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Application of a new method for nonlinear partial differential equations of fractional order arising in fluid mechanics

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Abstract

In this work, we established some exact solutions for the (2+1)-dimensional Zakharov-Kuznetsov, KdV and K(2,2) equations which are considered based on the improved Exp-function method, by utilizing Maple software. We use the fractional derivatives with fractional complex transform. We obtained new periodic solitary wave solutions. The obtained solutions include three classes of soliton wave solutions in terms of hyperbolic function, trigonometric function and rational function solutions. The obtained solutions and the exact solutions are shown graphically, highlighting the effects of non-linearity. Many other such types of nonlinear equations arising in fluid dynamics and nonlinear phenomena

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1. Introduction

Many nonlinear physical phenomena arise in various fields of engineering and science such as fluid dynamics, nuclear reactor dynamics, plasma physics, biology, optical fibres and solid state physics. To describe these complex physical phenomena, nonlinear differential equations play a significant role. Therefore, obtaining the solutions of these nonlinear equations is a topic of great interest in the study of many fields of science. To better understand the working of the physical problem, mathematical model came into picture in the form of nonlinear PDEs. The solutions of partial differential equations give the detailed summary about the nature of phenomena involved. Many numerical and analytical methods have been derived to deal with these kind of scientific problems. We need to adopt an effective and powerful method to investigate such type of mathematical model which gives the solutions upholding to physical reality. In most of the analytic techniques, linearization of the system is the main topic to focus on, and also, it is assumed that the nonlinearities are relatively insignificant. Sometimes, these assumptions made a strong affect on the solutions in respect to the real physics of the phenomena involved. Thus, finding the solutions of nonlinear ODEs and PDEs is still a significant problem. For this, we need new techniques to develop analytic and exact solutions.
In the past two decades, fractional calculus theory gained a great attention and popularity in various fields of science and engineering due to its demonstrated applications. These contributions to the fields of science and engineering are based on the mathematical analysis. It covers the widely known classical fields such as Abel's integral equation and viscoelasticity. Also, including the analysis of feedback amplifiers, fractional-order Chua-Hartley systems, electrode-electrolyte interface models, fractional-order models of neurons, electric conductance of biological systems, generalized voltage dividers, fitting of experimental data, capacitor theory and the fields of special functions [1–6].

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Several robust methods have been used to solve FDEs, fractional integro-differential equations and dynamic systems containing fractional derivatives. Some of the most important methods are the Adomian's decomposition method [7–9], the exp-function method [10], the He's the variational iteration method [11, 12], the fractional sub-equation method [13], the first integral method [14], the homotopy analysis method [15–17], the (G'/G)-expansion method [18], the generalized tanh-coth method [19], the $\tan(\phi/2)$-expansion method [20], the homotopy perturbation method [21, 22], spectral methods [23], the transform methods [24], and other methods [25–30].
A substantial amount of research work has been directed for the study of the nonlinear fractional Zakharov-Kuznetsov, KdV and K(2,2) equations given by

$$\frac{\partial^\alpha}{\partial t^\alpha}u+\frac{\partial}{\partial x}u^2+\frac{1}{8}\frac{\partial^3}{\partial x^3}u^2+\frac{1}{8}\frac{\partial^3}{\partial x\partial y^2}u^2=0, \tag{1.1}$$

and

$$\frac{\partial^\alpha}{\partial t^\alpha}u+\frac{\partial}{\partial x}u^3+2\frac{\partial^3}{\partial x^3}u^3+2\frac{\partial^3}{\partial x\partial y^2}u^3=0, \tag{1.2}$$

$$\frac{\partial^\alpha}{\partial t^\alpha}u-3\frac{\partial}{\partial x}u^2+\frac{\partial^3}{\partial x^3}u=0, \tag{1.3}$$

$$\frac{\partial^\alpha}{\partial t^\alpha}u+\frac{\partial}{\partial x}u^2+\frac{\partial^3}{\partial x^3}u^2=0. \tag{1.4}$$

The ZK equation, first obtained as a description of weakly nonlinear ion-acoustic modes in a strongly magnetized plasma, is of particular interest as it is the simplest equation that admits cylindrical and spherical solitary wave solutions in addition to the planar KdV soliton solutions [31]. Another powerful analytical method is called the Exp-function method (EFM), which was first presented by He [32]. The EFM has successfully been applied to many situations. For example, He [32] solved the nonlinear wave equations via the EFM. Abdou [33] solved generalized solitonary and periodic solutions for nonlinear partial differential equations by the EFM. For further information refer to vigorous references therein ([34–40]).
We will use the Jumarie's modified Riemann-Liouville derivative [45] of order α where is defined by the following expression:

$$D_t^\alpha u(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\int_0^t\left(t-\tau\right)^{-\alpha}\left(u(\tau)-u(0)\right)d\tau, & if \quad 0<\alpha\le 1,\\ \\ \left[u^{(n)}(t)\right]^{(\alpha-n)}, & if \quad n\le\alpha<n+1, n\ge 1.\end{cases} \tag{1.5}$$

The properties of the modified Riemann-Liouville derivative are given as:

$$(1) D^\alpha[f(t)g(t)]=f(t)D^\alpha g(t)+g(t)D^\alpha f(t),$$
$$(2) D^\alpha[f(g(t))]=f'_g(g(t))D^\alpha g(t),$$
$$(3) D^\alpha[f(g(t))]=D_g^\alpha f(g(t))[g'(t)]^\alpha,$$
$$(4) D_t^\alpha t^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(1+\alpha-\gamma)}t^{\gamma-\alpha},\ \gamma>0,$$

where Γ denotes the Gamma function.
Authors of [46] explained the generalized fifth-order KdV like equation with prime number $p=3$ via a generalized bilinear differential operator. N-lump was invstigated to the variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation [47]. Applications of $tan(\phi/2)$-expansion method for the Biswas-Milovic equation [48], the Gerdjikov-Ivanov model [49], the Kundu-Eckhaus equation [50] and the fifth-order integrable equations [51] were studied. Lump solutions were analyzed to the fractional generalized CBS-BK equation [52] and the (3+1)-D Burger system [53]. The approximations of one-dimensional hyperbolic equation with non-local integral conditions were constructed by reduced differential transform method [54]. The generalized Hirota bilinear strategy by the number prime was used to the

(2+1)-dimensional generalized fifth-order KdV like equation [55]. The traveling wave solutions and analytical treatment of the simplified MCH equation and the combined KdV-mKdV equations were studied [56].
Our objective here is to find exact solutions of some fractional nonlinear partial differential equations (FNLPDEs) under consideration the improved EFM for obtaining the new periodic solitary wave solutions. Discussion about the Improved Exp-function Method is given. Application of the improved EFM on FNLPDEs are investigated and derived exact solutions. In the continuation, we will present the graphical illustrations of some solutions of the aforementioned models. After that, we will deal with the investigation of solutions and we will end by a conclusion.

2. Improved Exp-function Method

To illustrate the basic idea of the improved EFM, we take the following nonlinear fractional partial differential equation in the form:

$$\mathcal{N}(\mathrm{u}, \mathrm{u_x}, \mathrm{u_y}, \mathrm{u_{xx}}, \mathrm{u_{yy}}, ..., \mathrm{D_t^\alpha u}, \mathrm{D_x^\alpha u}, \mathrm{D_{xx}^\alpha u}, ...) = 0, \qquad 0 < \alpha \leq 1. \tag{2.1}$$

Using a transformation

$$u(x, y, t) = u(\eta), \qquad \eta = kx + my + \frac{nt^\alpha}{\Gamma(\alpha+1)}, \tag{2.2}$$

where k, m and n are constants to be determined later, we can rewrite equation Eq. (2.1) in the following nonlinear ODE

$$\mathcal{M}(u, ku', mu', k^2u'', m^2u'', ..., nu', ...) = 0, \tag{2.3}$$

where the prime denotes derivative with respect to η. If possible, integrate Eq. (2.3) term by term one or more times. This yields constants of integration. For simplicity, the integration constants can be set to zero. Based on the to modified Exp-function method, the final solution can be presented as

$$\mathrm{u}(\eta) = \frac{\sum_{\mathrm{n}=1}^{2\mathrm{M}} \mathrm{a_n} \exp(\mathrm{n}\eta)}{\sum_{\mathrm{n}=1}^{2\mathrm{M}} \mathrm{b_n} \exp(\mathrm{n}\eta)} + \frac{\sum_{\mathrm{n}=1}^{2\mathrm{M}} \mathrm{a_{-n}} \exp(-\mathrm{n}\eta)}{\sum_{\mathrm{n}=1}^{2\mathrm{M}} \mathrm{b_{-n}} \exp(-\mathrm{n}\eta)}, \tag{2.4}$$

where M is integers which are unknown to be determine, a_n, b_n, a_{-n} and b_{-n} are unfound constants. To determine the value of M, we balance the linear term of highest order of Eq. (2.3) with the highest order nonlinear term. Plugging (2.4) into the Eq. (2.3), equating the coefficients of each power to zero gives the system of the algebraic equations for a_n, b_n, a_{-n} and b_{-n}, then solve the system to determine these constants.

3. Test Problems

In this section, we offer several examples to demonstrate the applicability of improved EFM to solve FNLPDEs.

3.1. The fractional ZK(2,2,2) equation.

Consider the (2+1)-dimensional ZK(2,2,2) equation of fractional order as follows [41]

$$\frac{\partial^\alpha}{\partial t^\alpha}u + \frac{\partial}{\partial x}u^2 + \frac{1}{8}\frac{\partial^3}{\partial x^3}u^2 + \frac{1}{8}\frac{\partial^3}{\partial x \partial y^2}u^2 = 0, \tag{3.1}$$

by utilizing the transformation $\eta = kx + my + \frac{nt^\alpha}{\Gamma(\alpha+1)}$, Eq. (3.1) is reduced to an ODE as

$$nu' + k(u^2)' + \frac{1}{8}k^3(u^2)''' + \frac{1}{8}km^2(u^2)''' = 0. \tag{3.2}$$

Integrating Eq. (3.2) once and setting the constant of integration equal to zero, results in

$$nu + ku^2 + \frac{1}{8}(k^3 + km^2)(u^2)'' = 0. \tag{3.3}$$

Balancing the $(u^2)''$ and u by employing the homogenous principle, we get

$$2M + 2 = M, \qquad \Rightarrow M = -2. \tag{3.4}$$

To get a closed form solution, we use the transformation

$$u(\eta) = v(\eta)^{-2}. \tag{3.5}$$

Substituting (3.5) into Eq. (3.3), we get

$$kv^2 + nv^4 + \frac{1}{2}(k^3 + km^2)(5v'^2 - vv'') = 0. \tag{3.6}$$

Balancing the vv'' and v^4, we get

$$2M + 2 = 4M, \qquad \Rightarrow M = 1. \tag{3.7}$$

Then the exact solution will be as

$$v(\eta) = \frac{a_1 \exp(\eta) + a_2 \exp(2\eta)}{b_1 \exp(\eta) + b_2 \exp(2\eta)} + \frac{a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)}. \tag{3.8}$$

For simplicity, we set $a_2 = b_2 = 1$ and $a_{-2} = b_{-2} = 1$, then (3.8) reduces to

$$v(\eta) = \frac{a_1 + \exp(\eta)}{b_1 + \exp(\eta)} + \frac{a_{-1} + \exp(-\eta)}{b_{-1} + \exp(-\eta)}. \tag{3.9}$$

Inserting (3.9) in to Eq. (3.6), we obtain

$$\left((\mathrm{b}_1 + \exp(\eta))^4 (\mathrm{b}_{-1} + \exp(-\eta))^4\right)^{-1} \sum_{\mathrm{n}=-4}^{4} \mathrm{C_n} \exp(\mathrm{n}\xi) = 0, \tag{3.10}$$

where $\mathrm{C_n}(-4 \leq \mathrm{n} \leq 4)$ are polynomial statements in terms of $\mathrm{a}_1, \mathrm{a}_{-1}, \mathrm{b}_1, \mathrm{b}_{-1}, \mathrm{k}, \mathrm{n}$ and m. Hence, solving the resulting system $\mathrm{C_n} = 0(-4 \leq \mathrm{n} \leq 4)$ simultaneously, we acquire the below set of parameters of solutions

Set I:

$$a_1 = -b_1, \quad a_{-1} = \frac{1}{b_1}, \quad b_{-1} = -\frac{1}{b_1}, \quad b_1 = b_1, \quad k = \sqrt{\frac{1+2m^2}{2}}i, \quad n = \frac{3}{16}k, \quad m = m. \tag{3.11}$$

We, therefore, gained the following generalized solitary solution

$$u_1(\eta) = \left(\frac{4b_1 e^{\eta}}{e^{2\eta} - b_1^2}\right)^{-2}, \qquad \eta = \sqrt{\frac{1+2m^2}{2}}ix + my + \frac{3}{16}\sqrt{\frac{1+2m^2}{2}}i\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.12}$$

If we choose $b_1 = 1$, then Eq. (3.12) get to

$$u_1(x,t) = \frac{1}{4}sinh^2\left(\sqrt{\frac{1+2m^2}{2}}ix + my + \frac{3}{16}\sqrt{\frac{1+2m^2}{2}}i\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right). \tag{3.13}$$

3.2. **The fractional ZK(3,3,3) equation.** As second example, assume the ZK(3,3,3) equation of fractional order [41] be as

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u + \frac{\partial}{\partial x}u^3 + 2\frac{\partial^3}{\partial x^3}u^3 + 2\frac{\partial^3}{\partial x \partial y^2}u^3 = 0, \tag{3.14}$$

by using the transformation $\eta = kx + my + \frac{nt^{\alpha}}{\Gamma(\alpha+1)}$, Eq. (3.14) is reduced to an ODE as

$$nu' + k(u^3)' + 2k^3(u^3)''' + 2km^2(u^3)''' = 0. \tag{3.15}$$

Integrating Eq. (3.15) once and setting the constant of integration equal to zero, results in

$$nu + ku^3 + 2(k^3 + km^2)(u^3)'' = 0. \tag{3.16}$$

Balancing the $(u^3)''$ and u, we achieve to $M = -1$. To obtain an exact solution, we get

$$u(\eta) = v(\eta)^{-1}. \tag{3.17}$$

Plugging (3.17) into Eq. (3.16), we obtain

$$kv^2 + nv^4 + 6(k^3 + km^2)(4v'^2 - vv'') = 0. \tag{3.18}$$

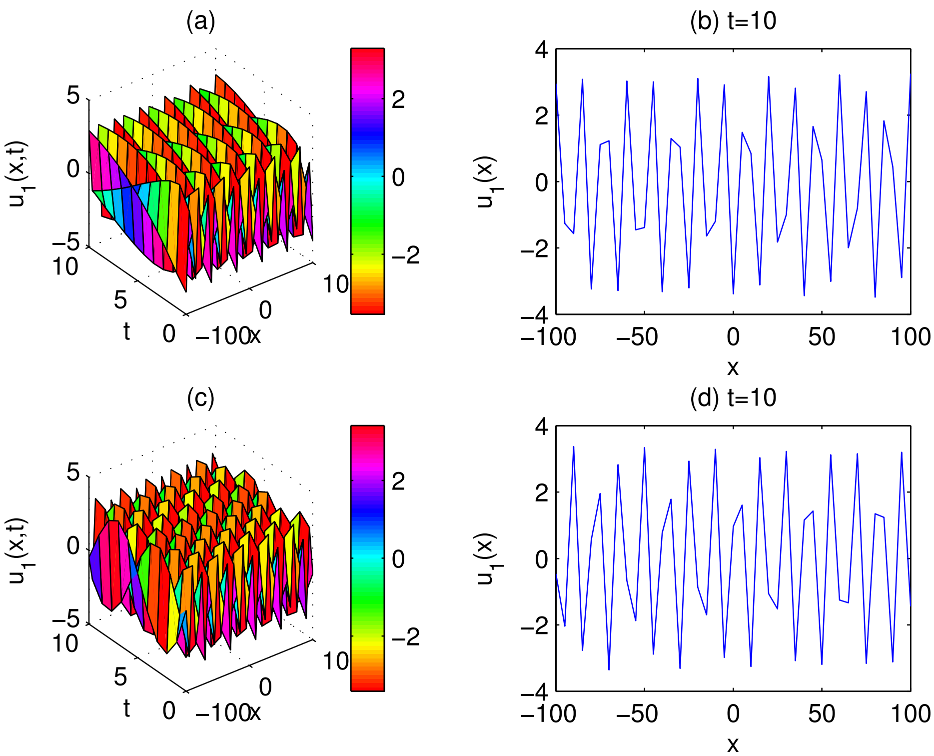

FIGURE 1. 3D and 2D plots of (3.13) (a) $\alpha = 0.5$, (c) $\alpha = 0.9$ when $-100 < x < 100$ and $0 < t < 10$ and (b) $\alpha = 0.5$, (d) $\alpha = 0.9$ when $y = 1$, $m = 2$, $-100 < x < 100$ and $t = 10$.

Balancing the vv'' and v^4, we get to $M = 1$. Therefore, the exact solution will be as

$$v(\eta) = \frac{a_1 \exp(\eta) + a_2 \exp(2\eta)}{b_1 \exp(\eta) + b_2 \exp(2\eta)} + \frac{a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)}. \tag{3.19}$$

For simplicity, we set $a_2 = b_2 = 1$ and $a_{-2} = b_{-2} = 1$, then (3.19) transforms to

$$v(\eta) = \frac{a_1 + \exp(\eta)}{b_1 + \exp(\eta)} + \frac{a_{-1} + \exp(-\eta)}{b_{-1} + \exp(-\eta)}. \tag{3.20}$$

Inserting (3.20) into Eq. (3.18), we acquire the below set of solutions

Set I:

$$a_1 = -b_1, \quad a_{-1} = \frac{1}{b_1}, \quad b_{-1} = -\frac{1}{b_1}, \quad b_1 = b_1, \quad k = \frac{1}{3}\sqrt{\frac{1+18m^2}{2}}i, \quad n = \frac{1}{6}k, \quad m = m, \tag{3.21}$$

$$u_1(x,t) = \left(\frac{4b_1}{b_1^2 e^{-\frac{1}{3}\sqrt{\frac{1+18m^2}{2}}ix - my - \frac{1}{18}\sqrt{\frac{1+18m^2}{2}}i\frac{t^\alpha}{\Gamma(\alpha+1)}} - e^{\frac{1}{3}\sqrt{\frac{1+18m^2}{2}}ix + my + \frac{1}{18}\sqrt{\frac{1+18m^2}{2}}i\frac{t^\alpha}{\Gamma(\alpha+1)}}} \right)^{-1}. \tag{3.22}$$

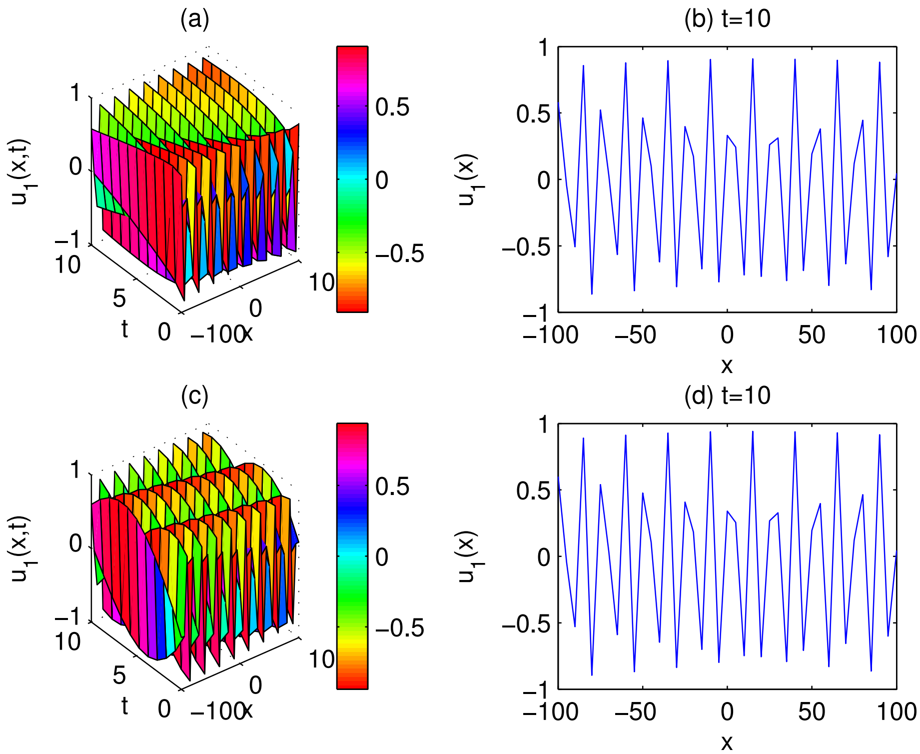

FIGURE 2. 3D and 2D plots of (3.23) (a) $\alpha = 0.5$, (c) $\alpha = 0.9$ when $-100 < x < 100$ and $0 < t < 10$ and (b) $\alpha = 0.5$, (d) $\alpha = 0.9$ when $y = 1$, $m = 2$, $-100 < x < 100$ and $t = 10$.

If we choose $b_1 = 1$, then equation (3.22) achieve to

$$u_1(x,t) = \frac{1}{4} sinh\left(\frac{1}{3}\sqrt{\frac{1+18m^2}{2}} ix + my + \frac{1}{18}\sqrt{\frac{1+18m^2}{2}} i \frac{t^\alpha}{\Gamma(\alpha+1)}\right). \tag{3.23}$$

3.3. **The fractional KdV equation.** As third example, we next consider (1+1)-dimensional fractional KdV equation [15]

$$\frac{\partial^\alpha}{\partial t^\alpha} u - 3\frac{\partial}{\partial x} u^2 + \frac{\partial^3}{\partial x^3} u = 0, \tag{3.24}$$

by using the transformation $\eta = kx + \frac{nt^\alpha}{\Gamma(\alpha+1)}$, Eq. (3.24) is reduced to an ODE as

$$nu' - 3k(u^2)' + k^3(u)''' = 0. \tag{3.25}$$

Integrating Eq. (3.25) once and setting the constant of integration equal to zero, concludes

$$nu - 3ku^2 + k^3u'' = 0. \tag{3.26}$$

Balancing the u'' and u^2, we obtain $M = 2$. Therefore, the exact solution can be written as

$$\mathrm{u}(\eta) = \frac{\sum_{\mathrm{n}=1}^{4} \mathrm{a_n} \exp(\mathrm{n}\eta)}{\sum_{\mathrm{n}=1}^{4} \mathrm{b_n} \exp(\mathrm{n}\eta)} + \frac{\sum_{\mathrm{n}=1}^{4} \mathrm{a_{-n}} \exp(-\mathrm{n}\eta)}{\sum_{\mathrm{n}=1}^{4} \mathrm{b_{-n}} \exp(-\mathrm{n}\eta)}. \tag{3.27}$$

For simplicity, we set $a_3 = b_3 = a_4 = b_4 = 1$ and $a_{-3} = b_{-3} = a_{-4} = b_{-4} = 1$, then (3.27) reduces to

$$u(\eta) = \frac{a_1 + a_2 e^{\eta} + e^{2\eta} + e^{3\eta}}{b_1 + b_2 e^{\eta} + e^{2\eta} + e^{3\eta}} + \frac{a_{-1} + a_{-2} e^{-\eta} + e^{-2\eta} + e^{-3\eta}}{b_{-1} + b_{-2} e^{-\eta} + e^{-2\eta} + e^{-3\eta}}. \tag{3.28}$$

Plugging (3.28) into Eq. (3.26), we gain the below set of solutions as

Set I:

$$a_{-1} = a_{-2} = \frac{4k^2 - 3}{3b_1}, \quad a_1 = a_2 = \frac{1}{3}(4k^2 - 3)b_1, \quad b_{-1} = b_{-2} = \frac{1}{b_1}, \quad b_1 = b_1, \quad k = k, \quad n = 4k^3, \tag{3.29}$$

$$u_1(x,t) = \frac{4}{3} \frac{k^2 \left[b_1^2 e^{-2k\left(x + \frac{4k^2 t^\alpha}{\Gamma(\alpha+1)}\right)} + e^{2k\left(x + \frac{4k^2 t^\alpha}{\Gamma(\alpha+1)}\right)} + 2b_1 \right]}{\left[e^{-2k\left(x + \frac{4k^2 t^\alpha}{\Gamma(\alpha+1)}\right)} b_1 + 1 \right] \left[e^{2k\left(x + \frac{4k^2 t^\alpha}{\Gamma(\alpha+1)}\right)} + b_1 \right]}. \tag{3.30}$$

Set II:

$$a_1 = a_2 = \frac{a_{-1}}{b_{-1}^2}, \quad a_{-1} = a_{-1} = a_{-2}, \quad b_{-1} = b_{-1} = b_{-2}, \quad b_1 = b_2 = \frac{1}{b_{-1}}, \quad k = k, \quad n = \frac{3k(b_{-1} + a_{-1})}{b_{-1}}, \tag{3.31}$$

$$u_2(x,t) = \frac{1}{b_{-1}} \left[\frac{(a_{-1} + b_{-1})e^{-\lambda} + 2b_{-1}^2 + a_{-1} b_{-1}^2 e^{\lambda} + b_{-1}^3 e^{\lambda}}{(e^{-\lambda} + b_{-1})(b_{-1} e^{\lambda} + 1)} \right], \tag{3.32}$$

where $\lambda = \frac{2k}{b_{-1}} \left(x b_{-1} + 3(b_{-1} + a_{-1}) \frac{t^\alpha}{\Gamma(\alpha+1)} \right)$. If we choose $b_{-1} = a_{-1} = 1$, then equation (3.32) get to

$$u_2(x,t) = \frac{e^{-\lambda} + 2 + 2e^{\lambda}}{e^{-\lambda} + 2 + e^{\lambda}}, \qquad \lambda = 2kx + \frac{12kt^\alpha}{\Gamma(\alpha+1)}. \tag{3.33}$$

Set III:

$$a_{-1} = a_{-2} = \frac{3k^2 - 3}{b_1}, \quad a_1 = a_2 = (3k^2 - 1)b_1, \quad b_{-1} = b_{-2} = \frac{1}{b_1}, \quad b_1 = b_1, \quad k = k, \quad n = 9k^3, \tag{3.34}$$

$$u_3(x,t) = \frac{3k^2 \left[b_1^2 e^{-2k\left(x + 9k^2 \frac{t^\alpha}{\Gamma(\alpha+1)}\right)} + e^{2k\left(x + 9k^2 \frac{t^\alpha}{\Gamma(\alpha+1)}\right)} + 2b_1 \right]}{\left[e^{-2k\left(x + 9k^2 \frac{t^\alpha}{\Gamma(\alpha+1)}\right)} b_1 + 1 \right] \left[e^{2k\left(x + 9k^2 \frac{t^\alpha}{\Gamma(\alpha+1)}\right)} + b_1 \right]}. \tag{3.35}$$

3.4. **The fractional K(2,2) equation.** The (1+1)-dimensional K(2,2) equation of fractional order is given as

$$\frac{\partial^\alpha}{\partial \mathrm{t}^\alpha} \mathrm{u} + \frac{\partial}{\partial \mathrm{x}} \mathrm{u}^2 + \frac{\partial^3}{\partial \mathrm{x}^3} \mathrm{u}^2 = 0, \tag{3.36}$$

by using the transformation $\eta = kx + \frac{nt^\alpha}{\Gamma(\alpha+1)}$, Eq. (3.36) is reduced to an ODE as

$$nu' + k(u^2)' + k^3(u^2)''' = 0. \tag{3.37}$$

Integrating Eq. (3.37) once and setting the constant of integration equal to zero, concludes

$$nu + ku^2 + k^3(u^2)'' = 0. \tag{3.38}$$

Balancing the $(u^2)''$ and u, we get $M = -2$. To get an exact solution, we obtain

$$u(\eta) = v(\eta)^{-2}. \tag{3.39}$$

Substituting Eq. (3.39) into Eq. (3.38), we get

$$kv^2 + nv^4 + 4k^3(5v'^2 - vv'') = 0. \tag{3.40}$$

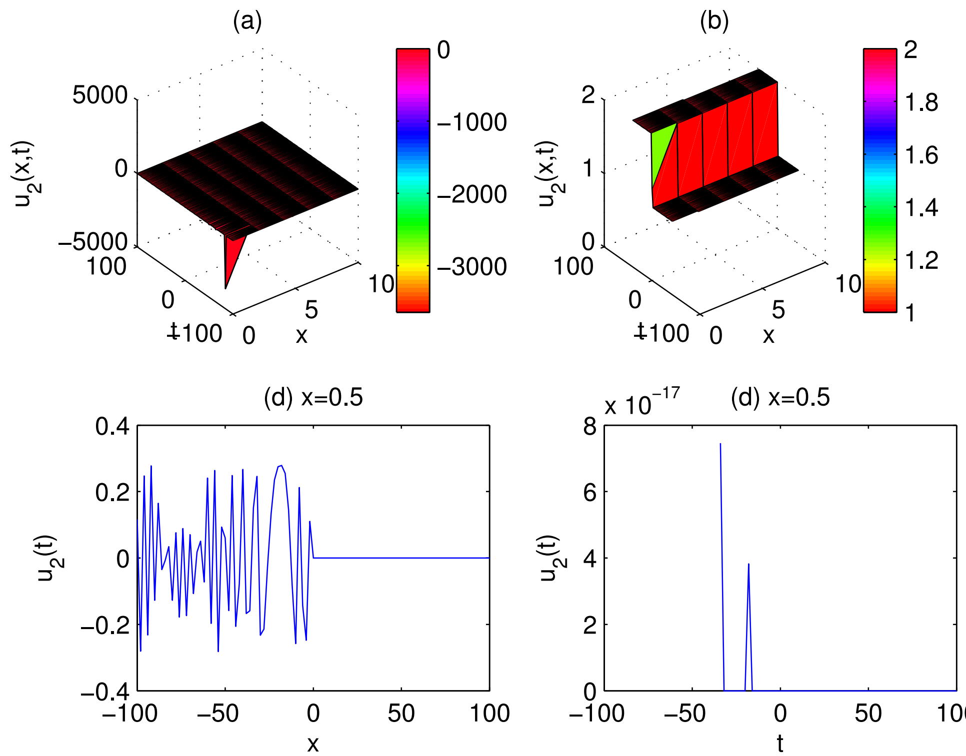

FIGURE 3. 3D and 2D plots of (3.33) (a) $\alpha = 0.5$, (b) $\alpha = 0.9$ when $0 < x < 10$ and $-100 < t < 100$ and (c) $\alpha = 0.5$, (d) $\alpha = 0.9$ when $k = 2$, $-100 < t < 100$ and $x = 0.5$.

Balancing the vv'' and v^4, we achieve to $M = 1$. Then the exact solution is as

$$v(\eta) = \frac{a_1 \exp(\eta) + a_2 \exp(2\eta)}{b_1 \exp(\eta) + b_2 \exp(2\eta)} + \frac{a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)}. \tag{3.41}$$

For simplicity, we set $a_2 = b_2 = 1$ and $a_{-2} = b_{-2} = 1$, then Eq. (3.41) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + \exp(2\eta)}{b_1 \exp(\eta) + \exp(2\eta)} + \frac{a_{-1} \exp(-\eta) + \exp(-2\eta)}{b_{-1} \exp(-\eta) + \exp(-2\eta)}. \tag{3.42}$$

Plugging (3.42) into Eq. (3.40), we gain the following set of solutions as
Set I:

$$a_1 = -b_1, \qquad a_{-1} = \frac{1}{b_1}, \qquad b_{-1} = -\frac{1}{b_1}, \qquad b_1 = b_1, \qquad k = \frac{1}{4}i, \qquad n = \frac{3}{64}i, \tag{3.43}$$

$$u_1(x,t) = \frac{1}{16b_1^2}\left(b_1 + e^{\frac{i}{64}\left[16x + 3\frac{t^\alpha}{\Gamma(\alpha+1)}\right]}\right)^2 \left(-1 + b_1 e^{-\frac{i}{64}\left[16x + 3\frac{t^\alpha}{\Gamma(\alpha+1)}\right]}\right)^2. \tag{3.44}$$

If we choose $b_1 = 1$, then equation (3.44) get to

$$u_1(x,t) = -\frac{I}{8}\sin\left[\frac{1}{64}\left(16x + \frac{3t^\alpha}{\Gamma(\alpha+1)}\right)\right], \qquad I = \sqrt{-1}. \tag{3.45}$$

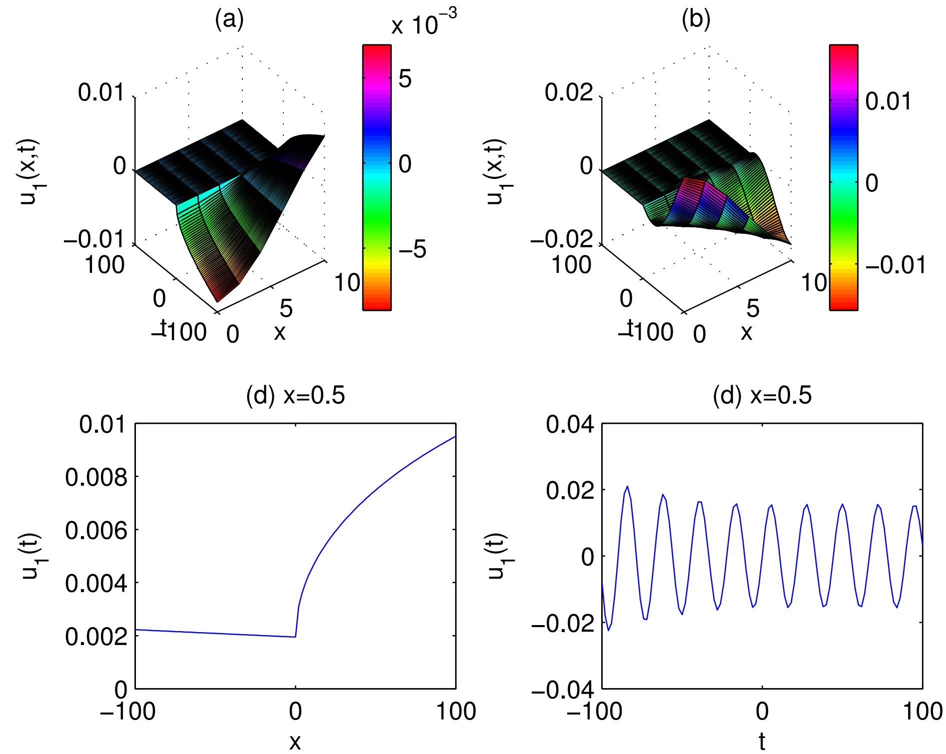

FIGURE 4. 3D and 2D plots of (3.45) (a) $\alpha = 0.5$, (b) $\alpha = 0.9$ when $0 < x < 10$ and $-100 < t < 100$ and (c) $\alpha = 0.5$, (d) $\alpha = 0.9$ when $-100 < t < 100$ and $x = 0.5$.

4. Physical Interpretations of the Solutions

Numerical simulation have been performed by using the Maple software. We depict some of the solutions to have an idea on the mechanism of the original equations. Particularly, we depict solutions of the fractional ZK(2,2,2,), ZK(3,3,3), KdV and K(2,2) equations by considering the suitable values of the parameters obtained. The graphical representations to this solution are presented in Figs. 1-4, respectively. Figures 1-4 represent the variation of some appropriated parameters. The curves 1-4 have been plotted for different values of appropriated parameters. In Figures 1-4, we plot two and three dimensional graphics of real values of (3.13), (3.23), (3.33) and (3.45), respectively.

5. Conclusion

In this paper, we employed the modified Exp-function method for deriving the exact solitary wave solutions for some of the fractional partial differential equations. As a consequence, we gained many new exact solitary solutions for the aforementioned equations which are expressed by a rational exponential function form. The modified Exp-function method has many applications in the field of engineering, mathematical science and physics for solving a large class of non-linear partial differential equations. We then think that these results will help for conducting future research in diverse areas of physics such as mathematical physics, nonlinear phenomena, fluid mechanics and other applied fields and so on. The method utilized here can be applied to other nonlinear fractional partial differential equations.

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