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q-Exponential fixed point theorem for mixed monotone operator with q-fractional problem

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Abstract

In this work, we examine the existence and uniqueness(EU) of q-Exponential positive solution (q-EPS) of the hybrid q-fractional boundary value problem (q-FBVP).
We prove the q-Exponential fixed point theorem (q-EFPT) with a new set ρ_{h,e_1} in the Banach space $\mathbf{E}$ to check the EU of q-EPS of the q-FBVP. In the long run, an exemplum is given to show the correctness of our results.

Keywords. q-Derivative, Positive solution, Mixed monotone operator, q-Integral, q-Exponential fixed point theorem.
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1. Introduction

Fractional differential equations (FDE) include fractional derivatives. Nowadays, there are several different concepts and definitions of fractional derivatives, fractional integrals and their applications in different fields of mathematics. In [8, 11], the existence and uniqueness of the positive solutions of the fractional boundary value problem have been investigated by using fixed point theorems in a set including $\varphi(h,e)$-concave operators. In [10], unique solutions of the fractional boundary value problem in ordered Banach spaces where the operators are mixed monotone were examined.

Two important types of these derivatives and integrals that have been of our interest are q-derivative, q-integral, Riemann-Liouville fractional derivative (RLFD) and Riemann-Liouville fractional integral (RLFI). In this work, we investigate the EU of the q-EPS of the problem below:

$$\begin{cases} -D_{\mathsf{q}} D^{\gamma}_{0+}\left(\frac{w(\tau)+3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)}\right) - [\chi(\tau,w(\tau),\vartheta(\tau)) + \varrho(\tau,w(\tau)) + \phi(\tau,\vartheta(\tau))] = 0, \\ 0<\tau<1,\ 2\le\gamma<3,\ 0<\mathsf{q}<1, \\ w(0)=w'(0)=e_1(0)=e_1'(0)=0, \\ \left[I^{\alpha}_{0+}\left(\frac{w(\tau)+3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)}\right)\right]_{\tau=1}=0,\ 1\le\alpha\le 2, \end{cases} \tag{1.1}$$

where λ is a positive real number, $e_1(\tau)$ is a differentiable function and $0<e_1(\tau)$, D^{γ}_{0+} is the RLFD of order γ, D_{q} is the q-derivative and I^{α}_{0+} is the RLFI of order α. Let's assume that $O=[0,1]$ and $\Upsilon=[0,+\infty)$. The functions $\chi: O\times\Upsilon\times\Upsilon\longrightarrow\Upsilon$ and $\varrho,\phi: O\times\Upsilon\longrightarrow\Upsilon$ are continuous.

Remark 1.1. In this work, the ϑ variable in the χ function equal to $\vartheta(\tau)=1-\frac{I_{\mathsf{q},0}w(\tau)}{\tau}$, $\mathsf{q},\tau\in(0,1)$.

We have extracted Definition 1.2 from reference [2].

Definition 1.2. The q-Exponential function $E_{\mathsf{q}}(\tau)$ is expressed as $E_{\mathsf{q}}(\tau)=\sum_{k=0}^{\infty}\frac{1}{\{k\}_{\mathsf{q}}!}\tau^k$ if $|\mathsf{q}|>1$ or $0<|\mathsf{q}|<1$, $|\tau|<|1-\mathsf{q}|^{-1}$. Where $\{k\}_{\mathsf{q}}!=\prod_{k=1}^{n}\frac{1-\mathsf{q}^k}{1-\mathsf{q}}$ and $\{0\}_{\mathsf{q}}=0$.

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In [1], A. Boutiara and M. Benbachir concerned the EU of solution of the q-system of nonlinear FDE on four points. Their analysis relies on two issues, one of them is topological degree and the other is the fixed point theorem (FPT) of Banach contraction principle.

$$D_{\mathsf{q}}^{\mathsf{q}_1} w_1(\mu) = \chi_1(\mu, w_1(\mu), w_2(\mu)), \ \mu \in O := [0,1],$$
$$D_{\mathsf{q}}^{\mathsf{q}_2} w_2(\mu) = \chi_2(\mu, w_1(\mu), w_2(\mu)),$$

with boundary conditions:

$$w_1(0) = a_1 I_q^{\beta_1} w(\eta_1), \ 0 < \eta_1 < 1, \ \beta_1 > 0,$$
$$w_1(1) = b_1 I_q^{\alpha_1} w(\sigma_1), \ 0 < \sigma_1 < 1, \ \alpha_1 > 0,$$
$$w_2(0) = a_2 I_q^{\beta_2} w(\eta_2), \ 0 < \eta_2 < 1, \ \beta_2 > 0,$$
$$w_2(1) = b_2 I_q^{\alpha_2} w(\sigma_2), \ 0 < \sigma_2 < 1, \ \alpha_2 > 0,$$

for $i = 1, 2$, $D_{\mathsf{q}}^{\mathsf{q}_i}$ are the q-derivative from the Caputo type of order $1 < \mathsf{q}_i \leq 2$.
In [12], C. Zhai and L. Wang proved the EU of positive solutions (PS) of the following FDE by using FPT:

$$D_{0+}^{\alpha} w(\tau) + \chi(\tau, w(\tau)) = b, \ 0 < \tau < 1,$$
$$w(0) = w'(0) = 0, \ w(1) = \beta \int_0^1 w(s) ds,$$

where $2 < \alpha \leq 3$, $0 < \beta < \alpha$ and $b > 0$ is a constants.
In [3], M. Gholami and A. Neamaty have proved the EU of solution of the problem below by using the λ-FPT of mixed monotone operator (MMO):

$$-D_{0+}^{\nu} w(\tau) = \lambda^{-1}(\chi(\tau, w(\tau), \vartheta(\tau)) + \varrho(\tau, w(\tau)) + \phi(\tau, \vartheta(\tau))), \qquad 0<\tau<1, \ 3\leq\nu\leq 4,$$
$$w(0) = w'(0) = w''(0) = 0,$$
$$[D_{0+}^{\rho} w(\tau)]_{\tau=1} = 0, \ 1 \leq \rho \leq 2.$$

This problem includes the RLFD and the positive number of λ. In [5], Y. Liu, Ch. Yan, and W. Jiang checked the existence of solution to the below FBVP:

$${}^{C}D_{1^-}^{\alpha} D_{0+}^{\beta} w(\tau) + \chi(\tau, w(\tau)) = b, \ 0 < \tau < 1,$$
$$w(0) = 0, \ w'(1) = D_{0+}^{\beta} w(1) = 0,$$

where ${}^{C}D_{1^-}^{\alpha}$ is the left-sided Capato derivative of order α, $0 < \alpha \leq 1$ and D_{0+}^{β} is the right RLFD of order β, $1 < \beta \leq 2$ and $\alpha + \beta \geq 2$. b is a constant.
In [9], Y. sang, and Y. Ren checked the EU of the PS to the following BVP using the method of the MMO:

$$w^{(4)}(\tau) = \chi(\tau, w(\tau), (Hw)(\tau)), \ \tau \in [0,1],$$
$$w(0) = w'(0) = 0,$$
$$w''(1) = 0, \ w'''(1) = \varrho(w(1)).$$

Our purpose is to survey the EU of the q-EPS of the hybrid q-FBVP (1.1) using the q-FPT with the new set ρ_{h,e_1}. We work in a real Banach space $(\mathbf{E}, \|.\|)$, with ρ cone. (Read [3] for more information).

2. Preliminaries

In the second part, we state the preliminaries that we need to prove the main results.
From the references [4] and [7], we have extracted Definitions below.

Definition 2.1. RLFI $I_{0+}^{\alpha}w$ of order $\alpha > 0$ on finite interval $\Omega = [0, b](0 < b < \infty)$ on the real axis $\Re$, is defined below:

$$I_{0+}^{\alpha}w(u) = \frac{1}{\Gamma(\alpha)}\int_0^u \frac{w(\tau)}{(u-\tau)^{1-\alpha}}d\tau,\ u > 0,\ \alpha > 0),$$

here $\Gamma(\alpha)$ is the Gamma function.

Definition 2.2. Let $\gamma > 0$. $D_{0+}^{\gamma}w$ is called RLFD of order $\gamma > 0$ and is expressed as below:

$$(D_{0+}^{\gamma}w)(u) := \left(\frac{d}{du}\right)^{\mathfrak{N}}(I_{0+}^{\mathfrak{N}-\gamma}w)(u) = \frac{1}{\Gamma(\mathfrak{N}-\gamma)}\left(\frac{d}{du}\right)^{\mathfrak{N}}\int_0^u \frac{w(\tau)}{(u-\tau)^{\gamma-\mathfrak{N}+1}}d\tau,\ (\mathfrak{N} = [\gamma]+1,\ u > 0),$$

where $[\gamma]$ means the integer part of γ.

Definition 2.3. Let $\varphi : [0, \infty) \to \Re$ be a function. $D_{\mathsf{q}}\varphi$ is called q-derivative, and it is defined as follows:

$$(D_{\mathsf{q}}\varphi)(u) = \frac{\varphi(u) - \varphi(\mathsf{q}(u))}{u - \mathsf{q}u},\ u \neq 0,\ \mathsf{q} \in \Re^+\backslash 1.$$

Definition 2.4. Let $\varphi : [0, \infty) \to \Re$ be a function. $I_{\mathsf{q},0}\varphi$ is called q-integral, and it is defined below:

$$(I_{\mathsf{q},0}\varphi)(u) = \int_0^u \varphi(\tau)d_{\mathsf{q}}(\tau) = u(1-\mathsf{q})\sum_{k=0}^{\infty}\varphi(u\mathsf{q}^k)\mathsf{q}^k,\ 0 < |\mathsf{q}| < 1,\ u \in \Re^+.$$

Also, we have:

$$(I_{\mathsf{q},a}\varphi)(u) = \int_a^u \varphi(\tau)d_{\mathsf{q}}(\tau) = \int_0^u \varphi(\tau)d_{\mathsf{q}}(\tau) - \int_0^a \varphi(\tau)d_{\mathsf{q}}\tau,\ a \in \Re^+,$$

$$(D_{\mathsf{q}}I_{\mathsf{q},a}\varphi)(u) = \varphi(u),\ a \in \Re^+,$$

$$(I_{\mathsf{q},a}D_{\mathsf{q}}\varphi)(u) = \varphi(u) - \varphi(a),\ a \in \Re^+. \tag{2.1}$$

The following Lemmas and Conclusion are extracted from references [4], [7] and [6].

Lemma 2.5. *For $\alpha, \beta \in \Re^+$, relation below is established:*

$$(I_{\mathsf{q},a}^{\beta}I_{\mathsf{q},a}^{\alpha}\varphi)(u) = (I_{\mathsf{q},a}^{\alpha+\beta}\varphi)(u),$$

Lemma 2.6. *For $\alpha \in \Re$ and $\lambda \in (-1, \infty)$, the following property is valid:*

$$I_{\mathsf{q},a}^{\alpha}((u-a)^{\lambda}) = \frac{\Gamma_{\mathsf{q}}(\lambda+1)}{\Gamma_{\mathsf{q}}(\alpha+\lambda+1)}(u-a)^{\alpha+\lambda},\ (0 < a < u < b).$$

Lemma 2.7. *If $v \in L^1[0, 1]$, $\eta > 0$ and $\kappa > 0$, then*

$$\begin{aligned} I_{0+}^{\kappa}I_{0+}^{\eta}v(\tau) &= I_{0+}^{\kappa+\eta}v(\tau), \\ D_{0+}^{\eta}I_{0+}^{\eta}v(\tau) &= v(\tau). \end{aligned}$$

Lemma 2.8. *If $\nu > -1$, $\alpha > 0$ and $\tau > 0$ thus*

$$D_{0^+}^{-\nu}\tau^{\alpha} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)}\tau^{\nu+\alpha}.$$

Remark 2.9. If v has a RLFD of order $\alpha > 0$ and $v \in \mathfrak{C}(0, 1) \cap \mathfrak{L}(0, 1)$, so

$$I_{0+}^{\alpha}D_{0+}^{\alpha}v(\tau) = v(\tau) - \sum_{j=1}^{n}\mathfrak{C}_j(\tau-a)^{\alpha-j},$$

for any $\mathfrak{C}_j \in \Re$ where $\mathfrak{N} = [\alpha] + 1$.

Lemma 2.10. *Let* $w, v \in \mathfrak{C}(0,1) \cap \mathfrak{L}(0,1)$, $D_{\mathsf{q}} D^{\gamma} w \in \mathfrak{L}^1[0,1]$, $I_{\mathsf{q},0} v \in \mathfrak{L}^1[0,1]$, $0 < \mathsf{q} < 1$ *and* $w(0) = 0$ *be given. Thus, the problem* $-D_{\mathsf{q}} D^{\gamma}_{0+} \left(\frac{w(\tau)+3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)}\right) = v(\tau)$ *has a general solution as follows:*

$$w(\tau) = E_{\mathsf{q}}(\lambda\tau) \left(\sum_{j=1}^{\mathfrak{N}} \mathfrak{C}_j \tau^{\gamma-j} - I^{\gamma}_{0+} I_{\mathsf{q},0} v(\tau) \right) - 3e_1(\tau),$$

where $\mathfrak{C}_j \in \Re (j = 1, 2, ..., n)$, $\mathfrak{N} = [\gamma]$.

Proof. First, we take the q-integral from the sides of the equation and we get

$$-I_{\mathsf{q},0} D_{\mathsf{q}} D^{\gamma}_{0+} \left(\frac{w(\tau) + 3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)} \right) = I_{\mathsf{q},0} v(\tau).$$

By using of (2.1), we attain $-D^{\gamma}_{0+} \left(\frac{w(\tau)+3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)}\right) = I_{\mathsf{q},0} v(\tau)$. Now we take the I^{γ}_{0+} from the sides, therefore,

$$-I^{\gamma}_{0+} D^{\gamma}_{0+} \left(\frac{w(\tau) + 3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)} \right) = I^{\gamma}_{0+} I_{\mathsf{q},0} v(\tau).$$

By using Remark 2.9, we obtain

$$\sum_{j=1}^{\mathfrak{N}} \mathfrak{C}_j \tau^{\gamma-j} - \left(\frac{w(\tau) + 3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)} \right) = I^{\gamma}_{0+} I_{\mathsf{q},0} v(\tau).$$

So, we have:

$$w(\tau) = E_{\mathsf{q}}(\lambda\tau) \left(\sum_{j=1}^{\mathfrak{N}} \mathfrak{C}_j \tau^{\gamma-j} - I^{\gamma}_{0+} I_{\mathsf{q},0} v(\tau) \right) - 3e_1(\tau).$$

□

Now we state a theorem in which we earn the Green function of the solution to the problem.

Theorem 2.11. *Assume* $v \in \mathfrak{C}0,1] \cap \mathfrak{L}^1[0,1]$, $D_{\mathsf{q}} D^{\gamma} w \in \mathfrak{L}^1[0,1]$, $I_{\mathsf{q},0} v \in \mathfrak{L}^1[0,1]$ *and* $0 < \mathsf{q} < 1$ *be given. Thus, the hybrid q-FBVP*

$$\begin{aligned} & - D_{\mathsf{q}} D^{\gamma}_{0+} \left(\frac{w(\tau) + 3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)} \right) = v(\tau),\ 2 \leq \gamma < 3,\ 0 < \mathsf{q} < 1, \\ & w(0) = w'(0) = e_1(0) = e_1'(0) = 0, \\ & \left[I^{\alpha}_{0+} \left(\frac{w(\tau) + 3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)} \right) \right]_{\tau=1} = 0,\ 1 \leq \alpha \leq 2, \end{aligned}$$

has a q-EPS as follows

$$w(\tau) = E_{\mathsf{q}}(\lambda\tau) \int_0^1 \mathfrak{G}(\tau,\varsigma) Y(\varsigma) d\varsigma - 3e_1(\tau) = E_{\mathsf{q}}(\lambda\tau) \int_0^1 \mathfrak{G}(\tau,\varsigma) I_{\mathsf{q},0} v(\varsigma) d\varsigma - 3e_1(\tau), \tag{2.2}$$

where $G(\tau,\varsigma)$ *a Green function is as below*

$$\mathfrak{G}(\tau,\varsigma) = \frac{1}{\Gamma(\gamma)} \begin{cases} \tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1} + (\tau-\varsigma)^{\gamma-1},\ 0 \leq \varsigma \leq \tau \leq 1, \\ \tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1},\ 0 \leq \tau \leq \varsigma \leq 1. \end{cases} \tag{2.3}$$

Proof. According to Lemma 2.10, the problem has a general solution as follows:

$$w(\tau) = E_{\mathsf{q}}(\lambda\tau) \left(\mathfrak{C}_1 \tau^{\gamma-1} + \mathfrak{C}_2 \tau^{\gamma-2} + \mathfrak{C}_3 \tau^{\gamma-3} - I^{\gamma}_{0+} I_{\mathsf{q},0} v(\tau) \right) - 3e_1(\tau).$$

Using the first boundary condition, we have that $\mathfrak{C}_2 = \mathfrak{C}_3 = 0$. Hence,

$$w(\tau) = E_{\mathsf{q}}(\lambda\tau) \left(\mathfrak{C}_1 \tau^{\gamma-1} - I^{\gamma}_{0+} I_{\mathsf{q},0} v(\tau) \right) - 3e_1(\tau).$$

Also, using the second boundary condition, we get the constant $\mathfrak{C}_1$:

$$I_{0^+}^{\alpha}\left(\frac{w(\tau)+3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)}\right)=\mathfrak{C}_1 I_{0^+}^{\alpha}\tau^{\gamma-1}-I_{0^+}^{\alpha}I_{0^+}^{\gamma}I_{\mathsf{q},0}v(\tau).$$

By using of Lemmas 2.5 and 2.8, and Definition 2.2 we obtain:

$$I_{0^+}^{\alpha}\left(\frac{w(\tau)+3e_1(\tau)}{E_{\mathsf{q}}(\lambda\tau)}\right)=\mathfrak{C}_1\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}t^{\alpha+\gamma-1}-\frac{1}{\Gamma(\alpha+\gamma)}\int_0^{\tau}(\tau-\varsigma)^{\gamma+\alpha-1}I_{\mathsf{q},0}v(\varsigma)d\varsigma.$$

By putting $\tau=1$, we get:

$$0=\mathfrak{C}_1\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}-\frac{1}{\Gamma(\alpha+\gamma)}\int_0^{1}(1-\varsigma)^{\gamma+\alpha-1}I_{\mathsf{q},0}v(\varsigma)d\varsigma,$$

we assume that $I_{\mathsf{q},0}v(\varsigma)=Y(\varsigma)$. Therefore, $\mathfrak{C}_1=\frac{1}{\Gamma(\gamma)}\int_0^1(1-\varsigma)^{\gamma+\alpha-1}Y(\varsigma)d\varsigma$. By placing $\mathfrak{C}_1$ and $\mathfrak{C}_2=\mathfrak{C}_3=0$ in $w(\tau)$, we have

$$\begin{aligned}
w(\tau)&=E_{\mathsf{q}}(\lambda\tau)(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)}\int_0^1(1-\varsigma)^{\gamma+\alpha-1}Y(\varsigma)d\varsigma-\frac{1}{\Gamma(\gamma)}\int_0^{\tau}(\tau-\varsigma)^{\gamma-1}Y(\varsigma)d\varsigma)-3e_1(\tau)\\
&=E_{\mathsf{q}}(\lambda\tau)(\frac{1}{\Gamma(\gamma)}\int_0^{\tau}(\tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}-(\tau-\varsigma)^{\gamma-1})Y(\varsigma)d\varsigma\\
&+\frac{1}{\Gamma(\gamma)}\int_{\tau}^1\tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}Y(\varsigma)d\varsigma)-3e_1(\tau).
\end{aligned}$$

Therefore, the Green function is as below:

$$\mathfrak{G}(\tau,\varsigma)=\frac{1}{\Gamma(\gamma)}\begin{cases}\tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}+(\tau-\varsigma)^{\gamma-1}, & 0\le\varsigma\le\tau\le 1,\\ \tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}, & 0\le\tau\le\varsigma\le 1.\end{cases}$$

So, we obtain that

$$w(\tau)=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)Y(\varsigma)d\varsigma-3e_1(\tau).$$

□

In the following Lemma, the characteristics of the Green function in (2.3) were stated and proved that we need in the main part.

Lemma 2.12. *The Green function defined in Theorem 2.11 has the following properties:*

(1) *for every* $\tau,\varsigma\in[0,1]$ *then,* $\mathfrak{G}(\tau,\varsigma)\ge 0$,
(2) *for any* $\tau,\varsigma\in[0,1]$ *so,*

$$\tau^{\gamma-1}[(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}]\le\Gamma(\gamma)\mathfrak{G}(\tau,\varsigma)\le\tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}.$$

Proof. (1). It can be concluded $\mathfrak{G}(0,0)=0$. Also, for any constant ς, the Green function $\mathfrak{G}$ is increasing with respect to τ. That is, for $0\le\tau\le\varsigma\le 1$

$$\frac{\partial\mathfrak{G}(\tau,\varsigma)}{\partial\tau}=\frac{(\gamma-1)\tau^{\gamma-2}(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)}\ge 0,$$

and for $0\le\varsigma\le\tau\le 1$, we have:

$$\frac{\partial\mathfrak{G}(\tau,\varsigma)}{\partial\tau}=\frac{(\gamma-1)\tau^{\gamma-2}(1-\varsigma)^{\gamma+\alpha-1}-(\gamma-1)\tau^{\gamma-2}(1-\frac{\varsigma}{\tau})^{\gamma-2}}{\Gamma(\gamma)}\ge 0.$$

That's mean, $\mathfrak{G}(\tau,\varsigma)\geq 0$. The verdict of the first part has been proven.
(2). It is obvious that $\Gamma(\gamma)\mathfrak{G}(\tau,\varsigma)\leq \tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}$. First, we prove the left side of the inequality, then

$$0\leq \tau-\varsigma\leq \tau-\tau\varsigma=\tau(1-\varsigma),$$

so, we get $(\tau-\varsigma)^{\gamma-1}\leq \tau^{\gamma-1}(1-\varsigma)^{\gamma-1}$. Therefore,

$$\begin{aligned}\Gamma(\gamma)\mathfrak{G}(\tau,\varsigma)&=\tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}-(\tau-\varsigma)^{\gamma-1}\\&\geq \tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}-\tau^{\gamma-1}(1-\varsigma)^{\gamma-1}\\&=\tau^{\gamma-1}[(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}],\end{aligned}$$

if $0\leq\tau\leq\varsigma\leq 1$ thus, we obtain:

$$\begin{aligned}\Gamma(\gamma)\mathfrak{G}(\tau,\varsigma)&=\tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}\\&\geq \tau^{\gamma-1}[(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}].\end{aligned}$$

□

3. Main results

In this section, we investigate EU of the q-EPS for the hybrid q-FBVP (1.1) using of the q-EFPT of MMO in cone ρ_{h,e_1}. Banach space $\mathbf{E}$ can be kit out with a partially order arranged with $u,v\in\mathfrak{C}[0,1]$, $u\leq v\Leftrightarrow u(\tau)\leq v(\tau)$ for $\tau\in[0,1]$. $\forall\ u,v\in\mathbf{E}$, the marking an equivalence relation $u\sim v$ that is, there exists $\mu_1>0$ and $\mu_2>0$, therefore $\mu_1u\leq v\leq\mu_2u$. If $h\in\mathbf{E}$ and $h>0$ are given, we define the $\rho_h=\{u\in\mathbf{E}|\ \exists\ \mu_1\leq\mu_2$ and $\mu_1h\leq u\leq\mu_2h\}$. It is obvious that $\rho_h\subseteq\rho$ for any $h\in\rho$. We consider ρ as a normal cone with a normality constant of 1. Let $e_1\in\rho$, $0\leq e_1\leq h$ be given and e_1 is differentiable . Then set ρ_{h,e_1} is expressed as below:

$$\begin{aligned}\rho_{h,e_1}&=\{u\in\mathbf{E}|\ \exists\ \mu_3\ and\ \mu_4,\ \mu_3h\leq u+e_1\leq\mu_4h\}\\&=\{u\in\mathbf{E}|\ u+e_1\in\mathbb{P}_h\}\\&=\{u\in\mathbf{E}|\exists\ \mu_3\ and\ \mu_4,\ \mu_3h+(\mu_3-1)e_1\leq u\leq\mu_4h+(\mu_4-1)e_1\}.\end{aligned}$$

It is clear that $\rho_h\subseteq\rho_{h,e_1}$ for any $h\in\rho_{h,e_1}$.

Definitions 3.1 and 3.2 are extracted from [12] and [9] respectively.

Definition 3.1. An operator $\mathcal{A}:\rho_{h,e_1}\times\rho_{h,e_1}\longrightarrow\rho_{h,e_1}$ is called a MMO if $\mathcal{A}(w,\vartheta)$ is decreasing relative to ϑ and increasing relative to w, means that, $w_i,\vartheta_i\in\rho_{h,e_1}$ $(i=1,2)$, $w_1\leq w_2$, $\vartheta_1\geq\vartheta_2$ imply

$$\mathcal{A}(w_1,\vartheta_1)\leq\mathcal{A}(w_2,\vartheta_2),$$

the element $u\in\rho_{h,e_1}$ is called a fixed point of $\mathcal{A}$ if $\mathcal{A}(u,u)=u$.

Definition 3.2. If $\mathcal{B}:\rho_{h,e_1}\longrightarrow\mathbf{E}$ is an operator. For each $u\in\rho_{h,e_1}$, there is a function $\varphi:[0,1]\rightarrow[0,1]$ such that for any $\lambda\in(0,1)$, $\varphi(\lambda)>\lambda$,

$$\mathcal{B}(\lambda u+(\lambda-1)e_1)\geq\varphi(\lambda)\mathcal{B}(u)+(\varphi(\lambda)-1)e_1.$$

Therefore $\mathcal{B}$ is named a generalized φ-(h,e_1)-concave.

Definition 3.3. If $\mathtt{T}$ is MMO in cone ρ_{h,e_1}, the element $u\in\rho_{h,e_1}$ is called a $\mathtt{q}$-Exponential fixed point ($\mathtt{q}$-EFP) of $\mathtt{T}$ if $E_{\mathtt{q}}(\lambda\tau)\mathtt{T}(u,u)=u$.

The following Lemmas are derived from [12].

Lemma 3.4. *If $u\in\rho_{h,e_1}$, then for $\lambda>0$,*

$$\lambda u+(\lambda-1)e_1\in\rho_{h,e_1}.$$

Lemma 3.5. *If $u, v \in \rho_{h,e_1}$, thus for $0 < \mu < 1$, $\nu > 1$ such that (S.T):*

$$\mu v + (\mu - 1)e_1 \leq v \leq \nu v + (\nu - 1)e_1.$$

So, we can elect a tiny $\epsilon \in (0,1)$ S.T

$$\epsilon v + (\epsilon - 1)e_1 \leq u \leq \epsilon^{-1} v + (\epsilon^{-1} - 1)e_1.$$

Now, we state and prove the main Lemma and Theorems. This Lemma and Theorems are specific to conical space of ρ_{h,e_1}, and this theorem is called the q-EFPT of MMO on cone ρ_{h,e_1}.

Lemma 3.6. *We assume that $\mathtt{T} : \rho \times \rho \longrightarrow \rho$ is a MMO, which we consider as a normal cone ρ in the real Banach space $\mathbf{E}$. There is a function $\iota : [0,1] \to [0,1]$ S.T for any τ, $\iota(\tau) \geq \tau$:*

$$\mathtt{T}\left(\tau w + (\tau - 1)e_1, \tau^{-1}\vartheta + (\tau^{-1} - 1)e_1\right) \geq \iota(\tau)\mathtt{T}(w, \vartheta) + (\iota(\tau) - 1)e_1, \ w, \vartheta \in \mathbb{P}_{h,e_1}. \tag{3.1}$$

(J) *For any $h_0 \in \rho_{h,e_1}$ that $h_0 \neq 0$, $\mathtt{T}(h_0, h_0) \in \rho_{h,e_1}$.*

Thus,

(1) $\mathtt{T} : \rho_{h,e_1} \times \rho_{h,e_1} \longrightarrow \rho_{h,e_1}$;
(2) *There is $w_0, \vartheta_0 \in \rho_{h,e_1}$ and $\epsilon \in (0,1)$ S.T $\epsilon\vartheta_0 \leq w_0 < E_{\mathtt{q}}(\lambda\tau)\vartheta_0$,*

$$w_0 \leq E_{\mathtt{q}}(\lambda\tau)\mathtt{T}(w_0, \vartheta_0) \leq E_{\mathtt{q}}(\lambda\tau)\mathtt{T}(\vartheta_0, w_0) \leq E_{\mathtt{q}}(\lambda\tau)\vartheta_0$$

(3) *$\mathtt{T}$ has a q-EFP w^* in ρ_{h,e_1};*
(4) *For each initial values $w_0, \vartheta_0 \in \rho_{h,e_1}$, constructing successively the sequences*

$$w_{\mathfrak{N}} = E_{\mathtt{q}}(\lambda\tau)\mathtt{T}(w_{\mathfrak{N}-1}, \vartheta_{\mathfrak{N}-1}), \ \mathfrak{N} = 1, 2, ...,$$
$$\vartheta_{\mathfrak{N}} = E_{\mathtt{q}}(\lambda\tau)\mathtt{T}(\vartheta_{\mathfrak{N}-1}, w_{\mathfrak{N}-1}), \ \mathfrak{N} = 1, 2, ...$$

in result $w_{\mathfrak{N}} \to w^*$ and $\vartheta_{\mathfrak{N}} \to \vartheta^*$ as $\mathfrak{N} \to \infty$.

Proof. According to the (3.1), we achieve:

$$\mathtt{T}\left(\tau^{-1}h_0 + (\tau^{-1} - 1)e_1, \tau h_0 + (\tau - 1)e_1\right) \geq \iota^{-1}(\tau)\mathtt{T}(h_0, h_0) + (\iota^{-1}(\tau) - 1)e_1. \tag{3.2}$$

Because $h_0 \in \rho_{h,e_1}$, there is a $\eta \in (0,1)$ S.T,

$$\eta h + (\eta - 1)e_1 \leq h_0 \leq \eta^{-1}h + (\eta^{-1} - 1)e_1. \tag{3.3}$$

According to the property of mixed monotone $\mathtt{T}$ and relation (3.1), (3.2), and (3.3) we get:

$$\begin{aligned}\mathtt{T}(h,h) &\geq \mathtt{T}\left(\eta h_0 + (\eta - 1)e_1, \eta^{-1}h_0 + (\eta^{-1} - 1)e_1\right)\\ &\geq \iota(\eta)\mathtt{T}(h_0, h_0) + (\iota(\eta) - 1)e_1,\\ \mathtt{T}(h,h) &\leq \mathtt{T}\left(\eta^{-1}h_0 + (\eta^{-1} - 1)e_1, \eta h_0 + (\eta - 1)e_1\right)\\ &\leq \iota^{-1}(\eta)\mathtt{T}(h_0, h_0) + (\iota^{-1}(\eta) - 1)e_1.\end{aligned}$$

Since $\mathtt{T}(h_0, h_0) \in \rho_{h,e_1}$, we conclude that $\mathtt{T}(h,h) \in \rho_{h,e_1}$.
Therefore, $\mathtt{T} : \rho_{h,e_1} \times \rho_{h,e_1} \longrightarrow \rho_{h,e_1}$. Because $\mathtt{T}(h,h) \in \rho_{h,e_1}$, we consider a number like $\tau_0 \in (0,1)$ S.T

$$\tau_0 h + (\tau_0 - 1)e_1 \leq \mathtt{T}(h,h) \leq \tau_0^{-1}h + (\tau_0^{-1} - 1)e_1. \tag{3.4}$$

Since $\tau_0 < \iota(\tau_0) \leq 1$ therefore, $\frac{\iota(\tau_0)}{\tau_0} \leq \frac{1}{\tau_0}$. We can find positive number like m that $(\frac{\iota(\tau_0)}{\tau_0})^m \geq \frac{1}{\tau_0}$ then,

$$\iota^m(\tau_0) \geq \tau_0^{m-1}. \tag{3.5}$$

We put

$$w_0 = \tau_0^m h + (\tau_0^m - 1)e_1,$$
$$\vartheta_0 = \tau_0^{-m}h + (\tau_0^{-m} - 1)e_1.$$

It is clear that, $w_0, \vartheta_0 \in \rho_{h,e_1}$ and

$$\begin{aligned} w_0 &= \tau_0^m h + (\tau_0^m - 1)e_1 < \tau_0^{-m} h + (\tau_0^{-m} - 1)e_1 \\ &= \vartheta_0 < E_{\mathsf{q}}(\lambda\tau)\vartheta_0, \end{aligned}$$

as a result, $w_0 < E_{\mathsf{q}}(\lambda\tau)\vartheta_0$. It is enough that $\mathbf{E} = \max E_{\mathsf{q}}(\lambda\tau)$, $\tau \in (0,1)$ and for large natural values $\mathfrak{N}$, we can be considered $\mathfrak{N}$ that $\epsilon = \frac{\mathbf{E}}{\mathfrak{N}} < 1$, consequently, $\epsilon\vartheta_0 \leq w_0$. Since $\mathtt{T}$ is a MMO, we get that $\mathtt{T}(w_0, \vartheta_0) \leq \mathtt{T}(\vartheta_0, w_0)$, by multiplying $E_{\mathsf{q}}(\lambda\tau)$ on the sides inequality, we obtain

$$E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_0, \vartheta_0) \leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\vartheta_0, w_0).$$

Using the relation (3.1), (3.4), and (3.5) we have:

$$\begin{aligned} E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_0, \vartheta_0) &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\left(\tau_0^m h + (\tau_0^m - 1)e_1, \tau_0^{-m} h + (\tau_0^{-m} - 1)e_1\right) \\ &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\{\tau_0[\tau_0^{m-1} h + (\tau_0^{m-1} - 1)e_1] + (\tau_0 - 1)e_1, \tau_0^{-1}[\tau_0^{1-m} h + (\tau_0^{1-m} - 1)e_1] \\ &+ (\tau_0^{-1} - 1)e_1\} \\ &\geq \iota(\tau_0)\mathtt{T}\left(\tau_0^{m-1} h + (\tau_0^{m-1} - 1)e_1, \tau_0^{1-m} h + (\tau_0^{1-m} - 1)e_1\right) + (\iota(\tau_0) - 1)\, e_1 \\ &= \iota(\tau_0)\mathtt{T}\{\tau_0[\tau_0^{m-2} h + (\tau 0^{m-2} - 1)e_1] + (\tau_0 - 1)e_1, \tau_0^{-1}[\tau_0^{2-m} h + (\tau_0^{2-m} - 1)e_1] \\ &+ (\tau_0^{-1} - 1)e_1\} + (\iota(\tau_0) - 1)\, e_1 \\ &\geq \iota^2(\tau_0)\mathtt{T}\left(\tau_0^{m-2} h + (\tau_0^{m-2} - 1)e_1, \tau_0^{2-m} h + (\tau_0^{2-m} - 1)e_1\right) + \left(\iota^2(\tau_0) - 1\right) e_1 \\ &\vdots \\ &\geq \iota^{m-1}(\tau_0)\mathtt{T}\left(\tau_0 h + [\tau_0 - 1]e_1, \tau_0^{-1} h + [\tau_0^{-1} - 1]e_1\right) + \left(\iota^{m-1}(\tau_0) - 1\right) e_1 \\ &\geq \iota^m(\tau_0)\mathtt{T}(h, h) + (\iota^m(\tau_0) - 1)\, e_1 \\ &\geq \iota^m(\tau_0)\left[\tau_0 h + (\tau_0 - 1)e_1\right] + (\iota^m(\tau_0) - 1)\, e_1 \\ &\geq \tau_0^{m-1}\left[\tau_0 h + (\tau_0 - 1)e_1\right] + (\tau_0^{m-1} - 1)e_1 \\ &= \tau_0^m h + \tau_0^m e_1 - \tau_0^{m-1} e_1 + \tau_0^{m-1} e_1 - e_1 \\ &= \tau_0^m h + (\tau_0^m - 1)e_1 = w_0. \end{aligned}$$

That is $E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_0, \vartheta_0) \geq w_0$. By using the relation (3.2), (3.4), and (3.5) we obtain:

$$\begin{aligned} E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\vartheta_0, w_0) &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\left(\tau_0^{-m} h + [\tau_0^{-m} - 1]e_1, \tau_0^m h + [\tau_0^m - 1]e_1\right) \\ &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\{\tau_0^{-1}[\tau_0^{1-m} h + (\tau_0^{1-m} - 1)e_1] + (\tau_0^{-1} - 1)e_1, \tau_0[\tau_0^{m-1} h + (\tau_0^{m-1} - 1)e_1] \\ &+ (\tau_0 - 1)e_1\} \\ &\leq E_{\mathsf{q}}(\lambda\tau)\left[\iota^{-1}(\tau_0)\mathtt{T}\left(\tau_0^{1-m} h + (\tau_0^{1-m} - 1)e_1, \tau_0^{m-1} h + (\tau_0^{m-1} - 1)e_1\right) + \left(\iota^{-1}(\tau_0) - 1\right) e_1\right] \\ &= E_{\mathsf{q}}(\lambda\tau)[\iota^{-1}(\tau_0)\mathtt{T}(\tau_0^{-1}\{\tau_0^{2-m} h + (\tau_0^{2-m} - 1)e_1\} + (\tau_0^{-1} - 1)e_1, \tau_0\{\tau_0^{m-2} h + (\tau_0^{m-2} - 1)e_1\} \\ &+ (\tau_0 - 1)e_1) + \left(\iota^{-1}(\tau_0) - 1\right) e_1] \\ &\leq E_{\mathsf{q}}(\lambda\tau)[\iota^{-2}(\tau_0)\mathtt{T}\left(\tau_0^{2-m} h + (\tau_0^{2-m} - 1)e_1, \tau_0^{m-2} h + \{\tau_0^{m-2} - 1\}e_1\right) + \left(\iota^{-2}(\tau_0) - 1\right) e_1] \\ &\vdots \\ &\leq E_{\mathsf{q}}(\lambda\tau)\left[\iota^{-m+1}(\tau_0)\mathtt{T}\left(\tau_0^{-1} h + (\tau_0^{-1} - 1)e_1, \tau_0 h + (\tau_0 - 1)e_1\right) + \left(\iota^{-m+1}(\tau_0) - 1\right) e_1\right] \end{aligned}$$

$$\begin{aligned} &\leq E_{\mathsf{q}}(\lambda\tau)\left[\iota^{-m}(\tau_0)\mathtt{T}(h, h) + (\iota^{-m}(\tau_0) - 1)e_1\right] \\ &\leq E_{\mathsf{q}}(\lambda\tau)\left[\iota^{-m}(\tau_0)(\tau_0^{-1} h + (\tau_0^{-1} - 1)e_1) + (\iota^{-m}(\tau_0) - 1)e_1\right] \\ &\leq E_{\mathsf{q}}(\lambda\tau)\left[\tau_0^{-m+1}\left(\tau_0^{-1} h + (\tau_0^{-1} - 1)e_1\right) + (\tau_0^{-m+1} - 1)e_1\right] \end{aligned}$$

$$= E_{\mathsf{q}}(\lambda\tau)\left[\tau_0^{-m}h + (\tau_0^{-m} - 1)e_1\right]$$
$$= E_{\mathsf{q}}(\lambda\tau)\vartheta_0.$$

That is $E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_0,\vartheta_0) \leq E_{\mathsf{q}}(\lambda\tau)\vartheta_0$. Then, we prove that

$$w_0 \leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_0,\vartheta_0) \leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\vartheta_0,w_0) \leq E_{\mathsf{q}}(\lambda\tau)\vartheta_0. \tag{3.6}$$

By constructing recursive sequences

$$w_{\mathfrak{N}} = E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_{\mathfrak{N}-1},\vartheta_{\mathfrak{N}-1}),$$
$$\vartheta_{\mathfrak{N}} = E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\vartheta_{\mathfrak{N}-1},w_{\mathfrak{N}-1}),\ \mathfrak{N}=1,2,....$$

By using the relation (3.6), $w_1 = E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_0,\vartheta_0) \leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\vartheta_0,w_0) = \vartheta_1$, ingeneral, we can take $w_{\mathfrak{N}} < \vartheta_{\mathfrak{N}}$, $(\mathfrak{N}=1,2,...)$, therefore, we get:

$$w_0 \leq w_1 \leq \cdots \leq w_{\mathfrak{N}} \leq \cdots \leq \vartheta_{\mathfrak{N}} \leq \cdots \leq \vartheta_1 \leq E_{\mathsf{q}}(\lambda\tau)\vartheta_0. \tag{3.7}$$

Since $w_0,\vartheta_0 \in \rho_{h,e_1}$, using Lemma 3.5, we choose a $\mu > 0$, S.T $w_0 \geq \mu(E_{\mathsf{q}}(\lambda\tau)\vartheta_0) + (\mu-1)e_1$, by using the relation (3.7), we obtain:

$$w_{\mathfrak{N}} \geq w_0 \geq \mu(E_{\mathsf{q}}(\lambda\tau)\vartheta_0) + (\mu-1)e_1 \geq \mu\vartheta_1 + (\mu-1)e_1,\ \mathfrak{N}=1,2,....$$

Suppose $\tau_{\mathfrak{N}} = \sup\{\tau > 0 | w_{\mathfrak{N}} \geq \tau\vartheta_{\mathfrak{N}} + (\tau-1)e_1\}$. We take from (3.7)

$$\tau_{\mathfrak{N}} \in (0,1),\ w_{\mathfrak{N}} \geq \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} + (\tau-1)e_1, \tag{3.8}$$

so, by using (3.7) and (3.8), we have:

$$w_{\mathfrak{N}+1} \geq w_{\mathfrak{N}} \geq \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} + (\tau_{\mathfrak{N}}-1)e_1 \geq \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}+1} + (\tau_{\mathfrak{N}}-1)e_1,\ \mathfrak{N}=1,2,....$$

In result of $\tau_{\mathfrak{N}+1} \geq \tau_{\mathfrak{N}}$ that is the sequence $\{\tau_{\mathfrak{N}}\}$ is an increasing. Suppose $\lim_{\mathfrak{N}\to\infty} \tau_{\mathfrak{N}} = \tau^*$. We show that $\tau^* = 1$. Differently, $0 < \tau^* < 1$. We examine two cases.

Case 1: $\exists\ N_0 \in N$, S.T $\tau_N = \tau^*$. For all $\mathfrak{N} > N_0$ we get $\tau_{\mathfrak{N}} = \tau^*$. Thus, for each $\mathfrak{N} \geq N_0$

$$\begin{aligned} w_{\mathfrak{N}+1} &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_{\mathfrak{N}},\vartheta_{\mathfrak{N}}) \geq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\left[\tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} + (\tau_{\mathfrak{N}}-1)e_1, \tau_{\mathfrak{N}}^{-1}w_{\mathfrak{N}} + (\tau_{\mathfrak{N}}^{-1}-1)e_1)\right] \\ &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\left[\tau^*\vartheta_{\mathfrak{N}} + (\tau^*-1)e_1, (\tau^*)^{-1}w_{\mathfrak{N}} + ((\tau^*)^{-1}-1)e_1)\right] \\ &\geq E_{\mathsf{q}}(\lambda\tau)\left[\iota(\tau^*)\mathtt{T}(\vartheta_{\mathfrak{N}},w_{\mathfrak{N}}) + (\iota(\tau^*)-1)\,e_1\right], \\ &= E_{\mathsf{q}}(\lambda\tau)\iota(t^*)\vartheta_{\mathfrak{N}+1} + E_{\mathsf{q}}(\lambda\tau)(\iota(\tau^*)-1)e_1 \\ &\geq \iota(\tau^*)\vartheta_{\mathfrak{N}+1} + (\iota(\tau^*)-1)\,e_1. \end{aligned}$$

According to the increasing sequence $\{\tau_{\mathfrak{N}}\}$, we have $\tau_{\mathfrak{N}+1} \geq, \iota(\tau^*) > \tau^*$ by We take limits from both sides as $\mathfrak{N} \to \infty$, we obtain $\tau^* \geq \iota(\tau^*) > \tau^*$, which is contrary to our assumption.

Case 2: $\forall\ \mathfrak{N} \in N$, $\tau_{\mathfrak{N}} < \tau^*$. So, we get:

$$\begin{aligned} w_{\mathfrak{N}+1} &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_{\mathfrak{N}},\vartheta_{\mathfrak{N}}) \geq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\left(\tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} + (\tau_{\mathfrak{N}}-1)e_1, \tau_{\mathfrak{N}}^{-1}w_{\mathfrak{N}} + (\tau_{\mathfrak{N}}^{-1}-1)e_1)\right) \\ &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\Big[\frac{\tau_{\mathfrak{N}}}{\tau^*}\{\tau^*\vartheta_{\mathfrak{N}} + (\tau^*-1)e_1\} + (\frac{\tau_{\mathfrak{N}}}{\tau^*}-1)e_1, (\frac{\tau_{\mathfrak{N}}}{\tau^*})^{-1}\{(\tau^*)^{-1}w_{\mathfrak{N}} + (\tau^*)^{-1}-1)e_1\} + \{(\frac{\tau_{\mathfrak{N}}}{\tau^*})^{-1}-1\}e_1\Big] \\ &\geq E_{\mathsf{q}}(\lambda\tau)\left[\iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})\mathtt{T}\{\tau^*\vartheta_{\mathfrak{N}} + (\tau^*-1)e_1, (\tau^*)^{-1}w_{\mathfrak{N}} + ((\tau^*)^{-1}-1)e_1\} + (\iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})-1)e_1\right] \\ &\geq E_{\mathsf{q}}(\lambda\tau)\left[\iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})\left(\iota(\tau^*)\mathtt{T}(w_{\mathfrak{N}},\vartheta_{\mathfrak{N}}) + (\iota(\tau^*)-1)e_1\right) + (\iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})-1)e_1\right] \\ &= E_{\mathsf{q}}(\lambda\tau)\iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})\iota(\tau^*)\vartheta_{\mathfrak{N}+1} + E_{\mathsf{q}}(\lambda\tau)(\iota(\tau^*)-1)e_1 + E_{\mathsf{q}}(\lambda\tau)(\iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})-1)e_1 \\ &= E_{\mathsf{q}}(\lambda\tau)\iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})\iota(\tau^*)\vartheta_{\mathfrak{N}+1} + E_{\mathsf{q}}(\lambda\tau)\left(\iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})\iota(\tau^*)-1\right)e_1. \end{aligned}$$

By the sequence $\{\tau_{\mathfrak{N}}\}$, we have:

$$\tau_{\mathfrak{N}+1} \geq \iota(\frac{\tau_{\mathfrak{N}}}{\tau^*})\iota(\tau^*) \geq \frac{\tau_{\mathfrak{N}}}{\tau^*}\iota(\tau^*).$$

We take limits from both sides as $\mathfrak{N} \to \infty$, we get:

$$\tau^* \geq \frac{\tau^*}{\tau^*}\iota(\tau^*) \geq \frac{\tau_{\mathfrak{N}}}{\tau^*}\iota(\tau^*),$$

which is against our hypothesis, in result of $\tau^* = 1$. For every ρ that is a natural number, we obtain:

$$\begin{aligned}
&0 \leq w_{\mathfrak{N}+\rho} - w_{\mathfrak{N}} \leq \vartheta_{\mathfrak{N}} - w_{\mathfrak{N}} \leq \vartheta_{\mathfrak{N}} - \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} - (\tau_{\mathfrak{N}} - 1)e_1 \\
&\quad = (1-\tau_{\mathfrak{N}})\vartheta_{\mathfrak{N}} + (1-\tau_{\mathfrak{N}})e_1 \leq (1-\tau_{\mathfrak{N}})E_{\mathsf{q}}(\lambda\tau)\vartheta_0 + (1-\tau_{\mathfrak{N}})e_1.
\end{aligned}$$

Therefore

$$\begin{aligned}
&0 \leq \vartheta_{\mathfrak{N}} - \vartheta_{\mathfrak{N}+\rho} \leq \vartheta_{\mathfrak{N}} - w_{\mathfrak{N}} \leq \vartheta_{\mathfrak{N}} - \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} - (\tau_{\mathfrak{N}} - 1)e_1 \\
&\quad = (1-\tau_{\mathfrak{N}})\vartheta_{\mathfrak{N}} + (1-\tau_{\mathfrak{N}})e_1 \leq (1-\tau_{\mathfrak{N}})E_{\mathsf{q}}(\lambda\tau)\vartheta_0 + (1-\tau_{\mathfrak{N}})e_1.
\end{aligned}$$

For constant normal N and ρ normal cone we get:

$$\begin{aligned}
&\parallel w_{\mathfrak{N}+\rho} - w_{\mathfrak{N}} \parallel \leq N(1-\tau_{\mathfrak{N}}) \parallel E_{\mathsf{q}}(\lambda\tau)\vartheta_0 + e_1 \parallel \to 0, \ (as\ \mathfrak{N} \to \infty), \\
&\parallel \vartheta_{\mathfrak{N}} - \vartheta_{\mathfrak{N}+\rho} \parallel \leq N(1-\tau_{\mathfrak{N}}) \parallel E_{\mathsf{q}}(\lambda\tau)\vartheta_0 + e_1 \parallel \to 0, \ (as\ \mathfrak{N} \to \infty),
\end{aligned}$$

then, $\{w_{\mathfrak{N}}\}$ and $\{\vartheta_{\mathfrak{N}}\}$ are Cauchy sequences. Since $\mathbf{E}$ is complete, there exists w^* and ϑ^* S.T $w_{\mathfrak{N}} \to w^*$ and $\vartheta_{\mathfrak{N}} \to \vartheta^*$ as $\mathfrak{N} \to \infty$. Using the relation (3.8) We know that:

$$w_0 \leq w_{\mathfrak{N}} \leq w^* \leq \vartheta^* \leq \vartheta_{\mathfrak{N}} \leq E_{\mathsf{q}}(\lambda\tau)\vartheta_0.$$

Then, $w^*, \vartheta^* \in \rho_{h,e_1}$ and

$$0 \leq \vartheta^* - w^* \leq \vartheta_{\mathfrak{N}} - w_{\mathfrak{N}} \leq (1-\tau_{\mathfrak{N}})E_{\mathsf{q}}(\lambda\tau)\vartheta_0 + e_1.$$

We conclude that:

$$\parallel \vartheta^* - w^* \parallel \leq N(1-\tau_{\mathfrak{N}}) \parallel E_{\mathsf{q}}(\lambda\tau)\vartheta_0 + e_1 \parallel \to 0, \ (as\ \mathfrak{N} \to \infty),$$

that is $w^* = \vartheta^*$, if $u^* := w^* = \vartheta^*$, we have:

$$w_{\mathfrak{N}+1} = E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_{\mathfrak{N}}, \vartheta_{\mathfrak{N}}) \leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u^*, u^*) \leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\vartheta_{\mathfrak{N}}, w_{\mathfrak{N}}) = \vartheta_{\mathfrak{N}+1}.$$

As $\mathfrak{N} \to \infty$ we get that $E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u^*, u^*) = u^*$. That is u^* is a q-EFP of $\mathtt{T}$ in ρ_{h,e_1}. Now we display that u^* is an uniquely q-EFP of $\mathtt{T}$ in ρ_{h,e_1}. Assume that $\overline{u}$ is a q-EFP of $\mathtt{T}$ in ρ_{h,e_1}. By using of Lemma 3.5, there are $s_1, s_1^{-1} > 0$ S.T

$$s_1\overline{u} + (s_1 - 1)e_1 \leq u^* \leq s_1^{-1}\overline{u} + (s_1^{-1} - 1)e_1,$$

we put $s_2 = \sup\{s_1 > 0 | s_1\overline{u} + (s_1 - 1)e_1 \leq u^* \leq s_1^{-1}\overline{u} + (s_1^{-1} - 1)e_1\}$. We show that $s_2 \geq 1$. If $0 < s_2 < 1$ thus,

$$s_2\overline{u} + (s_2 - 1)e_1 \leq u^* \leq s_2^{-1}\overline{u} + (s_2^{-1} - 1)e_1,$$

therefore,

$$\begin{aligned}
u^* &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\left(u^*, u^*\right) \geq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(s_2\overline{u} + [s_2 - 1]e_1, s_2^{-1}\overline{u} + [s_2^{-1} - 1]e_1\big) \\
&\geq \iota(s_2)E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\overline{u}, \overline{u}) + [\iota(s_2) - 1]e_1 = \iota(s_2)\overline{u} + [\iota(s_2) - 1]e_1, \\
u^* &= E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u^*, u^*) \leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}\left(s_2^{-1}\overline{u} + [s_2^{-1} - 1]e_1, s_2\overline{u} + [s_2 - 1]e_1\right) \\
&\leq \iota^{-1}(s_2)E_q(\lambda\tau)\mathtt{T}(\overline{u}, \overline{u}) + [\iota^{-1}(s_2) - 1]e_1 = \iota^{-1}(s_2)\overline{u} + (\iota^{-1}(s_2) - 1)e_1.
\end{aligned}$$

By using of definition s_2, we cane take that $s_2 > \iota(s_2) > s_2$ and $s_2^{-1} < \iota^{-1}(s_2) < s_2^{-1}$, which is against our hypothesis. Therefore, $s_2 \geq 1$ and

$$\begin{aligned}
u^* &\geq \iota(s_2)\overline{u} + (\iota(s_2) - 1)e_1 \geq s_2\overline{u} + (s_2 - 1)e_1 \geq s_2\overline{u} \geq \overline{u}, \\
u^* &\leq \iota^{-1}(s_2)\overline{u} + (\iota^{-1}(s_2) - 1)e_1 \leq s_2^{-1}\overline{u} + (s_2^{-1} - 1)e_1 \leq s_2^{-1}\overline{u} \leq \overline{u}.
\end{aligned}$$

That's mean, $\overline{u} \leq u^* \leq \overline{u}$, thus, $u^* = \overline{u}$. Therefore, u^* is an uniquely q-EFP of $\mathtt{T}$ in ρ_{h,e_1}. For every $u_0, v_0 \in \rho_{h,e_1}$, we choose tiny number $s_3, s_4 \in (0,1)$ S.T:

$$s_2 h + (s_2 - 1)e_1 \leq u_0 \leq s_2^{-1} h + (s_2^{-1} - 1)e_1,$$
$$s_3 h + (s_3 - 1)e_1 \leq v_0 \leq s_3^{-1} h + (s_3^{-1} - 1)e_1.$$

If $s^* = \min\{s_3, s_4\}$, $s^* \in (0,1)$ and

$$s^* h + (s^* - 1)e_1 \leq u_0,\ v_0 \geq (s^*)^{-1} h + ((s^*)^{-1} - 1)e_1.$$

Assume that

$$\overline{w}_0 = (s^*)^m h + ((s^*)^m - 1)e_1,\ \overline{\vartheta}_0 = (s^*)^{-m} h + ((s^*)^{-m} - 1)e_1.$$

We see that, according Lemma 3.4, $\overline{w}_0, \overline{\vartheta}_0 \in \rho_{h,e_1}$, $\overline{w}_0 < u_0$ and $\overline{\vartheta}_0 > v_0$. If

$$\overline{w}_0 = E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\overline{w}_{\mathfrak{N}-1}, \overline{\vartheta}_{\mathfrak{N}-1}),\ \mathfrak{N} - 1, 2, ...,$$
$$\overline{\vartheta}_0 = E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\overline{\vartheta}_{\mathfrak{N}-1}, \overline{w}_{\mathfrak{N}-1}),\ \mathfrak{N} - 1, 2, ...$$

Similarly to, there is $v^* \in \rho_{h,e_1}$, S.T

$$E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(v^*, v^*) = v^*,\ \lim_{\mathfrak{N}\to\infty} \overline{w}_{\mathfrak{N}} = \lim_{\mathfrak{N}\to\infty} \overline{\vartheta}_{\mathfrak{N}} = v^*.$$

Since fixed point $\mathtt{T}$ is unique in ρ_{h,e_1}, we have $u^* = v^*$, and we get by analysis

$$\overline{w}_{\mathfrak{N}} \leq u_{\mathfrak{N}} \leq \overline{\vartheta}_{\mathfrak{N}},\ \overline{w}_{\mathfrak{N}} \leq v_{\mathfrak{N}} \leq \overline{\vartheta}_{\mathfrak{N}}.$$

Because ρ is normal, we obtain:

$$\lim_{\mathfrak{N}\to\infty} \overline{u}_{\mathfrak{N}} = u^*,\ \lim_{\mathfrak{N}\to\infty} \overline{v}_{\mathfrak{N}} = u^*.$$

□

Theorem 3.7. *Suppose that $0 < \alpha(\tau) < 1$, $0 < \sigma(\tau) < 1$, $0 < \beta(\tau) < 1$, $0 < \alpha(\tau) + \beta(\tau) < 1$. Also, if ρ is a normal cone in $\mathbf{E}$ and $\mathcal{A} : \rho \times \rho \longrightarrow \rho$ is a MMO, $\mathcal{B}, \mathcal{C} : \rho \longrightarrow \rho$. $\mathcal{B}$ is a generalized ψ-(h, e_1)-concave operator and increasing, and $\mathcal{C}$ is a decreasing and they apply in the following conditions:*

(J_1) *There is function $\varphi : [0,1] \to [0,1]$ S.T for any $\tau \in (0,1)$, $\varphi(\tau) = \tau^{\alpha(\tau)+\beta(\tau)} > \tau$:*

$$\mathcal{A}\left(\tau u + (\tau - 1)e_1, \tau^{-1} v + (\tau^{-1} - 1)e_1\right) \geq \tau^{\alpha(\tau)+\beta(\tau)}\mathcal{A}(u,v) + (\tau^{\alpha(\tau)+\beta(\tau)} - 1)e_1, \forall u, v \in \rho_{h,e_1}; \tag{3.9}$$

(J_2) *There is function $\psi : [0,1] \to [0,1]$ S.T for any $\tau \in (0,1)$, $\psi(\tau) = \tau^{\sigma(\tau)} > \tau$, operator $\mathcal{B}$ applies in the following relation:*

$$\mathcal{B}(\tau u + (\tau - 1)e_1) \geq \tau^{\sigma(\tau)}\mathcal{B}(u) + (\tau^{\sigma(\tau)} - 1)e_1,\ \forall u \in \rho_{h,e_1}, \tag{3.10}$$

operator $\mathcal{C}$ is decreasing and it applies in the following relation:

$$\mathcal{C}(\tau^{-1} v + (\tau^{-1} - 1)e_1) \geq \tau^{\sigma(\tau)}\mathcal{C}(v) + (\tau^{\sigma(\tau)} - 1)e_1,\ \forall v \in \rho_{h,e_1}; \tag{3.11}$$

(J_3) *Exists $h_0 \in \rho_h \subseteq \rho_{h,e_1}$ S.T $\mathcal{A}(h_0, h_0) \in \rho_{h,e_1}$, $\mathcal{B}(h_0) \in \rho_{h,e_1}$ and $\mathcal{C}(h_0) \in \rho_{h,e_1}$;*

(J_4) *Exists a constant $\delta > 0$, S.T for each $u, v \in \rho_{h,e_1}$*

$$\mathcal{A}(u,v) + e_1 \geq \delta(\mathcal{B}(u) + \mathcal{C}(v) + 2e_1) + (\delta - 1)e_1.$$

Thus,

(1) *$\mathcal{A} : \rho_{h,e_1} \times \rho_{h,e_1} \to \rho_{h,e_1}$ and $\mathcal{B}, \mathcal{C} : \rho_{h,e_1} \to \rho_{h,e_1}$;*

(2) *There is $w_0, \vartheta_0 \in \rho_{h,e_1}$ and $\epsilon \in (0,1)$ S.T $\epsilon\vartheta_0 \leq w_0 < E_{\mathsf{q}}(\lambda\tau)\vartheta_0$,*

$$\begin{aligned} w_0 &\leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_0, \vartheta_0) = \mathcal{A}(w_0, \vartheta_0) + \mathcal{B}(w_0) + \mathcal{C}(\vartheta_0) + 3e_1 \\ &\leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\vartheta_0, w_0) = \mathcal{A}(\vartheta_0, w_0) + \mathcal{B}(\vartheta_0) + \mathcal{C}(w_0) + 3e_1 \leq E_{\mathsf{q}}(\lambda\tau)\vartheta_0; \end{aligned}$$

(3) *$\mathtt{T}$ has an unique q-EFP u^* in ρ_{h,e_1};*

(4) *For any initial value $u_0, v_0 \in \rho_{h,e_1}$, there are recursive sequences*

$$u_{\mathfrak{N}+1} = E_q(\lambda\tau)\mathtt{T}(u_{\mathfrak{N}}, v_{\mathfrak{N}}) = \mathcal{A}(u_{\mathfrak{N}}, v_{\mathfrak{N}}) + \mathcal{B}(u_{\mathfrak{N}}) + \mathcal{C}(v_{\mathfrak{N}}) + 3e_1, \ \mathfrak{N} = 1, 2, ...$$
$$v_{\mathfrak{N}+1} = E_q(\lambda\tau)\mathsf{T}(v_{\mathfrak{N}}, u_{\mathfrak{N}}) = \mathcal{A}(v_{\mathfrak{N}}, u_{\mathfrak{N}}) + \mathcal{B}(v_{\mathfrak{N}}) + \mathcal{C}(u_{\mathfrak{N}}) + 3e_1, \ \mathfrak{N} = 1, 2, ...$$

we obtain $u_{\mathfrak{N}} \to u^*$ and $v_{\mathfrak{N}} \to u^*$ as $\mathfrak{N} \to \infty$.

Proof. Using the (J_1), (J_2) and relations (3.9), (3.10), and (3.11), for any $\tau \in (0, 1)$ and $u, v \in \rho_{h,e_1}$ we get:

$$\mathcal{A}\left(\tau^{-1}u + (\tau^{-1} - 1)e_1, \tau v + (\tau - 1)e_1\right) \leq \frac{1}{\tau^{\alpha(\tau)+\beta(\tau)}}\mathcal{A}(u, v) + (\frac{1}{\tau^{\alpha(\tau)+\beta(\tau)}} - 1)e_1, \tag{3.12}$$

$$\mathcal{B}(\tau^{-1}u + (\tau^{-1} - 1)e_1) \leq \frac{1}{\tau^{\sigma(\tau)}}\mathcal{B}(u) + (\frac{1}{\tau^{\sigma(\tau)}} - 1)e_1, \tag{3.13}$$

$$\mathcal{C}(\tau v + (\tau - 1)e_1) \leq \frac{1}{\tau^{\sigma(\tau)}}\mathcal{C}(v) + (\frac{1}{\tau^{\sigma(t)}} - 1)e_1. \tag{3.14}$$

Now, we disply that $\mathcal{A} : \rho_{h,e_1} \times \rho_{h,e_1} \to \rho_{h,e_1}$, $\mathcal{B}, \mathcal{C} : \rho_{h,e_1} \to \rho_{h,e_1}$. Since $\mathcal{A}(h_0, h_0) \in \rho_h$, $\mathcal{B}(h_0) \in \rho_h$ and $\mathcal{C}(h_0) \in \rho_h$, exists constant $\gamma_i > 0$ and $\mu_i > 0$, $(i = 1, 2, ...)$ S.T:

$$E_{\mathsf{q}}(\lambda\tau)\gamma_1 h \leq \mathcal{A}(h_0, h_0) \leq E_{\mathsf{q}}(\lambda\tau)\mu_1 h, \tag{3.15}$$
$$E_{\mathsf{q}}(\lambda\tau)\gamma_2 h \leq \mathcal{B}(h_0) \leq E_{\mathsf{q}}(\lambda\tau)\mu_2 h, \tag{3.16}$$
$$E_{\mathsf{q}}(\lambda\tau)\gamma_3 h \leq \mathcal{C}(h_0) \leq E_{\mathsf{q}}(\lambda\tau)\mu_3 h. \tag{3.17}$$

Because $h_0 \in \rho_h \subseteq \rho_{h,e_1}$, there exist constant $\tau_0 \in (0, 1)$ S.T:

$$\tau_0 h + (\tau_0 - 1)e_1 \leq h_0 \leq \frac{1}{\tau_0}h + (\frac{1}{\tau_0} - 1)e_1.$$

By using of relation (3.9)-(3.17), we have:

$$\begin{aligned}\mathcal{A}(h, h) &\leq \mathcal{A}\left(\frac{1}{\tau_0}h_0 + (\frac{1}{\tau_0} - 1)e_1, \tau_0 h_0 + (\tau_0 - 1)e_1\right)\\ &\leq \frac{1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}}\mathcal{A}(h_0, h_0) + (\frac{1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} - 1)e_1\\ &\leq \frac{E_{\mathsf{q}}(\lambda\tau)\mu_1 h}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} + (\frac{1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} - 1)e_1,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\mathcal{A}(h, h) &\geq \mathcal{A}\left(\tau_0 h_0 + (\tau_0 - 1)e_1, \frac{1}{\tau_0}h_0 + (\frac{1}{\tau_0} - 1)e_1\right)\\ &\geq \tau_0^{\alpha(\tau_0)+\beta(\tau_0)}\mathcal{A}(h_0, h_0) + (\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} - 1)e_1\\ &\geq E_{\mathsf{q}}(\lambda\tau)\gamma_1 h\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} + (\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} - 1)e_1,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\mathcal{B}(h) &\leq \mathcal{B}\left(\frac{1}{\tau_0}h_0 + (\frac{1}{\tau_0} - 1)e_1\right) \leq \frac{1}{\tau_0^{\sigma(\tau_0)}}\mathcal{B}(h_0) + (\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1)e_1\\ &\leq \frac{E_{\mathsf{q}}(\lambda\tau)\mu_2 h}{\tau_0^{\sigma(\tau_0)}} + (\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1)e_1,\end{aligned} \tag{3.20}$$

$$\begin{aligned}\mathcal{B}(h) &\geq \mathcal{B}\left(\tau_0 h_0 + (\tau_0 - 1)e_1\right) \geq \tau_0^{\sigma(\tau_0)}\mathcal{B}(h_0) + (\tau_0^{\sigma(\tau_0)} - 1)e_1\\ &\geq E_{\mathsf{q}}(\lambda\tau)\gamma_2 h\tau_0^{\sigma(\tau_0)} + (\tau_0^{\sigma(\tau_0)} - 1)e_1.\end{aligned} \tag{3.21}$$

$$\begin{aligned}\mathcal{C}(h) &\leq \mathcal{C}\left(\tau_0 h_0 + (\tau_0 - 1)e_1\right)\\ &\leq \frac{1}{\tau_0^{\sigma(\tau_0)}}\mathcal{C}(h_0) + (\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1)e_1\end{aligned}$$

$$\leq \frac{E_{\mathsf{q}}(\lambda\tau)\mu_3 h}{\tau_0^{\sigma(\tau_0)}} + (\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1)e_1, \tag{3.22}$$

$$\begin{aligned} \mathcal{C}(h) &\geq \mathcal{C}\left(\frac{1}{\tau_0^{\sigma(\tau_0)}} h_0 + (\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1)e_1\right) \\ &\geq \tau_0^{\sigma(\tau_0)}\mathcal{C}(h_0) + (\tau_0^{\sigma(\tau_0)} - 1)e_1 \\ &\geq E_{\mathsf{q}}(\lambda\tau)\gamma_3 h\tau_0^{\sigma(\tau_0)} + (\tau_0^{\sigma(\tau_0)} - 1)e_1. \end{aligned} \tag{3.23}$$

It is obvious that $\frac{\mu_1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}}$, $\gamma_1\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}$, $\frac{\mu_2}{\tau_0^{\sigma(\tau_0)}}$, $\gamma_2\tau_0^{\sigma(\tau_0)}$, $\frac{\mu_3}{\tau_0^{\sigma(\tau_0)}}$ and $\gamma_3\tau_0^{\sigma(\tau_0)}$ are greater than zero. As a result, we proved that $\mathcal{A}(h,h) \in \rho_{h,e_1}$, $\mathcal{B}(h) \in \rho_{h,e_1}$ and $\mathcal{C}(h) \in \rho_{h,e_1}$. For every $u, v \in \rho_{h,e_1}$, we select a sufficiently tiny number like $\xi \in (0,1)$ S.T:

$$\xi h + (\xi - 1)e_1 \leq u \leq \frac{1}{\xi}h + (\frac{1}{\xi} - 1)e_1, \tag{3.24}$$

$$\xi h + (\xi - 1)e_1 \leq v \leq \frac{1}{\xi}h + (\frac{1}{\xi} - 1)e_1. \tag{3.25}$$

By using of relations (3.9)-(3.14) and (3.18)-(3.25), we have:

$$\begin{aligned} \mathcal{A}(u,v) &\geq \mathcal{A}\left(\xi h + (\xi - 1)e_1, \frac{1}{\xi}h + (\frac{1}{\xi} - 1)e_1\right) \\ &\geq \xi^{\alpha(\xi)+\beta(\xi)}\mathcal{A}(h,h) + (\xi^{\alpha(\xi)+\beta(\xi)} - 1)e_1 \\ &\geq \xi^{\alpha(\xi)+\beta(\xi)}\left(E_{\mathsf{q}}(\lambda\tau)\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}\gamma_1 h + (\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} - 1)e_1\right) + (\xi^{\alpha(\xi)+\beta(\xi)} - 1)e_1 \\ &= E_{\mathsf{q}}(\lambda\tau)\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}\gamma_1\xi^{\alpha(\xi)+\beta(\xi)}h + (\xi^{\alpha(\xi)+\beta(\xi)}\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} - 1)e_1, \end{aligned}$$

$$\begin{aligned} \mathcal{A}(u,v) &\leq \mathcal{A}\left(\frac{1}{\xi}h + (\frac{1}{\xi} - 1)e_1, \xi h + (\xi - 1)e_1\right) \\ &\leq \frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}}\mathcal{A}(h,h) + (\frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}} - 1)e_1 \\ &\leq \frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}}\left(\frac{E_{\mathsf{q}}(\lambda\tau)\mu_1 h}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} + (\frac{1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} - 1)e_1\right) + (\frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}} - 1)e_1 \\ &= \frac{E_{\mathsf{q}}(\lambda\tau)\mu_1 h}{\xi^{\alpha(\xi)+\beta(\xi)}\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} + (\frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} - 1)e_1. \end{aligned}$$

Thus, $\mathcal{A}(u,v) \in \rho_{h,e_1}$.

$$\begin{aligned} \mathcal{B}(u) &\geq \mathcal{B}\left(\xi h + (\xi - 1)e_1\right) \geq \xi^{\sigma(\xi)}\mathcal{B}(h) + (\xi^{\sigma(\xi)-1})e_1 \\ &\geq \xi^{\sigma(\xi)}\left(E_{\mathsf{q}}(\lambda\tau)\gamma_2 h\tau_0^{\sigma(\tau_0)} + (\tau_0^{\sigma(\tau_0)} - 1)e_1\right) + (\xi^{\sigma(\xi)} - 1)e_1 \\ &= E_{\mathsf{q}}(\lambda\tau)\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}\gamma_2 h + (\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)} - 1)e_1, \end{aligned}$$

$$\begin{aligned} \mathcal{B}(u) &\leq \mathcal{B}\left(\frac{1}{\xi}h + (\frac{1}{\xi} - 1)e_1\right) \leq \frac{1}{\xi^{\sigma(\xi)}}\mathcal{B}(h) + (\frac{1}{\xi^{\sigma(\xi)}} - 1)e_1 \\ &\leq \frac{1}{\xi^{\sigma(\xi)}}\left(\frac{E_q(\lambda\tau)\mu_2 h}{\tau_0^{\sigma(\tau_0)}} + (\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1)e_1\right) + (\frac{1}{\xi^{\sigma(\xi)}} - 1)e_1 \\ &= \frac{E_{\mathsf{q}}(\lambda\tau)\mu_2 h}{\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}} + (\frac{1}{\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}} - 1)e_1. \end{aligned}$$

Then, $\mathcal{B}(u) \in \rho_{h,e_1}$.

$$\begin{aligned}
\mathcal{C}(v) &\geq \mathcal{C}\left(\frac{1}{\xi}h + (\frac{1}{\xi}-1)e_1\right) \geq \xi^{\sigma(\xi)}\mathcal{C}(h) + (\xi^{\sigma(\xi)-1})e_1 \\
&\geq \xi^{\sigma(\xi)}\left(E_{\mathsf{q}}(\lambda\tau)\gamma_3 h\tau_0^{\sigma(\tau_0)} + (\tau_0^{\sigma(\tau_0)}-1)e_1\right) + (\xi^{\sigma(\xi)}-1)e_1 \\
&= E_{\mathsf{q}}(\lambda\tau)\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}\gamma_3 h + (\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}-1)e_1, \\
\mathcal{C}(v) &\leq \mathcal{C}\left(\xi h + (\xi-1)e_1\right) \leq \frac{1}{\xi^{\sigma(\xi)}}\mathcal{C}(h) + (\frac{1}{\xi^{\sigma(\xi)}}-1)e_1 \\
&\leq \frac{1}{\xi^{\sigma(\xi)}}\left(\frac{E_{\mathsf{q}}(\lambda\tau)\mu_3 h}{\tau_0^{\sigma(\tau_0)}} + (\frac{1}{\tau_0^{\sigma(\tau_0)}}-1)e_1\right) + (\frac{1}{\xi^{\sigma(\xi)}}-1)e_1 \\
&= \frac{E_q(\lambda\tau)\mu_3 h}{\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}} + (\frac{1}{\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}}-1)e_1.
\end{aligned}$$

Therefore, $\mathcal{C}(v) \in \rho_{h,e_1}$. That is $\mathcal{A}: \rho_{h,e_1} \times \rho_{h,e_1} \longrightarrow \rho_{h,e_1}$, $\mathcal{B}, \mathcal{C}: \rho_{h,e_1} \longrightarrow \rho_{h,e_1}$. We define operator

$$E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v) = \mathcal{A}(u,v) + \mathcal{B}(u) + \mathcal{C}(v) + 3e_1, \ \forall u, v \in \rho_{h,e_1}.$$

According to $\mathcal{A}(h,h) \in \rho_{h,e_1}$, $\mathcal{B}(h) \in \rho_{h,e_1}$ and $\mathcal{C}(h) \in \rho_{h,e_1}$ we have,

$$E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(h,h) = \mathcal{A}(h,h) + \mathcal{B}(h) + \mathcal{C}(h) + 3e_1 \in \rho_{h,e_1}.$$

Now we demonstrate that for any $\tau \in (0,1)$ there exist $\iota(\tau) \in (\tau, 1]$, S.T for each $u, v \in \rho_{h,e_1}$,

$$\mathtt{T}\left(u + (\tau-1)e_1, \tau^{-1}v + (\tau^{-1}-1)e_1\right) \geq \iota(\tau)\mathtt{T}(u,v) + (\iota(\tau)-1)e_1. \tag{3.26}$$

For any $u, v \in \rho_{h,e_1}$ and by using of (J_4) there is constant $\delta > 0$ S.T:

$$(\mathcal{A}+e_1) + \delta(\mathcal{A}+e_1) \geq \delta(\mathcal{B}+\mathcal{C}+2e_1) + \delta(\mathcal{A}+e_1) + (\delta-1)e_1.$$

Thus, we obtain:

$$(\mathcal{A}+e_1)(1+\delta) \geq \delta[\mathcal{A}+\mathcal{B}+\mathcal{C}+3e_1] + (\delta-1)e_1, \ \forall u, v \in \rho_{h,e_1}.$$

We divide the sides by the $(1+\delta)$:

$$\begin{aligned}
(\mathcal{A}+e_1) &\geq \frac{\delta}{1+\delta}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v) + \frac{\delta-1}{1+\delta}e_1 \\
&\geq \frac{\delta}{1+\delta}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v) - \frac{e_1}{1+\delta}, \ \forall u, v \in \rho_{h,e_1}.
\end{aligned}$$

Using relations(3.9), (3.10), (3.11), and (3.26), we get:

$$\begin{aligned}
&E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\tau u + (\tau-1)e_1, \tau^{-1}v + (\tau^{-1}-1)e_1) - \tau^{\sigma(\tau)}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v) \\
&= \mathcal{A}(\tau u + (\tau-1)e_1, \tau^{-1}v + (\tau^{-1}-1)e_1) + \mathcal{B}(\tau u + (\tau-1)e_1) + \mathcal{C}(\tau^{-1}v + (\tau^{-1}-1)e_1) \\
&+ 3e_1 - \tau^{\sigma(\tau)}[\mathcal{A}(u,v) + \mathcal{B}(u) + \mathcal{C}(v) - 3e_1] \\
&\geq \tau^{\alpha(\tau)+\beta(\tau)}\mathcal{A}(u,v) + (\tau^{\alpha(\tau)+\beta(\tau)}-1)e_1 + \tau^{\sigma(\tau)}\mathcal{B}(u) + (\tau^{\sigma(\tau)}-1)e_1 \\
&+ \tau^{\sigma(\tau)}\mathcal{C}(v) + (\tau^{\sigma(\tau)}-1)e_1 + 3e_1 - \tau^{\sigma(\tau)}\mathcal{A}(u,v) - \tau^{\sigma(\tau)}\mathcal{B}(u) - \tau^{\sigma(\tau)}\mathcal{C}(v) - 3\tau^{\sigma(\tau)}e_1 \\
&= (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})(\mathcal{A}+e_1) + 3e_1(1-\tau^{\sigma(\tau)}) \\
&\geq (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})(\mathcal{A}+e_1) \\
&\geq (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})(\mathcal{A}+e_1) + (\tau^{\alpha(\tau)+\beta(\tau)}-1)e_1 \\
&\geq (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})[\frac{\delta}{1+\delta}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v) - \frac{e_1}{1+\delta}] + (\tau^{\alpha(\tau)+\beta(\tau)}-1)e_1.
\end{aligned}$$

So, we have:

$$E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\tau u+(\tau-1)e_1,\tau^{-1}v+(\tau^{-1}-1)e_1)\geq(\tau^{\alpha(\tau)+\beta(\tau)}-\tau^{\sigma(\tau)})[\frac{\delta}{1+\delta}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v) \tag{3.27}$$
$$-\frac{e_1}{1+\delta}]+(\tau^{\alpha(\tau)+\beta(\tau)}-1)e_1+\tau^{\sigma(\tau)}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v)$$
$$=\frac{\delta\tau^{\alpha(\tau)+\beta(\tau)}-\delta\tau^{\sigma(\tau)}}{1+\delta}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v)+\frac{\delta\tau^{\sigma(\tau)}+\tau^{\sigma(\tau)}}{1+\delta}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v)$$
$$-\frac{\tau^{\alpha(\tau)+\beta(\tau)}+\tau^{\sigma(\tau)}}{1+\delta}e_1+\frac{\delta\tau^{\alpha(\tau)+\beta(\tau)}+\tau^{\alpha(\tau)+\beta(\tau)}-(1+\delta)}{1+\delta}e_1$$
$$=\frac{\tau^{\alpha(\tau)+\beta(\tau)}+\tau^{\sigma(\tau)}}{1+\delta}E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u,v)+(\frac{\tau^{\alpha(\tau)+\beta(\tau)}+\tau^{\sigma(\tau)}}{1+\delta}-1)e_1,\ \forall u,v\in\rho_{h,e_1}. \tag{3.28}$$

If $\iota(\tau)=\frac{\tau^{\alpha(\tau)+\beta(\tau)}+\tau^{\sigma(\tau)}}{1+\delta}$ thus, $\iota(\tau)\in(\tau,1],\tau\in(0,1)$. By using (3.28), We draw conclusions

$$\mathtt{T}(\tau u+(\tau-1)e_1,\tau^{-1}v+(\tau^{-1}-1)e_1)\geq\iota(\tau)\mathtt{T}(u,v)+(\iota(\tau)-1)e_1,\ \forall u,v\in\rho_{h,e_1}.$$

Therefore, condition (J) of Lemma 3.6 is true. Therefore, using Lemma 3.6, we draw conclusions that

(1) There are $w_0,\vartheta_0\in\rho_{h,e_1}$ and $r\in(0,1)$ S.T $r\vartheta_0\leq w_0<E_{\mathsf{q}}(\lambda\tau)\vartheta_0$;

$$w_0\leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w_0,\vartheta_0)\leq E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(\vartheta_0,w_0)\leq E_{\mathsf{q}}(\lambda\tau)\vartheta_0;$$

(2) $\mathtt{T}$ has an unique q-EFP u^* in ρ_{h,e_1};

(3) For any primitive value $u_0,v_0\in\rho_{h,e_1}$, by making recursive sequences

$$u_{\mathfrak{N}}=E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(u_{\mathfrak{N}-1},v_{\mathfrak{N}-1}),$$
$$v_{\mathfrak{N}}=E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(v_{\mathfrak{N}-1},u_{\mathfrak{N}-1}),\ \mathfrak{N}=1,2,...,$$

we get $u_{\mathfrak{N}}\to u^*$ and $v_{\mathfrak{N}}\to u^*$ as $\mathfrak{N}\to\infty$. Means that, conditions 1-3 are fulfilled. □

Theorem 3.8. *Assume that the conditions* $(J_1')-(J_5')$ *in* (1.1) *are satisfied :*

(J_1') $\chi:O\times\Upsilon\times\Upsilon\to\Upsilon$ *and* $\varrho,\phi:O\times\Upsilon\to\Upsilon$ *are continuous.* $\chi(\tau,0,1)\not\equiv0$, $\varrho(\tau,0)\not\equiv0$ *and* $\phi(\tau,1)\not\equiv0$;

(J_2') *For each constant* $\tau\in O$ *and* $\vartheta\in\Upsilon$ *in* $w\in\Upsilon$, *function* $\chi(\tau,w(\tau),\vartheta(\tau))$ *is increasing and for every constant* $\tau\in(0,1)$ *and* $w\in\Upsilon$ *in* $\vartheta\in\Upsilon$ *function* $\chi(\tau,w(\tau),\vartheta(\tau))$ *is decreasing. For constant* $\tau\in O$ *in* $w\in\Upsilon$, *the function* $\varrho(\tau,w)$ *is increasing and for constant* $\tau\in O$ *in* $\vartheta\in\Upsilon$, *the function* $\phi(\tau,\vartheta)$ *is decreasing;*

(J_3') *For any* $\tau,\eta\in(0,1)$, $w,\vartheta\in\Upsilon$ *and the continuous functions* $0<\alpha(\eta)<1$, $0<\beta(\eta)<1$ *and* $0<\alpha(\eta)+\beta(\eta)<1$. *There is a function* $\eta^{\alpha(\eta)+\beta(\eta)}>\eta$ *S.T:*

$$\chi(\tau,\eta w+(\eta-1)e_1,\eta^{-1}\vartheta+(\eta^{-1}-1)e_1)\geq\eta^{\alpha(\eta)+\beta(\eta)}\chi(\tau,w,\vartheta)+(\eta^{\alpha(\eta)+\beta(\eta)}-1)e_1;$$

(J_4') *For any* $\tau,\eta\in(0,1)$, $w,\vartheta\in\Upsilon$ *and continuous function* $0<\sigma(\eta)<1$, *there is a function* $\eta^{\sigma(\eta)}>\eta$, *S.T*

$$\varrho(\tau,\eta w+(\eta-1)e_1)\geq\eta^{\sigma(\eta)}\varrho(\tau,w)+(\eta^{\sigma(\eta)}-1)e_1,$$
$$\phi(\tau,\eta^{-1}\vartheta+(\eta^{-1}-1)e_1)\geq\eta^{\sigma(\eta)}\phi(\tau,\vartheta)+(\eta^{\sigma(\eta)}-1)e_1;$$

(J_5') *For every* $\tau,\eta\in(0,1)$, $w,\vartheta\in\Upsilon$ *exists a constant* $\delta>0$, *S.T:*

$$\chi(\tau,w(\tau),\vartheta(\tau))\geq\delta[\varrho(\tau,w(\tau))+\phi(\tau,\vartheta(\tau))]+(\delta-1)e_1.$$

Thus,

(1) *There are* $\tau\in O$, $w_0,\vartheta_0\in\rho_{h,e_1}$ *and* $\epsilon\in(0,1)$, *S.T* $\epsilon\vartheta_0\leq w_0<E_{\mathsf{q}}(\lambda\tau)\vartheta_0$ *and*

$$w_0(\tau)\leq E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}[\chi(\varsigma,w_0(\varsigma),\vartheta_0(\varsigma))+\varrho(\varsigma,w_0(\varsigma))+\phi(\varsigma,\vartheta_0(\varsigma))]d\varsigma-3e_1,$$
$$E_{\mathsf{q}}(\lambda\tau)\vartheta_0\geq E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}[\chi(\varsigma,\vartheta_0(\varsigma),w_0(\varsigma))+\varrho(\varsigma,w_0(\varsigma))+\phi(\varsigma,w_0(\varsigma))]d\varsigma-3e_1,$$

where $h(\tau)=\tau^{\gamma-1}$, $\tau\in[0,1]$*;*

(2) *The hybrid q-FBVP* (1.1) *has an unique* q*-EFP* w^* *in* ρ_{h,e_1}*;*

(3) *For any* $u_0,v_0\in\rho_{h,e_1}$*, by making recursive sequences*

$$u_{\mathfrak{N}+1}=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}[\chi(\varsigma,u_{\mathfrak{N}}(\varsigma),v_{\mathfrak{N}}(\varsigma))+\varrho(\varsigma,u_{\mathfrak{N}}(\varsigma))+\phi(\varsigma,v_{\mathfrak{N}}(\varsigma))]d\varsigma-3e_1,$$

$$v_{\mathfrak{N}+1}=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}[\chi(\varsigma,v_{\mathfrak{N}}(\varsigma),u_{\mathfrak{N}}(\varsigma))+\varrho(\varsigma,v_{\mathfrak{N}}(\varsigma))+\phi(\varsigma,u_{\mathfrak{N}}(\varsigma))]d\varsigma-3e_1,$$

$\mathfrak{N}=0,1,2,....$

we have $\|u_{\mathfrak{N}}-w^*\|\longrightarrow 0$ *and* $\|w_{\mathfrak{N}}-w^*\|\longrightarrow 0$ *as* $\mathfrak{N}\longrightarrow\infty$*.*

Proof. The hybrid q-FBVP (1.1) has a solution as below according to Theorem 2.11:

$$w(\tau)=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}[\chi(\varsigma,w(\varsigma),\vartheta(\varsigma))+\varrho(\varsigma,w(\varsigma))+\phi(\varsigma,\vartheta(\varsigma))]d\varsigma-3e_1,$$

where $\mathfrak{G}(\tau,\varsigma)$ is the same as (2.3).
We display that w is a **q**-EPS of problem (1.1) $\leftrightarrow$ $w=\mathcal{A}(w,\vartheta)+\mathcal{B}(w)+\mathcal{C}(\vartheta)+3e_1=E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w,\vartheta)$. We express operators $\mathcal{A}:\rho\times\rho\longrightarrow\mathbf{E}$ and $\mathcal{B},\mathcal{C}:\rho\longrightarrow\mathbf{E}$ as below:

$$\mathcal{A}(w,\vartheta)(\tau)=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma,w(\varsigma),\vartheta(\varsigma))d\varsigma-e_1,$$

$$\mathcal{B}(w)(\tau)=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma,w(\varsigma))d\varsigma-e_1,$$

$$\mathcal{C}(\vartheta)(\tau)=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\phi(\varsigma,\vartheta(\varsigma))d\varsigma-e_1.$$

For $w_i,\vartheta_i\in\rho_{h,e_1}$, $(i=1,2)$ with $w_1\le w_2$ and $\vartheta_1\ge\vartheta_2$, we know that $\vartheta_1(\tau)\ge\vartheta_2(\tau)$, $w_1(\tau)\le w_2(\tau)$. We show that operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ apply to the following relations, respectively.

$$\mathcal{A}(w_1,\vartheta_1)(\tau)\le\mathcal{A}(w_2,\vartheta_2)(\tau).$$
$$\mathcal{B}(w_1)(\tau)\le\mathcal{B}(w_2)(\tau),$$
$$\mathcal{C}(\vartheta_1)(\tau)\le\mathcal{C}(\vartheta_2)(\tau).$$

Using of (J_2'), we have:

$$\begin{aligned}\mathcal{A}(w_1,\vartheta_1)(\tau)&=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma,w_1(\varsigma),\vartheta_1(\varsigma))d\varsigma-e_1\\&\le E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma,w_2(\varsigma),\vartheta_2(\varsigma))d\varsigma-e_1\\&=\mathcal{A}(w_2,\vartheta_2)(\tau).\end{aligned}$$

That is, $\mathcal{A}(w_1,\vartheta_1)(\tau)\le\mathcal{A}(w_2,\vartheta_2)(\tau)$. Similarly, for $\mathcal{B}$ and $\mathcal{C}$ we obtain.

$$\begin{aligned}\mathcal{B}(w_1)(\tau)&=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma,w_1(\varsigma))d\varsigma-e_1\\&\le E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma,w_2(\varsigma))d\varsigma-e_1\\&=\mathcal{B}(w_2)(\tau).\end{aligned}$$

Means that, $\mathcal{B}(w_1)(\tau)\le\mathcal{B}(w_2)(\tau)$.

$$\mathcal{C}(\vartheta_1)(\tau)=E_{\mathsf{q}}(\lambda\tau)\int_0^1\mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\phi(\varsigma,\vartheta_1(\varsigma))d\varsigma-e_1$$

$$\leq E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\phi(\varsigma,\vartheta_2(\varsigma))d\varsigma - e_1$$
$$= \mathcal{C}(\vartheta_2)(\tau).$$

That is, $\mathcal{C}(\vartheta_1)(\tau) \leq \mathcal{C}(\vartheta_2)(\tau)$. Now, for each $w, \vartheta \in \rho_{h,e_1}$, $\eta \in (0,1)$ and $\tau \in O$ using of (J_3'), we show that operator $\mathcal{A}$ satisfy in condition (3.9).

$$\begin{aligned}
&\mathcal{A}(\eta w + (\eta-1)e_1, \eta^{-1}\vartheta + (\eta^{-1}-1)e_1)(\tau)\\
&= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma, \eta w(\varsigma) + (\eta-1)e_1, \eta^{-1}\vartheta(\varsigma) + (\eta^{-1}-1)e_1)d\varsigma - e_1\\
&\geq \eta^{\alpha(\eta)+\beta(\eta)}E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma, w(\varsigma), \vartheta(\varsigma))d\varsigma + (\eta^{\alpha(\eta)+\beta(\eta)}-1)e_1 - e_1\\
&= \eta^{\alpha(\eta)+\beta(\eta)}\mathcal{A}(w,\vartheta)(\tau) + (\eta^{\alpha(\eta)+\beta(\eta)}-1)e_1.
\end{aligned}$$

That's mean, the operator $\mathcal{A}$ satisfies in condition (3.9). Now, for any $w, \vartheta \in \rho_{h,e_1}$, $\eta \in (0,1)$ and $\tau \in [0,1]$, by using of (J_4'), we show that $\mathcal{B}$ and $\mathcal{C}$ apply in conditions (3.10) and (3.11).

$$\begin{aligned}
\mathcal{B}(\eta w + (\eta-1)e_1) &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma, \eta w(\varsigma) + (\eta-1)e_1)d\varsigma - e_1\\
&\geq E_{\mathsf{q}}(\lambda\tau)[\eta^{\sigma(\eta)}E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma, w(\varsigma))d\varsigma + (\eta^{\sigma(\eta)}-1)e_1] - e_1\\
&= \eta^{\sigma(\eta)}\mathcal{B}(w)(\tau) + (\eta^{\sigma(\eta)}-1)e_1.
\end{aligned}$$

That is, the operator $\mathcal{B}$ applies in condition (3.10).

$$\begin{aligned}
\mathcal{C}(\eta^{-1}\vartheta + (\eta^{-1}-1)e_1 &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\phi(\varsigma, \eta^{-1}\vartheta(\varsigma) + (\eta^{-1}-1)e_1)d\varsigma - e_1\\
&\geq E_{\mathsf{q}}(\lambda\tau)[\eta^{\sigma(\eta)}\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\phi(\varsigma, \vartheta(\varsigma))d\varsigma + (\eta^{\sigma(\eta)}-1)e_1] - e_1\\
&= \eta^{\sigma(\eta)}\mathcal{C}(\vartheta)(\tau) + (\eta^{\sigma(\eta)}-1)e_1.
\end{aligned}$$

That's mean, the operator $\mathcal{C}$ satisfies in condition (3.11). Now we need to demonstrate that $\mathcal{A}(h,h) \in \rho_{h,e_1}$, $\mathcal{B}(h) \in \rho_{h,e_1}$ and $\mathcal{C}(h) \in \rho_{h,e_1}$. So we need to prove $\mathcal{A}(h,h) + e_1 \in \rho_h$, $\mathcal{B}(h) + e_1 \in \rho_h$ and $\mathcal{C}(h) + e_1 \in \rho_h$. By using of (J_1') and (J_2'), we have:

$$\begin{aligned}
\mathcal{A}(h,h)(\tau) + e_1 &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma, h(\varsigma), h(\varsigma))d\varsigma\\
&= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma, \varsigma^{\gamma-1}, -I_{\mathsf{q},0}\varsigma^{\gamma-1})d\varsigma
\end{aligned}$$

According to Lemma 2.6, we know that if $(\lambda = \gamma-1, a = 0, \alpha = 1)$, thus,
$-I_{\mathsf{q},0}\varsigma^{\gamma-1} = \frac{-\Gamma_{\mathsf{q}}(\gamma)}{\Gamma_{\mathsf{q}}(\gamma+1)}\varsigma^{\gamma}$. Therefor, by using of Lemma 2.12 we get:

$$\begin{aligned}
\mathcal{A}(h,h) + e_1 &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma, \varsigma^{\gamma-1}, \frac{-\Gamma_{\mathsf{q}}(\gamma)}{\Gamma_{\mathsf{q}}(\gamma+1)}\varsigma^{\gamma})d\varsigma\\
&\leq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\chi(\varsigma, \varsigma^{\gamma-1}, \frac{-\Gamma_{\mathsf{q}}(\gamma)}{\Gamma_{\mathsf{q}}(\gamma+1)}\varsigma^{\gamma})d\varsigma\\
&\leq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\chi(\varsigma, 1, 0)d\varsigma,
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}(h,h)(\tau)+e_1 &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma,h(\varsigma),h(\varsigma))d\varsigma\\
&= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma,\varsigma^{\gamma-1},-I_{\mathsf{q},0}\varsigma^{\gamma-1})d\varsigma\\
&= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma,\varsigma^{\gamma-1},\frac{-\Gamma_{\mathsf{q}}(\gamma)}{\Gamma_{\mathsf{q}}(\gamma+1)}\varsigma^{\gamma})d\varsigma\\
&\geq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\chi(\varsigma,\varsigma^{\gamma-1}\\
&,\frac{-\Gamma_{\mathsf{q}}(\gamma)}{\Gamma_{\mathsf{q}}(\gamma+1)}\varsigma^{\gamma})d\varsigma\\
&\geq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1})}{\Gamma(\gamma)}I_{\mathsf{q},0}\chi(\varsigma,0,1)d\varsigma.
\end{aligned}$$

From (J_1') and (J_2'), we obtain:

$$\chi(\varsigma,0,1)\not\equiv 0,\ \ \chi(\varsigma,1,0)\geq\chi(\varsigma,0,1).$$

So, $\int_0^1 \chi(\varsigma,1,0)d\varsigma \geq \int_0^1 \chi(\varsigma,0,1)d\varsigma$. Consequently,

$$\begin{aligned}
&E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\chi(\varsigma,0,1)d\varsigma\\
&\leq \mathcal{A}(h,h)(\tau)+e_1 \leq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\chi(\varsigma,1,0)d\varsigma.
\end{aligned}$$

So, we define that:

$$\begin{aligned}
L_1 &:= E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\chi(\varsigma,0,1)d\varsigma\\
L_2 &:= E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\chi(\varsigma,1,0)d\varsigma.
\end{aligned}$$

Therefore, for $L_1\tau^{\gamma-1}\leq\mathcal{A}(h,h)(\tau)+e_1\leq L_2\tau^{\gamma-1}$, $\tau\in O$ where $h(\tau)=\tau^{\gamma-1}$. Then, we obtain $L_1h(\tau)\leq\mathcal{A}(h,h)(\tau)+e_1\leq L_2h(\tau)$ that's mean, $\mathcal{A}(h,h)+e_1\in\rho_h$. Thus, according to define ρ_{h,e_1}, we conclude that $\mathcal{A}(h,h)\in\rho_{h,e_1}$. By using of (J_1'), (J_2') and Lemma 2.6, we get:

$$\begin{aligned}
\mathcal{B}(h)+e_1 &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma,h(\varsigma))d\varsigma\\
&= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma,\varsigma^{\gamma-1})d\varsigma\\
&\geq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \mathfrak{G}(\tau,\varsigma)\frac{(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\varrho(\varsigma,0)d\varsigma,
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}(h)+e_1 &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma,h(\varsigma))d\varsigma\\
&= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\varrho(\varsigma,\varsigma^{\gamma-1})d\varsigma\\
&\leq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\varrho(\varsigma,1)d\varsigma.
\end{aligned}$$

Because $\varrho(\varsigma, 0) \not\equiv 0$ and $\varrho(\varsigma, 1) \geq \varrho(\varsigma, 0)$, consequently

$$E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \mathfrak{G}(\tau,\varsigma)\frac{(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\varrho(\varsigma,0)d\varsigma$$
$$\leq \mathcal{B}(h)(\tau)+e_1 \leq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\varrho(\varsigma,1)d\varsigma,$$

that's mean, $L_1h(\tau) \leq \mathcal{B}(h)(\tau) + e_1 \leq L_2h(\tau)$. So, according to define ρ_{h,e_1}, we get $\mathcal{B}(h) \in \rho_{h,e_1}$. Similarly, since $\phi(\varsigma, 1) \not\equiv 0$, we have:

$$E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \mathfrak{G}(\tau,\varsigma)\frac{(1-\varsigma)^{\gamma+\alpha-1}-(1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\phi(\varsigma,1)d\varsigma$$
$$\leq \mathcal{C}(h)(\tau)+e_1 \leq E_{\mathsf{q}}(\lambda\tau)\tau^{\gamma-1}\int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)}I_{\mathsf{q},0}\phi(\varsigma,0)d\varsigma,$$

that's mean, $L_1h(\tau) \leq \mathcal{C}(h)(\tau)+e_1 \leq L_2h(\tau)$ and $\mathcal{C}(h)+e_1 \in \rho_h$ consequently, $\mathcal{C}(h) \in \rho_{h,e_1}$. For any $w, \vartheta \in \rho_{h,e_1}$ and every $\tau \in O$ by using of (J_5') there is a constant $\delta > 0$ S.T

$$\begin{aligned}\mathcal{A}(w,\vartheta)(\tau)+e_1 &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\chi(\varsigma,w(\varsigma),\vartheta(\varsigma))d\varsigma\\ &\geq \delta[E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\{\varrho(\varsigma,w(\varsigma))+\phi(\varsigma,\vartheta(\varsigma))\}d\varsigma]+(\delta-1)e_1\\ &= \delta[\mathcal{B}(w)(\tau)+\mathcal{C}(\vartheta)(\tau)+2e_1]+(\delta-1)e_1,\end{aligned}$$

so, we obtain

$$\mathcal{A}(w,\vartheta)(\tau)+e_1 \geq \delta[\mathcal{B}(w)(\tau)+\mathcal{C}(\vartheta)(\tau)+2e_1]+(\delta-1)e_1.$$

Finally, using Theorem 3.7, we conclude that there exists $w_0, \vartheta_0 \in \rho_{h,e_1}$ and $\epsilon \in (0,1)$ S.T $\epsilon\vartheta_0 \leq w_0 < E_{\mathsf{q}}(\lambda\tau)\vartheta_0$ and

$$\begin{aligned}w_0(\tau) &\leq E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}[\chi(\varsigma,w_0(\varsigma),\vartheta_0(\varsigma))+\varrho(\varsigma,w_0(\varsigma))+\phi(\varsigma,\vartheta_0(\varsigma))]d\varsigma-3e_1\\ &\leq E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}[\chi(\varsigma,\vartheta_0(\varsigma),w_0(\varsigma))+\varrho(\varsigma,\vartheta_0(\varsigma))+\phi(\varsigma,w_0(\varsigma))]d\varsigma-3e_1\\ &\leq E_{\mathsf{q}}(\lambda\tau)\vartheta_0.\end{aligned}$$

Operator equation $w = \mathcal{A}(w,\vartheta)+\mathcal{B}(w)+\mathcal{C}(\vartheta)+3e_1 = E_{\mathsf{q}}(\lambda\tau)\mathtt{T}(w,\vartheta)$ has a unique solution q-EFP w^* in ρ_{h,e_1}. For any primitive value $u_0, v_0 \in \rho_{h,e_1}$ by making recursive sequences

$$\begin{aligned}u_{\mathfrak{N}} &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\left[\chi(\varsigma,u_{\mathfrak{N}-1}(\varsigma),v_{\mathfrak{N}-1}(\varsigma))+\varrho(\varsigma,u_{\mathfrak{N}-1}(\varsigma))+\phi(\varsigma,v_{\mathfrak{N}-1}(\varsigma))\right]d\varsigma-3e_1,\\ v_{\mathfrak{N}} &= E_{\mathsf{q}}(\lambda\tau)\int_0^1 \mathfrak{G}(\tau,\varsigma)I_{\mathsf{q},0}\left[\chi(\varsigma,v_{\mathfrak{N}-1}(\varsigma),u_{\mathfrak{N}-1}(\varsigma))+\varrho(\varsigma,v_{\mathfrak{N}-1}(\varsigma))+\phi(\varsigma,u_{\mathfrak{N}-1}(\varsigma))\right]d\varsigma-3e_1,\\ \mathfrak{N} &= 1,2,....\end{aligned}$$

We obtain $u_{\mathfrak{N}} \to w^*$ and $v_{\mathfrak{N}} \to w^*$ as $\mathfrak{N} \to \infty$. Therefore, conditions Theorem 3.8 is satisfied. □

4. Examples

Example 4.1. We consider the following problem in the case where $q > \frac{1}{2}$ and $\lambda = 2$.

$$D_{\mathsf{q}}D_{0+}^{\frac{5}{2}}\left(\frac{w(\tau)+3e_1(\tau)}{E_{\mathsf{q}}(2\tau)}\right) = [w(\tau)+e_1(\tau)]^{\frac{1}{3}\cos\frac{\pi}{4}\tau}+[(1-\frac{I_{\mathsf{q},0}w(\tau)}{\tau})+e_1(\tau)]^{\frac{-1}{3}\sin\frac{\pi}{4}\tau}$$

$$\begin{aligned} &+ \quad [w(\tau)+e_1(\tau)]^{\frac{1}{3}\sin\frac{\pi}{4}\tau}+[\vartheta(\tau)+e_1(\tau)]^{\frac{-1}{3}\sin\frac{\pi}{4}\tau}, \qquad (4.1)\\ w(0) &= w'(0)=e_1(0)=e_1'(0)=0,\\ I^{\frac{1}{2}}_{0+}\left(\frac{w(1)+3e_1(1)}{E_{\mathsf{q}}(2)}\right) &= 0. \end{aligned}$$

In this problem $\gamma=\frac{5}{2}$, $\alpha=\frac{1}{2}$, $\alpha(\tau)=\frac{1}{3}\cos\frac{\pi}{4}\tau$, $\beta(\tau)=\sigma(\tau)=\frac{1}{3}\sin\frac{\pi}{4}\tau$ and $\vartheta(\tau)=1-\frac{I_{\mathsf{q},0}w(\tau)}{\tau}$. Thus, we have:

$$\begin{aligned} \chi(\tau,w,\vartheta) &= [w(\tau)+e_1(\tau)]^{\frac{1}{3}\cos\frac{\pi}{4}\tau}+[\vartheta(\tau)+e_1(\tau)]^{\frac{-1}{3}\sin\frac{\pi}{4}\tau},\\ \varrho(\tau,w) &= [e_1(\tau)w(\tau)+e_1(\tau)]^{\frac{1}{3}\sin\frac{\pi}{4}\tau},\\ \phi(\tau,\vartheta) &= [e_1(\tau)\vartheta(\tau)+e_1(\tau)]^{\frac{-1}{3}\sin\frac{\pi}{4}\tau}. \end{aligned}$$

Obviously, the functions $\chi: O\times\Upsilon\times\Upsilon\to\Upsilon$ and $\varrho,\phi: O\times\Upsilon\to\Upsilon$ are continuous with $\chi(\tau,0,1)\not\equiv 0$, $\varrho(\tau,0)\not\equiv 0$ and $\phi(\tau,1)\not\equiv 0$. The function $\chi(\tau,w,\vartheta)$ is increasing with respect to $w\in\Upsilon$ for fixed $\tau\in O$ and $\vartheta\in\Upsilon$, and function χ is decreasing with respect to $\vartheta\in\Upsilon$ for fixed $\tau\in O$ and $w\in\Upsilon$. The function $\varrho(\tau,w)$ is increasing with respect to $w\in\Upsilon$ for fixed $\tau\in O$ and $\phi(\tau,\vartheta)$ is decreasing with respect to $\vartheta\in\Upsilon$ for fixed $\tau\in O$. For each $\eta\in(0,1)$, $\tau\in O$, $w,\vartheta\geq 0$, we have

$$\begin{aligned} \chi\left(\tau,\eta w+(\eta-1)e_1,\eta^{-1}\vartheta+(\eta^{-1}-1)e_1\right) &= [e_1(\tau)\left(\eta w(\tau)+(\eta-1)e_1\right)+e_1(\tau)]^{\frac{1}{3}\cos\frac{\pi}{4}\tau}\\ &+\left[e_1(\tau)\left(\eta^{-1}\vartheta(\tau)+(\eta^{-1}-1)e_1\right)+e_1(\tau)\right]^{\frac{-1}{3}\sin\frac{\pi}{4}\tau}\\ &\geq \eta^{\frac{1}{3}(\sin\frac{\pi}{4}\tau+cos\frac{\pi}{4}\tau)}\{\,[e_1(\tau)w(\tau)+e_1(\tau)]^{\frac{1}{3}\cos\frac{\pi}{4}\tau}\\ &+[e_1(\tau)(\vartheta(\tau))+e_1(\tau)]^{\frac{-1}{3}\sin\frac{\pi}{4}\tau}\,\}+(\eta^{\frac{1}{3}(\sin\frac{\pi}{4}\tau+\cos\frac{\pi}{4}\tau)}-1)e_1, \end{aligned}$$

$$\begin{aligned} \varrho(\tau,\eta w+(\eta-1)e_1) &= \{e_1(\tau)\left[\eta w(\tau)+(\eta-1)e_1\right]+e_1(\tau)\}^{\frac{1}{3}\sin\frac{\pi}{4}\tau}\\ &\geq \eta^{\frac{1}{3}\sin\frac{\pi}{4}\tau}\left[(e_1(\tau)w(\tau)+e_1(\tau))^{\frac{1}{3}\sin\frac{\pi}{4}\tau}\right]+(\eta^{\frac{1}{3}\sin\frac{\pi}{4}\tau}-1)e_1, \end{aligned}$$

$$\begin{aligned} \phi(\tau,\eta^{-1}\vartheta+(\eta^{-1}-1)e_1) &= \left[e_1(\tau)\left(\eta^{-1}\vartheta(\tau)+(\eta^{-1}-1)e_1\right)+e_1(\tau)\right]^{\frac{-1}{3}\sin\frac{\pi}{4}\tau},\\ &\geq \eta^{\frac{1}{3}\sin\frac{\pi}{4}\tau}\left[(e_1(\tau)\vartheta(\tau)+e_1(\tau))^{\frac{-1}{3}\sin\frac{\pi}{4}\tau}\right]+(\eta^{\frac{1}{3}\sin\frac{\pi}{4}\tau}-1)e_1. \end{aligned}$$

For $w,\vartheta\geq 0$ there is a constant $\delta=\frac{1}{6}$ S.T

$$\begin{aligned} \chi(\tau,w,\vartheta) &= [e_1(\tau)w(\tau)+e_1(\tau)]^{\frac{1}{3}\cos\frac{\pi}{4}\tau}+[e_1(\tau)\vartheta(\tau)+e_1(\tau)]^{\frac{-1}{3}\sin\frac{\pi}{4}\tau}\\ &\geq \frac{1}{6}\left[(e_1(\tau)w(\tau)+e_1(\tau))^{\frac{1}{3}\sin\frac{\pi}{4}\tau}+(e_1(\tau)\vartheta(\tau)+e_1(\tau))^{\frac{-1}{3}\sin\frac{\pi}{4}\tau}\right]-\frac{5}{6}e_1\\ &= \delta\left[\varrho(\tau,w)+\phi(\tau,\vartheta)\right]-\frac{5}{6}e_1. \end{aligned}$$

Therefore, we conclude that the condition of Theorem 3.7 is satisfied and Theorem 3.8 guarantees that the problem (4.1) has a unique q-EPS in ρ_{h,e_1}, where $h(\tau)=\tau^{\frac{3}{2}}$, $\tau\in O$.

Example 4.2. Now, for a better understanding of the issue, we consider another example with a different lambda and q and check the existence and uniqueness of its positive answer.
We consider the following problem assuming that $\lambda=\frac{3}{2}$ and $q>\frac{1}{3}$.

$$\begin{aligned} D_{\mathsf{q}}D^{\frac{7}{3}}_{0+}\left(\frac{w(\tau)+3e_1(\tau)}{E_{\mathsf{q}}(\frac{3}{2}\tau)}\right) &= [w(\tau)+e_1(\tau)]^{\frac{1}{3}\tau}+[(1-\frac{I_{\mathsf{q},0}w(\tau)}{\tau})+e_1(\tau)]^{\frac{-1}{6}\tau}\\ &+ \quad [w(\tau)+e_1(\tau)]^{\frac{1}{6}\tau}+[\vartheta(\tau)+e_1(\tau)]^{\frac{-1}{6}\tau}, \qquad (4.2)\\ w(0) &= w'(0)=e_1(0)=e_1'(0)=0, \end{aligned}$$

$$I_{0^+}^{\frac{4}{3}}\left(\frac{w(1)+3e_1(1)}{E_{\mathsf{q}}(\frac{3}{2})}\right) = 0.$$

In this problem $\gamma=\frac{7}{3}$, $\alpha=\frac{4}{3}$, $\alpha(\tau)=\frac{1}{3}\tau$, $\beta(\tau)=\sigma(\tau)=\frac{1}{6}\tau$ and $\vartheta(\tau)=1-\frac{I_{\mathsf{q},0}w(\tau)}{\tau}$. Thus, we have:

$$\begin{aligned}
\chi(\tau,w,\vartheta) &= [w(\tau)+e_1(\tau)]^{\frac{1}{3}\tau}+[\vartheta(\tau)+e_1(\tau)]^{\frac{-1}{6}\tau}\\
\varrho(\tau,w) &= [e_1(\tau)w(\tau)+e_1(\tau)]^{\frac{1}{6}\tau}\\
\phi(\tau,\vartheta) &= [e_1(\tau)\vartheta(\tau)+e_1(\tau)]^{\frac{-1}{6}\tau}.
\end{aligned}$$

Obviously, the functions $\chi: O\times\Upsilon\times\Upsilon\to\Upsilon$ and $\varrho,\phi: O\times\Upsilon\to\Upsilon$ are continuous with $\chi(\tau,0,1)\not\equiv 0$, $\varrho(\tau,0)\not\equiv 0$ and $\phi(\tau,1)\not\equiv 0$. The function $\chi(\tau,w,\vartheta)$ is increasing with respect to $w\in\Upsilon$ for fixed $\tau\in O$ and $\vartheta\in\Upsilon$, and function χ is decreasing with respect to $\vartheta\in\Upsilon$ for fixed $\tau\in O$ and $w\in\Upsilon$. The function $\varrho(\tau,w)$ is increasing with respect to $w\in\Upsilon$ for fixed $\tau\in O$ and $\phi(\tau,\vartheta)$ is decreasing with respect to $\vartheta\in\Upsilon$ for fixed $\tau\in O$. For each $\eta\in(0,1)$, $\tau\in O$, $w,\vartheta\geq 0$, we have

$$\begin{aligned}
\chi\left(\tau,\eta w+(\eta-1)e_1,\eta^{-1}\vartheta+(\eta^{-1}-1)e_1\right) &= [e_1(\tau)\left(\eta w(\tau)+(\eta-1)e_1\right)+e_1(\tau)]^{\frac{1}{3}\tau}\\
&\quad+\left[e_1(\tau)\left(\eta^{-1}\vartheta(\tau)+(\eta^{-1}-1)e_1\right)+e_1(\tau)\right]^{\frac{-1}{6}\tau}\\
&\geq \eta^{\frac{1}{2}\tau}\{\ [e_1(\tau)w(\tau)+e_1(\tau)]^{\frac{1}{3}\tau}\\
&\quad+[e_1(\tau)(\vartheta(\tau))+e_1(\tau)]^{\frac{-1}{6}\tau}\}+(\eta^{\frac{1}{2}\tau}-1)e_1,
\end{aligned}$$

$$\begin{aligned}
\varrho(\tau,\eta w+(\eta-1)e_1) &= \{e_1(\tau)\left[\eta w(\tau)+(\eta-1)e_1\right]+e_1(\tau)\}^{\frac{1}{6}\tau}\\
&\geq \eta^{\frac{1}{6}\tau}\left[(e_1(\tau)w(\tau)+e_1(\tau))^{\frac{1}{6}}\right]+(\eta^{\frac{1}{6}\tau}-1)e_1,
\end{aligned}$$

$$\begin{aligned}
\phi(\tau,\eta^{-1}\vartheta+(\eta^{-1}-1)e_1) &= \left[e_1(\tau)\left(\eta^{-1}\vartheta(\tau)+(\eta^{-1}-1)e_1\right)+e_1(\tau)\right]^{\frac{-1}{6}\tau},\\
&\geq \eta^{\frac{1}{6}\tau}\left[(e_1(\tau)\vartheta(\tau)+e_1(\tau))^{\frac{-1}{6}\tau}\right]+(\eta^{\frac{1}{6}\tau}-1)e_1.
\end{aligned}$$

For $w,\vartheta\geq 0$ there is a constant $\delta=\frac{1}{5}$ S.T

$$\begin{aligned}
\chi(\tau,w,\vartheta) &= [e_1(\tau)w(\tau)+e_1(\tau)]^{\frac{1}{3}\tau}+[e_1(\tau)\vartheta(\tau)+e_1(\tau)]^{\frac{-1}{6}\tau}\\
&\geq \frac{1}{6}\left[(e_1(\tau)w(\tau)+e_1(\tau))^{\frac{1}{6}\tau}+(e_1(\tau)\vartheta(\tau)+e_1(\tau))^{\frac{-1}{6}\tau}\right]-\frac{4}{5}e_1\\
&= \delta\left[\varrho(\tau,w)+\phi(\tau,\vartheta)\right]-\frac{4}{5}e_1.
\end{aligned}$$

Therefore, we conclude that the condition of Theorem 3.7 is satisfied and Theorem 3.8 guarantees that the problem (4.2) has a unique q-EPS in ρ_{h,e_1}, where $h(\tau)=\tau^{\frac{4}{3}}$, $\tau\in O$.

5. Conclusion

In this paper, we firstly consider existence and uniqueness of q-Exponential positive solution of the hybrid q-fractional boundary value problem on Banach space E with a new set $\phi(h,e)$. Using the abstract result, we give some sufficient under which q-fractional boundary value problem (1.1) has a uniqueness positive solution. In addition, we also construct two iterative sequences to approximate the unique solution.

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