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Solving a class of Volterra integral equations with M-derivative

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Abstract

In this current article, the well-known Neumann method for solving the time M-fractional Volterra integral equations of the second kind is developed. In the several theorems existence and uniqueness of the solution and convergence of the proposed approach are also studied. The Neumann method for this class of the time M-fractional Volterra integral equations has been called the M-fractional Neumann method (MFNM). The results obtained demonstrate the efficiency of the proposed method for the time M-fractional Volterra integral equations. Several illustrative numerical examples have presented the ability and adequacy of the MFNM for a class of fractional integral equations.

Keywords. Local M-fractional integral, M-fractional Volterra integral equations, M-fractional Neumann method, Existence and uniqueness of solution, Theorem of convergence.

2000 Mathematics Subject Classification. [2020] 45Axx-45Dxx-45Exx.

1. Introduction

Thanks to the efforts of mathematicians over the past few decades, fractional calculus is as well known to everyone as ordinary calculus. Famous mathematicians who have made significant efforts in this field include Riemann, Liouville, Grenville, Caputo, and other of mathematicians are cited [1]. All fractional derivatives are generally divided into local and non-local categories. However, some of those definitions take care of drawbacks that caused their application to confront difficulties such as satisfying the derivative product rule, the derivative quotient rule, and the chain rule. In 2017, Sousa et al. introduced an M-fractional derivative involving a Mittag-Leffler function with one parameter that also satisfies the properties of integer-order calculus [2, 3]. In this sense, Sousa and Oliveira introduced a truncated M-fractional derivative type that unifies four existing fractional derivative types mentioned above and which also satisfied the classical properties of integer-order calculus [4]. The truncated M-fractional derivative is one of the types of local fractional derivatives, so in the mode of derivation from the positive integer order, it completely follows the ordinary derivative, and in the fractional mode, it has almost all the properties of the ordinary derivative. Many researchers have used the M-fraction derivative in their research [9, 10].

Definition 1.1. Given a function $f : [0, \infty) \to \mathbb{R}$. Then the truncated M-fractional derivative of f of order α is defined by

$$ {}_i\mathcal{D}_M^{\alpha,\beta} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t_i\mathbb{E}_\beta(\varepsilon t^{-\alpha})\right) - f(t)}{\varepsilon}, \tag{1.1} $$

for all $t > 0$, $\alpha \in (0, 1)$, where ${}_i\mathbb{E}_\beta(\cdot)$, $\beta > 0$ is the Mittag-Leffler function with one parameter as defined by in Eq. (1.1) [4].

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Note that if f is α- differentiable in some $(0,a)$, $a \geq 0$, and $\lim_{t\to 0^+} {}_i\mathcal{D}_M^{\alpha,\beta} f(t)$ exists, then one can define [4]

$$ {}_i\mathcal{D}_M^{\alpha,\beta} f(0) = \lim_{t\to 0^+} {}_i\mathcal{D}_M^{\alpha,\beta} f(t). $$

Definition 1.2. Given a function $f : [0,\infty) \to \mathbb{R}$, $a \geq 0$. Then local M-fractional integral of f order α is defined by

$$ {}_M\mathcal{T}_a^{\alpha,\beta} f(t) = \Gamma(\beta+1) \int_a^t \frac{f(s)}{s^{1-\alpha}} ds \tag{1.2} $$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$, and $\beta > 0$ [4]. One of the well results is the following [3, 4]. Let $\alpha \in (0,1)$, and f be α-differentiable at a point $t > 0$, then

H: (Invers theorem) ${}_i\mathcal{D}_M^{\alpha,\beta}\left({}_M\mathcal{T}_a^{\alpha,\beta} f(t)\right) = f(t)$,

I: (Fundamental theorem of calculus) ${}_M\mathcal{T}_a^{\alpha,\beta}\left({}_i\mathcal{D}_M^{\alpha,\beta} f(t)\right) = f(t) - f(0)$.

Most phenomena in our real world are described by time fractional functional equations (FFEs). Although having the exact solution of FFEs in analyzing the phenomena is important, there are many FFEs that cannot be resolved accurately. Due to this fact, finding approximate solutions to time-fractional functional equations is clearly required. In recent years, many effective methods have been proposed for approximate solutions of time-fractional integral equations [12]-[17].

Özturk used Examination of Sturm-Liouville problem with proportional derivative in control theory [18]. Duran et al. obtained Discussion of numerical and analytical techniques for the emerging fractional order Murnaghan model in materials science [19]. Tariq et al. found Some integral inequalities via new family of preinvex functions. Bonayah et al. created A robust study on the listeriosis disease by adopting fractal-fractional operators [20]. Bonyah et al. proposed A robust study on the listeriosis disease by adopting fractal-fractional operators.[21]. Zada et al. obtained New approximate-analytical solutions to partial differential equations via auxiliary function method [21]. Youssri et al. used Numerical spectral LEGENDRE-GALERKIN algorithm for solving time fractional Telegraph equation [23]. Hafez et al. proposed Shifted Jacobi collocation scheme for multidimensional time-fractional order telegraph equation [24]. Atta et al. found Advanced shifted first-kind Chebyshev collocation approach for solving the nonlinear time-fractional partial integro-differential equation with a weakly singular kernel [25]. Abdelghany et al. gained A Tau Approach for Solving Time-Fractional Heat Equation Based on the Shifted Sixth-Kind Chebyshev Polynomials [26]. Youssri et al. obtained Petrov-Galerkin Lucas Polynomials Procedure for the Time-Fractional Diffusion Equation [27]. Moustafa et al. used Explicit Chebyshev Petrov–Galerkin scheme for time-fractional fourth-order uniform Euler–Bernoulli pinned–pinned beam equation [28].

In this study, the Neumann method is advanced for time M-fractional Volterra integral equations of the second kind, and convergence of the proposed approach is studied and this method is utilized to find an approximate solution of the time M-fractional Volterra integral equations.

The organization of the paper is as follows: In section 2 and 3, the several primary definitions and essential concepts and convergence study related to the local time M-fractional Volterra integral equations of the second kind are given. In section 4, the local M-fractional Neumann method is presented to solve a class of fractional integral equations. In section 5, several illustrative examples are provided to show the efficiency of the method. Finally, conclusion is appointed in section 6.

2. The several primary definitions and essential concepts

Suppose the local time M-fractional Volterra integral equations (MFVIEs) as the following form

$$ f(t) = g(t) + \eta\, {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)f(s)), \qquad \forall\alpha \in (0,1), \beta > 0, \tag{2.1} $$

where g and K are known functions, η and a are constants, and f is an unknown function [5]-[8]. Applying the local M-fractional integral definition on Equation (2.1), results in

$$ f(t) = g(t) + \eta \int_a^t \frac{\Gamma(\beta+1)K(t,s)f(s)}{s^{1-\alpha}} ds. \tag{2.2} $$

By considering

$$K_{\beta}^{\alpha}(t,s) = \frac{\Gamma(\beta+1)K(t,s)}{s^{1-\alpha}}, \tag{2.3}$$

as the new Volterra kernel, and substituting (2.3) into (2.2), we obtain

$$f(t) = g(t) + \eta \int_a^t K_{\beta}^{\alpha}(t,s) f(s) ds. \tag{2.4}$$

According to Equation (2.4), the operator form of MFVIEs (2.1), can be denoted as follows

$$f = g + \eta K_{\beta}^{\alpha} f, \qquad \forall \alpha \in (0,1), \beta > 0, \tag{2.5}$$

or

$$L_{\beta}^{\alpha} f = (I - \eta K_{\beta}^{\alpha}) f = g, \qquad \forall \alpha \in (0,1), \beta > 0, \tag{2.6}$$

Definition 2.1. Let's consider $\eta = \eta_0, \alpha = \alpha_0, \beta_0 = \beta$, and $(L_{\beta_0}^{\alpha_0})^{-1}$ as an $\mathcal{L}^2$ operator exists and satisfies

$$(L_{\beta_0}^{\alpha_0})^{-1} L_{\beta_0}^{\alpha_0} = L_{\beta_0}^{\alpha_0} (L_{\beta_0}^{\alpha_0})^{-1} = I \tag{2.7}$$

then η_0 is called a regular value of the local M-fractional operator $K_{\beta_0}^{\alpha_0}$ [5]-[8].

Theorem 2.2. *If for a given $\alpha = \alpha_0$, $\beta = \beta_0$, and $\eta = \eta_0$, the operator $(L_{\beta_0}^{\alpha_0})^{-1}$ exists, then it is unique [5]-[8].*

Proof. Suppose that $(L_{\beta_0}^{\alpha_0})^{-1}$ and $(\tilde{L}_{\beta_0}^{\alpha_0})^{-1}$ are two $\mathcal{L}^2$ operators that satisfy Eq. (2.3), and let

$$H = (L_{\beta_0}^{\alpha_0})^{-1} - (\tilde{L}_{\beta_0}^{\alpha_0})^{-1}$$

Regarding Eq. (2.7), one has

$$\begin{aligned} &(L_{\beta_0}^{\alpha_0})^{-1} L_{\beta_0}^{\alpha_0} = L_{\beta_0}^{\alpha_0} (L_{\beta_0}^{\alpha_0})^{-1} = I \\ &(\tilde{L}_{\beta_0}^{\alpha_0})^{-1} L_{\beta_0}^{\alpha_0} = L_{\beta_0}^{\alpha_0} (\tilde{L}_{\beta_0}^{\alpha_0})^{-1} = I \end{aligned} \tag{2.8}$$

and subtracting these two relations results in

$$H L_{\beta_0}^{\alpha_0} = L_{\beta_0}^{\alpha_0} H = 0. \tag{2.9}$$

applying Eq. (2.9) by the local M-fractional fractional operator $(L_{\beta_0}^{\alpha_0})^{-1}$ and regarding Eq. (2.8), we get $H = 0$. ☐

Theorem 2.3. *If η is a regular value of the local M-fractional fractional operator K_{β}^{α}, with inverse the local M-fractional fractional operator $(L_{\beta}^{\alpha})^{-1}$, then for any $\mathcal{L}^2$ function g, Eq. (2.6) has a unique $\mathcal{L}^2$ solution, say, f, satisfying [5]-[8].*

$$f = (L_{\beta}^{\alpha})^{-1} g. \tag{2.10}$$

Proof. By Substitution of Equation (2.10) into Equation (2.2), we have

$$L_{\beta}^{\alpha} (L_{\beta}^{\alpha})^{-1} g = g, \tag{2.11}$$

and since $L_{\beta}^{\alpha}(L_{\beta}^{\alpha})^{-1} = I$, thus the function f, defined by Eq. (2.10), is a solution of Eq. (2.6). To show the uniqueness, let's f_1 and f_2 be two different solutions of (2.6), then

$$L_{\beta}^{\alpha}(f_1 - f_2) = 0,$$

hence

$$(L_{\beta}^{\alpha})^{-1} L_{\beta}^{\alpha}(f_1 - f_2) = 0.$$

So

$$f_1 = f_2,$$

which completes the proof. If η is a regular value of the local M-fractional operator K^{α}_{β}, then the Eq. (2.6) has a unique solution

$$f = (L^{\alpha}_{\beta})^{-1} g = (I - \eta K^{\alpha}_{\beta})^{-1} g.$$

So

$$(L^{\alpha}_{\beta})^{-1} = (I - \eta K^{\alpha}_{\beta})^{-1} = I + \eta K^{\alpha}_{\beta} + (\eta K^{\alpha}_{\beta})^2 + (\eta K^{\alpha}_{\beta})^3 + (\eta K^{\alpha}_{\beta})^4 + \cdots,$$

$$(L^{\alpha}_{\beta})^{-1} = I + \sum_{n=1}^{\infty} (\eta K^{\alpha}_{\beta})^n, \qquad \forall \alpha \in (0,1), \beta > 0, \tag{2.12}$$

where Eq. (2.12) is called the local M-fractional Neumann series for the invers the local M-fractional operator $(L^{\alpha}_{\beta})^{-1}$. We set

$$f_0 = g,$$
$$f_1 = g + \eta K^{\alpha}_{\beta} f_0 = g + \eta K^{\alpha}_{\beta} g,$$
$$f_2 = g + \eta K^{\alpha}_{\beta} f_1 = g + \eta K^{\alpha}_{\beta} g + (\eta K^{\alpha}_{\beta})^2 g,$$
$$f_3 = g + \eta K^{\alpha}_{\beta} f_2 = g + \eta K^{\alpha}_{\beta} g + (\eta K^{\alpha}_{\beta})^2 g + (\eta K^{\alpha}_{\beta})^3 g,$$
$$\vdots$$

so, the nth approximation to f, can be presented as below

$$f_n = g + \eta K^{\alpha}_{\beta} f_{n-1} = g + \sum_{i=1}^{n} \eta (\eta K^{\alpha}_{\beta})^i g.$$

Therefore, if the sequence of functions f_n have a limit as n, tends to infinity, then

$$f = \lim_{n\to\infty} f_n = g + \sum_{i=1}^{\infty} (\eta K^{\alpha}_{\beta})^i g, \qquad \forall \alpha \in (0,1), \beta > 0, \tag{2.13}$$

where Eq. (2.13) is called the local M-fractional Neumann series for the solution x of MFVIEs (2.3) [7, 8]. □

3. The convergence study

Theorem 3.1. *The local M-fractional Neumann series* (2.12)*, for* $(L^{\alpha}_{\beta})^{-1} \alpha \in (0,1)$ *and* $\beta > 0$*, is strong convergence if* $\|\eta K^{\alpha}_{\beta}\| < 1, \forall \alpha \in (0,1), \beta > 0$*,* [5]-[8].

Proof. Assume that $\alpha \in (0,1), \beta > 0$ is given and considered as a constant throughout the proof. Define

$$S_n = \sum_{i=0}^{n} \eta (K^{\alpha}_{\beta})^i, \tag{3.1}$$

and take $n > m$. Regarding Equation (3.1), we have

$$\|S_n - S_m\| \le \sum_{i=m+1}^{n} \|\eta K^{\alpha}_{\beta}\|^i = \frac{\|\eta K^{\alpha}_{\beta}\| (\|\eta K^{\alpha}_{\beta}\|^m - \|\eta K^{\alpha}_{\beta}\|^n)}{1 - |\eta K^{\alpha}_{\beta}|}. \tag{3.2}$$

Since $\|\eta K^{\alpha}_{\beta}\| < 1$, thus

$$\lim_{n\to\infty} \|\eta K^{\alpha}_{\beta}\|^n = 0, \tag{3.3}$$

by considering Equations (3.2) and (3.3), we derive

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0. \tag{3.4}$$

So, the sequence S_n is a Cauchy sequence, so the limit S_n exists.

Now, let's consider the residual R_n as the following form

$$R_n = I - (I - \eta K_\beta^\alpha) S_n. \tag{3.5}$$

Setting Equation (3.1) in Equation (3.5), results in

$$R_n = \eta K_\beta^\alpha)^{n+1},$$
$$\|R_n\| \leq |\eta K_\beta^\alpha|^{n+1}.$$

Since $\|\eta K_\beta^\alpha\| < 1$, therefore

$$\lim_{n\to\infty} \|R_n\| = 0.$$

Then, the local M-fractional operator $(L_\beta^\alpha)^{-1}$ is a right inverse of L_β^α, a similar proof shows that it is also a left inverse of the local M-fractional operator L_β^α. □

Lemma 3.2. *Whenever K_β^α is an $\mathcal{L}^2$ the local M-fractional Volterra operator for a given $\alpha \in (0,1), \beta > 0$, and $b > a$, then*

$$|(K_\beta^\alpha)^{n+1}(t,s)| \leq \frac{\|K_\beta^\alpha\|_E^{n+1}}{[(n-1)!]^{\frac{1}{2}}} K_1(t) K_2(s),$$

where $K_1(t) = [\int_a^t |K_\beta^\alpha(t,s)|^2 ds]^{\frac{1}{2}}$, and $K_2(s) = [\int_s^b |K_\beta^\alpha(t,s)|^2 dt]^{\frac{1}{2}}$.

Proof. For $\alpha, \beta = 1$, refer to books [5]-[7]. □

Theorem 3.3. *If K_β^α is an $\mathcal{L}^2$ the local M-fractional Volterra operator for a given $\alpha \in (0,1), \beta > 0$, the local M-fractional Neumann series (2.12), converges strongly for all η to the inverse the local M-fractional operator of K_β^α [5]-[8].*

Proof: According to Eq. (3.3), for $n > m$, we obtain

$$\|S_n - S_m\|_E \leq \sum_{i=m+1}^{n} \|\eta K_\beta^\alpha\|_E^i. \tag{3.6}$$

But from lemma 3.2, and Euclidean norm, we get

$$\|\eta K_\beta^\alpha\|_E^i \leq |\eta|^i \frac{\|K_\beta^\alpha\|_E^i}{[(i-2)!]^{\frac{1}{2}}},$$

and hence, for all η,

$$\lim_{i\to\infty} |\eta K_\beta^\alpha|_E^i = 0. \tag{3.7}$$

By considering Equations (3.6) and (3.7), we persuade the sequence S_n is Cauchy, so the local M-fractional Neumann series (2.12), is strong convergence for all η to the inverse the local M-fractional operator of K_β^α

4. The summary of local M-fractional Neumann method (MFNM) for applying in MFVIEs

Suppose local time M-fractional Volterra integral equations of the second kind as follows form

$$f(t) = g(t) + \eta\, {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)g(s)), \ \forall \alpha \in (0,1), \beta > 0,$$

where g, K are known functions and η, a are constants and f an unknown function. We define

$$\begin{aligned} f_0(t) &= g(t), \\ f_1(t) &= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)f_0(s)) = g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)g(s)), \\ f_2(t) &= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)f_1(s)) \\ &= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)g(s)) + \eta^2 {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s) {}_M\mathcal{T}_a^{\alpha,\beta}(K(s,s_1)g(s_1))), \\ &\ \vdots \end{aligned}$$

moreover, the nth approximations f_n, to x, will be as

$$\begin{aligned} f_n(t) &= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)f_{n-1}(s)) \\ &= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)g(s)) + \eta^2 {}_M\mathcal{T}_a^{\alpha,\beta}\left(K(t,s) {}_M\mathcal{T}_a^{\alpha,\beta}(K(s,s_1)g(s_1))\right) + \cdots \\ &+ \eta^n {}_M\mathcal{T}_a^{\alpha,\beta}\left(K(t,s) {}_M\mathcal{T}_a^{\alpha,\beta}\left(K(s,s_1)\cdots({}_M\mathcal{T}_a^{\alpha,\beta}(K(s_{n-1},s_n)g(s_n)\right)\right). \end{aligned}$$

The solution of MFVIEs is

$$f(t) = \lim_{n\to\infty} f_n(t).$$

5. Examples

In this section, the several illustrative examples are provided to demonstrate the efficiency of the method in solving the local time M-fractional Volterra integral equations of second kind.

Example 5.1. Consider the following local time M-fractional Volterra integral equation

$$f(t) = 1 - {}_M\mathcal{T}_0^{\alpha,\beta}((t-s)f(s)), \ \forall\alpha \in (0,1), \beta > 0, \tag{5.1}$$

where the exact solution of this MFVIE (5.1), for non-fractional case is as follows [29]

$$f(t) = \cos(t).$$

According to the proposed the local M-fractional Neumann method, we have

$$\begin{aligned} f_0(t) =&1, \\ f_1(t) =&1 - {}_M\mathcal{T}_a^{\alpha,\beta}(t-s), \\ f_2(t) =&1 - {}_M\mathcal{T}_a^{\alpha,\beta}(t-s) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) {}_M\mathcal{T}_a^{\alpha,\beta}(s-s_1)\right), \\ f_3(t) =&1 - {}_M\mathcal{T}_a^{\alpha,\beta}(t-s) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) {}_M\mathcal{T}_a^{\alpha,\beta}(s-s_1)\right) \\ &- {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) {}_M\mathcal{T}_a^{\alpha,\beta}(s-s_1)\right)\right) \\ &\ \vdots \end{aligned} \tag{5.2}$$

By solving this sequence of integral equations, the solution of Equation (5.2), can be obtained as the following form

$$\begin{aligned} f_0(t) =&1, \\ f_1(t) =&1 - \frac{\Gamma(\beta+1)}{\alpha(1+\alpha)} t^{1+\alpha}, \\ &\ \vdots \end{aligned}$$

The seven-terms approximate solutions of Eq. (5.1), for different α, β, will be obtained, as follows

$$\begin{aligned} f_6(t) =& 1 - \frac{\Gamma(\beta+1)}{\alpha(1+\alpha)} t^{1+\alpha} + \frac{\Gamma(\beta+1)^2}{\alpha(1+2\alpha)(1+\alpha)^2} \frac{t^{2+2\alpha}}{2!} \\ &- \frac{\Gamma(\beta+1)^3}{\alpha(1+2\alpha)(2+3\alpha)(1+\alpha)^3} \frac{t^{3+3\alpha}}{3!} + \frac{\Gamma(\beta+1)^4}{\alpha(1+2\alpha)(2+3\alpha)(3+4\alpha)(1+\alpha)^4} \frac{t^{4+4\alpha}}{4!} \\ &- \frac{\Gamma(\beta+1)^5}{\alpha(1+2\alpha)(2+3\alpha)(3+4\alpha)(4+5\alpha)(1+\alpha)^5} \frac{t^{5+5\alpha}}{5!} \\ &+ \frac{\Gamma(\beta+1)^6}{\alpha(1+2\alpha)(2+3\alpha)(3+4\alpha)(4+5\alpha)(5+6\alpha)(1+\alpha)^6} \frac{t^{6+6\alpha}}{6!}. \end{aligned}$$

In Figures 1, the seventh-order approximate solution of Local M-fractional Volterra integral equation for different Values α, β, and exact solution for $\alpha, \beta = 1$ are plotted

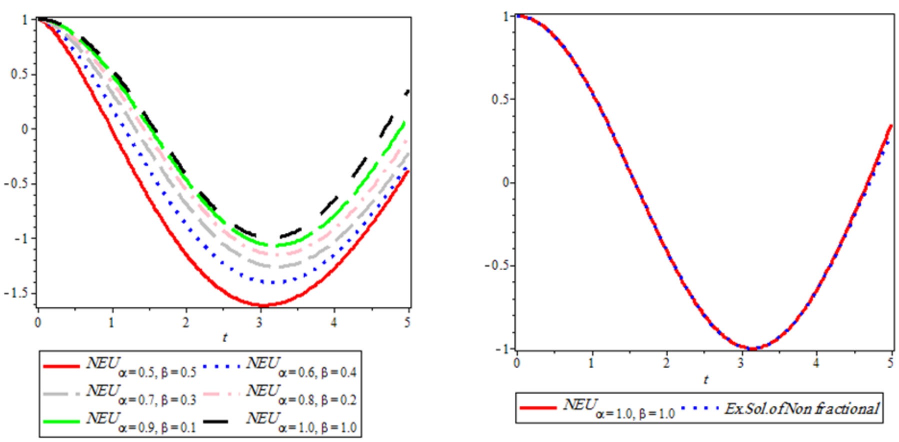

FIGURE 1. The 7th-order approximation of MFNM for different values α and β (left) and for $\alpha, \beta = 1$, versus exact solution of Non-fractional Volterra integral equation (right).

Example 5.2. Consider the following local time M-fractional Volterra integral equation

$$f(t) = 2 + t^2 + {}_M\mathcal{T}_\alpha^{\alpha,\beta}((t-s)f(s)), \ \forall \alpha \in (0,1), \beta > 0, \tag{5.3}$$

where for $\alpha = 1, \beta = 1$, the exact solution of Eq. (5.3) is as follows [29]

$$f(t) = 4\cosh(t) - 2.$$

Applying to the proposed local M-fractional Neumann method, results in

$$\begin{aligned} f_0(t) =& 2 + t^2, \\ f_1(t) =& 2 + t^2 + {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)(2+s^2)), \\ f_2(t) =& 2 + t^2 + {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)(2+s^2)) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}((s-s_1)(2+s_1^2))\right), \\ f_3(t) =& 2 + t^2 + {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)(2+s^2)) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}((s-s_1)(2+s_1^2))\right) \\ &+ {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}((s-s_1)(2+s_1^2))\right)\right), \\ &\vdots \end{aligned} \tag{5.4}$$

The corresponding solutions of these sequences (5.4) are as below

$$
\begin{aligned}
f_0(t) =& 2 + t^2,\\
f_1(t) =& 2 + t^2 + \frac{\Gamma(\beta+1)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} t^{\alpha}(t^3\alpha^2 + t^3\alpha + 2t\alpha^2 + 10t\alpha + 12t),\\
f_2(t) =& 2 + t^2 + \frac{\Gamma(\beta+1)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} t^{\alpha}(t^3\alpha^2 + t^3\alpha + 2t\alpha^2 + 10t\alpha + 12t)\\
&+ \frac{\Gamma(\beta+1)^2}{2\alpha(\alpha+1)(\alpha+2)(2\alpha+3)(\alpha^3+6\alpha^2+11\alpha+6)} t^{1+2\alpha}(2t^3\alpha^4 + 5t^3\alpha^3 + 4t^3\alpha^2\\
&+ 4t\alpha^4 + t^3\alpha + 34t\alpha^3 + 106t\alpha^2 + 144t\alpha + 72t),\\
&\vdots
\end{aligned}
$$

In Figures 2, the seventh-order approximate solution of Local M-fractional Volterra integral equation for different Values α, β, and exact solution for $\alpha, \beta = 1$ are plotted.

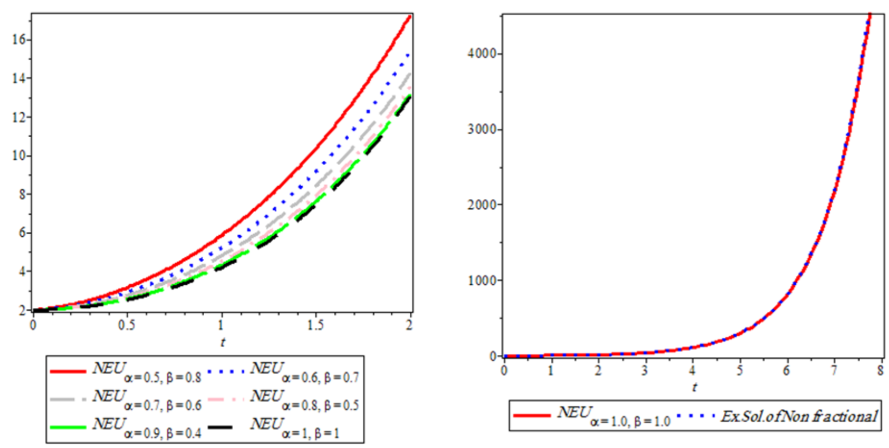

FIGURE 2. The 7th-order approximation of MFNM for different values α and β (left) and for $\alpha, \beta = 1$, versus exact solution of Non-fractional Volterra integral equation (right).

Example 5.3. Consider the following MFVIE

$$
f(t) = \exp(t) + {}_M\mathcal{T}_0^{\alpha,\beta}(\exp((t-s)f(s)), \qquad \forall\alpha \in (0,1), \beta > 0, \tag{5.5}
$$

whit the exact solution of this local M-fractional Volterra integral Equation (5.5), for non-fractional case is as follows [29]

$$
f(t) = \exp(2t).
$$

According to the MFNM approach, we have

$$
\begin{aligned}
f_0(t) =& \exp(t),\\
f_1(t) =& \exp(t) + {}_M\mathcal{T}_a^{\alpha,\beta}(\exp(t-s)\exp(s))\\
f_2(t) =& \exp(t) + {}_M\mathcal{T}_a^{\alpha,\beta}(\exp(t-s)\exp(s)) + {}_M\mathcal{T}_a^{\alpha,\beta}\left(\exp(t-s){}_M\mathcal{T}_a^{\alpha,\beta}\left(\exp(s-s_1)\exp(s_1)\right)\right),\\
f_3(t) =& \exp(t) + {}_M\mathcal{T}_a^{\alpha,\beta}(\exp(t-s)\exp(s)) + {}_M\mathcal{T}_a^{\alpha,\beta}\left(\exp(t-s){}_M\mathcal{T}_a^{\alpha,\beta}\left(\exp(s-s_1)\exp(s_1)\right)\right)\\
&+ {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s){}_M\mathcal{T}_a^{\alpha,\beta}\left((s-s_1)\exp(s_1)\right)\right)\\
&+ {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s){}_M\mathcal{T}_a^{\alpha,\beta}\left((s-s_1){}_M\mathcal{T}_a^{\alpha,\beta}((s_1-s_2)\exp(s_2))\right)\right),
\end{aligned} \tag{5.6}
$$

By solving this sequence of integral Equations (5.8), seventh-order approximation of Eq. (5.5) is

$$\begin{aligned} f_6(t) =& e^t\Big[1 + \frac{\Gamma(\beta+1)}{1!}(\frac{t^\alpha}{\alpha}) + \frac{\Gamma(\beta+1)^2}{2!}(\frac{t^\alpha}{\alpha})^2 + \frac{\Gamma(\beta+1)^3}{3!}(\frac{t^\alpha}{\alpha})^3 + \frac{\Gamma(\beta+1)^4}{4!}(\frac{t^\alpha}{\alpha})^4 \\ &+ \frac{\Gamma(\beta+1)^5}{5!}(\frac{t^\alpha}{\alpha})^5 + \frac{\Gamma(\beta+1)^6}{6!}(\frac{t^\alpha}{\alpha})^6\Big]. \end{aligned}$$

In Figures 3, the seventh-order approximate solution of Local M-fractional Volterra integral equation for different Values α, β, and exact solution for $\alpha, \beta = 1$ are plotted.

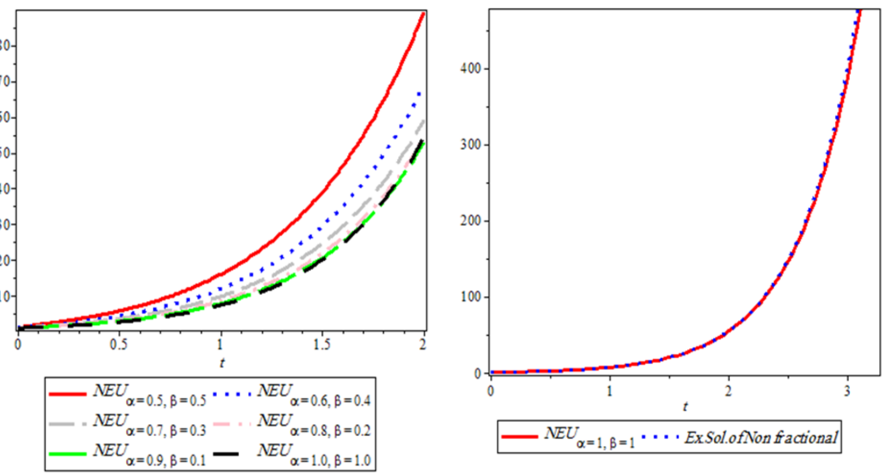

FIGURE 3. The 7th-order approximation of MFNM for different values α and β (left) and for $\alpha, \beta = 1$, versus exact solution of Non-fractional Volterra integral equation (right).

Example 5.4. Consider the time fractional integral equation as follows

$$f(t) = 3\sin(2t) - {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)f(s)), \qquad \forall\alpha \in (0,1), \beta > 0, \tag{5.7}$$

with the exact solution of this local M-fractional Volterra integral equation for non-fractional case is [29]

$$f(t) = 4\sin(2t) - 2\sin(t).$$

By using the proposed MFNM approach, we gain

$$\begin{aligned} f_0(t) =& 3\sin(2t), \\ f_1(t) =& 3\sin(2t) - {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)3\sin(2s)), \\ f_2(t) =& 3\sin(2t) + {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)3\sin(2s)) \\ &+ {}_M\mathcal{T}_a^{\alpha,\beta}\big((t-s)_M\mathcal{T}_a^{\alpha,\beta}((s-s_1)3\sin(2s_1))\big) \\ &\vdots \end{aligned} \tag{5.8}$$

By solving above sequences of integral Equations, the second-order approximate solution of Equation (5.8), can be obtained that in Figures 4, for different Values α, β, and exact solution for $\alpha, \beta = 1$ are plotted.

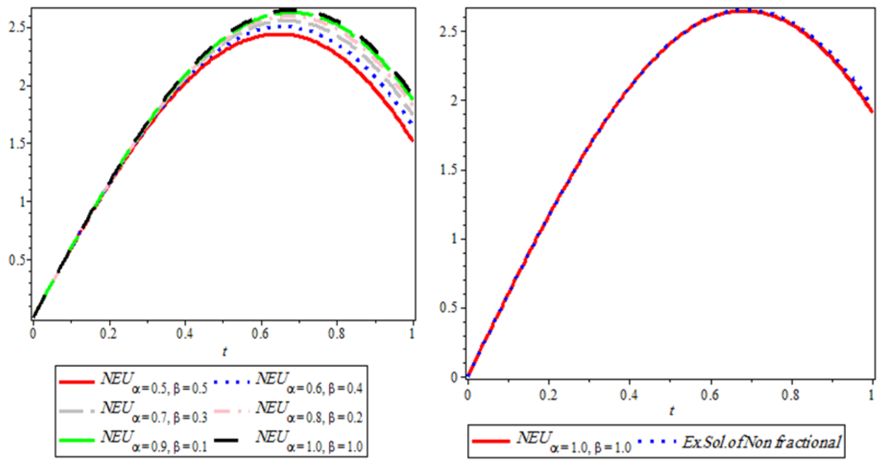

FIGURE 4. The 7th-order approximation of MFNM for different values α and β (left) and for $\alpha, \beta = 1$, versus exact solution of Non-fractional Volterra integral equation (right).

6. Conclusion

In the course of the present investigation, was presented the solving for a class of fractional Volterra integral equations in the sense of the truncated M-fractional derivative. For this aim, the well-recognized Neumann method was successfully expanded and the several theorems related to conditions for existence and uniqueness and also sufficient condition for convergence of solution were proved. The proposed method was called the M-fractional Neumann method (MFNM). Since for $\alpha = 1$, and $\beta = 1$, MFVIEs is changed into a Volterra integral equations, thus not unexpected that M-fractional Neumann method have had the same accuracy and efficiency the Neumann method for Volterra integral equations. The several illustrative examples also were presented, corroborating the satisfactory implementation of the method in solving the local M-fractional Volterra integral equations. In this study the norm $\|\cdot\|_2$, was utilized.

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