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An efficient high-order compact finite difference scheme for Lane-Emden-type equations

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Abstract

In this paper, an efficient high-order compact finite difference (HOCFD) scheme is introduced for solving generalized Lane-Emden equations. For nonlinear types, it is shown that a combined quasilinearization and HOCFD scheme gives excellent results, while a few quasilinear iterations are needed. Then the proposed method is developed for solving the system of linear and nonlinear Lane-Emden equations. Some numerical examples are provided, and the obtained results of the proposed method are then compared with previous well-established methods. The numerical experiments show the accuracy and efficiency of the proposed method.

Keywords. Lane-Emden equations, Compact finite difference scheme, Quasilinearization method, High-order accuracy.

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1. Introduction

The main objective in this paper is to find the numerical solution to generalized Lane-Emden equation

$$u''(x) + p(x)u'(x) + F(u,x) = h(x), \quad a \le x \le b, \tag{1.1}$$

with boundary conditions

$$u(a) = u_a, \quad u(b) = u_b, \tag{1.2}$$

and the system of Lane-Emden equations

$$u'' + p_1(x)u' + F_1(u,v) = h_1(x), \tag{1.3}$$
$$v'' + p_2(x)v' + F_2(u,v) = h_2(x), \tag{1.4}$$

with known boundary conditions $u(a_1,y)$, $u(b_1,y)$, $u(x,a_2)$ and $u(x,b_2)$ for $a_1 \le x \le b_1$ and $a_2 \le y \le b_2$.

The Lane-Emden equation was first studied by astrophysicists J. H. Lane and R. Emden [9] and is categorized as a singular initial value problem. A lot of researches containing both analytical as well as numerical techniques have been presented for the solution of Lane-Emden equations and system of Lane-Emden equations, such as Adomian decomposition method [30, 31], series solutions [25], wavelets methods [28, 33], differential transform method [12], Bernstein and Legendre operational matrix of differentiation [20, 21], rational Legendre pseudospectral approach [24], Homotopy analysis method [29], modified Adomian decomposition method [8], Hermit functions collocation method [23], B-spline expansion and collocation approach [13, 27], a Jacobi-Gauss collocation method [3],collocation method based on cubic Hermit spline functions [18], Chebyshev neural network-based model approach [15], Picard-reproducing kernel Hilbert space method [1], generalized Chebyshev function methods [22], compact finite difference method [4], and Laguerre collocation method [35].

High-Order Compact Finite Difference (HOCFD) schemes [7, 14, 17] have been studied to approximate the function derivatives in grid points. The HOCFD schemes give high and better resolution characteristics as compared to classical

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finite difference schemes for the same number of grid points. This feature brings them closer to the spectral methods while the freedom in choosing the mesh geometry and the boundary conditions is maintained. Moreover, as compared to classical finite difference schemes, the HOCFD schemes have a simpler stencil, less computation cost and higher efficiency. This paper considers the HOCFD scheme for solving the generalized Lane-Emden equation and the system of Lane-Emden equations with known boundary conditions. In order to solve boundary value problems, we need to adjust HOCFD formulas with known boundary conditions. Hence, an HOCFD scheme is presented such that the function derivatives are considered only in grid points while the function values are known at boundary points. We show that applying the HOCFD scheme on a linear Lane-Emden equation leads to solving a linear system. The quasilinearization method (QLM) is an iterative method and was originally introduced by Bellman and Kalaba [2, 11] as a generalization of the Newton–Raphson method [5] to solve individual or systems of nonlinear ordinary and partial differential equations. Hence, in this paper, the nonlinear Lane-Emden equation is linearized by the qausilinearization method. Then, the proposed HOCFD method is used to solve the nonlinear Lane-Emden equations. However, the proposed method applied on nonlinear Lane-Emden equations yields to a linear system in every QLM iteration, but it is shown that a few iterations already provide suitable solutions. The second main goal in this paper is to extend the proposed method for solving the system of Lane-Emden equations. The merit of the proposed method is simplicity in implementation, high accuracy, and high convergence speed. The numerical experiments show the efficiency of the proposed method.

2. Compact finite difference scheme

Consider the function $u(x)$ on the interval $[a, b]$ with grid points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \tag{2.1}$$

with equal distance $h = (b-a)/n$. The classical HOCFD schemes are implicit and have a form of [14, 17]

$$A_i^{(k)} \begin{pmatrix} u^{(k)}(x_0) \\ \vdots \\ u^{(k)}(x_n) \end{pmatrix} = B_i^{(k)} \begin{pmatrix} u(x_0) \\ \vdots \\ u(x_n) \end{pmatrix}, \tag{2.2}$$

where $A_i^{(k)}$ and $B_i^{(k)}$ are corresponding $(n+1) \times (n+1)$ matrices to the k-th derivative of $u(x)$ of order i. But for solving the boundary value problems, we have to obtain the HOCFD formulas as

$$A_i^{(k)} \begin{pmatrix} u^{(k)}(x_1) \\ \vdots \\ u^{(k)}(x_{n-1}) \end{pmatrix} = B_i^{(k)} \begin{pmatrix} u(x_1) \\ \vdots \\ u(x_{n-1}) \end{pmatrix} + b_i^{(k)}, \tag{2.3}$$

where $A_i^{(k)}$ and $B_i^{(k)}$ are the corresponding $(n-1) \times (n-1)$ matrices to the k-th derivative of $u(x)$ of order i, and the $(n-1)$ vector $b_i^{(k)}$ is a known vector contain the boundary values $u(x_0)$ and $u(x_n)$. Moreover, for simplicity in numerical computations and also the stability of the method, we can choose the coefficient matrix $A_i^{(k)}$ as a symmetric diagonally dominant Toeplitz matrix. For this purpose, we use the method of undetermined coefficients which was introduced by Lele [14]. For first derivative of order 4 in the interior points (2.1), we have [14]

$$\frac{1}{4}u'_{i-1} + u'_i + \frac{1}{4}u'_{i+1} = \frac{3}{4h}(u_{i+1} - u_{i-1}), \quad i = 2, 3, \ldots, n-2.$$

For imposing the boundary conditions, we can write [7]

$$\begin{aligned} u'_1 + \frac{1}{4}u'_2 &= \frac{1}{h}\left[a_0u_0 + a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4\right] + o(h^4), \\ \frac{1}{4}u'_{n-2} + u'_{n-1} &= \frac{1}{h}\left[b_nu_n + b_{n-1}u_{n-1} + b_{n-2}u_{n-2} + b_{n-3}u_{n-3} + b_{n-4}u_{n-4}\right] + o(h^4). \end{aligned}$$

By matching the Taylor expansion coefficients of both sides of the above equations, we have $A_4^{(1)}\mathbf{u}' = B_4^{(1)}\mathbf{u} + b_4^{(1)}$, where

$$A_4^{(1)} = \begin{pmatrix} 1 & \frac{1}{4} & & & \\ \frac{1}{4} & 1 & \frac{1}{4} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{4} & 1 & \frac{1}{4} \\ & & & \frac{1}{4} & 1 \end{pmatrix}, \quad B_4^{(1)} = \frac{1}{h}\begin{pmatrix} -1 & \frac{3}{2} & -\frac{1}{3} & \frac{1}{16} & \\ -\frac{3}{4} & 0 & \frac{3}{4} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{3}{4} & 0 & \frac{3}{4} \\ & -\frac{1}{16} & \frac{1}{3} & -\frac{3}{2} & 1 \end{pmatrix},$$

and

$$b_4^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{11}{48}u_0 & 0 & \dots & 0 & \frac{11}{48}u_n \end{pmatrix}^T.$$

Similar to the above scheme, for the second derivative, we can write $A_4^{(2)}\mathbf{u}'' = B_4^{(2)}\mathbf{u} + b_4^{(2)}$ where

$$A_4^{(2)} = \begin{pmatrix} 1 & \frac{1}{10} & & & \\ \frac{1}{10} & 1 & \frac{1}{10} & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & \frac{1}{10} \\ & & & \frac{1}{10} & 1 \end{pmatrix}, \quad B_4^{(2)} = \frac{1}{h^2}\begin{pmatrix} -\frac{23}{15} & \frac{1}{4} & \frac{7}{15} & -\frac{11}{120} & & & \\ \frac{6}{5} & -\frac{12}{5} & \frac{6}{5} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \frac{6}{5} & -\frac{12}{5} & \frac{6}{5} \\ & & & -\frac{11}{120} & \frac{7}{15} & \frac{1}{4} & -\frac{23}{15} \end{pmatrix},$$

and $b_4^{(2)} = \frac{1}{h^2}\begin{pmatrix} \frac{109}{120}u_0 & 0 & \dots & 0 & \frac{109}{120}u_n \end{pmatrix}^T$. Also $A_6^{(1)}\mathbf{u}' = B_6^{(1)}\mathbf{u} + b_6^{(1)}$, where

$$A_6^{(1)} = \begin{pmatrix} 1 & \frac{1}{3} & & & \\ \frac{1}{3} & 1 & \frac{1}{3} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{3} & 1 & \frac{1}{3} \\ & & & \frac{1}{3} & 1 \end{pmatrix}, \quad B_6^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{17}{12} & \frac{83}{36} & -\frac{11}{9} & \frac{2}{3} & -\frac{37}{180} & \frac{1}{36} & & \\ -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} & & & & \\ -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} \\ & & & & -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} \\ & & & -\frac{1}{36} & \frac{37}{180} & -\frac{2}{3} & \frac{11}{9} & -\frac{83}{36} & \frac{17}{12} \end{pmatrix},$$

and $b_6^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{7}{45}u_0 & -\frac{1}{36}u_0 & 0 & \dots & 0 & \frac{1}{36}u_n & \frac{7}{45}u_n \end{pmatrix}^T$. Also $A_6^{(2)}u'' = B_6^{(2)}u + b_6^{(2)}$, where

$$A_6^{(2)} = \begin{pmatrix} 1 & \frac{2}{11} & & & \\ \frac{2}{11} & 1 & \frac{2}{11} & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & \frac{2}{11} \\ & & & \frac{2}{11} & 1 \end{pmatrix},$$

$$B_6^{(2)} = \frac{1}{h^2}\begin{pmatrix} -\frac{129}{220} & -\frac{81}{44} & \frac{557}{198} & -\frac{69}{44} & \frac{111}{220} & -\frac{39}{1980} & & & & & & \\ \frac{12}{11} & -\frac{51}{22} & \frac{12}{11} & \frac{3}{44} & & & & & & & & \\ \frac{3}{44} & \frac{12}{11} & -\frac{51}{22} & \frac{12}{11} & \frac{3}{44} & & & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & & & & & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & \frac{3}{44} & \frac{12}{11} & -\frac{51}{22} & \frac{12}{11} & \frac{3}{44} & & \\ & & & & & & \frac{3}{44} & \frac{12}{11} & -\frac{51}{22} & \frac{12}{11} & & \\ & & & & -\frac{39}{1980} & \frac{111}{220} & -\frac{69}{44} & \frac{557}{198} & -\frac{81}{44} & -\frac{129}{220} & & \end{pmatrix},$$

and

$$b_6^{(2)} = \frac{1}{h^2}\begin{pmatrix} \frac{1481}{1980}u_0 & \frac{3}{44}u_0 & 0 & \dots & 0 & \frac{3}{44}u_n & \frac{1481}{1980}u_n \end{pmatrix}^T.$$

Moreover, we have $A_8^{(1)}\mathbf{u}' = B_8^{(1)}\mathbf{u} + b_8^{(1)}$ where

$$A_8^{(1)} = \begin{pmatrix} 1 & \frac{4}{9} & \frac{1}{36} & & & & & \\ \frac{4}{9} & 1 & \frac{4}{9} & \frac{1}{36} & & & & \\ \frac{1}{36} & \frac{4}{9} & 1 & \frac{4}{9} & \frac{1}{36} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \frac{1}{36} & \frac{4}{9} & 1 & \frac{4}{9} & \frac{1}{36} \\ & & & & \frac{1}{36} & \frac{4}{9} & 1 & \frac{4}{9} \\ & & & & & \frac{1}{36} & \frac{4}{9} & 1 \end{pmatrix},$$

$$B_8^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{481}{280} & \frac{1103}{360} & -\frac{1889}{720} & \frac{115}{48} & -\frac{317}{216} & \frac{641}{1080} & -\frac{239}{1680} & \frac{99}{6466} & & & & \\ -\frac{26}{27} & \frac{7}{18} & \frac{2}{9} & \frac{65}{108} & -\frac{14}{45} & \frac{7}{54} & -\frac{2}{63} & \frac{1}{288} & & & & \\ -\frac{25}{216} & -\frac{20}{27} & 0 & \frac{20}{27} & \frac{25}{216} & & & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & & & & & & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & & -\frac{25}{216} & -\frac{20}{27} & 0 & \frac{20}{27} & \frac{25}{216} & \\ & & & & -\frac{1}{288} & \frac{2}{63} & -\frac{7}{54} & \frac{14}{45} & -\frac{65}{108} & -\frac{2}{9} & -\frac{7}{18} & \frac{26}{27} \\ & & & & -\frac{99}{6466} & \frac{239}{1680} & -\frac{641}{1080} & \frac{317}{216} & -\frac{115}{48} & \frac{1889}{720} & -\frac{1103}{360} & \frac{481}{280} \end{pmatrix},$$

and

$$b_8^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{709}{6048}u_0 & -\frac{1217}{30240}u_0 & 0 & \dots & 0 & \frac{1217}{30240}u_n & \frac{709}{6048}u_n \end{pmatrix}^T.$$

Also, we have $A_8^{(2)}\mathbf{u}'' = B_8^{(2)}\mathbf{u} + b_8^{(2)}$ where

$$A_8^{(2)} = \begin{pmatrix} 1 & \frac{344}{11799} & \frac{23}{2358} & & & & & \\ \frac{344}{1179} & 1 & \frac{344}{1179} & \frac{23}{2358} & & & & \\ \frac{23}{2358} & \frac{344}{1179} & 1 & \frac{344}{1179} & \frac{23}{2358} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \frac{23}{2358} & \frac{344}{1179} & 1 & \frac{344}{1179} & \frac{23}{2358} \\ & & & & \frac{23}{2358} & \frac{344}{1179} & 1 & \frac{344}{1179} \\ & & & & & \frac{23}{2358} & \frac{344}{1179} & 1 \end{pmatrix},$$

$$B_8^{(2)} = \frac{1}{h^2}\begin{pmatrix} \frac{7685}{22008} & -\frac{1983151}{424440} & \frac{1630439}{212220} & \frac{44051}{6288} & \frac{1892837}{424440} & -\frac{156895}{84888} & \frac{6197}{13755} & -\frac{583309}{11884320} \\ \frac{44663}{41265} & -\frac{6887}{2620} & \frac{89269}{53055} & -\frac{12173}{18864} & \frac{1081}{1965} & -\frac{49289}{212220} & \frac{2369}{41265} & -\frac{2783}{440160} \\ \frac{155}{786} & \frac{320}{393} & -\frac{265}{131} & \frac{320}{393} & \frac{155}{786} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \frac{155}{786} & \frac{320}{393} \\ & & & & & & \frac{1081}{1965} & -\frac{12173}{18864} \\ & & & & & & \frac{1892837}{424440} & \frac{44051}{6288} \end{pmatrix}$$

$$\begin{pmatrix} \ddots & \ddots & \ddots \\ -\frac{265}{131} & \frac{320}{393} & \frac{155}{786} \\ \frac{89269}{53055} & -\frac{6887}{2620} & \frac{44663}{41265} \\ \frac{1630439}{212220} & -\frac{1983151}{424440} & \frac{7685}{22008} \end{pmatrix},$$

and

$$b_8^{(2)} = \frac{1}{h^2}\begin{pmatrix} \frac{7525099}{11884320}u_0 & \frac{1664387}{11884320}u_0 & 0 & \dots & 0 & \frac{1664387}{11884320}u_n & \frac{7525099}{11884320}u_n \end{pmatrix}^T.$$

3. Fourier analysis of error

Fourier analysis provides an effective way to analysis the error of difference schemes and is widely used to obtain the dispersion and dissipation errors which quantifies the resolution characteristics of difference approximations [14]. We assume that $u(x)$ is periodic over the domain $[0, L]$, then it can be written as

$$u(x) = \sum_{k=-N/2}^{N/2} \hat{u}_k e^{\frac{2\pi lkx}{L}}, \tag{3.1}$$

where l is imaginary unit and N is the number of mesh points with domain step size $h = L/N$. By defining a scaled wave number $\omega = \frac{2\pi kh}{L} = \frac{2\pi k}{N}$ and a scaled coordinate $s = \frac{x}{h}$, the Fourier modes simplify to $e^{i\omega s}$. The domain of ω is then $[0, \pi]$. The exact first and second derivatives of (3.1) (with respect to s) provide a function with Fourier coefficients

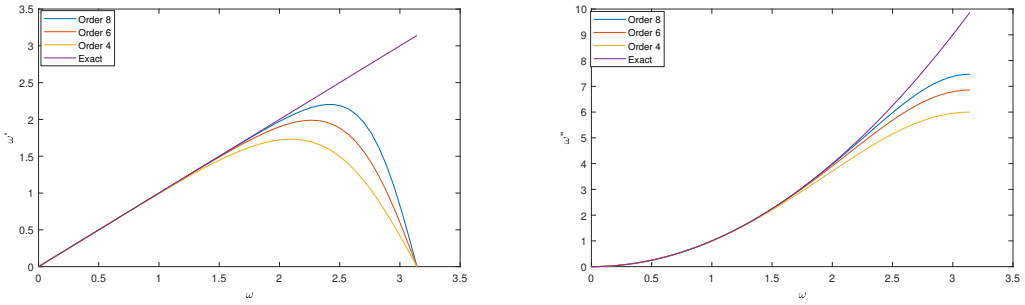
$$\hat{u}_k' = l\omega\hat{u}_k, \quad \hat{u}_k'' = -\omega^2\hat{u}_k, \tag{3.2}$$

respectively. The difference errors of the first and second derivative schemes are obtained by comparing the Fourier coefficients of the derivatives obtained from the differencing schemes

$$(\hat{u}_k')_{fd} = l\omega'\hat{u}_k, \quad (\hat{u}_k'')_{fd} = -\omega''\hat{u}_k,$$

with the exact Fourier coefficients (3.2), where ω' and ω'' represent the modified wavenumbers for tha first and second derivatives, respectively. The finite difference scheme for the first and second derivatives, correspond to functions $\omega'(\omega)$ and $\omega''(\omega)$, respectively. Spectral methods provide $\omega' = \omega$ for $\omega \neq \pi$ ($\omega' = 0$ for $\omega = \pi$) and also $\omega'' = \omega^2$. Hence difference between $\omega'(\omega)$ and ω, and also difference between $\omega''(\omega)$ and ω^2 are two measures of errors.
The compact finite difference schemes for the first derivative is corresponded to

$$\omega'(\omega) = \frac{a\sin(\omega) + (b/2)\sin(2\omega) + (c/3)\sin(3\omega)}{1 + 2\alpha\cos(\omega) + 2\beta\cos(2\omega)},$$



(a) (b)

FIGURE 1. The modified wavenumbers for fourth, sixth and eighth order compact finite difference schemes for first derivative approximation (a) and second derivative approximation (b).

where

$$\begin{cases} \alpha = \frac{1}{4}, \beta = 0, a = \frac{3}{2}, b = 0, c = 0, & \text{for fourth order compact scheme,} \\ \alpha = \frac{1}{3}, \beta = 0, a = \frac{14}{9}, b = \frac{1}{9}, c = 0, & \text{for sixth order compact scheme,} \\ \alpha = \frac{4}{9}, \beta = \frac{1}{36}, a = \frac{40}{27}, b = \frac{25}{54}, c = 0, & \text{for eighth order compact scheme,} \end{cases}$$

Also for the second derivative, we can write

$$\omega''(\omega) = \frac{2a(1-\cos(\omega)) + (b/2)(1-\cos(2\omega)) + (2c/9)(1-\cos(3\omega))}{1 + 2\alpha\cos(\omega) + 2\beta\cos(2\omega)},$$

where

$$\begin{cases} \alpha = \frac{1}{10}, \beta = 0, a = \frac{6}{5}, b = 0, c = 0, & \text{for fourth order compact scheme,} \\ \alpha = \frac{2}{11}, \beta = 0, a = \frac{12}{11}, b = \frac{3}{11}, c = 0, & \text{for sixth order compact scheme,} \\ \alpha = \frac{344}{1179}, \beta = \frac{23}{2358}, a = \frac{320}{393}, b = \frac{310}{393}, c = 0, & \text{for eighth order compact scheme.} \end{cases}$$

Figure 1 shows the modified wavenumbers ω' and ω'' for the fourth, sixth and eighth compact finite difference schemes.

In the following, we use the HOCFD formulas (2.3) to solve the Lane-Emden Equation (1.1) with boundary conditions (1.2).

4. Lane-Emden equations

Consider the generalized Lane-Emden equation

$$u''(x) + p(x)u'(x) + F(u,x) = h(x), \quad a \le x \le b, \tag{4.1}$$

with boundary conditions

$$u(a) = u_a, \quad u(b) = u_b. \tag{4.2}$$

We discretize the interval $[a,b]$ to $(n+1)$ grid points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \tag{4.3}$$

with equal distance $h = (b-a)/n$. We consider the Lane-Emden Equation (4.1) in cases of linear and nonlinear function $F(u,x)$.

TABLE 1. The maximum absolute error and CPU time for Example 4.1.

h	Order 4		Order 6		Order 8	
	Error	CPU time	Error	CPU time	Error	CPU time
$h = \frac{1}{10}$	6.3297e-16	0.002068 s	1.5266e-16	0.002143 s	6.4705e-16	0.002377 s

4.1. **Linear Lane-Emden equations.** Without loss generality, we consider the linear Lane-Emden Equation (4.1) as

$$u''(x) + p(x)u'(x) + g(x)u(x) = h(x), \quad a \le x \le b, \tag{4.4}$$

with known boundary conditions $u(a) = u_a$ and $u(b) = u_b$. Let us U be the vector of function values $u(x)$ in the interior grid points (4.3) as

$$U = \begin{pmatrix} u(x_1), & \ldots, & u(x_{n-1}) \end{pmatrix}^T. \tag{4.5}$$

Then, the Equation (4.4) can be written in interior grid points (4.3) as

$$U'' + diag_{i=1}^{n-1}(p(x_i))U' + diag_{i=1}^{n-1}(g(x_i))U = diag_{i=1}^{n-1}(h(x_i)), \tag{4.6}$$

where

$$diag_{i=1}^{n-1}(f(x_i)) = \begin{pmatrix} f(x_1) & & \\ & \ddots & \\ & & f(x_{n-1}) \end{pmatrix}. \tag{4.7}$$

By using HOCFD formulas (2.3) for the first and second derivatives of U and substituting in (4.6), we have the following linear system

$$M^{(k)}U = R^{(k)}, \tag{4.8}$$

where $M^{(k)}$ and $R^{(k)}$ are the matrices corresponding to approximate solution U of order $k = 4, 6, 8$ as

$$M^{(k)} = \left(A_2^{(k)}\right)^{-1} B_2^{(k)} + diag_{i=1}^{n-1}(p(x_i)) \left(A_1^{(k)}\right)^{-1} B_1^{(k)} + diag_{i=1}^{n-1}(g(x_i)), \tag{4.9}$$

$$R^{(k)} = diag_{i=1}^{n-1}(h(x_i)) - \left(A_2^{(k)}\right)^{-1} b_k^{(2)} - diag_{i=1}^{n-1}(p(x_i)) \left(A_1^{(k)}\right)^{-1} b_k^{(1)}. \tag{4.10}$$

The linear system (4.8) gives an approximate solution of the Lane-Emden Equation (4.4) of orders $k = 4, 6, 8$. Here, we consider some linear Lane-Emden equations from the literature. Through this paper, all numerical experiments where done using MATLAB 2018a on a computer with configuration: Intel(R) Core(TM) i5-5300U @2.30 GHz processor. For the same spatial and time step sizes, the rate of convergence of the proposed method is defined as

$$R_{oc} = \frac{\log(\frac{Error_{new}}{Error_{old}})}{\log(\frac{h_{new}}{h_{old}})}.$$

Example 4.1. *[10, 31, 33] Consider the Lane-Emden Equation (4.4) on $x \in [0, 1]$ with parameters $p(x) = \frac{8}{x}$, $g(x) = x$, $h(x) = -30x + 44x^2 - x^4 + x^5$ and boundary conditions $u_0 = u_1 = 0$. The exact solution of this boundary-value problem is $u(x) = x^4 - x^3$. Table 1 shows the maximum absolute error and CPU times for $h = 1/10$. It can be seen that the obtained solutions by the proposed method are in excellent agreement with the exact solution (up to machine epsilon).*

Example 4.2. *[1, 9, 10, 20–22, 30] We consider the linear Lane-Emden equation*

$$u''(x) + \frac{2}{x}u'(x) + u^m(x) = 0, \quad 0 \le x \le 1, \tag{4.11}$$

with known boundary conditions u_0 and u_1. This problem is modeled in the thermal behavior of a spherical cloud of gas action under the mutual attraction of its molecules and is subject to classical laws of thermodynamics.

For $m = 0$, the Equation (4.11) is a version of (4.4) with parameters $p(x) = \frac{2}{x}$, $g(x) = 0$, $h(x) = -1$. The exact solution for this problem is $u(x) = 1 - \frac{x^2}{6}$. Table 2 gives the maximum absolute errors, CPU times, and the rate of

TABLE 2. The maximum absolute error and CPU time for $m = 0$ for Example 4.2 by the proposed method for given step size h.

h	Order 4		Order 6		Order 8	
	Error	CPU time	Error	CPU time	Error	CPU time
$h = \frac{1}{10}$	4.8850e-15	0.000331 s	3.4417e-15	0.000348 s	8.6597e-15	0.000542 s

TABLE 3. The maximum absolute error and CPU time for $m = 1$ for Example 4.2 by the proposed method with given step sizes h.

h	Order 4			Order 6			Order 8		
	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}
$h = \frac{1}{10}$	1.5540e-07	0.000416 s	–	7.0026e-10	0.000450 s	–	4.6463e-12	0.000496 s	–
$h = \frac{1}{20}$	4.5213e-09	0.000465 s	5.1	6.3736e-12	0.000468 s	6.7	4.6074e-14	0.000469 s	6.6
$h = \frac{1}{40}$	4.3651e-10	0.000489 s	3.3	4.3077e-14	0.000611 s	7.2	4.6629e-15	0.000624 s	3.3

TABLE 4. Comparison of the maximum absolute error for $m = 1$ in Example 4.2 acquired by the proposed the method of order 8 with step size $h = \frac{1}{20}$ with previous works [10, 20, 21].

Proposed method	BOMD [21]	LOMD [20]	LDG [10]
4.6074e-14	5.0e-10	3.0e-07	1e-08

convergence by the proposed method and shows that the obtained solutions are in excellent agreement with the exact solution (up to machine epsilon). As shown in Equation (4.8), the proposed method for solving linear Lane-Emden equations leads to a linear system. But as it can be seen in (4.9), the coefficient matrix $M^{(k)}$ is dependent on functions $p(x)$ and $g(x)$. Hence, for any linear Lane-Emden problem, we have to consider eigenvalues of the corresponding coefficient matrix $M^{(k)}$. For existing the numerical solution in this example, we plot the eigenvalues of coefficient matrix $M^{(k)}$ for orders $k = 4, 6, 8$ in Figure 2 that show the real parts of eigenvalues are negative.

Also, for case $m = 1$, the Equation (4.11) is a homogeneous version of (4.4) on $x \in [0, 1]$ with parameters $p(x) = \frac{2}{x}$, $g(x) = 1$, $h(x) = 0$. The exact solution for this problem is $\frac{sinx}{x}$. We set the initial condition as $u_0 = \lim_{x \longrightarrow 0} \frac{sinx}{x}$. Table 3 shows the maximum absolute errors, CPU times, and the rate of convergence by the proposed method. Moreover, for $m = 1$, comparing between the exact and approximate solutions by the proposed method for $h = \frac{1}{10}$ is shown in Figure 3. In Table 4, the maximum absolute error by the proposed method is compared to existing numerical methods in literature, such as Bernstein's operational matrix of differentiation (BOMD) method from [21], Legendre operational matrix of differentiation (LOMD) method from [20] and the local discontinuous Galerkin (LDG) method from [10].

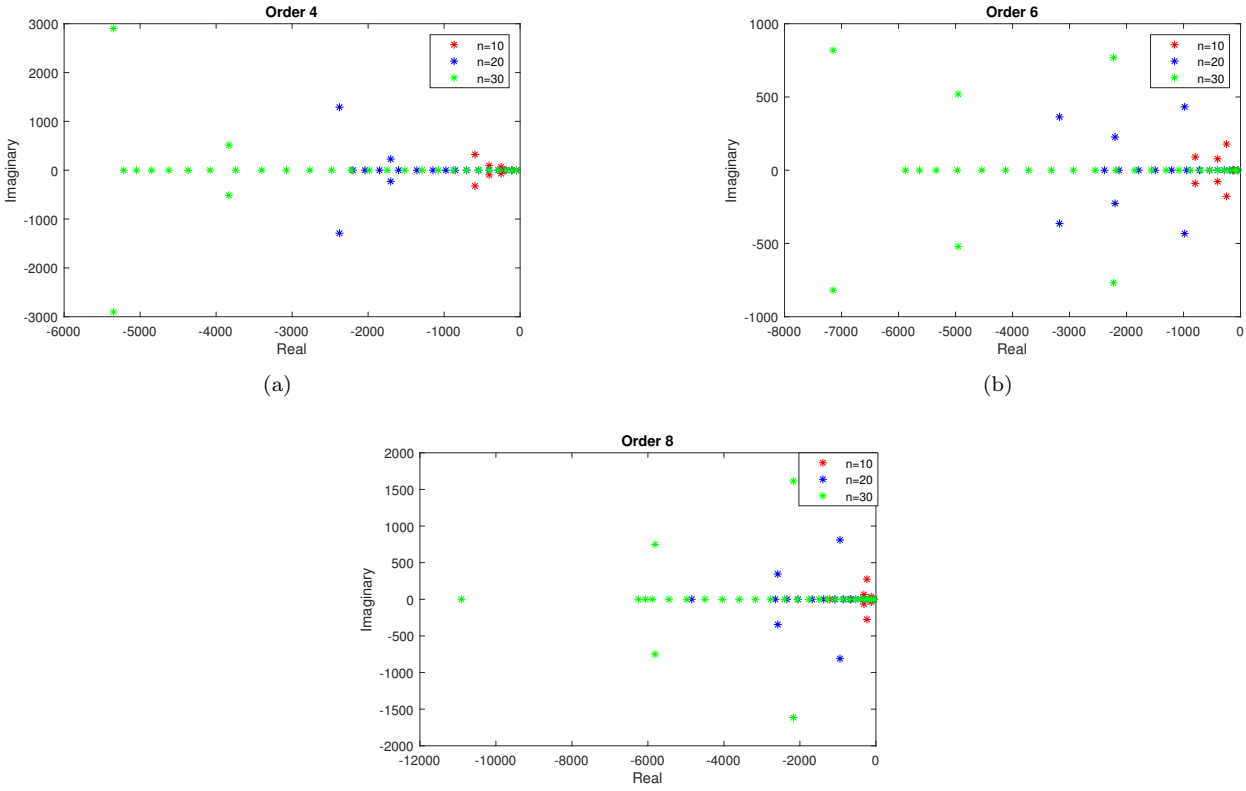
4.2. **Non-Linear Lane-Emden equations.** Consider the following nonlinear Lane-Emden equation

$$u''(x) + p(x)u'(x) + F(u, x) = h(x), \quad a \le x \le b, \tag{4.12}$$

with boundary conditions $u(a) = u_a$ and $u(b) = u_b$. Without lost of generality, we consider $F(u, x) = F(u)$ as

$$F(u) = L(u) + N(u),$$

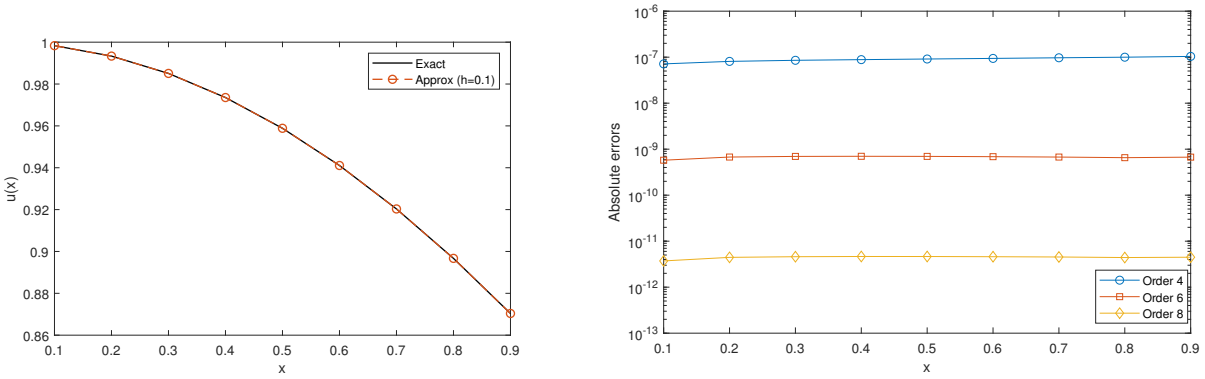
where L and N are linear and nonlinear operators, respectively. The quasilinearization method (QLM) was originally introduced by Bellman and Kalaba [2, 11] as a generalization of the Newton–Raphson method [5] to solve individual or systems of nonlinear ordinary and partial differential equations. It was shown that the difference between the exact solution $u(x)$ and rth iteration $u_r(x)$ of the QLM is decreasing quadratically and the QLM iterations converge uniformly to the exact solution [16]. It is important to stress that in view of the quadratic convergence of the QLM, convergence of two subsequent QLM iterations leads to convergence of the QLM iteration sequence to the exact solution. Also, Once the quasilinear iteration sequence at some interval starts to converge, it will always continue to do so. Unlike an asymptotic perturbation series, the QLM yields the required precision once a successful initial guess generates convergence after a few steps. In order to solve the nonlinear Lane-Emden Equation (4.12), we linearize the nonlinear operator N by QLM. Then, the HOCFD scheme is used to solve the problem. Let us assume that the



(a) (b)

(c)

FIGURE 2. The eigenvalues of coefficient matrix by the proposed method for the case $m = 0$ in Example 4.2.



(a) (b)

FIGURE 3. Comparing the obtained solution by the HOCFD and exact solutions (a) and the corresponding absolute errors (b) for $m = 1$ in Example 2 using $h = \frac{1}{10}$.

TABLE 5. The maximum absolute error and CPU time for Example 4.3.

h	Order 4			Order 6			Order 8		
	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}
$h=\frac{1}{10}$	9.9914e-07	–	0.000487 s	4.5128e-08	0.000503 s	–	9.8868e-09	0.000515 s	–
$h=\frac{1}{20}$	3.3462e-08	0.000548 s	4.9	4.7650e-10	0.000612 s	6.5	6.7806e-11	0.000621 s	7.1
$h=\frac{1}{40}$	3.7025e-09	0.001609 s	3.1	5.1402e-12	0.002987 s	6.5	1.1535e-13	0.002998 s	9.1

TABLE 6. Comparison of the maximum absolute error by the proposed method of order 8 with step size $h=\frac{1}{20}$ for Example 4.3.

Proposed method	ADM [30]	BOMD [21]	LOMD [20]	LDG [10]
4.6074e-14	3.0000e-8	5.0e-10	3.0e-07	1e-08

difference between $u_{r+1}-u_r$ is small, then we can approximate the nonlinear operator N using the linear terms of Taylor series as

$$N(u)\approx N(u_r)+(\frac{\partial N}{\partial u})_r(u_{r+1}-u_r), \tag{4.13}$$

where r and $r+1$ denote previous and current iterations, respectively. Hence, we can write the nonlinear Lane-Emden Equation (4.12) in the current iteration $r+1$ as

$$u''_{r+1}+p(x)u'_{r+1}+g(x)L(u_{r+1})+g(x)N(u_{r+1})=h(x).$$

By (4.13) we have

$$u''_{r+1}+p(x)u'_{r+1}+g(x)L(u_{r+1})+g(x)N(u_r)+g(x)(\frac{\partial N}{\partial u})_r(u_{r+1}-u_r)=h(x),$$

thus

$$u''_{r+1}+p(x)u'_{r+1}+g(x)\left(L(u_{r+1})+(\frac{\partial N}{\partial u})_r u_{r+1}\right)=h(x)-g(x)N(u_r)+g(x)(\frac{\partial N}{\partial u})_r u_r. \tag{4.14}$$

Hence, by substituting the grid points and by using the HOCFD formulas (2.3) in the above equation, the following linear system is derived in each iteration of QLM:

$$M^{(k)}U_{r+1}=R_r^{(k)}(U_r),\quad r=0,1,\dots \tag{4.15}$$

where $M^{(k)}$ and $R^{(k)}$ are the matrices corresponding to approximate solution U of order $k=4,6,8$. Hence, by using an initial vector U_0, we can obtain the solutions of (4.14) for $r=0,1,\dots$. However, the HOCFD scheme with QLM for nonlinear Lane-Emden equation leads to an iterative method, but in numerical experiments, we show that a few iterations is needed to obtain a suitable solution.

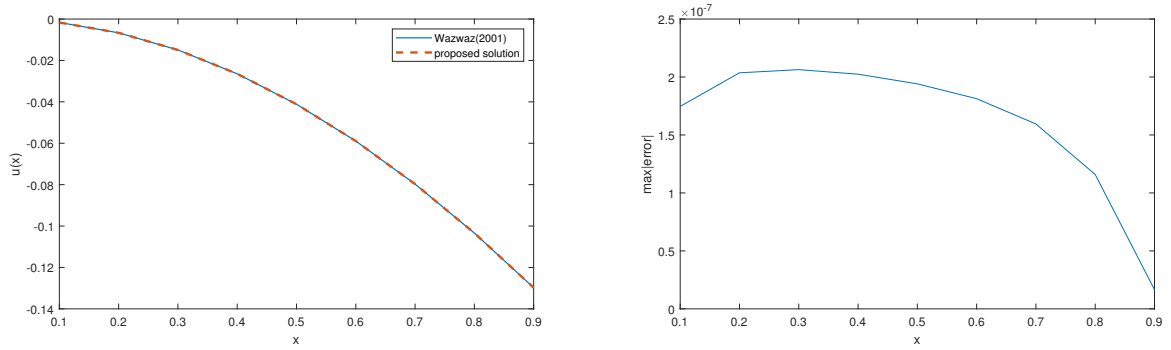
Example 4.3. *[1, 9, 10, 20–22] We consider the nonlinear Lane-Emden equation*

$$u''(x)+\frac{2}{x}u'(x)+u^5(x)=0,\quad 0\le x\le 1, \tag{4.16}$$

with known boundary conditions u_0 and u_1. The exact solution is $\frac{1}{\sqrt{1+\frac{x^2}{3}}}$. Table 5 shows the maximum absolute error and CPU time by the proposed method of orders 4, 6, and 8 for given step size h and iteration numbers $r=5$. In Table 6, the maximum absolute error by the proposed method is compared to Adomian decomposition method (ADM) from [30], BOMD [21], LOMD [20] and LDG [10].

Example 4.4. *We consider the Isothermal gas spheres equation [6, 10, 20, 21, 30]*

$$u''(x)+\frac{2}{x}u'(x)+e^{u(x)}=0,\quad 0\le x\le 1, \tag{4.17}$$



(a) (b)

FIGURE 4. The graph of isothermal gas sphere equation in comparison with [30] (a), and corresponding maximum absolute error (b).

TABLE 7. The maximum absolute error and CPU time for Example 4.5.

h	Order 4			Order 6			Order 8		
	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}
$h = \frac{1}{10}$	3.1187e-04	0.000972 s	–	3.4750e-05	0.002730 s	–	4.5614e-06	0.003461 s	–
$h = \frac{1}{20}$	1.4932e-05	0.002224 s	4.3	5.3850e-07	0.002812 s	6.0	2.3554e-08	0.004372 s	7.5
$h = \frac{1}{40}$	5.8248e-07	0.005492 s	4.6	6.0244e-09	0.006216 s	6.4	7.6953e-11	0.006503 s	8.2

with boundary conditions u_0 and u_1. For this problem, the series solution using the ADM [30] is given as

$$u(x) \approx -\frac{x^2}{6} + \frac{x^4}{5 \times 4!} - \frac{8x^6}{21 \times 6!} + \frac{122x^8}{81 \times 8!} - \frac{61 \times 67x^{10}}{495 \times 10!} + \dots \tag{4.18}$$

The graph of isothermal gas sphere Equation (4.17) by the proposed method of order 4 for $h = \frac{1}{10}$ and $r = 5$ in comparison with the approximate series solution (4.18) is shown in Figure 4(a). Also, Figure 4(b) shows the corresponding absolute error.

Example 4.5. *[23, 26, 34] Consider the nonlinear Lane-Emden equation*

$$u''(x) + \frac{2}{x}u'(x) - 6u(x) = 4u(x)\ln(u(x)), \quad 0 \le x \le 1, \tag{4.19}$$

with known boundary conditions $u_0 = 1$ and $u_1 = e$, which has the exact solution $u(x) = e^{x^2}$. Table 7 shows the maximum absolute error, CPU time, and the rate of convergence by the proposed method of orders 4, 6, and 8 for given step size h and iteration numbers $r = 5$.

Example 4.6. *This example corresponds to the following Lane-Emden equation [13, 23, 30]*

$$u''(x) + \frac{2}{x}u'(x) + \sinh(u(x)) = 0, \quad 0 \le x \le 1, \tag{4.20}$$

with boundary conditions u_0 and u_1. For this problem, the series solution using the ADM [30] is given as

$$\begin{aligned} u(x) \approx\ & 1 - \frac{(e^2-1)x^2}{12e} + \frac{(e^4-1)x^4}{480e^2} - \frac{(2e^6+3e^2-3e^4-2)x^6}{30240e^3} \\ & + \frac{(61e^8-104e^6+104e^2-61)x^8}{26127360e^4}. \end{aligned} \tag{4.21}$$

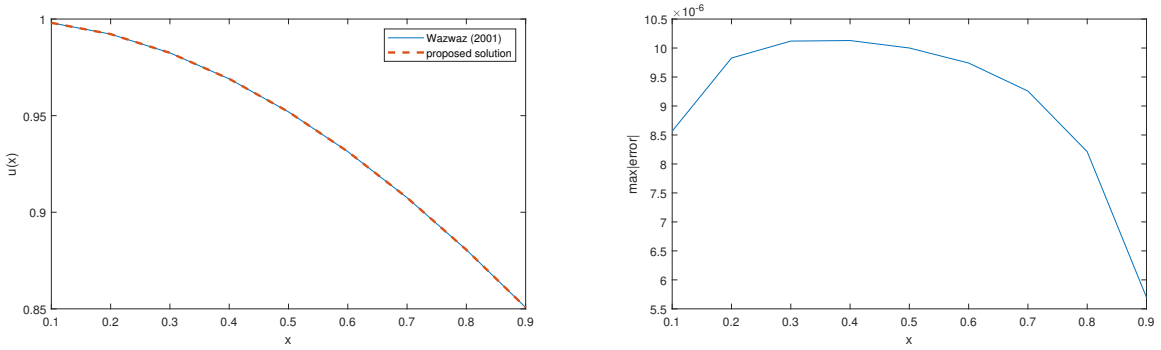


FIGURE 5. Comparison between the solutions obtained by the proposed method and ADM [30] (a), and corresponding maximum absolute error (b).

TABLE 8. The maximum absolute error and CPU time for Example 5.1.

h	For	Order 4		Order 6		Order 8	
		Error	CPU time	Error	CPU time	Error	CPU time
$h = \frac{1}{10}$	$u(x)$	3.3079e-04	0.000822 s	3.6939e-05	0.001275 s	4.8520e-06	0.001279 s
	$v(x)$	1.1231e-05		1.4332e-06		1.8836e-07	
$h = \frac{1}{20}$	$u(x)$	1.5354e-05	0.000773 s	5.5503e-07	0.001465 s	2.4281e-08	0.001449 s
	$v(x)$	4.6211e-07		1.9751e-08		8.5071e-10	
$h = \frac{1}{40}$	$u(x)$	5.8976e-07	0.001556 s	6.1161e-09	0.002700 s	7.8109e-11	0.002973 s
	$v(x)$	1.3063e-08		2.1111e-10		2.6114e-12	

The graph of the approximate solution of Equation (4.20) *by the proposed method of order 4 for* $h = \frac{1}{10}$ *and* $r = 5$ *in comparison with the approximate series solution* (4.21) *is shown in Figure 5(a) . Also, Figure 5(b) shows the corresponding absolute error.*

In the next section, we will study the linear and nonlinear systems of Lane-Emden equations.

5. System of Lane-Emden equations

In this section, we give some examples for systems of Lane–Emden equations (linear and nonlinear) that show the proposed method can be easily extended to solve the system of Lane-Emden equations (linear and nonlinear).

Example 5.1. *We consider the non-homogeneous linear systems of Lane–Emden equations, which describes polytropes in hydrostatic equilibrium as simple models of a star [19]*

$$u'' + \frac{2}{x}u' - (4x^2 + 6)u + v = x^4 - x^3, \tag{5.1}$$

$$v'' + \frac{8}{x}v' + xv + u = e^{x^2} + x^5 - x^4 + 44x^2 - 30x, \tag{5.2}$$

where $x, y \in [0, 1]$ *and* u_0, u_1, v_0 *and* v_1 *are known.*

The exact solutions are $u(x) = e^{x^2}$ and $v(x) = x^4 - x^3$. Table 8 shows the maximum absolute error with the corresponding CPU time by the proposed method for given step sizes of orders 4, 6, and 8. Also, comparisons between exact and approximate solutions for $u(x)$ and $v(x)$ are shown in Tables 9 and 10, respectively.

TABLE 9. Comparison between exact and approximate solution $u(x)$ with $h = \frac{1}{40}$ for Example 5.1.

x	Order 4	Order 6	Order 8	Exact
0.1	1.010050236235647	1.010050169393248	1.010050167112353	1.010050167084168
0.2	1.040810851077961	1.040810776633562	1.040810774222218	1.040810774192388
0.3	1.094174371892875	1.094174286285983	1.094174283736813	1.094174283705210
0.4	1.173510976475032	1.173510873759354	1.173510871025806	1.173510870991810
0.5	1.284025548102475	1.284025419706214	1.284025416724985	1.284025416687741
0.6	1.433329584541152	1.433329417910964	1.433329414601939	1.433329414560340
0.7	1.632316447425718	1.632316223740882	1.632316220002771	1.632316219955379
0.8	1.896481193204513	1.896480883656438	1.896480879360031	1.896480879304952
0.9	2.247908432063972	2.247907991762387	2.247907986741762	2.247907986676472

TABLE 10. Comparison between exact and aproximate solution $v(x)$ with $h = \frac{1}{40}$ for Example 5.1.

x	Order 4	Order 6	Order 8	Exact
0.1	-0.000899987752263	-0.000899999797720	-0.000899999997460	-0.0009
0.2	-0.006399987798657	-0.006399999801294	-0.006399999997508	-0.0064
0.3	-0.018899988027302	-0.018899999808291	-0.018899999997593	-0.0189
0.4	-0.038399988396586	-0.038399999818686	-0.038399999997721	-0.0384
0.5	-0.062499988959514	-0.062499999832999	-0.062499999997897	-0.0625
0.6	-0.086399989800275	-0.086399999851960	-0.086399999998131	-0.0864
0.7	-0.102899991050239	-0.102899999876557	-0.102899999998437	-0.1029
0.8	-0.102399992915670	-0.102399999908112	-0.102399999998832	-0.1024
0.9	-0.072899995723731	-0.072899999948388	-0.072899999999340	-0.0729

TABLE 11. The maximum absolute error and CPU time for Example 5.2.

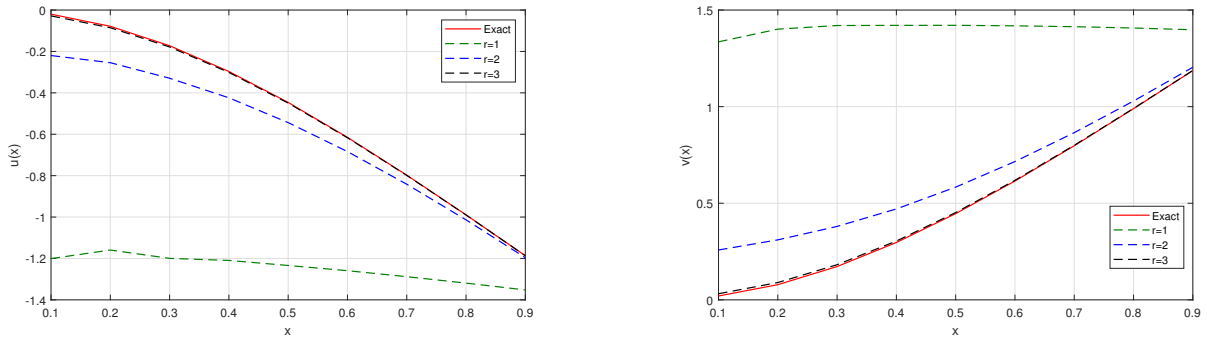
h	For	Order 4		Order 6		Order 8	
		Error	CPU time	Error	CPU time	Error	CPU time
$h = \frac{1}{10}$	$u(x)$	4.4490e-05	0.001405 s	1.414e-05	0.001464 s	7.5261e-06	0.001657 s
	$v(x)$	4.4848e-05		1.4955e-05		7.84283-06	
$h = \frac{1}{20}$	$u(x)$	2.4205e-06	0.002059 s	7.1466e-08	0.001638 s	1.0758e-09	0.001777 s
	$v(x)$	2.6123e-06		7.4963e-08		1.1073e-09	
$h = \frac{1}{40}$	$u(x)$	1.2389e-07	0.005400 s	3.5722e-10	0.005563 s	4.2613e-12	0.005691 s
	$v(x)$	1.3539e-07		3.9147e-10		3.7076e-12	

Example 5.2. *Consider the nonlinear system of Lane–Emden equations [32]*

$$u'' + \frac{5}{x}u' + 8(e^u + 2e^{-\frac{v}{2}}) = 0, \tag{5.3}$$

$$v'' + \frac{3}{x}v' - 8(e^{-v} + e^{\frac{u}{2}}) = 0, \tag{5.4}$$

where $x, y \in [0, 1]$ and with known boundary conditions. The exact solutions are $u(x) = -2ln(1 + x^2)$ and $v(x) = 2ln(1 + x^2)$. Table 11 shows the maximum absolute error with corresponding CPU time by the proposed method for $r = 6$ and given step sizes. Also, Figure 6 shows the convergence of the proposed method with an increase in the iterations r of QLM for $h = 1/10$ for order 4. Moreover, a comparison between the exact and approximate solutions for $r = 6$ and $h = \frac{1}{40}$ for $u(x)$ and $v(x)$ are shown in Tables 12 and 13, respectively.



(a) (b)

FIGURE 6. Convergence of the proposed method with increase in the QLM iterations r for Example 5.2.

TABLE 12. Comparison between exact and approximate solution $u(x)$ with $h = \frac{1}{40}$ for Example 5.2.

x	Order 4	Order 6	Order 8	Exact
0.1	-0.019900543321239	-0.019900662050199	-0.019900661709269	-0.019900661706336
0.2	-0.078441325283406	-0.078441426659778	-0.078441426309448	-0.078441426306563
0.3	-0.172355314327941	-0.172355392838920	-0.172355392484839	-0.172355392482105
0.4	-0.296839954615990	-0.296840010580882	-0.296840010239055	-0.296840010236547
0.5	-0.446287065352555	-0.446287102944218	-0.446287102630656	-0.446287102628420
0.6	-0.614969375024299	-0.614969399773190	-0.614969399497878	-0.614969399495921
0.7	-0.797552223180955	-0.797552240150563	-0.797552239916426	-0.797552239914736
0.8	-0.989392470901493	-0.989392483868766	-0.989392483673663	-0.989392483672214
0.9	-1.186653679338796	-1.186653690717454	-1.186653690556702	-1.186653690555469

TABLE 13. Comparison between exact and approximate solution $v(x)$ with $h = \frac{1}{40}$ for Example 5.2.

x	Order 4	Order 6	Order 8	Exact
0.1	0.019900529460760	0.019900662085841	0.019900661709699	0.019900661706336
0.2	0.078441311559095	0.078441426696000	0.078441426309791	0.078441426306563
0.3	0.172355301438630	0.172355392872056	0.172355392485144	0.172355392482105
0.4	0.296839943070497	0.296840010610382	0.296840010239320	0.296840010236547
0.5	0.446287055568406	0.446287102969845	0.446287102630880	0.446287102628420
0.6	0.614969367273717	0.614969399794541	0.614969399498056	0.614969399495921
0.7	0.797552217566634	0.797552240167020	0.797552239916556	0.797552239914736
0.8	0.989392467377548	0.989392483879650	0.989392483673740	0.989392483672214
0.9	1.186653677756701	1.186653690722175	1.186653690556727	1.186653690555469

Example 5.3. *Consider the nonlinear systems of Lane–Emden equations [32]*

$$u'' + \frac{8}{x}u' + (18u - 4ln(v)) = 0, \tag{5.5}$$

$$v'' + \frac{4}{x}v' + (4vln(u) - 10v) = 0, \tag{5.6}$$

where $x, y \in [0, 1]$ and with known boundary conditions. The exact solutions are $u(x) = e^{-x^2}$ and $v(x) = e^{x^2}$. Table 14 shows the maximum absolute error with corresponding CPU time by the proposed method for $r = 6$ and given step sizes.

TABLE 14. The maximum absolute error and CPU time for Example 5.3.

h	For	Order 4		Order 6		Order 8	
		Error	CPU time	Error	CPU time	Error	CPU time
$h = \frac{1}{10}$	$u(x)$	5.5210e-05	0.001422 s	1.4777e-06	0.001508 s	1.2227e-06	0.001592 s
	$v(x)$	3.4417e-04		3.9047e-05		5.0273e-06	
$h = \frac{1}{20}$	$u(x)$	6.4219e-07	0.002168 s	6.8614e-08	0.003951 s	1.8672e-09	0.004517 s
	$v(x)$	1.5732e-05		5.6870e-07		2.4940e-08	
$h = \frac{1}{40}$	$u(x)$	3.0090e-08	0.007460 s	7.7963e-010	0.007847 s	6.7902e-012	0.007945 s
	$v(x)$	5.9655e-07		6.1932e-09		7.9213e-11	

6. Conclusions

In this paper, we have considered an efficient high-order compact finite difference (HOCFD) scheme for solving generalized Lane-Emden and system of Lane-Emden equations. For nonlinear types, it is shown that a combined quasilinearization and HOCFD scheme gives excellent results while a few quasilinear iterations is needed. Some numerical examples have been provided, and the obtained results of the proposed method have been compared with previous well-established methods. The numerical experiments with low CPU time show the accuracy and efficiency of the proposed method.

References

[1] B. Azarnavid, F. Parvaneh, and S. Abbasbandy, *Picard-reproducing kernel Hilbert space method for solving generalized singular nonlinear Lane-Emden type equations*, Mathematical Modelling and Analysis, *20*(6) (2015), 754–767.

[2] R. E. Bellman and R. K. Kalaba, *Nonlinear Boundary-Value Problems*, American Elsevier Publishing Co., Inc., New York, 1965.

[3] A. H. Bhrawy and A. S. Alofi, *A Jacobi–Gauss collocation method for solving nonlinear Lane–Emden type equations*, Communications in Nonlinear Science and Numerical Simulation, *17*(1) (2012), 62–70.

[4] M. Bisheh Niasar, *A Computational Method for Solving the Lane-Emden Initial Value Problems*, Computational Methods for Differential Equations, *8*(4) (2020), 673-684.

[5] S. D. Conte and C. De Boor, *Elementary Numerical Analysis: an Algorithmic Approach*, Society for Industrial and Applied Mathematics, 2017.

[6] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, US Government Printing Office, 1961.

[7] R. Doostaki, M. M. Hosseini, and A. Salemi, *A new simultaneously compact finite difference scheme for high-dimensional time-dependent PDEs*, Mathematics and Computers in Simulation, *212* (2023), 504-523.

[8] Y. Q. Hasan and L. M. Zhu, *Solving singular boundary value problems of higher-order ordinary differential equations by modified Adomian decomposition method*, Communications in Nonlinear Science and Numerical Simulation, *14*(6) (2009), 2592–2596.

[9] G. P. Horedt, *Polytropes: applications in astrophysics and related fields*, Springer Science and Business Media, 2004.

[10] M. Izadi, *A discontinuous finite element approximation to singular Lane-Emden type equations*, Applied Mathematics and Computation, *401* (2021), 126115.

[11] R. E. Kalaba, *On Nonlinear Differential Equations, the Maximum Operation, and Monotone Convergence*, New York University, 1957.

[12] A. R. Kanth and K. Aruna, *Solution of singular two-point boundary value problems using differential transformation method*, Physics Letters A, *372*(26) (2008), 4671–4673.

[13] M. Lakestani and M. Dehghan, *Four techniques based on the B-spline expansion and the collocation approach for the numerical solution of the Lane-Emden equation*, Mathematical Methods in the Applied Sciences, *36*(16) (2013), 2243–2253.

[14] S. K. Lele, *Compact finite difference schemes with spectral-like resolution*, Journal of computational physics, *103*(1) (1992), 16–42.
[15] S. Mall and S. Chakraverty, *Chebyshev neural network based model for solving Lane-Emden type equations*, Applied Mathematics and Computation, *247* (2014), 100–114.
[16] V. Mandelzweig and F. Tabakin, *Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs*, Computer Physics Communications, *141*(2) (2001), 268–281.
[17] M. P. Mehra and K. S. Patel, *Algorithm 986: a suite of compact finite difference schemes*, ACM Transactions on Mathematical Software, *44*(2) (2017), 1–31.
[18] R. Mohammadzadeh, M. Lakestani, and M. Dehghan, *Collocation method for the numerical solutions of Lane–Emden type equations using cubic Hermite spline functions*, Mathematical Methods in the Applied Sciences, *37*(9) (2014), 1303-1717.
[19] Y. Öztürk, *solution for the system of Lane–Emden type equations using Chebyshev polynomials*, Mathematics, *5*(10) (2018), 181.
[20] R. K. Pandey, N. Kumar, and A. Bhardwaj, *Solution of Lane–Emden type equations using Legendre operational matrix of differentiation*, Applied Mathematics and Computation, *218*14 (2012), 7629–7637.
[21] R. K. Pandey and N. Kumar, *Solution of Lane–Emden type equations using Bernstein operational matrix of differentiation*, New Astronomy, *17*(3) (2012), 303–308.
[22] K. Parand and M. Delkhosh, *An effective numerical method for solving the nonlinear singular Lane-Emden type equations of various orders*, Jurnal Teknologi, *79*(1) (2017).
[23] K. Parand, M. Dehghan, A. Rezaei, and S. Ghaderi, *An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method*, Computer Physics Communications, *181*(6) (2010), 1096–1108.
[24] K. Parand, M. Shahini, and M. Dehghan, *Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane-Emden type*, Journal of Computational Physics, *228*(23) (2009), 8830–8840.
[25] J. Ramos, *Series approach to the Lane-Emden equation and comparison with the homotopy perturbation method*, Chaos, Solitons & Fractals, *38*(2) (2008), 400–408.
[26] J. I. Ramos, *Linearization techniques for singular initial-value problems of ordinary differential equations*, Applied Mathematics and Computation, *161*(2) (2005), 525–542.
[27] P. Roul, K. Thula, and V. P. Goura, *An optimal sixth-order quartic B-spline collocation method for solving Bratu-type and Lane-Emden type problems*, Mathematical Methods in the Applied Sciences, *42*(8) (2019), 2613–2630.
[28] S. Shiralashetti and S. Kumbinarasaiah, *Theoretical study on continuous polynomial wavelet bases through wavelet series collocation method for nonlinear Lane-Emden type equations*, Applied Mathematics and computation, *315* (2017), 591–602.
[29] O. P. Singh, R. K. Pandey, and V. K. Singh, *An analytic algorithm of Lane-Emden type equations arising in astrophysics using modified homotopy analysis method*, Computer Physics Communications, *180*(7) (2009), 1116–1124.
[30] A. M. Wazwaz, *A new algorithm for solving differential equations of Lane-Emden type*, Applied mathematics and computation, *118*(2-3) (2001), 287–310.
[31] A. M. Wazwaz, *A new method for solving singular initial value problems in the second-order ordinary differential equations*, Applied Mathematics and computation, *128*(1) (2002), 45–57.
[32] A. M. Wazwaz, R. Rach, and J. S. Duan, *A study on the systems of the Volterra integral forms of the Lane-Emden equations by the Adomian decomposition method*, Mathematical Methods in the Applied Sciences, *37*(1) (2014), 10–19.
[33] S. A. Yousefi, *Legendre wavelets method for solving differential equations of Lane-Emden type*, Applied Mathematics and Computation, *181*(2) (2006), 1417–1422.
[34] A. Yildirim, T. Öziş, *Solutions of singular IVPs of Lane-Emden type by the variational iteration method*, Nonlinear Analysis: Theory, Methods & Applications, *70*(6) (2009), 2480–2484.
[35] A. Zamiri, A. Borhanifar, and A. Ghannadiasl, *Laguerre collocation method for solving Lane-Emden type equations*, Computational Methods for Differential Equations, *9*(4) (2021), 1176-1197.