma=0},\; i = 1, 2\;. \tag{3.23}$$

By calculation we derive

$$\begin{aligned}
B_1(U,V) &= \Big(\frac{-2r}{k} + \frac{mv_* p u_*^{p-1}((u_*^p + c_*)(p+1) - 2pu_*^p)}{(u_*^p + c_*)^3}\Big) u_1 v_1 - \Big(\frac{m(u_*^p + c_* - pu_*^p)}{(u_*^p + c_*)^2}\Big) \\
&\quad \times (u_1 v_2 + u_2 v_1), \\
B_2(U,V) &= -\Big(\frac{2d^3 v_*^2}{e^2}\Big) u_1 v_1 + \Big(\frac{2d^2 v_*}{e}\Big)(u_1 v_2 + u_2 v_1), \\
C_1(U,V,W) &= \Big(\frac{mv_* u_*^{p-2}(p^3 - p)}{(u_*^p + c_*)^2} - \frac{6mv_* u_*^{2p-2} p^3}{(u_*^p + c_*)^3} + \frac{6mv_* u_*^{3p-2} p^3}{(u_*^p + c_*)^4}\Big) u_1 v_1 \omega_1 \\
&\quad + \Big(\frac{mpu_*^{p-1}((c_* + u_*^p)(p+1) - 2pu_*^p)}{(u_*^p + c_*)^3}\Big)(u_1 v_1 \omega_2 + u_1 v_2 \omega_1 + u_2 v_1 \omega_1), \\
C_2(U,V,W) &= \Big(\frac{6d^4 v_*^2}{e^3}\Big) u_1 v_1 \omega_1 - \Big(\frac{4d^3 v_*}{e^2}\Big)(u_1 v_1 \omega_2 + u_1 v_2 \omega_1 + u_2 v_1 \omega_1) \\
&\quad + \Big(\frac{2d^2}{e}\Big)(u_1 v_2 \omega_2 + u_2 v_1 \omega_2 + u_2 v_2 \omega_1).
\end{aligned}$$

Therefore, system (3.17) restricted to the center manifold in z and $\bar{z}$ coordinates is as follows,

$$\frac{dz}{dt} = i\beta_0 z + < q^*, f^{\vee} >= i\beta_0 z + \frac{h_{20}}{2} z^2 + h_{11} z\bar{z} + \frac{h_{02}}{2}\bar{z}^2 + \frac{h_{21}}{2} z^2 \bar{z} + O(\mid z \mid^4), \tag{3.24}$$

where

$$\begin{aligned}
h_{20} =< q^*, B(q,q) > &= \frac{1}{2}\Bigg[\Big(\frac{-2r}{k} + \frac{mv_* p u_*^{p-1}((u_*^p + c_*)(p+1) - 2pu_*^p)}{(u_*^p + c_*)^3}\Big)\Big(\frac{e\beta_0 i}{d^2 v_*^2}\Big) \\
&\quad - \frac{2m(u_*^p + c_* - pu_*^p)}{(u_*^p + c_*)^2} + \frac{2\beta_0^2}{dv_*^2} + \frac{4\beta_0 i}{v_*}\Bigg],
\end{aligned} \tag{3.25}$$

$$h_{11} =< q^*, B(q,\bar{q}) > = \frac{1}{2}\Bigg[\Big(\frac{-2r}{k} + \frac{mv_* p u_*^{p-1}((u_*^p + c_*)(p+1) - 2pu_*^p)}{(u_*^p + c_*)^3}\Big)\Big(\frac{-e\beta_0 i}{d^2 v_*^2}\Big) - \frac{2\beta_0^2}{dv_*^2}\Bigg], \tag{3.26}$$

$$\begin{aligned}
h_{21} =< q^*, C(q,q,\bar{q}) > &= \frac{1}{2}\Bigg[\Big(\frac{mv_* u_*^{p-2}(p^3 - p)}{(u_*^p + c_*)^2} - \frac{6mv_* u_*^{2p-2} p^3}{(u_*^p + c_*)^3} + \frac{6mv_* u_*^{3p-2} p^3}{(u_*^p + c_*)^4}\Big)\Big(\frac{e^2\beta_0^2}{d^4 v_*^4}\Big) \\
&\quad + \Big(\frac{mpu_*^{p-1}((u_*^p + c_*)(p+1) - 2pu_*^p)}{(u_*^p + c_*)^3}\Big)\Big(\frac{-e\beta_0 i}{d^2 v_*^2}\Big) + \frac{6\beta_0^3 i}{d^2 v_*^4} - \frac{4\beta_0^2}{dv_*^3} + \frac{2\beta_0 i}{v_*^2}\Bigg].
\end{aligned} \tag{3.27}$$

According to [6], we have

$$\begin{aligned}
Re\, c_1(c_*) &= Re\Big(\frac{i}{2\beta_0}(h_{20}h_{11} - 2 \mid h_{11} \mid^2 - \frac{1}{3} \mid h_{02} \mid^2) + \frac{h_{21}}{2}\Big) = Re\Big(\frac{i}{2\beta_0} h_{20}h_{11} + \frac{h_{21}}{2}\Big) \\
&= \frac{-1}{2\beta_0} Im(h_{20}h_{11}) + \frac{1}{2} Re(h_{21}).
\end{aligned} \tag{3.28}$$

By a simple calculation, from (3.25)-(3.27), we get

$$\begin{aligned}
Im(h_{20}h_{11}) &= \frac{1}{4}\Bigg[\Big(\frac{-2r}{k} + \frac{mv_* p u_*^{p-1}((u_*^p + c_*)(p+1) - 2pu_*^p)}{(u_*^p + c_*)^3}\Big)\Bigg(\Big(\frac{2m(u_*^p + c_* - pu_*^p)}{(u_*^p + c_*)^2}\Big)\Big(\frac{e\beta_0}{d^2 v_*^2}\Big) \\
&\quad - \frac{4e\beta_0^3}{d^3 v_*^4}\Bigg) - \frac{8\beta_0^3}{dv_*^3}\Bigg],
\end{aligned} \tag{3.29}$$

and

$$Re(h_{21}) = \frac{1}{2}\Bigg[\Big(\frac{mv_* u_*^{p-2}(p^3 - p)}{(u_*^p + c_*)^2} - \frac{6mv_* u_*^{2p-2} p^3}{(u_*^p + c_*)^3} + \frac{6mv_* u_*^{3p-2} p^3}{(u_*^p + c_*)^4}\Big)\Big(\frac{e^2\beta_0^2}{d^4 v_*^4}\Big) - \frac{4\beta_0^2}{dv_*^3}\Bigg]. \tag{3.30}$$

Then, we obtain

$$
\begin{aligned}
Re\, c_1(c_*) &= \frac{-1}{8\beta_0}\Big[\Big(\frac{-2r}{k} + \frac{mv_* p u_*^{p-1}((u_*^p + c_*)(p+1) - 2pu_*^p)}{(u_*^p + c_*)^3}\Big)\Bigg(\Big(\frac{2m(u_*^p + c_* - pu_*^p)}{(u_*^p + c_*)^2}\Big)\Big(\frac{e\beta_0}{d^2 v_*^2}\Big) \\
&- \frac{4e\beta_0^3}{d^3 v_*^4}\Bigg) - \frac{8\beta_0^3}{dv_*^3}\Big] + \frac{1}{4}\Big[\Big(\frac{mv_* u_*^{p-2}(p^3 - p)}{(u_*^p + c_*)^2} - \frac{6mv_* u_*^{2p-2} p^3}{(u_*^p + c_*)^3} + \frac{6mv_* u_*^{3p-2} p^3}{(u_*^p + c_*)^4}\Big)\Big(\frac{e^2\beta_0^2}{d^4 v_*^4}\Big) - \frac{4\beta_0^2}{dv_*^3}\Big] \\
&= \frac{1}{4}\Big(\frac{e\beta_0^2}{d^3 v_*^4}\Big)\Bigg[\Big(\frac{e}{d}\Big)\Big(\frac{mv_*(p^3 - p^2) u_*^{p-2}}{(u_*^p + c_*)^2} + \frac{mv_*(p^2 - 5p^3) u_*^{2p-2}}{(u_*^p + c_*)^3} + \frac{4mv_* p^3 u_*^{3p-2}}{(u_*^p + c_*)^4}\Big) \\
&+ \frac{2mv_*(p^2 - p) u_*^{p-1}}{(u_*^p + c_*)^2} - \frac{4mv_* p^2 u_*^{2p-1}}{(u_*^p + c_*)^3}\Bigg] \\
&= \frac{mpv_* u_*^{p-2}}{4(u_*^p + c_*)^2}\Big(\frac{e\beta_0^2}{d^3 v_*^4}\Big)\Bigg[p(u_* + a)\big(p - 1 - \frac{u_*}{p(k - u_*)}(5p - 1 - \frac{4u_*}{(k - u_*)})\big) \\
&+ 2u_*\big(p - 1 - \frac{2u_*}{k - u_*}\big)\Bigg] \\
&= \frac{mpv_* u_*^{p-2}}{4(u_*^p + c_*)^2}\Big(\frac{e\beta_0^2}{d^3 v_*^4}\Big)\Bigg[(pu_* + pa + 2u_*)\big(p - 1 - \frac{2u_*}{(k - u_*)}\big) - \frac{u_*(u_* + a)}{(k - u_*)} \\
&\times (3p - 1 - \frac{4u_*}{(k - u_*)})\Bigg].
\end{aligned}
$$

If

$$
\frac{3(p+1)u_*}{3p-1} < k < \frac{(p+1)u_*}{p-1}, \tag{3.31}
$$

then

$$
p - 1 - \frac{2u_*}{(k - u_*)} < 0,\ 3p - 1 - \frac{4u_*}{(k - u_*)} > 0. \tag{3.32}
$$

Therefor, $Rec_1(c_*) < 0$. Note that according to (3.1), $\alpha_0'(c_*) < 0$ when $k < 3u_*$ and $\alpha_0'(c_*) > 0$ when $k > 3u_*$. Furthermore by Theorem 2.1 in [21], we can determine the direction of bifurcation. Hence The above calculations prove the following theorem.

Theorem 3.2. *Let (3.31)be satisfied. Assume (2.19) in the case $\beta > 0$ and (2.21) in the case $\beta < 0$ hold. Then for $c = c_*$, system (3.2) has a Hopf bifurcation at U_* and*

- (*i*) *if $k > 3u_*$, then the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are asymptotically orbitally stable;*
- (*ii*) *if $k < 3u_*$, then the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are asymptotically orbitally stable.*

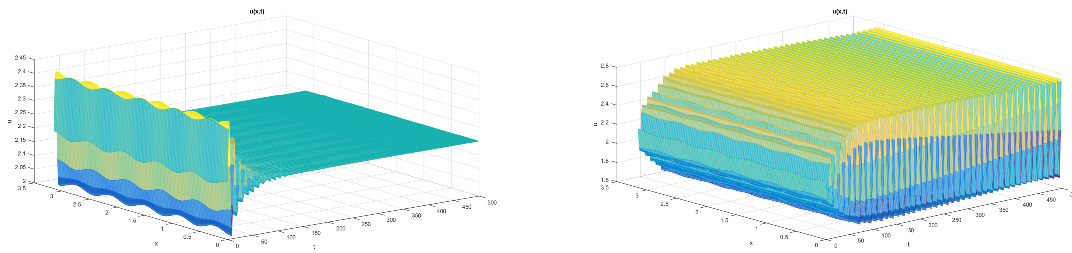
Remark 3.3. From a biological perspective, when the system (3.2) has a stable periodic solution, the populations of both prey and predator species change in a cyclical manner. This implies that the predator and prey populations are able to coexist and keep a stable relationship. In other words, all species are able to coexist without any being wiped out, ensuring the balance and survival of the ecosystem.

4. Numerical simulations

In this section, we perform some numerical simulations of the system (1.3) to illustrate our theoretical results. Since the distributions of prey and predator are of the same type, we only give the numerical simulation of the prey.

Example 4.1. Consider system (1.3) with $\Omega = (0, \pi)$ and

$$
\begin{aligned}
&\alpha = 1.4,\ \beta = 2.1,\ \mu = 2,\ r = 3.1,\ e = 0.5, \\
&d = 0.2,\ a = 0.3,\ k = 4.1,\ p = 2,\ m = 8.
\end{aligned} \tag{4.1}
$$



(a) Graph of $u(x,t)$ for $c = 4 > c_*$. (b) Graph of $u(x,t)$ for $c = 3.2 < c_*$.

FIGURE 1. Numerical simulations of $u(x,t)$ for system (1.3) under (4.1).

Then the positive constant steady state solution of system (1.3) is $(u_*, v_*) = (2.2, 1.5)$ and $c_* = 3.52$. Also, we have $3u_* = 6.6 > k = 4.1$ and $1 = \lambda_1 > \frac{r}{\mu}(1 - \frac{u_*}{k}) = 0.72$. From Theorem 3.1-(i) , a Hopf bifurcation occurs at $(u_*, v_*) = (2.2, 1.5)$. In addition, $\frac{3(p+1)u_*}{3p-1} = 3.96$ and $\frac{(p+1)u_*}{p-1} = 6.6$. Hence, the inequality (3.31) is satisfied. Therefore, from Theorem 3.2-(ii) , the bifurcating periodic solutions are asymptotically stable and the bifurcation direction is subcritical. By choosing $c = 4 > c_*$, then the condition (2.19) in Theorem 2.2 is satisfied. So, system (1.3) is asymptotically stable at (u_*, v_*). This is shown in Figure 1(a) , by taking the initial condition as

$$(u_0(x), v_0(x)) = (2.2 + 0.01\cos(10x), 1.5 + 0.01\cos(10x)). \tag{4.2}$$

By choosing $c = 3.2 < c_*$ and the initial condition (4.2), as we can see in Figure 1(b) , a stable periodic solution is formed.

Example 4.2. Consider system (1.3) with $\Omega = (0, 6\pi) \times (0, 6\pi)$ and

$$\alpha = \mu = 1,\ \beta = 500,\ r = 9.2,\ e = 5,\ d = 2.8,\ a = 0.6,\ k = 4.3,\ p = 3,\ m = 15. \tag{4.3}$$

Then $c_* = 11.47$ and $(u_*, v_*) = (1.18, 5.83)$. The eigenvalues of $-\Delta$ with the Neumann boundary condition in $\Omega = (0, 6\pi) \times (0, 6\pi)$ are determined by

$$\lambda_{mn} = \frac{1}{36}(m^2 + n^2),\ m, n = 0, 1, 2, ...$$

Then the first positive eigenvalue is $\lambda_1(\Omega) := \lambda_{01} = \frac{1}{36}$. In addition, we choose $c = 14 > c_*$. Under the mentioned parameters, we get $\frac{1}{36} = \lambda_1 < \frac{r}{\mu}(1 - \frac{2u_*}{k}) = 4.13$ and $\frac{-F_1}{E_1} = 330.74$. Then (2.20) is satisfied and by Theorem 2.2-(ii) the Turing instability occurs at $(u_*, v_*) = (1.18, 6.96)$ for system (1.3). In Figure 2, we take the following initial condition

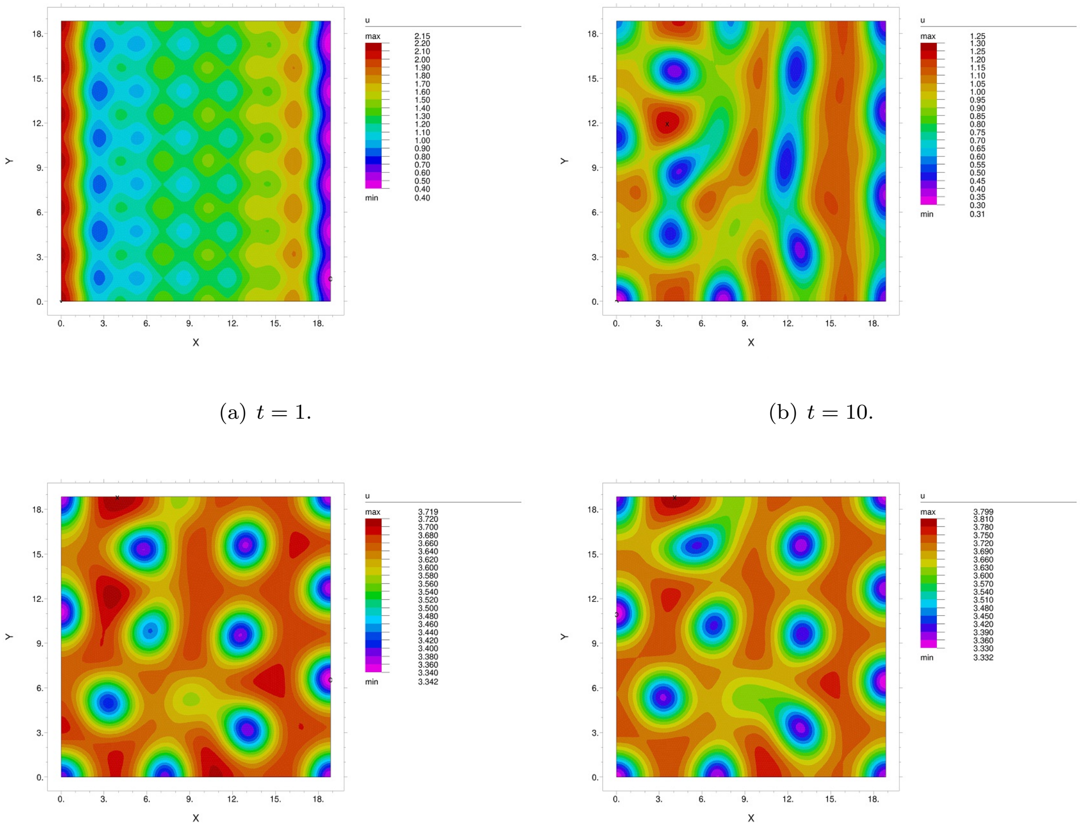
$$(u_0(x,y), v_0(x,y)) = (1.2 + 0.01(\sin(2x) + \cos(2y)), 7 + 0.01(\sin(2x) + \cos(2y))).$$

We see in Figure 2, that the Turing instability leads to the formation of a spot pattern.

5. CONCLUSION

We studied a nonlinear cross-diffusion prey-predator system involving a nonmonotonic functional response, under the Neumann boundary condition. We obtained sufficient conditions of the Turing instability and the Hopf bifurcation for system (1.3). We used the idea of Turing to show that the cross-diffusion changes the stability of the system (1.3).

The stability of the system (1.3) with and without cross-diffusion term was determined. Based on Theorem 2.2, when $\beta = 0$, system (1.3) is stable. Moreover, for $\beta > 0$, under condition (2.20), and for $\beta < 0$ under condition (2.22), system (1.3) is unstable. We found that the prey-predator system (1.3) under condition (3.31) has stable periodic



(a) $t = 1$.

(b) $t = 10$.

(c) $t = 20$.

(d) $t = 100$.

FIGURE 2. The process of Turing pattern of $u(x,t)$ for system (1.3) under (4.3) and $c = 14 > c_*$.

solutions through the Hopf bifurcation. Also under the condition $3u_* \neq k$, the direction of bifurcating periodic solutions has been determined in Theorem 3.2.

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