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Soliton solutions to the DS and generalized DS system via an analytical method

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Abstract

In this article, the exact solutions for nonlinear Drinfeld-Sokolov (DS) and generalized Drinfeld-Sokolov (gDS) equations are established. The rational Exp-function method (EFM) is used to construct solitary and soliton solutions of nonlinear evolution equations. This method is developed for searching exact traveling wave solutions of nonlinear partial differential equations. Also, exact solutions with solitons and periodic structures are obtained. The obtained results are not only presented numerically but are also accompanied by insightful physical interpretations, enhancing the understanding of the complex dynamics described by these mathematical models. The utilization of the rational EFM and the broad spectrum of obtained solutions contribute to the depth and significance of this research in the field of nonlinear wave equations.

Keywords. Exp–function method, Nonlinear partial differential equation, Drinfeld-Sokolov system, Generalized Drinfeld-Sokolov, Solitons and periodic structures, Traveling wave solution.

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1. Introduction

Drinfeld-Sokolov system was introduced by Drinfeld and Sokolov as an example of a system of nonlinear equations possessing Lax pairs of a special form [13]. This system used by [37, 39, 41] as follows

$$u_t + (v^2)_x = 0, \qquad v_t - av_{xxx} + 3bu_x v + 3kuv_x = 0, \tag{1.1}$$

where a, b and k are constants. The main aim of the present paper is to contribute to the research in this direction. As mentioned in [41] the term soliton was devised by Zabusky and Kruskal [44], who performed numerical studies of the KdV equation. There has been an enormous number of examples of solitons equations [12, 20, 36, 38]. We consider a family of generalizations of the DS system and a variant of the DS system given by [41]

$$\begin{aligned} &u_t + (v^n)_x = 0, \qquad v_t - av_{xxx} + 3bu_x v + 3kuv_x = 0, \\ &u_t + (v^{-n})_x = 0, \qquad v_t - av_{xxx} + 3bu_x v + 3kuv_x = 0,\ n > 2, \end{aligned} \tag{1.2}$$

respectively, where a, b, n and k are constants. Here our aim is the determination of traveling wave solutions with compact and noncompact structures for the DS system, a generalized form of the DS system, and one type different of the DS system. Recently, the investigation of exact traveling wave solutions to nonlinear partial differential equations plays an important role in the study of nonlinear modelling physical phenomena. The study of the traveling wave

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solutions plays an important role in nonlinear science. Meanwhile, a variety of powerful methods for seeking the explicit and exact solutions of nonlinear evolution equations have been proposed and developed. Among them are the Hirota's bilinear method ([18]), homotopy analysis method ([6, 7]), variational iteration method ([8, 17]), homotopy perturbation method ([5]), sine-cosine method ([42]), tanh-coth method ([25]), Bäcklund transformation ([33]), ($\frac{G'}{G}$)-expansion method ([11]), Exp-function method ([9, 10, 24]), modified simple equation method ([19]) and so on. Here, we use of two effective methods for constructing a range of exact solutions for the following nonlinear partial differential equations that in this article we developed solutions as well. The standard tanh method is well-known analytical method which first presented by Malfliet's ([22]) and developed in ([22, 23]). In ([25]), we applied the generalized tanh-coth method in for solving some nonlinear partial differential equations. Also in ([35]), the new approach of generalized (G'/G)-expansion method to obtain exact traveling wave solutions of NLEEs is presented. In this paper we explain methods which are called the generalized tanh–coth and generalized (G'/G)-expansion methods are presented to look for traveling wave solutions of nonlinear evolution equations. Authors of ([26]), obtained exact solutions for the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota system by the generalized tanh–coth and generalized (G'/G)-expansion methods. The basic idea of the Exp-function method was proposed by J. H. He [15]. Some illustrative examples in references [16, 43] show that this method is very effective to search for various solitary and periodic solutions of nonlinear equations. The Exp-function method has successfully been applied to many situations. For example, the Exp-function method along with Hirota's and tanh-coth methods have been applied for solving solitary wave solutions of the generalized shallow water wave equation by Wazwaz [40]. Abdou [1] solved generalized solitonary and periodic solutions for nonlinear partial differential equations by the Exp-function method. Boz and Bekir [4] applied the Exp-function method for (3+1)-dimensional nonlinear evolution equations.

Authors of [31] explained the generalized fifth-order KdV like equation with prime number $p = 3$ via a generalized bilinear differential operator. N-lump was investigated to the variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation [30]. Applications of $tan(\phi/2)$-expansion method for the Biswas-Milovic equation [27], the Gerdjikov-Ivanov model [29], the Kundu-Eckhaus equation [28] and the fifth-order integrable equations [21] were studied. Lump solutions were analyzed to the fractional generalized CBS-BK equation [45] and the (3+1)-D Burger system [14]. The approximations of one-dimensional hyperbolic equation with non-local integral conditions were constructed by reduced differential transform method [34]. The generalized Hirota bilinear strategy by the number prime was used to the (2+1)-dimensional generalized fifth-order KdV like equation [32]. The traveling wave solutions and analytical treatment of the simplified MCH equation and the combined KdV-mKdV equations were studied [2].

Our aim of this paper is to obtain analytical solutions of the DS and generalized DS equations, and to determine the accuracy of the EFM in solving these kind of problems. The article is organized as follows: In section 2, we briefly give the steps of the Exp-function method. In sections 3, 4, and 5 the DS system, gDS system and a variant of the DS system respectively will be introduced briefly and obtained exact solutions for related equations. Also, a conclusion is given in section 6.

2. Basic idea of the Exp–function method

We first consider nonlinear equation of form

$$\mathcal{N}(\mathrm{u}, \mathrm{u_t}, \mathrm{u_x}, \mathrm{u_{xx}}, \mathrm{u_{tt}}, \mathrm{u_{tx}}, ...) = 0, \tag{2.1}$$

and introduce a transformation

$$\mathrm{u(x,t) = u(\eta)}, \qquad \xi = \mathrm{x - ct}, \tag{2.2}$$

where c is constant to be determined later. Therefore Eq. (2.1) is reduced to an ODE as follows

$$\mathcal{M}(\mathrm{u}, -\mathrm{cu}', \mathrm{u}', \mathrm{u}'', ...) = 0. \tag{2.3}$$

The EFM is based on the assumption that traveling wave solutions as in [15] can be expressed in the form

$$\mathrm{u}(\xi) = \frac{\sum_{\mathrm{n=-c}}^{\mathrm{d}} \mathrm{a_n} \exp(\mathrm{n}\xi)}{\sum_{\mathrm{m=-p}}^{\mathrm{q}} \mathrm{b_m} \exp(\mathrm{m}\xi)}, \tag{2.4}$$

where c, d, p, and q are positive integers which could be freely chosen, a_n's and b_m's are unknown constants to be determined. To determine the values of c and p, we balance the linear term of highest order in Eq. (2.3) with the highest order nonlinear term. Also to determine the values of d and q, we balance the linear term of lowest order in Eq. (2.3) with the lowest order nonlinear term.

3. The Drinfeld-Sokolov Equation

We first consider the DS system with the EFM as follows

$$u_t + (v^2)_x = 0, \qquad v_t - av_{xxx} + 3bu_x v + 3kuv_x = 0, \tag{3.1}$$

where a, b and k are constants. Using the wave variable $\eta = x - ct$ carries the system (3.1) into the system of ODE

$$-cu' + (v^2)' = 0, \qquad cv' + av''' - 3bu'v - 3kuv' = 0, \tag{3.2}$$

where by integrating the first equation in the system Eq. (3.2) and neglecting the constant of integration we obtain

$$cu = v^2. \tag{3.3}$$

Substituting (3.3) into the second equation of the system (3.2) and integrating we find

$$c^2v + acv'' - (2b+k)v^3 = 0. \tag{3.4}$$

In order to determine values of c and p, we balance the linear term of the highest order v'' with the highest order nonlinear term v^3 in Eq. (3.4), we get

$$v'' = \frac{c_1 \exp((c+3p)\eta) + ...}{c_2 \exp(4p\eta) + ...}, \tag{3.5}$$

$$v^3 = \frac{c_3 \exp(3c\eta) + ...}{c_4 \exp(3p\eta) + ...} = \frac{c_3 \exp((3c+p)\eta) + ...}{c_4 \exp(4p\eta) + ...}, \tag{3.6}$$

respectively. Balancing highest order of the Exp–function in (3.5) and (3.6), we get

$$c + 3p = 3c + p, \tag{3.7}$$

which leads to the result $c = p$. Similarly to determine values of d and q, for the terms v'' and v^3 in Eq. (3.4) by simple calculation, we obtain

$$v'' = \frac{... + d_1 \exp(-(d+3q)\eta)}{... + d_2 \exp(-4q\eta)}, \tag{3.8}$$

$$v^3 = \frac{... + d_3 \exp(-3d\eta)}{... + d_4 \exp(-3q\eta)} = \frac{... + d_3 \exp(-(3d+q)\eta)}{... + d_4 \exp(-4q\eta)}, \tag{3.9}$$

respectively. Balancing lowest order of the Exp–function in Eqs. (3.8) and (3.9), we have

$$-(d+3q) = -(3d+q), \tag{3.10}$$

which leads to the result $d = q$. For simplicity, we set $p = c = 1$ and $d = q = 1$, then (2.4) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{3.11}$$

Substituting (3.11) into Eq. (3.4), we get an equation of the form

$$\left([\mathrm{b}_{-1} \exp(-\eta) + \mathrm{b}_0 + \mathrm{b}_1 \exp(\eta)]^3\right)^{-1} \sum_{\mathrm{n}=-3}^{3} \mathrm{C_n} \exp(\mathrm{n}\xi) = 0, \tag{3.12}$$

where $C_n(-3 \leq n \leq 3)$ are polynomial expressions in terms of $a_1, a_0, a_{-1}, b_0, b_{-1}$ and b_1, c, β. Thus, solving the resulting system $\mathrm{C_n} = 0(-3 \leq \mathrm{n} \leq 3)$ simultaneously, we obtain the following set of algebraic equations:

(I) The first set is:

$$a_1 = a_1,\ a_{-1} = a_{-1},\ ac = ac,\ b_0 = 0,\ b_{-1} = -\frac{a_{-1}b_1}{a_1},\ b_1 = b_1, \tag{3.13}$$

$$c^2 = 2ac,\ a_0 = 0,\ -k - 2b = -\frac{2acb_1^2}{a_1^2},\ c = 2a,$$

$$v_1(x,t) = \frac{a_1}{b_1}\frac{a_{-1}\exp(-x+ct) + a_1\exp(x-ct)}{-a_{-1}\exp(-x+ct) + a_1\exp(x-ct)}.$$

If we choose $a_1 = a_{-1}$, then the solutions (3.13) along with (3.3) give (cf. Eq. (4.12) in [3])

$$v_1(x,t) = \frac{2a}{\sqrt{k+2b}}\coth(x-2at), \qquad u_1(x,t) = \frac{2a}{k+2b}\coth^2(x-2at).$$

(II) The second set is:

$$a_1 = a_1,\ a_0 = a_0,\ ac = ac,\quad a_{-1} = 0,\ b_{-1} = 0, \tag{3.14}$$

$$b_1 = -\frac{b_0 a_1}{a_0},\ c^2 = \frac{ac}{2},\ b_0 = b_0,\ -k - 2b = \frac{-1}{2}\frac{acb_0^2}{a_0^2},\quad c = \frac{a}{2},$$

$$v_2(x,t) = \frac{a_0}{b_0}\frac{a_0 + a_1\exp(x-ct)}{a_0 - a_1\exp(x-ct)}.$$

If we choose $a_0 = a_1$, then the solutions (3.14) along with (3.3) give (cf. Eq. (72) in [41])

$$v_2(x,t) = \frac{-a}{2\sqrt{k+2b}}\coth\left(\frac{2x-at}{4}\right),$$

$$u_2(x,t) = \frac{a}{2(k+2b)}\coth^2\left(\frac{2x-at}{4}\right).$$

(III) The third set is:

$$a_1 = a_1,\ b_0 = b_0,\ ac = -c^2,\ a_{-1} = 0,\ b_{-1} = 0,\ b_1 = b_1,\ c^2 = c^2, \tag{3.15}$$

$$a_0 = 0,\ -k - 2b = -\frac{c^2 b_1^2}{a_1^2},\quad v_3(x,t) = \frac{a_1\exp(x-ct)}{b_0 + b_1\exp(x-ct)},\qquad c = -a.$$

If we choose $b_0 = b_1$, then we can obtain

$$v_3(x,t) = \frac{-a}{2\sqrt{k+2b}}\left[1 + \tanh\left(\frac{x+at}{2}\right)\right],$$

$$u_3(x,t) = \frac{-a}{4(k+2b)}\left(1 + \tanh\left[\frac{x+at}{2}\right]\right)^2.$$

(IV) The fourth set is:

$$a_1 = a_1,\ b_{-1} = b_{-1},\ ac = -\frac{1}{2}\frac{(-k-2b)a_1^2}{b_1^2},\ c^2 = -\frac{(-k-2b)a_1^2}{b_1^2}, \tag{3.16}$$

$$b_1 = b_1,\ a_{-1} = 0,\ b_0 = 0,\ -k - 2b = -k - 2b,\ a_0 = 0,\ c = 2a,$$

$$v_4(x,t) = \frac{a_1\exp(x-ct)}{b_{-1}\exp(-x+ct) + b_1\exp(x-ct)}.$$

If we choose $b_1 = b_{-1}$, then we can obtain

$$v_4(x,t) = \frac{a}{\sqrt{k+2b}}\left[1 + \tanh(x-2at)\right],$$

$$u_4(x,t) = \frac{1}{2}\left(\frac{a}{k+2b}\right)\left[1 + \tanh(x-2at)\right]^2.$$

(V) The fifth set is:

$$a_{-1} = a_{-1},\ b_{-1} = b_{-1},\ ac = ac,\ b_0 = 0,\ c^2 = -4ac, \tag{3.17}$$
$$-k - 2b = \frac{4acb_{-1}^2}{a_{-1}^2},\ b_1 = b_1,\ a_0 = 0,\ a_1 = 0,\ c = -4a,$$
$$v_5(x,t) = \frac{a_{-1}\exp(-x+ct)}{b_{-1}\exp(-x+ct) + b_1\exp(x-ct)}.$$

If we choose $b_1 = b_{-1}$, then we can obtain

$$v_5(x,t) = \frac{2a}{\sqrt{k+2b}}\left[1 - \tanh(x+4at)\right],$$
$$u_5(x,t) = \frac{-a}{k+2b}\left[1 - \tanh(x+4at)\right]^2.$$

(VI) The sixth set is:

$$a_{-1} = a_{-1},\ b_{-1} = 0,\ c^2 = c^2,\ b_0 = b_0,\ c^2 = -ac,\ -k-2b = 0, \tag{3.18}$$
$$a_0 = a_0,\ b_1 = -\frac{1}{5}\frac{a_0 b_0}{a_{-1}},\ a_1 = 0,\ c = -a,$$
$$v_6(x,t) = \frac{5a_{-1}}{b_0}\frac{a_{-1}\exp(-x+ct) + a_0}{5a_{-1} - a_0\exp(x-ct)}.$$

If we choose $a_0 = 5a_{-1}$, then we can obtain

$$\begin{aligned} v_6(x,t) &= \frac{-a_{-1}}{2b_0}\left\{\mathrm{csch}\left(\frac{\mathrm{x+at}}{2}\right)\left[\cosh\left(\frac{3x+3at}{2}\right) - \sinh\left(\frac{3x+3at}{2}\right)\right.\right. \\ &+ \left.\left. 5\cosh\left(\frac{x+at}{2}\right) - 5\sinh\left(\frac{x+at}{2}\right)\right]\right\}, \\ u_6(x,t) &= \frac{-a_{-1}}{4ab_0}\left\{\mathrm{csch}\left(\frac{\mathrm{x+at}}{2}\right)\left[\cosh\left(\frac{3x+3at}{2}\right) - \sinh\left(\frac{3x+3at}{2}\right)\right.\right. \\ &+ \left.\left. 5\cosh\left(\frac{x+at}{2}\right) - 5\sinh\left(\frac{x+at}{2}\right)\right]\right\}^2. \end{aligned}$$

(VII) The seventh set is:

$$a_{-1} = 0,\ b_{-1} = 0,\ c^2 = -\frac{(-k-2b)a_0^2}{b_0^2},\ b_0 = b_0,\ = a_0, \tag{3.19}$$
$$ac = -\frac{2(-k-2b)a_0^2}{b_0^2},\ a_0,\ b_1 = b_1,\ -k-2b = -k-2b,\ a_1 = 0,\ c = \frac{a}{2},$$
$$v_7(x,t) = \frac{a_0}{b_1\exp(x-ct) + b_0}.$$

If we choose $b_0 = b_1$, then we can obtain

$$v_7(x,t) = \frac{a}{4\sqrt{k+2b}}\left(1 - \tanh\left[\frac{2x-at}{4}\right]\right),$$
$$u_7(x,t) = \frac{a}{8(k+2b)}\left(1 - \tanh\left[\frac{2x-at}{4}\right]\right)^2.$$

(VIII) The eighth set is:

$$a_{-1} = 0,\ b_{-1} = b_{-1},\ c^2 = -ac,\ b_0 = 0,\ ac = ac,\ a_1 = 0, \tag{3.20}$$
$$a_0 = a_0,\ b_1 = b_1,\ -k-2b = \frac{8acb_1 b_{-1}}{a_0^2},\quad c = -a,$$

$$v_8(x,t) = \frac{a_0}{b_1 \exp(x-ct) + b_{-1}\exp(-x+ct)},$$

If we choose $b_1 = b_{-1}$, then the solutions (3.20) along with (3.3) give (cf. Eqs. (28) and (30) in [41])

$$v_8(x,t) = \frac{\sqrt{2}a}{\sqrt{k+2b}} \operatorname{sech}(x+at), \quad u_8(x,t) = \frac{-2a}{k+2b} \operatorname{sech}^2(x+at).$$

(IX) The ninth set is:

$$\begin{aligned} &a_{-1} = a_1 = 0,\ b_{-1} = b_{-1},\ c^2 = c^2, \quad b_0 = b_0,\ ac = -c^2,\ a_0 = a_0, \\ &b_1 = 0,\ k+2b = \frac{c^2 b_0^2}{a_0^2},\ v_9(x,t) = \frac{a_0}{b_{-1}\exp(-x+ct) + b_0},\ c = -a. \end{aligned} \tag{3.21}$$

If we choose $b_0 = b_{-1}$, then we can get

$$v_9(x,t) = \frac{a}{2\sqrt{k+2b}}\left[1 + \tanh\left(\frac{x+at}{2}\right)\right],$$

$$u_9(x,t) = \frac{-a}{4(k+2b)}\left[1 + \tanh\left(\frac{x+at}{2}\right)\right]^2.$$

It is obvious that nine pairs of solutions were obtained by using the Exp-function method, whereas two and four pairs of solutions were obtained in [3, 41] respectively.

4. A generalized Drinfeld-Sokolov system

In this section we apply the EFM to the generalized DS system of the form

$$u_t + (v^n)_x = 0, \qquad v_t - av_{xxx} + 3bu_x v + 3kuv_x = 0, \tag{4.1}$$

where a, b, n and k are constants. Using the wave variable $\eta = x - ct$ carries system (4.1) to

$$-cu' + (v^n)' = 0, \qquad cv' + av''' - 3bu'v - 3kuv' = 0. \tag{4.2}$$

As before by integrating the first equation in the system Eq. (4.2) and neglecting the constant of integration we obtain

$$cu = v^n. \tag{4.3}$$

Substituting (4.3) into the second equation of the system Eq. (4.2) and integrating we get

$$c^2 v + acv'' - \frac{3(nb+k)}{n+1} v^{n+1} = 0. \tag{4.4}$$

To get a closed form solution, we use the transformation

$$v(\eta) = w(\eta)^{\frac{1}{n}}, \tag{4.5}$$

that will carry Eq. (4.4) into an ODE

$$c^2 n^2 (n+1) w^2 - 3n^2(k+bn) w^3 + acn(n+1) ww'' - ac(n^2-1)(w')^2 = 0, \tag{4.6}$$

we set

$$w(\eta) = \frac{a_c \exp(c\eta) + \ldots + a_{-d}\exp(-d\eta)}{b_p \exp(p\eta) + \ldots + b_{-q}\exp(-q\eta)}. \tag{4.7}$$

By the same manipulation as illustrated in the previous section, we can determine values of c and p by balancing (ww'') and w^3 in Eq. (4.6)

$$ww'' = \frac{c_1 \exp((2c+3p)\eta) + \ldots}{c_2 \exp(5p\eta) + \ldots}, \tag{4.8}$$

$$w^3 = \frac{c_3 \exp(3c\eta) + \ldots}{c_4 \exp(3p\eta) + \ldots} = \frac{c_3 \exp((3c+2p)\eta) + \ldots}{c_4 \exp(5p\eta) + \ldots}, \tag{4.9}$$

respectively. Balancing highest order of the Exp–function in (4.8) and (4.9), we have

$$2c+3p=3c+2p, \tag{4.10}$$

which leads to the result $c=p$. By a similar derivation as illustrated in above, we obtain $d=q$. For simplicity, we set $p=c=1$ and $d=q=1$, then Eq. (4.7) reduces to

$$w(\eta)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)}. \tag{4.11}$$

Substituting (4.11) into Eq. (4.6), we get an equation of the form

$$\left([\mathrm{b}_{-1}\exp(-\eta)+\mathrm{b}_0+\mathrm{b}_1\exp(\eta)]^4\right)^{-1}\sum_{\mathrm{n}=-4}^{4}\mathrm{C_n}\exp(\mathrm{n}\xi)=0, \tag{4.12}$$

where $\mathrm{C_n}(-4\le \mathrm{n}\le 4)$ are polynomial expressions in terms of $a_1,a_0,a_{-1},b_0,b_{-1},b_1,c,$ and β. Thus, solving the resulting system $C_n=0(-4\le n\le 4)$ simultaneously, we obtain the following set of algebraic equations
(I) The first set is:

$$a_0=a_0,\ a_1=b_1==0,\ a_{-1}=0,\ acn(n+1)=\frac{3(k+bn)n^2a_0}{b_0},\ b_0=b_0, \tag{4.13}$$

$$b_{-1}=b_{-1},\ c=\frac{a}{n},\ c^2n^2(n+1)=\frac{3(k+bn)n^2a_0}{b_0}, ac(n^2-1)=\frac{6(k+bn)n^2a_0}{b_0},$$

$$w_1(x,t)=\frac{a_0}{b_{-1}\exp(-x+ct)+b_0}, \qquad .$$

If we choose $b_0=b_{-1}$, then we can obtain

$$v_1(x,t)=\sqrt[n]{\frac{a^2(n+1)}{6(k+nb)n^2}}\left[1+\tanh\left(\frac{nx-at}{2n}\right)\right]^{\frac{1}{n}},$$

$$u_1(x,t)=\frac{a(n+1)}{6(k+nb)n}\left[1+\tanh\left(\frac{nx-at}{2n}\right)\right].$$

(II) The second set is:

$$a_1=0,\ a_{-1}=0,\ acn(n+1)=\frac{-3(k+bn)n^2a_0+2c^2n^2(n+1)b_0}{b_0},\ b_0=b_0, \tag{4.14}$$

$$b_{-1}=\frac{1}{4}\frac{b_0^2}{b_1},\ c^2n^2(n+1)=c^2n^2(n+1),\ a_0=a_0, b_1=b_1,$$

$$-ac(n^2-1)=\frac{-3(k+bn)n^2a_0+3c^2n^2(n+1)b_0}{b_0},$$

$$w_2(x,t)=\frac{4a_0b_1}{b_0^2\exp(-x+ct)+4b_0b_1+4b_1^2\exp(x-ct)}, \qquad c=-\frac{2n-1}{n^2}a.$$

If we choose $b_0=2b_1$, then we can obtain

$$v_2(x,t)=\sqrt[n]{\frac{(10n^2-9n+2)(n+1)a^2}{6(k+nb)n^4}}\left[\operatorname{sech}^2\left(\frac{n^2x+(2n-1)at}{2n^2}\right)\right]^{\frac{1}{n}},$$

$$u_2(x,t)=-\frac{(5n-2)(n+1)a}{6(k+nb)n^2}\operatorname{sech}^2\left(\frac{n^2x+(2n-1)at}{2n^2}\right).$$

(III) The third set is:

$$a_1=a_{-1}=0,\ acn(n+1)=c^2n^2(n+1),\ b_0=b_0,\ b_{-1}=b_{-1},\ b_1=b_1, \tag{4.15}$$

$$c^2n^2(n+1)=c^2n^2(n+1),\ a_0=\frac{1}{3}\frac{c^2(n+1)b_0}{k+bn},\ ac(n^2-1)=2c^2n^2(n+1),$$

$$w_3(x,t) = \frac{1}{3}\frac{c^2(n+1)b_0}{(k+bn)[b_{-1}\exp(-x+ct)+b_0+b_1\exp(x-ct)]}, \ c = \frac{a}{n}.$$

If we choose $b_0 = 2b_1$ and $b_1 = b_{-1}$ then can be found

$$v_3(x,t) = \sqrt[n]{\frac{a^2(n+1)}{6(k+nb)n^2}}\left[\operatorname{sech}^2\left(\frac{nx-at}{2n}\right)\right]^{\frac{1}{n}},$$

$$u_3(x,t) = \frac{a(n+1)}{6(k+nb)n}\operatorname{sech}^2\left(\frac{nx-at}{2n}\right).$$

(IV) The fourth set is:

$$a_0 = 0,\ a_1 = a_1, \quad a_{-1} = 0,\ acn(n+1) = \frac{3}{4}\frac{(k+bn)n^2a_1}{b_1},\ b_0 = 0, \tag{4.16}$$

$$b_{-1} = b_{-1},\ b_1 = b_1,\ c = \frac{4a}{n},\ c^2n^2(n+1) = \frac{3(k+bn)n^2a_1}{b_1},$$

$$ac(n^2-1) = \frac{3(k+bn)n^2a_1}{2b_1},\ w_4(x,t) = \frac{a_1\exp(x-ct)}{b_{-1}\exp(-x+ct)+b_1\exp(x-ct)},$$

If we choose $b_1 = b_{-1}$, then can be found

$$v_4(x,t) = \sqrt[n]{\frac{16a^2(n+1)}{6n^2(k+nb)}}\left[1+\tanh\left(\frac{nx-4at}{n}\right)\right]^{\frac{1}{n}},$$

$$u_4(x,t) = \frac{4a(n+1)}{6n(k+nb)}\left[1+\tanh\left(\frac{nx-4at}{n}\right)\right].$$

(V) The fifth set is:

$$a_1 = a_1,\ a_{-1} = 0,\ acn(n+1) = \frac{3(k+bn)n^2a_1}{b_1},\ b_0 = b_0,\ b_{-1} = \frac{1}{4}\frac{b_0^2}{b_1}, \tag{4.17}$$

$$b_1 = b_1,\ c^2n^2(n+1) = \frac{3(k+bn)n^2a_1}{b_1},$$

$$a_0 = 0,\ -ac(n^2-1) = -\frac{15(k+bn)n^2a_1}{4b_1},$$

$$w_5(x,t) = \frac{4a_1b_1\exp(x-ct)}{b_0^2\exp(-x+ct)+4b_0b_1+4b_1^2\exp(x-ct)}, \qquad c = \frac{a}{n}.$$

If we choose $b_0 = 2b_1$, then we can obtain

$$v_5(x,t) = \sqrt[n]{\frac{a^2(n+1)}{12(k+nb)n^2}}\left[1+\tanh\left(\frac{nx-at}{2n}\right)\right]^{\frac{2}{n}},$$

$$u_5(x,t) = \frac{a(n+1)}{12(k+nb)n}\left[1+\tanh\left(\frac{nx-at}{2n}\right)\right]^2.$$

(VI) The sixth set is:

$$a_{-1} = a_{-1},\ acn(n+1) = \frac{3(k+bn)n^2a_{-1}}{b_{-1}},\ b_0 = b_0,\ b_{-1} = b_{-1}, \tag{4.18}$$

$$b_1 = 0,\ c^2n^2(n+1) = \frac{3(k+bn)n^2a_{-1}}{b_{-1}},\ a_1 = a_0 = 0,$$

$$-ac(n^2-1) = -\frac{6(k+bn)n^2a_{-1}}{b_{-1}},\ w_6(x,t) = \frac{a_{-1}\exp(-x+ct)}{b_{-1}\exp(-x+ct)+b_0},\ c = \frac{a}{n}.$$

If we choose $b_0 = b_{-1}$, then we can obtain

$$v_6(x,t) = \sqrt[n]{\frac{a^2(n+1)}{6(k+nb)n^2}}\left[1+\tanh\left(\frac{nx-at}{2n}\right)\right]^{\frac{1}{n}},$$

$$u_6(x,t) = \frac{a(n+1)}{6(k+nb)n}\left[1+\tanh\left(\frac{nx-at}{2n}\right)\right].$$

(VII) The seventh set is:

$$a_0 = a_1 = 0,\ a_{-1} = a_{-1},\ acn(n+1) = \frac{12(k+bn)n^2a_{-1}b_1}{b_0^2},\ b_{-1} = \frac{b_0^2}{4b_1}, \tag{4.19}$$

$$c^2(n+1) = \frac{12(k+bn)a_{-1}b_1}{b_0^2},\ -ac(n^2-1) = -\frac{15(k+bn)n^2a_{-1}b_1}{b_0^2},\ b_1 = b_1,$$

$$w_7(x,t) = \frac{a_{-1}\exp(-x+ct)}{\frac{1}{4}\frac{b_0^2}{b_1}\exp(-x+ct)+b_0+b_1\exp(x-ct)},\ c = \frac{a}{n}.$$

If we choose $b_0 = 2b_1$, then we can obtain

$$v_7(x,t) = \sqrt[n]{\frac{a^2(n+1)}{12(k+nb)n^2}}\left[1-\tanh\left(\frac{nx-at}{2n}\right)\right]^{\frac{2}{n}},$$

$$u_7(x,t) = \frac{a(n+1)}{12(k+nb)n}\left[1-\tanh\left(\frac{nx-at}{2n}\right)\right]^2.$$

(VIII) The eighteenth set is:

$$a_1 = a_1,\ a_{-1} = \frac{1}{9}\frac{b_0^2a^2c^2(n^2-1)^2}{(k+bn)^2n^4a_1},\ acn(n+1) = 2ac(n^2-1),\ b_0 = b_0, \tag{4.20}$$

$$b_{-1} = \frac{1}{6}\frac{b_0^2ac(n^2-1)}{(k+bn)n^2a_1},\ c^2n^2(n+1) = 2ac(n^2-1),\ c = \frac{a}{n},$$

$$a_0 = \frac{-2}{3}\frac{b_0ac(n^2-1)}{(k+bn)n^2},\ -ac(n^2-1) = -ac(n^2-1),\quad b_1 = \frac{3}{2}\frac{(k+bn)n^2a_1}{ac(n^2-1)},$$

$$w_8(x,t) = \frac{2ac(n^2-1)}{3(k+bn)n^2}\frac{\left[b_0ac(n^2-1)\exp(\frac{-x+ct}{2})-3a_1(k+bn)n^2\exp(\frac{x-ct}{2})\right]^2}{\left[b_0ac(n^2-1)\exp(\frac{-x+ct}{2})+3a_1(k+bn)n^2\exp(\frac{x-ct}{2})\right]^2}.$$

If we choose $\frac{b_0(n^2-1)a^2}{3a_1(k+bn)n^3} = 1$, then we can obtain

$$v_8(x,t) = \sqrt[n]{\frac{2a^2(n^2-1)}{3(k+nb)n^3}}\ \tanh^{\frac{2}{n}}\left(\frac{nx-at}{2n}\right),\ u_8(x,t) = \frac{2a(n^2-1)}{3(k+nb)n^2}\ \tanh^2\left(\frac{nx-at}{2n}\right).$$

It is obvious that eight pairs of solutions were obtained by using the Exp-function method, whereas two pairs of solutions were obtained in [41].

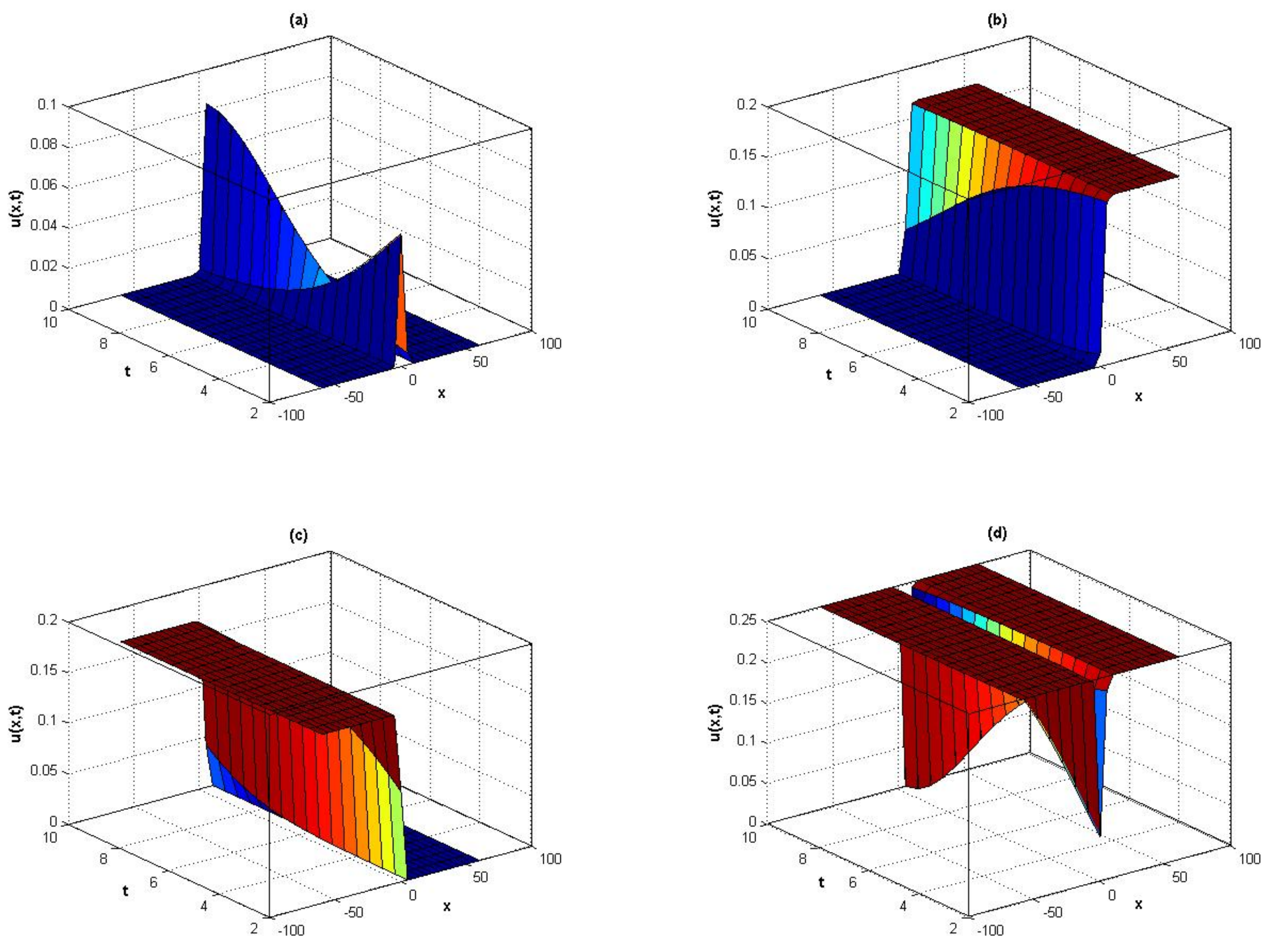

Figure 1. The exact traveling wave solution of the generalized Drinfeld-Sokolov system (a) Eq. (4.15), (b) Eq. (4.17), (c) Eq. (4.19), and (d) Eq. (4.20) with a fixed values $a = 1$, $b = 1$, $n = 2$ and $k = 1$.

5. A different type of generalized Drinfeld-Sokolov system

We next consider a different type of the DS system with negative exponent with the EFM as follows

$$u_t + (v^{-n})_x = 0, \qquad v_t - av_{xxx} + 3bu_x v + 3kuv_x = 0, \qquad n > 2, \tag{5.1}$$

where a, b, n and k are constants. Using the wave variable $\eta = x - ct$ carries the system (5.1) to

$$-cu' + (v^{-n})' = 0, \qquad cv' + av''' - 3bu'v - 3kuv' = 0, \tag{5.2}$$

where by integrating the first equation in the system Eq. (5.2) and neglecting the constant of integration we obtain

$$cu = v^{-n}. \tag{5.3}$$

Substituting (5.3) into the second equation of the system Eq. (5.2) and integrating we find

$$c^2 v + acv'' + \frac{3(nb-k)}{1-n} v^{-n+1} = 0. \tag{5.4}$$

We use of (4.5) that will carry Eq. (5.4) into the ODE

$$c^2n^2(1-n)w^2+3n^2(nb-k)w+acn(1-n)ww''+ac(1-n)^2(w')^2=0. \tag{5.5}$$

Substituting (4.11) into Eq. (5.5), we obtain the following sets of solutions

(I) The first set is:

$$\begin{aligned}
&a_1=0,\quad a_{-1}=0,\quad acn(1-n)=-3(nb-k)n^2-c^2n^2(1-n),\quad b_0=b_0,\quad b_{-1}=b_{-1},\\
&c^2n^2(1-n)=c^2n^2(1-n),\ a_0=a_0,\ ac(1-n)^2=\frac{b_0[3(nb-k)n^2-2c^2n^2(1-n)]}{a_0},\\
&b_1=\frac{1}{4}\frac{b_0^2}{b_{-1}},\quad w_1(x,t)=\frac{a_0}{b_{-1}\exp(-x+ct)+b_0+\frac{1}{4}\frac{b_0^2}{b_{-1}}\exp(x-ct)}.
\end{aligned}\tag{5.6}$$

If we choose $b_0=2b_{-1}$, then we can obtain

$$v_1(x,t)=\sqrt[n]{\frac{3(nb-k)n^2-2c^2n^2(1-n)}{2ac(1-n)^2}}\ \operatorname{sech}^{\frac{2}{n}}\left(\frac{x-ct}{2}\right),$$

$$u_1(x,t)=\frac{2a(1-n)^2}{3(nb-k)n^2-2c^2n^2(1-n)}\ \cosh^2\left(\frac{x-ct}{2}\right).$$

(II) The second set is:

$$\begin{aligned}
&a_1=0,\quad a_{-1}=0,\quad acn(1-n)=-6(nb-k)n^2,\quad b_0=b_0,\quad b_{-1}=b_{-1},\quad b_1=b_1,\\
&c^2n^2(1-n)=3(nb-k)n^2,\qquad a_0=a_0,\qquad ac(1-n)^2=-\frac{3(nb-k)n^2b_0}{a_0},\\
&w_2(x,t)=\frac{a_0}{b_{-1}\exp(-x+ct)+b_0+b_1\exp(x-ct))},\ c=-\frac{a}{2n}.
\end{aligned}\tag{5.7}$$

If we choose $b_1=b_{-1}$ and $b_0=2b_1$ then can be found

$$v_2(x,t)=\sqrt[n]{\frac{3n^3(nb-k)}{a^2(1-n)^2}}\ \left[\operatorname{sech}^2\left(\frac{2nx+at}{4n}\right)\right]^{\frac{1}{n}},$$

$$u_2(x,t)=\frac{-2a(1-n)^2}{3n^2(nb-k)}\ \cosh^2\left(\frac{2nx+at}{4n}\right).$$

(III) The third set is:

$$\begin{aligned}
&a_0=a_1=0,\ a_{-1}=a_{-1},\ acn(1-n)=-6(nb-k)n^2,\ b_0=0,\\
&b_{-1}=b_{-1},\ b_1=b_1,\ \ c^2(1-n)=12(nb-k),\ c=\frac{-2a}{n},\\
&ac(1-n)^2=-\frac{12(nb-k)n^2b_{-1}}{a_{-1}},\ \ w_3(x,t)=\frac{a_{-1}\exp(-x+ct)}{b_{-1}\exp(-x+ct)+b_1\exp(x-ct)},
\end{aligned}\tag{5.8}$$

If we choose $b_1=b_{-1}$, then can be found

$$v_3(x,t)=\sqrt[n]{\frac{3(nb-k)n^3}{a^2(1-n)^2}}\left[1-\tanh\left(\frac{nx+2at}{n}\right)\right]^{\frac{1}{n}},$$

$$u_3(x,t)=\frac{-a(1-n)^2}{12(nb-k)n^2}\left[1+\cosh\left(\frac{2nx+4at}{n}\right)+\sinh\left(\frac{2nx+4at}{n}\right)\right].$$

(IV) The fourth set is:

$$a_1=0,\quad a_{-1}=a_{-1},\quad acn(1-n)=\frac{15(bn-k)n^2}{4},\quad b_0=b_0,\quad b_1=\frac{1}{4}\frac{b_0^2}{b_{-1}}, \tag{5.9}$$

$$b_{-1} = b_{-1}, \quad c^2n^2(1-n) = 3(nb-k)n^2,\ a_0 = 0,\ ac(1-n)^2 = -\frac{3(nb-k)n^2b_{-1}}{a_{-1}},$$

$$w_4(x,t) = \frac{a_{-1}\exp(-x+ct)}{b_{-1}\exp(-x+ct) + b_0 + \frac{1}{4}\frac{b_0^2}{b_{-1}}\exp(x-ct)}, \qquad c = \frac{4a}{5n}.$$

If we choose $b_0 = 2b_{-1}$, then we can obtain

$$v_4(x,t) = \sqrt[n]{\frac{-15(nb-k)n^3}{16a^2(1-n)^2}}\left[1-\tanh\left(\frac{5nx-4at}{10n}\right)\right]^{\frac{2}{n}},$$

$$u_4(x,t) = \frac{-a(1-n)^2}{3(nb-k)n^2}\left[1+\cosh\left(\frac{5nx-4at}{5n}\right)+\sinh\left(\frac{5nx-4at}{5n}\right)\right]^2.$$

(V) The fifth set is:

$$a_1 = a_1, \quad a_{-1} = 0, \quad acn(1-n) = -6(nb-k)n^2, \quad b_0 = 0, \quad b_{-1} = b_{-1}, \quad b_1 = b_1, \tag{5.10}$$

$$a_0 = 0,\ c^2n^2(1-n) = 12(nb-k)n^2,\ ac(1-n)^2 = -\frac{12(nb-k)n^2b_1}{a_1},$$

$$w_5(x,t) = \frac{a_1\exp(x-ct)}{b_{-1}\exp(-x+ct)+b_1\exp(x-ct)}, \qquad c = \frac{-2a}{n}.$$

If we choose $b_1 = b_{-1}$, then can be found

$$v_5(x,t) = \sqrt[n]{\frac{3(nb-k)n^3}{a^2(1-n)^2}}\left[1+\tanh\left(\frac{nx+2at}{n}\right)\right]^{\frac{1}{n}},$$

$$u_5(x,t) = \frac{-a(1-n)^2}{6(nb-k)n^2}\cosh\left(\frac{nx+2at}{n}\right)\left[\cosh\left(\frac{nx+2at}{n}\right)-\sinh\left(\frac{nx+2at}{n}\right)\right].$$

(VI) The sixth set is:

$$a_1 = a_1, \quad a_{-1} = 0, \quad acn(1-n) = -\frac{15(bn-k)n^2}{4}, \quad b_0 = b_0, \quad b_1 = \frac{1}{4}\frac{b_0^2}{b_{-1}}, \tag{5.11}$$

$$b_{-1} = b_{-1}, \quad c^2n^2(1-n) = 3(nb-k)n^2,\ a_0 = 0,\ ac(1-n)^2 = -\frac{3}{4}\frac{(nb-k)n^2b_0^2}{b_{-1}a_1},$$

$$w_6(x,t) = \frac{a_1\exp(x-ct)}{b_{-1}\exp(-x+ct)+b_0+\frac{1}{4}\frac{b_0^2}{b_{-1}}\exp(x-ct)}, \qquad c = \frac{4a}{5n}.$$

If we choose $b_0 = 2b_{-1}$, then we can obtain

$$v_6(x,t) = \sqrt[n]{\frac{15(nb-k)n^3}{16a^2(1-n)^2}}\left[1+\tanh\left(\frac{5nx-4at}{10n}\right)\right]^{\frac{2}{n}},$$

$$u_6(x,t) = \frac{-4a(1-n)^2}{3(nb-k)n^2}\cosh^2\left(\frac{5nx-4at}{10n}\right)\left[\cosh\left(\frac{5nx-4at}{10n}\right)-\sinh\left(\frac{5nx-4at}{10n}\right)\right]^2.$$

(VII) The seventh set is:

$$a_1 = -\frac{a_0b_1}{b_0}, \quad a_{-1} = -\frac{1}{4}\frac{a_0b_0}{b_1}, \quad acn(1-n) = -\frac{3(nb-k)n^2}{2}, \quad b_0 = b_0, \quad b_1 = b_1, \tag{5.12}$$

$$b_{-1} = \frac{1}{4}\frac{b_0^2}{b_1}, \quad c^2n^2(1-n) = 3(nb-k)n^2,\ a_0 = a_0,\ ac(1-n)^2 = \frac{3(nb-k)n^2b_0}{a_0},$$

$$w_7(x,t) = -\frac{a_0}{b_0}\frac{b_0^2\exp(-x+ct)-4b_0b_1+4b_1^2\exp(x-ct)}{b_0^2\exp(-x+ct)+4b_0b_1+4b_1^2\exp(x-ct)}, \qquad c = \frac{-2a}{n}.$$

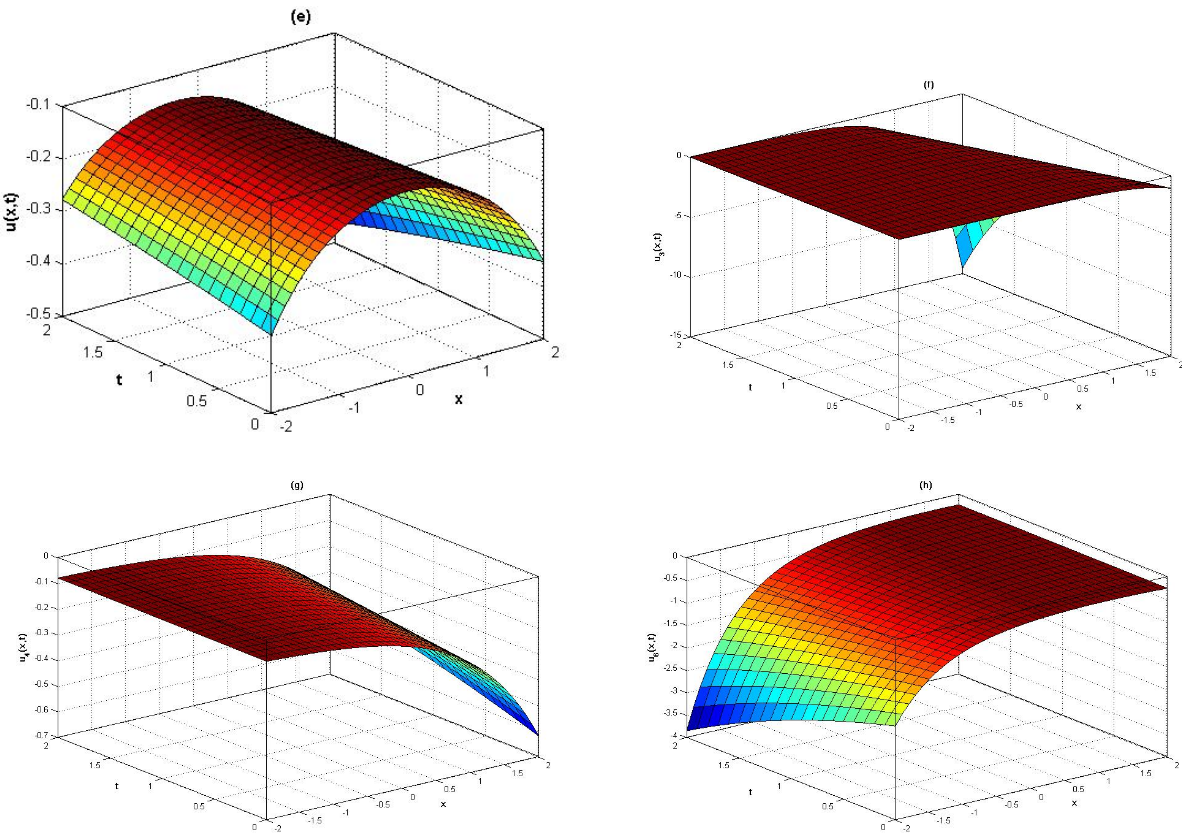

FIGURE 2. The exact traveling wave solution a variant of the Drinfeld-Sokolov system (e) Eq. (5.7), (f) Eq. (5.8), (g) Eq. (5.9), and (h) Eq. (5.11) with a fixed values $a = 1$, $b = 1$, $n = 3$ and $k = 1$.

If we choose $b_0 = 2b_1$, then we can obtain

$$v_7(x,t) = \sqrt[n]{\frac{3(nb-k)n^3}{2a^2(1-n)^2}}\tanh^{\frac{2}{n}}\left(\frac{nx+2at}{2n}\right), \quad u_7(x,t) = \frac{-2a(1-n)^2}{3(nb-k)n^2}\coth^2\left(\frac{nx+2at}{2n}\right).$$

Also, it is obvious that seven pairs of solutions were obtained by using the EFM, whereas two pairs of solutions were obtained in [41].

Remark 5.1. We obtained analytical solutions by Exp-function method. To show the properties of the solutions for Drinfeld-Sokolov and generalized Drinfeld-Sokolov equations, we take some solutions, as illustrative samples and draw theirs plots (see Figures 1-3).

6. Conclusion

In this paper, we applied the Exp-function method for constructing exact traveling wave solutions of nonlinear partial differential equations. The validity of the method was successfully applied to study three types of nonlinear equations such as the Drinfeld-Sokolov system, generalized the Drinfeld-Sokolov system and a different type of generalized Drinfeld-Sokolov system. We can successfully recover the previously known solitary wave solutions that were found by other methods. In addition, this method allows us to perform complicated and tedious algebraic calculation on the computer. Some of the results are in agreement with the results reported by others in the literature, and new results

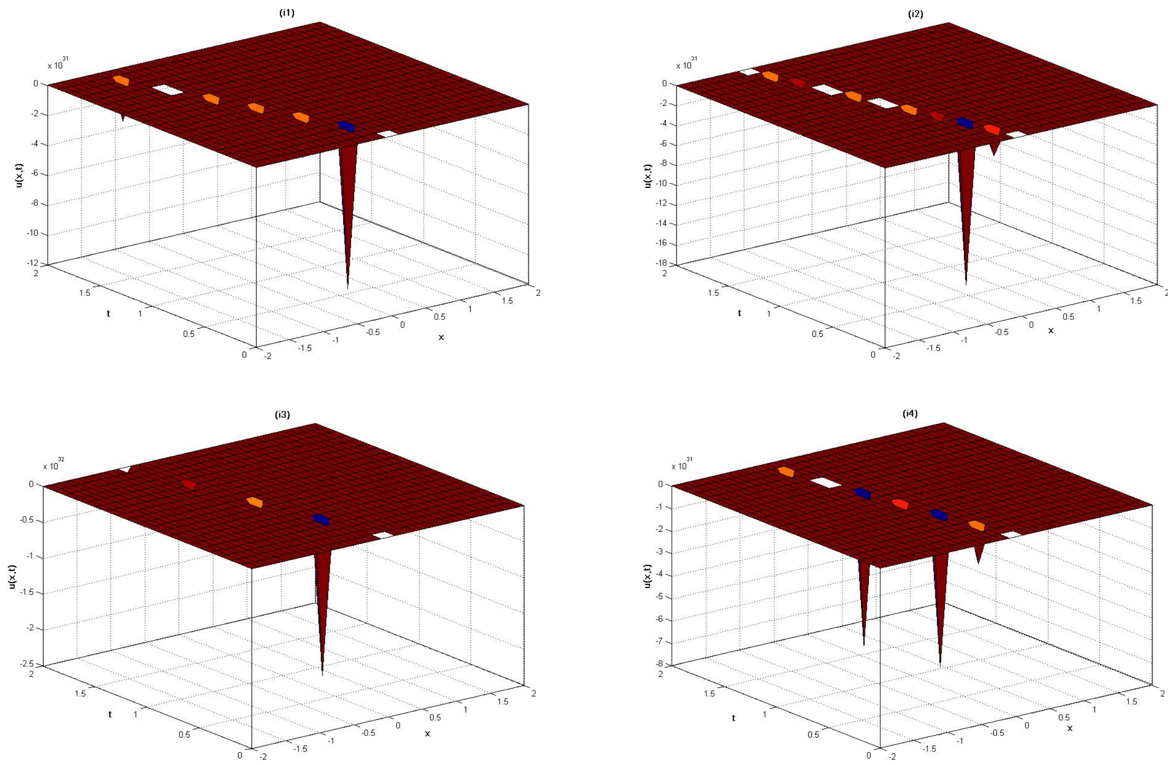

FIGURE 3. The exact traveling wave solution a variant of the Drinfeld-Sokolov system (i1) $n = 3$, (i2) $n = 4$, (i3) $n = 5$ and (i4) $n = 6$ for Eq. (5.12) with a fixed values $a = 1$, $b = 1$, and $k = 1$.

were formally developed in this work. It can be concluded that the rational Exp-function method was a very powerful and efficient technique in finding exact solutions for wide classes of problems. The solution procedure was very simple, and the obtained solution was very concise.

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