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Symmetries of the minimal lagrangian hypersurfaces on cylindrically symmetric static space-times

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Abstract

In this work, we study a hypersurface immersed in specific types of cylindrically symmetric static space-times, then we identify the gauge fields of the Lagrangian that minimizes the area besides the Noether symmetries. We show that these symmetries are part of the Killing algebra of cylindrically symmetric static space-times. By using Noether's theorem, we construct the conserved vector fields for the minimal hypersurface.

Keywords. Conserved vector fields, Cylindrically space-times, Minimal Lagrangian, Symmetries.
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1. Introduction

The geodesic equations describe the path of a particle moving along a curve in a Riemannian manifold, with the curve being defined as the shortest path between two points. These equations have symmetries, which refer to the invariance of the equations under certain transformations. Symmetries play a crucial role in understanding the properties of the geodesic equations, such as conserved quantities, stability, and singularities. One important type of symmetry is the Noether symmetry, which is associated with the existence of a conserved quantity along the geodesic. The study of the symmetries of the geodesic equations can provide valuable insight into the underlying geometry of the Riemannian manifold and can also have important implications in physics and engineering. Numerous authors have studied the geodesic equations' symmetries, and the classification results have been extensively disseminated [? ? ?]. Noether symmetries are a type of symmetry that arise in the context of Lagrangian mechanics. They are named after the mathematician Emmy Noether, who showed that for every continuous symmetry of a Lagrangian system, there exists a corresponding conserved quantity. In other words, if a physical system is invariant under certain transformations, then the corresponding Noether symmetry generates a conserved quantity. This conserved quantity can often be interpreted as a physical quantity, such as energy, momentum, or angular momentum. Noether symmetries have important applications in physics and engineering, as they provide a tool for understanding the underlying structure of physical systems and for making predictions about their behavior. Additionally, the study of Noether symmetries can provide insight into the geometric structure of the underlying space and can lead to the discovery of new phenomena.

Symmetries of action integrals are linked to Noether symmetries, and thus they can be applied to any problem involving action integrals, regardless of its connection to the geodesic equation. Aslam and Qadir [?] attempted to establish the geometric technique for PDEs, and investigated several specific spaces based on this intriguing idea and discovered the Noether symmetries of the minimal hypersurfaces. The minimal hypersurface equations are the Euler-Lagrange equations for the area-minimizing action. Additionally, the Noether symmetries of the area-minimizing surface Lagrangian with constant volume in some spaces have been studied [?].

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Here, we will continue the same idea of [? ? ?] to provide the Noether symmetries that arise from minimal Lagrangian of cylindrically symmetric static space-time. We will also present the relevant conservation laws. These space-times, which are translationally symmetric and axis symmetric, have been thoroughly studied in the cosmology literature. To implement the geodesic Lagrangian method over this space led to the discovery of the Noether symmetries and accompanying conservation laws [? ?]; hence, here via intriguing generalization, we will focus on the minimum Lagrangian to illustrate the isometries of these space-times.
This article is devised as follows: In section 2, some definitions and explanation of minimal Lagrangian hypersurface along with an overview of the mathematical framework and tools used in the analysis, are discussed. Section 3, deals with the study of the isometries of minimal Lagrangian hypersurfaces under the constraint of fixed volume, and discussion of the results and insights gained from the analysis. In section 4, classification of Noether symmetry is investigated. Discussion of the conserved field associated with the Noether symmetry is considered in section 5. The last section, contains some conclusions and final remarks.

2. Preliminaries

Here, we review the Noether symmetries'fundamental definitions and prerequisites from [? ? ?].

Let the dependent variable ϖ^A depend on variables x^i, $\varpi^A_{(.)}$ represents the partial derivative of arbitrary order of ϖ^A and $H(x^i, \varpi^A, \varpi^A(1), \dots, \varpi^A(r)) = 0$ be a partial differential equation. The n-th order extension of a symmetry $\Gamma = \xi^i(x^K, \varpi^B)\dfrac{\partial}{\partial x^i} + \eta^A(x^K, \varpi^B)\dfrac{\partial}{\partial \varpi^A}$ is formulated by

$$\Gamma^{[r]} = \Gamma + \sum_{1\leq s\leq r} \eta^A_{i_1\dots i_s}\frac{\partial}{\partial \varpi^A_{i_1\dots i_s}},$$

where

$$\begin{aligned}&\eta^A_i = D_i(\eta^A - \xi^j\varpi^A_j) + \xi^j\varpi^A_{ij},\\ &\eta^A_{i_1\dots i_s} = D_{i_1}\cdots D_{i_s}(\eta^A - \xi^j\varpi^A_j) + \xi^j\varpi^A_{ji_1\dots i_s},\ \ s>1,\end{aligned}$$

where $D_i = \dfrac{\partial}{\partial x^i} + \varpi^A_i\dfrac{\partial}{\partial \varpi^A} + \varpi^A_{ij}\dfrac{\partial}{\partial \varpi^A_j} + \cdots$.

Assume that $\Upsilon(x^i, \varpi^A, \varpi^A_{(1)}, \dots, \varpi^A_{(r)})$ is a Lagrangian related to $H(x^i, \varpi^A, \varpi^A_{(1)}, \dots, \varpi^A_{(r)}) = 0$. If Euler-Lagrange operator $\dfrac{\delta}{\delta\varpi^A}$ is given by

$$\frac{\delta}{\delta\varpi^A} = \frac{\partial}{\partial\varpi^A} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial\varpi^A_{i_1\dots i_s}},$$

then the Lagrangian Υ has to satisfy the following Euler-Lagrange equation

$$\frac{\delta\Upsilon}{\delta\varpi^A} = 0.$$

Examining the invariance of the variational integral $I = \int \Upsilon(x^i, \varpi^A, \varpi^A(1)\dots, \varpi^A(r))$ under the influence of a gauge vector field $\mathbf{A} = A^i\partial_i = A^i(x^k, \varpi^A)\dfrac{\partial}{\partial x_i}$ results in the definition of a Noether symmetry Γ of the Lagrangian Υ. For Γ to be considered a Noether symmetry with respect to gauge field $\mathbf{A}$, it must meet the following requirements:

$$\Gamma^{[r-1]}\Upsilon + \Upsilon D_i\xi^i = D_iA^i. \tag{2.1}$$

The components of the conserved vector field $\mathbf{T} = T^i\partial_i$ are specified as follows:

$$T^i = A^i + (\xi^k\varpi_{,k} - \eta)\frac{\partial\Upsilon}{\partial\varpi_{,i}} - \xi^i\Upsilon. \tag{2.2}$$

Killing fields generates the infinitesimally small continuous isometries. If the Lie derivative of the metric tensor g respect to Γ is equal to zero, then vector field $\Gamma = x^i \frac{\partial}{\partial x^i}$ is a Killing field, i.e.

$$\mp_{\Gamma} g = 0,$$

so, we have

$$g_{ab,c}x^c + g_{bc}x^c_{,a} + g_{ac}x^c_{,b} = 0. \tag{2.3}$$

3. Isometries of minimal Lagrangian hypersurface with fixed volume

Suppose that the hypersurface $p = p(t, \theta, z)$ embedded in the following cylindrical symmetric static space-time (CSSS)

$$ds^2 = e^{\upsilon(p)}dt^2 - dp^2 - e^{\lambda(p)}a^2 d\theta^2 - e^{\mu(p)}dz^2, \tag{3.1}$$

where the constant a has units of length. The Lagrangian for an area-minimizing hypersurface $p = p(t, \theta, z)$ with a fixed volume is described as follows [?]:

$$\Upsilon = a\sqrt{|h|}\sqrt{e^{-\nu}p_{,t}^2 - (e^{-\lambda}/a^2)p_{,\theta}^2 - e^{-\mu}p_{,z}^2 - 1} + a\gamma\int\sqrt{|h|}dp, \tag{3.2}$$

where $|h| = e^{\nu+\lambda+\mu}$. Consequently, the Noether symmetry $\Gamma = \xi^1\partial_t + \xi^2\partial_\theta + \xi^3\partial_z + \eta\partial_p$ satisfying the condition (**??**) respect to the Lagrangian (**??**) gives:

$$\begin{aligned}
&\mathbf{E_1}: \eta(e^{-\nu}p_{,t}^2(|h|_{,p} - |h|\nu') - |h|_{,p}) + 2|h|e^{-\nu}p_{,t}(\eta_{,t} + p_{,t}\eta_{,p} - \xi^1_{,t}p_{,t} - \xi^2_{,t}p_{,\theta} - \xi^3_{,t}p_{,z}\\
&- \xi^1_{,p}p_{,t}^2 - \xi^2_{,p}p_{,t}p_{,\theta} - \xi^3_{,p}p_{,t}p_{,z}) + 2(|h|e^{-\nu}p_{,t}^2 - |h|)(\xi^1_{,t} + \xi^2_{,\theta} + \xi^3_{,z} + p_{,t}\xi^1_{,p}\\
&+ p_{,\theta}\xi^2_{,p} + p_{,z}\xi^3_{,z}) = 0,\\
&\mathbf{E_2}: \eta(\frac{1}{a^2}e^{-\lambda}p_{,\theta}^2(\lambda'|h| - |h|_{,p}) - |h|_{,p}) - \frac{2}{a^2}e^{-\lambda}p_{,\theta}(\eta_{,\theta} + p_{,\theta}\eta_{,p} - \xi^1_{,\theta}p_{,t} - \xi^2_{,\theta}p_{,\theta}\\
&- \xi^3_{,\theta}p_{,z} - \xi^1_{,p}p_{,\theta}p_{,t} - \xi^2_{,p}p_{,\theta}^2 - \xi^3_{,p}p_{,\theta}p_{,z}) - 2|h|(1 + \frac{1}{a^2}e^{-\lambda}p_{,\theta}^2)(\xi^1_{,t} + \xi^2_{\theta} + \xi^3_{,z}\\
&+ p_{,t}\xi^1_{,p} + p_{,\theta}\xi^2_{,p} + p_{,z}\xi^3_{,p}) = 0,\\
&\mathbf{E_3}: \eta(e^{-\mu}p_{,z}^2(\mu'|h| - |h|_{,p}) - |h|_{,p}) - 2e^{-\mu}p_{,z}(\eta_{,z} + p_{,z}\eta_{,p} - \xi^1_{,z}p_{,t} - \xi^2_{,z}p_{,\theta} - \xi^3_{,z}p_{,z}\\
&- \xi^1_{,p}p_{,z}p_{,t} - \xi^2_{,p}p_{,z}p_{,\theta} - \xi^3_{,p}p_{,z}^2) - 2|h|(1 + e^{-\mu}p_{,z}^2)(\xi^1_{,t} + \xi^2_{,\theta} + \xi^3_{,z} + p_{,t}\xi^1_{,p} + p_{,\theta}\xi^2_{,p}\\
&+ p_{,z}\xi^3_{,p}) = 0,\\
&\mathbf{E_4}: -\eta|h|_{,p} + |h|e^{-\nu}p_{,t}(\eta_{,t} + p_{,t}\eta_{,p} - \xi^1_{,t}p_{,t} - \xi^2_{,t}p_{,\theta} - \xi^3_{,t}p_{,z} - \xi^1_{,p}p_{,t}^2 - \xi^2_{,p}p_{,t}p_{,\theta}\\
&- \xi^3_{,p}p_{,t}p_{,z}) - \frac{1}{a^2}e^{-\lambda}p_{,\theta}(\eta_{,\theta} + p_{,\theta}\eta_{,p} - \xi^1_{,\theta}p_{,t} - \xi^2_{\theta}p_{,\theta} - \xi^3_{,\theta}p_{,z} - \xi^1_{,p}p_{,t}p_{,\theta} - \xi^2_{,p}p_{,\theta}^2\\
&- \xi^3_{,p}p_{,\theta}P_{,z}) - 2|h|(\xi^1_{,t} + \xi^2_{,\theta} + \xi^3_{,z} + p_{,t}\xi^1_{,p} + p_{,\theta}\xi^2_{,p} + p_{,z}\xi^3_{,p}) = 0,\\
&\mathbf{E_5}: -\eta|h|_{,p} + e^{-\nu}|h|p_{,t}(\eta_{,t} + p_{,t}\eta_{,p} - \xi^1_{,t}p_{,t} - \xi^2_{,t}p_{,\theta} - \xi^3_{,t}p_{,z} - \xi^1_{,p}p_{,t}^2 - \xi^2_{,p}p_{,t}p_{,\theta} - \xi^3_{,p}p_{,t}p_{,z})\\
&- e^{-\mu}|h|p_{,z}(\eta_{,z} + p_{,z}\eta_{,p} - \xi^1_{,z}p_{,t} - \xi^2_{,z}p_{,\theta} - \xi^3_{,z}p_{,z} - \xi^1_{,p}p_{,t}p_{,z} - \xi^2_{,p}p_{,\theta}p_{,z} - \xi^3_{,p}p_{,z}^2)\\
&- 2|h|(\xi^1_{,t} + \xi^2_{,\theta} + \xi^3_{,z} + p_{,t}\xi^1_{,p} + p_{,\theta}\xi^2_{,p} + p_{,z}\xi^3_{,p}) = 0,
\end{aligned}$$

$$\mathbf{E_6}: \eta|h|_{,p}+\frac{1}{a^2}e^{-\lambda}|h|p_{,\theta}(\eta_{,\theta}+p_{,\theta}\eta_{,p}-\xi^1_{\theta}p_{,t}-\xi^2_{,\theta}p_{,\theta}-\xi^3_{,\theta}p_{,z}-\xi^1_{,p}p_{,t}p_{,\theta}-\xi^2_{,p}p^2_{,\theta}$$
$$-\xi^3_{,p}p_{,\theta}p_{,z})+e^{-\mu}|h|p_{,z}(\eta_{,z}+p_{,z}\eta_{,p}-\xi^1_{,z}p_{,t}-\xi^2_{,z}p_{,\theta}-\xi^3_{,z}p_{,z}-\xi^1_{,p}p_{,t}p_{,z}-\xi^2_{,p}p_{,\theta}p_{,z}$$
$$-\xi^3_{,p}p^2_{,z})+2|h|(\xi^1_{,t}+\xi^2_{,\theta}+\xi^3_{,z}+p_{,t}\xi^1_{,p}+p_{,\theta}\xi^2_{,p}+p_{,z}\xi^3_{,p})=0,$$
$$\mathbf{E_7}: \gamma\eta\sqrt{|h|}+\gamma\int\sqrt{|h|}dp(\xi^1_{,t}+\xi^2_{,\theta}+\xi^3_{,z}+p_{,t}\xi^1_{,p}+p_{,\theta}\xi^2_{,p}+p_{,z}\xi^3_{,p})=A^1_{,t}+A^2_{,\theta}+A^3_{,z}$$
$$+p_{,t}A^1_{,p}+p_{,\theta}A^2_{,p}+p_{,z}A^3_{p}.$$

Now we find that the following relations by equating the constant coefficients and the coefficients of $p_{,t}, p_{,\theta}, p_{,z}$ into the expressions $E_1,\dots,E_7$

$$|h|_{,p}\eta+2|h|(\xi^1_{,t}+\xi^2_{,\theta}+\xi^3_{,z})=0, \tag{3.3a}$$
$$e^{-\nu}\eta_{,t}-\xi^1_{,p}=0, \tag{3.3b}$$
$$\frac{1}{a^2}e^{-\lambda}\eta_{,\theta}+\xi^2_{,p}=0, \tag{3.3c}$$
$$e^{-\mu}\eta_{,z}+\xi^3_{,p}=0. \tag{3.3d}$$

Also, by Comparing the 2nd power coefficients of $p_{,t}, p_{,\theta}, p_{,z}$ in the phrases $E_1,\dots,E_7$ and applying the formula (**??**), we get that

$$2\eta_{,p}-2\xi^1_{,t}-\nu'\eta=0, \tag{3.4a}$$
$$\lambda'\eta-2\eta_{,p}+2\xi^2_{,\theta}=0, \tag{3.4b}$$
$$\mu'\eta-2\eta_{,p}+2\xi^2_{,z}=0, \tag{3.4c}$$
$$\eta_{,p}(e^{-\nu}-\frac{1}{a^2}e^{-\lambda})-e^{-\nu}\xi^2_{,t}+\frac{1}{a^2}e^{-\lambda}\xi^1_{,\theta}=0, \tag{3.4d}$$
$$\eta_{,p}(e^{-\nu}-e^{-\mu})-e^{-\nu}\xi^3_{,t}+e^{-\mu}\xi^1_{,z}=0, \tag{3.4e}$$
$$\eta_{,p}(\frac{1}{a^2}e^{-\lambda}+e^{-\mu})-\frac{1}{a^2}e^{-\lambda}\xi^3_{,\theta}-e^{-\mu}\xi^2_{,z}=0. \tag{3.4f}$$

Adding the Equations (**??**) and (**??**) together then subtracting with (**??**), we get that

$$|h|_{,p}\eta-6|h|\eta_{,p}+2|h|(\xi^1_{,t}+\xi^2_{,\theta}+\xi^3_{,z})=0.$$

According to the above equation and relation (**??**) we get that $\eta_{,p}=0$, in other words η isn't a function of p. Consequently, we find the following determining equations by rewriting the relations (**??**)-(**??**)

$$\eta_{,p}=0, \quad \eta_{,t}-e^{\nu}\xi^1_{,p}=0, \quad \eta_{,\theta}+a^2e^{\lambda}\xi^2_{,p}=0, \quad \eta_{,z}+e^{\mu}\xi^3_{,p}=0,$$
$$\nu'\eta+2\xi^1_{,t}=0, \quad \lambda'\eta+2\xi^2_{,\theta}=0, \quad \mu'\eta+2\xi^3_{,z}=0, \quad e^{-\lambda}\xi^1_{,\theta}-a^2e^{-\nu}\xi^2_{,t}=0,$$
$$e^{-\mu}\xi^1_{,z}-e^{-\nu}\xi^3_{,t}=0, \quad e^{-\lambda}\xi^3_{,\theta}+a^2e^{-\mu}\xi^2_{,z}=0.$$

Now by applying (**??**), we can easily deduce that the Noether symmetry $\Gamma=\xi^1\partial_t+\xi^2\partial_\theta+\xi^3\partial_z+\eta\partial_p$ is Killing field.

Similarly, gauge vector field satisfies the following relations by comparing the coefficients of the condition E_7, $\mathbf{A} = A^1(t,\theta,z,p)\partial_t + A^2(t,\theta,z,p)\partial_\theta + A^3(t,\theta,z,p)\partial_z$:

$$\gamma\sqrt{|h|}\eta + \gamma\int\sqrt{|h|}dp(\xi^1_{,t} + \xi^2_{,\theta} + \xi^3_{,z}) = A^1_{,t} + A^2_{,\theta} + A^3_{,z}, \tag{3.5a}$$

$$\gamma\xi^1_{,p}\int\sqrt{|h|}dp = A^1_{,p}, \tag{3.5b}$$

$$\gamma\xi^2_{,p}\int\sqrt{|h|}dp = A^2_{,p}, \tag{3.5c}$$

$$\gamma\xi^3_{,p}\int\sqrt{|h|}dp = A^3_{,p}. \tag{3.5d}$$

4. Classification of Noether symmetry

Here, we obtain the Killing fields of some particular types of the CSSS metric by solving the determining equations. Then, assuming that some gauge fields satisfy the Eqs. (??)-(??), we may derive the Noether symmetries and related gauge vector fields for minimal Lagrangian with fixed volume among these isometries.

Case I: $\mu = \nu = \lambda = 0$.

The first interesting case of the CSSS metric is called wrapped Minkowski space (see [?]). In this case, all of the Killing fields Γ^I_{1-10} are accepted as the Noether symmetries. Since the maximum number of linearly-independent Killing vectors is equal to $\frac{1}{2}n(n+1) = 10$, where $n = 4$ is the dimension of the CSSS, the wrapped Minkowski space is maximally symmetric, so it has the constant curvature [?].

$$\begin{aligned}
&\Gamma^I_1 = \partial_t, && \text{with zero gauge field,}\\
&\Gamma^I_2 = \partial_\theta, && \text{with zero gauge field,}\\
&\Gamma^I_3 = \partial_z, && \text{with gauge field } \mathbf{A^I_3} = \gamma p\partial_z,\\
&\Gamma^I_4 = \partial_p, && \text{with gauge field } \mathbf{A^I_4} = \frac{\gamma}{3}(t\partial_t + \theta\partial_\theta + z\partial_z),\\
&\Gamma^I_5 = \theta\partial_t + \frac{t}{a^2}\partial_\theta, && \text{with zero gauge field,}\\
&\Gamma^I_6 = z\partial_t + t\partial_z, && \text{with zero gauge field,}\\
&\Gamma^I_7 = \theta\partial_z - \frac{z}{a^2}\partial_\theta, && \text{with zero gauge field,}\\
&\Gamma^I_8 = p\partial_t + t\partial_p, && \text{with gauge field } \mathbf{A^I_8} = \frac{\gamma}{2}(p^2 + t^2)\partial_t,\\
&\Gamma^I_9 = -\frac{p}{a^2}\partial_\theta + \theta\partial_p, && \text{with gauge field } \mathbf{A^I_9} = \frac{\gamma}{2}(\theta^2 - \frac{p^2}{a^2})\partial_\theta,\\
&\Gamma^I_{10} = -p\partial_z + z\partial_p, && \text{with gauge field } \mathbf{A^I_{10}} = \frac{\gamma}{2}(z^2 - p^2)\partial_z.
\end{aligned}$$

In the following the constant p_0 has units of length.

Case II: $\mu = \ln(\frac{p}{p_0})^2,\ \nu = \lambda = 0$.

The Lagrangian for a minimal hypersurface with a fixed volume exhibits the following six-dimensional algebra of Noether symmetries in the given metric.

$$\begin{aligned}
\Gamma_1^{II} &= \partial_t, && \text{with zero gauge field,}\\
\Gamma_2^{II} &= \partial_\theta, && \text{with zero gauge field,}\\
\Gamma_3^{II} &= \partial_z, && \text{with zero gauge field,}\\
\Gamma_4^{II} &= \frac{t}{a^2}\partial_\theta + \theta\partial_t, && \text{with zero gauge field,}\\
\Gamma_5^{II} &= -\cos(\frac{z}{p_0})\partial_p + \frac{p_0}{p}\sin(\frac{z}{p_0})\partial_z, && \text{with gauge field } \mathbf{A_5^{II}} = -\frac{\gamma p}{2}\sin(\frac{z}{p_0})\partial_z,\\
\Gamma_6^{II} &= \sin(\frac{z}{p_0})\partial_p + \frac{p_0}{p}\cos(\frac{z}{p_0})\partial_z, && \text{with gauge field } \mathbf{A_6^{II}} = -\frac{\gamma p}{2}\cos(\frac{z}{p_0})\partial_z.
\end{aligned}$$

The rest of Killing fields do not satisfy the relations (**??**)-(**??**) so they are not Noether symmetries. They are listed below:

$$\begin{aligned}
\Gamma_7^{II} &= \frac{p_0 t}{p}\cos(\frac{z}{p_0})\partial_z + t\sin(\frac{z}{p_0})\partial_p + p\sin(\frac{z}{p_0})\partial_t,\\
\Gamma_8^{II} &= \frac{p_0 t}{p}\sin(\frac{z}{p_0})\partial_z - t\cos(\frac{z}{p_0})\partial_p - p\cos(\frac{z}{p_0})\partial_t,\\
\Gamma_9^{II} &= \frac{p}{a^2}\cos(\frac{z}{p_0})\partial_\theta - \theta\cos(\frac{z}{p_0})\partial_p + \frac{p_0\theta}{p}\sin(\frac{z}{p_0})\partial_z,\\
\Gamma_{10}^{II} &= \frac{p}{a^2}\sin(\frac{z}{p_0})\partial_\theta - \theta\sin(\frac{z}{p_0})\partial_p - \frac{p_0\theta}{p}\cos(\frac{z}{p_0})\partial_z.
\end{aligned}$$

So we have a ten of linearly-independent Killing vectors, therefore the space is maximally symmetric.

Case III: $\mu = \nu = \lambda = \ln(\frac{p}{p_0})^{2l}$, where l is real constant and $(l \neq 0, 1)$.

The third case gives the Segrá type [1,(11)1], Petrov type D which renders a tachyonic fluid and we can exegesis again as an anisotropic fluid with a proper cosmological constant [**?**]. All of the following Killing fields are accepted as the strict Noether symmetries of Lagrangian that minimizes the area. Therefore, we obtain the following six-dimensional algebra of strict Noether symmetry

$$\begin{aligned}
\Gamma_1^{III} &= \partial_t, && \text{with zero gauge field,}\\
\Gamma_2^{III} &= \partial_\theta, && \text{with zero gauge field,}\\
\Gamma_3^{III} &= \partial_z, && \text{with zero gauge field,}\\
\Gamma_4^{III} &= \frac{t}{a^2}\partial_\theta + \theta\partial_t, && \text{with zero gauge field,}\\
\Gamma_5^{III} &= t\partial_z + z\partial_t, && \text{with zero gauge field,}\\
\Gamma_6^{III} &= \frac{z}{a^2}\partial_\theta - \theta\partial_z, && \text{with zero gauge field.}
\end{aligned}$$

Case IV: $\nu = \ln(\frac{p}{p_0})^2,\ \mu = \lambda = 0.$

The Lagrangian for a minimal hypersurface with a constant volume has been shown to admit a six-dimensional algebra of Noether symmetry within this metric.

$$
\begin{aligned}
&\Gamma_1^{IV} = \partial_t, && \text{with zero gauge field},\\
&\Gamma_2^{IV} = \partial_\theta, && \text{with zero gauge field},\\
&\Gamma_3^{IV} = \partial_z, && \text{with zero gauge field},\\
&\Gamma_4^{IV} = \theta\partial z - \frac{z}{a^2}\partial_\theta, && \text{with zero gauge field},\\
&\Gamma_5^{IV} = \cosh(\frac{t}{p_0})\partial_p - \frac{p_0}{p}\sinh(\frac{t}{p_0})\partial_t, && \text{with gauge field } \mathbf{A_5^{IV}} = \frac{\gamma p}{2}\sinh(\frac{t}{p_0})\partial_t,\\
&\Gamma_6^{IV} = \sinh(\frac{t}{p_0})\partial_p - \frac{p_0}{p}\cosh(\frac{t}{p_0})\partial_t, && \text{with gauge field } \mathbf{A_6^{IV}} = \frac{\gamma p}{2}\cosh(\frac{t}{p_0})\partial_t.
\end{aligned}
$$

The rest of Killing fields do not satisfy the relations (**??**)-(**??**) so they are not Noether symmetries. They are listed below:

$$
\begin{aligned}
\Gamma_7^{IV} &= z\sinh(\frac{t}{p_0})\partial_p - \frac{p_0 z}{p}\cosh(\frac{t}{p_0})\partial_t - p\sinh(\frac{t}{p_0})\partial_z,\\
\Gamma_8^{IV} &= z\cosh(\frac{t}{p_0})\partial_p - \frac{p_0 z}{p}\sinh(\frac{t}{p_0})\partial_t - p\cosh(\frac{t}{p_0})\partial_z,\\
\Gamma_9^{IV} &= \theta\cosh(\frac{t}{p_0})\partial_p - \frac{p}{a^2}\cosh(\frac{t}{p_0})\partial_\theta - \frac{p_0\theta}{p}\sinh(\frac{t}{p_0})\partial_t,\\
\Gamma_{10}^{IV} &= \theta\sinh(\frac{t}{p_0})\partial_p - \frac{p}{a^2}\sinh(\frac{t}{p_0})\partial_\theta - \frac{p_0\theta}{p}\cosh(\frac{t}{p_0})\partial_t.
\end{aligned}
$$

So we have a ten of linearly-independent Killing vectors, therefore the space is maximally symmetric.

The Lagrangian for a minimal hypersurface with a constant volume has been determined to accept all four Killing fields as strict Noether symmetries in each of the four cases considered. As a result, in each case, we obtain a 4-dimensional algebra with strict Noether symmetry. The first two cases give the Segrá type [1,111], Petrov type D space, the third case gives the Segrá type [(1,1)(11)], Petrov type D space and the last case gives the Segrá type [1,(11)1], Petrov type D space [**?**].

Case V: $\nu = \ln(\frac{p}{p_0})^2,\ \mu = 0,\ \lambda = \ln(\frac{p}{a})^2$.

$$
\begin{aligned}
&\Gamma_1^{V} = \partial_t, && \text{with zero gauge field},\\
&\Gamma_2^{V} = \partial_\theta, && \text{with zero gauge field},\\
&\Gamma_3^{V} = \partial_z, && \text{with zero gauge field},\\
&\Gamma_4^{V} = \frac{t}{p_0^2}\partial_\theta + \theta\partial_t, && \text{with zero gauge field}.
\end{aligned}
$$

Case VI: $\nu = \mu = \ln(\frac{p}{p_0})^2,\ \lambda = 0$.

$$
\begin{aligned}
&\Gamma_1^{VI} = \partial_t, && \text{with zero gauge field},\\
&\Gamma_2^{VI} = \partial_\theta, && \text{with zero gauge field},\\
&\Gamma_3^{VI} = \partial_z, && \text{with zero gauge field},\\
&\Gamma_4^{VI} = t\partial_z + z\partial_t, && \text{with zero gauge field}.
\end{aligned}
$$

Case VII: $\nu = 0,\ \mu = \lambda = \ln(\frac{p}{a})^2$.

$\Gamma_1^{VII} = \partial_t,$ with zero gauge field,

$\Gamma_2^{VII} = \partial_\theta,$ with zero gauge field,

$\Gamma_3^{VII} = \partial_z,$ with zero gauge field,

$\Gamma_4^{VII} = \theta\partial_z - \dfrac{z}{a^2}\partial_\theta,$ with zero gauge field.

Case VIII: $\nu = \ln(\frac{p}{p_0})^{2l},\ \mu = \lambda = \ln(\frac{p}{p_0})^{2r}\ (l \neq r,\ l, r \neq 0, 1)$.

$\Gamma_1^{VIII} = \partial_t,$ with zero gauge field,

$\Gamma_2^{VIII} = \partial_\theta,$ with zero gauge field,

$\Gamma_3^{VIII} = \partial_z,$ with zero gauge field,

$\Gamma_4^{VIII} = \theta\partial_z - \dfrac{z}{a^2}\partial_\theta,$ with zero gauge field.

5. Conserved field

In this section, we will use the (**??**), formula to find the conserved field $\mathbf{T} = T^1\partial_t + T^2\partial_\theta + T^3\partial_z$ for various Noether symmetries with non-zero gauge fields.

• For the Noether symmetry $\Gamma_4^I = \partial_p$ with gauge field $\mathbf{A_4^I} = \dfrac{\gamma}{3}(t\partial_t + \theta\partial_\theta + z\partial_z)$, we find the following conserved field:

$$\mathbf{T_4^I} = (\frac{\gamma}{3}t - \frac{ap_{,t}}{\Upsilon^I})\partial_t + (\frac{\gamma}{3}\theta - \frac{p_{,\theta}}{a\Upsilon^I})\partial_\theta + (\frac{\gamma}{3}z - \frac{ap_{,z}}{\Upsilon^I})\partial_z,$$

where $\Upsilon^I = \sqrt{p_{,t}^2 + \dfrac{1}{a^2}p_{,\theta}^2 - p_{,z}^2 - 1}$.

• For the Noether symmetry $\Gamma_8^I = p\partial_t + t\partial_p$ with gauge field $\mathbf{A_8^I} = \dfrac{\gamma}{2}(p^2 + t^2)\partial_t$, the following conserved field resulted:

$$\mathbf{T_8^I} = \left(\frac{\gamma}{2}(p^2 + t^2) + \frac{ap_{,t}}{\Upsilon^I}(pp_{,t} - t) - p(a\Upsilon^I + \gamma p)\right)\partial_t + \frac{p_{,\theta}}{a\Upsilon^I}(pp_{,t} - t)\partial_\theta - \frac{ap_{,z}}{\Upsilon^I}(pp_{,t} - t)\partial_z.$$

• For the Noether symmetry $\Gamma_5^{II} = \dfrac{p_0}{p}\sin(\dfrac{z}{p_0})\partial_z - \cos(\dfrac{z}{p_0})\partial_p$ with gauge field $\mathbf{A_5^{II}} = -\dfrac{\gamma}{2}p\sin(\dfrac{z}{p_0})\partial_z$, we have the following conserved field:

$$\mathbf{T_5^{II}} = \frac{app_{,t}}{p_0\Upsilon^{II}}\left(\frac{p_0}{p}\sin(\frac{z}{p_0})p_{,z} + \cos(\frac{z}{p_0})\right)\partial_t + \frac{pp_{,\theta}}{ap_0\Upsilon^{II}}\left(\frac{p_0}{p}\sin(\frac{z}{p_0})p_{,z} + \cos(\frac{z}{p_0})\right)\partial_\theta$$
$$\left(-\frac{\gamma}{2}p\sin(\frac{z}{p_0}) + \frac{p_0p_{,z}}{p\Upsilon^{II}}\left(\frac{p_0}{p}\sin(\frac{z}{p_0}) + \cos(\frac{z}{p_0})\right) + a(\Upsilon^{II} + \frac{\gamma}{2}p)\sin(\frac{z}{p_0})\right)\partial_z,$$

where $\Upsilon^{II} = \sqrt{p_{,t}^2 + \dfrac{1}{a^2}p_{,\theta}^2 - (\dfrac{p_0}{p})^2p_{,z}^2 - 1}$.

• For the Noether symmetry $\Gamma_5^{IV} = \cosh(\dfrac{t}{p_0})\partial_t - \dfrac{p_0}{p}\sinh(\dfrac{t}{p_0})\partial_t$ with gauge field $\mathbf{A_5^{IV}} = \dfrac{\gamma}{2}p\sinh(\dfrac{t}{p_0})\partial_t$, we get that:

$$\mathbf{T_5^{IV}} = \left(\frac{\gamma}{2}p\sinh(\frac{t}{p_0}) - \frac{ap_0p_{,t}}{p\Upsilon^{IV}}\left(\frac{p_0p_{,t}}{p}\sinh(\frac{t}{p_0}) + \cosh(\frac{t}{p_0})\right) + a(\Upsilon^{IV} + \frac{\gamma}{2}p)\sinh(\frac{t}{p_0})\right)\partial_t$$
$$- \frac{pp_{,\theta}}{ap_0\Upsilon^{IV}}\left(\frac{p_0p_{,t}}{p}\sinh(\frac{t}{p_0}) + \cosh(\frac{t}{p_0})\right)\partial_\theta + \frac{app_{,z}}{p_0\Upsilon^{IV}}\left(\frac{p_0p_{,t}}{p}\sinh(\frac{t}{p_0}) + \cosh(\frac{t}{p_0})\right)\partial_z,$$

where $\Upsilon^{IV} = \sqrt{(\frac{p_0}{p})^2 p_{,t}^2 + \frac{1}{a^2} p_{,\theta}^2 - p_{,z}^2 - 1}.$

6. Conclusion

We have shown the area-minimizing hypersurface Lagrangian in cylindrically symmetric static space-time and shown that the related Lagrangian's Noether symmetries are components of the Killing algebra in CSSS space. By reducing the order of the differential equation, Noether symmetries are useful tools for identifying conservation laws that can be utilized to obtain the solution of the area-minimizing hypersurface equation. For example if we choose the first case of CSSS, the area-minimizing hypersurface equation which follows from the Lagrangian $\Upsilon = a(\sqrt{p_t^2 + \frac{1}{a^2} p_\theta^2 - p_z^2 - 1} + \gamma p)$ is a PDE of order two with independents variables $\{t, \theta, z\}$. By choosing ∂_t, ∂_z as the Noether symmetries and the reduction of the equation by these symmetries, we find that the following ODE

$$a q_{,\theta\theta} - \gamma(q_{,\theta}^2 - a^2)\sqrt{q_{,\theta}^2 - a^2} = 0,$$

where $p(t, \theta, z) = q(\theta)$. To convert the aforementioned equation to quadratures, one might use the remaining Noether symmetries in case I. For further information on how to reduce the area-minimizing hypersurface equation, see [**?** **?**].

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