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On an efficient method for the fractional nonlinear Newell-Whithead-Segel equations

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Abstract

In this study, the time-fractional Newell-Whitehead-Segel (NWS) equation and its different nonlinearity cases are investigated. Schemes obtained by the Newtonian linearization method are used to numerically solve different cases of the time-fractional Newell-Whitehead-Segel (NWS) equation. Stability and convergence conditions of the Newtonian linearization method have been determined for the related equation. The numerical results obtained as a result of the appropriate stability criteria are compared with the help of tables and graphs with exact solutions for different fractional values.

Keywords. Newell-Whitehead-Segel equation (NWS), Newtonian type linearization method (NTLM), CFL condition, Stability and convergence criteria.
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1. Introduction

The Newell-Whitehead-Segel (NWS) equation is the modeling of the interaction of the nonlinear effect of the reaction term and the effect of the diffusion term. NWS also determines the dynamic action near the bifurcation point of Rayleigh-Bénard convection of binary fluid combinations. Rayleigh-Bénard convection (RBC) rises in a smooth fluid heated from below. Here the fluid creates a consistent arrangement of cells called Bénard cells. These forms are the best studied example of self-organizing nonlinear systems. The time-fractional NWS equation is defined as

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = k\frac{\partial^2 u}{\partial x^2} + cu - du^p, \\ u(x, t_0) = \xi(x), \\ u(x_0, t) = \psi(t), u(x_1, t) = \eta(t), \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, $c, d, k > 0$, $2 < p \in \mathbb{N}$. [1] In recent years, it has been done a lot of research on the NWS equation. Zahra et al. [21] used the cubic B-spline collocation algorithm to solve the NWS equation numerically. Singh and Sharma [14] used the homotopy perturbation Sumudu transform method to numerically solve the NWS equation, convergence and error analysis was performed. Prakash and Verma [11]used the Adomian decomposition method for the fractional NWS equation. Saadeh et al. [12] used the fractional residual power series algorithm to solve the fractional Newell–Whitehead–Segel equation. Singh and Shama [15] used the homotopy perturbation transformation with the homotopy perturbation Elzaki transform method to solve a relevant equation. Jneid and Chaouk [5] used the conformable reduced differential transform method for solving fractional Newell-Whitehead-Segel equation. Kheiri et al. [7] used the Homotopy analysis and homotopy Padé methods to solve the NWS equation. The Newtonian linearization method is a linearized finite-difference method. There are many studies on the finite-difference method in the literature. Some of them; Yokus and Bulut [19] conducted a numerical study on the Cahn-Allen equation by using the finite-difference method. Suleiman et al. [16] used the finite-difference method to solve the Sharma-Tasso-Olver equation. Yokus et al. [20] conducted a study on the exact and numerical solution of a nonlinear model arising in

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mathematical biology. Sanjaya and Mungkasi [13] used accurate explicit finite-difference method for the advection-diffusion equation. Khankishiyev [6] examined a linear loaded differential parabolic equation by finite-difference method. Messikh et al. [8] examined numerical solution for the chemotaxis model by finite- difference method. In this study, the finite-difference schemes obtained by Newtonian linearization method for the time fractional NWS equation are examined. The stability and convergence criteria of the Newtonian linearization method for different nonlinearity cases in Equation (1.1) are evaluated. The numerical results for the different nonlinear cases in Equation (1.1) are compared with the exact solution under appropriate stability and convergence conditions. It is visualized with both tables and graphs that the numerical solutions are stable and highly compatible with the exact solution.

2. Methods and Notations

In this section, it is introduced some essential definitions of fractional calculus and the main idea of the method used in our working.

2.1. Grunwald-Letnikov Approximation. Basically, we consider the following fractional ordinary differential equation

$$\begin{cases} D^{\alpha}u(t)=f(t,u(t)), 0\leq t\leq T,\\ u(t_0)=0, \end{cases} \tag{2.1}$$

here $n-1<\alpha\leq n, n\in\mathbb{N}$ and D^{α} denotes the Riemann-Liouville fractional derivative, defined by

$$D^{\alpha}u(t)=J^{n-\alpha}\left\{D^{\alpha}u(t)\right\}, \tag{2.2}$$

and J^n denotes the n th order Riemann-Liouville fractional integral operator, defined for $t>0$ as follows.[9]

$$J^{n}u(t)=\frac{1}{\Gamma(n)}\int_0^t(t-\tau)^{n-1}u(\tau)d\tau. \tag{2.3}$$

As shown at [10], to apply finite-difference schemes, we have chosen Grunwald-Letnikov method approximation for the R-L fractional derivative as follows:

$$D^{\alpha}u(t)=\lim_{h\to 0}h^{-a}\sum_{j=0}^{N}(-1)^j\binom{\alpha}{j}u(t-jh), \tag{2.4}$$

here $N=\frac{t}{h}$ and h is the step size. Therefore, Equation (2.1) is discretized (finite-difference) as:

$$h^{-\alpha}\sum_{j=0}^{N}\omega_j^{\alpha}u(t_{n-j})=f(t_n,u(t_n)), n=1,2,3,..., \tag{2.5}$$

here $t_n=nh$ and $\omega_j^{\alpha}$are the Grunwald-Letnikov coefficients defined as:

$$\omega_j^{\alpha}=(1-\frac{\alpha+1}{j}).\omega_{j-1}^{\alpha}, \omega_0^{\alpha}=1, j=1,2,3,... \tag{2.6}$$

2.2. Newtonian Type Linearization Method. While posing the finite-difference method for nonlinear ordinary, partial or fractional differential equations, one of the alternative ways is to linearize the nonlinear equation to get a finite-difference scheme. For this, the following procedure is followed: Let $u_0,u_1,...,u_N$ be dependent variables, $f_i, i=1,2,...,N$ functions.A system of nonlinear equations consisting of N equations can be given as follows

$$f_i(u_1,u_2,...,u_N)=0, i=1,2,...,N. \tag{2.7}$$

Let $V_i, i=1,2,..N$ be the known approximations for u_i values. By $u_i=V_i+\varepsilon_i$ transforming and Taylor series around V_i for $f_i, i=1,2,\dots N$ functions can be written as

$$f_i(u_1,u_2,...,u_N)=f_i(V_1,V_2,...,V_N)+[\frac{\partial f_i}{\partial u_1}.\varepsilon_1+\frac{\partial f_i}{\partial u_2}.\varepsilon_2+...+\frac{\partial f_i}{\partial u_N}.\varepsilon_N]_{u_i=V_i}+..., \tag{2.8}$$

By ignoring after the first derivative, the following difference scheme can be obtained.

$$f_i(u_1,u_2,...,u_N)=f_i(V_1,V_2,...,V_N)+[\frac{\partial f_i}{\partial u_1}.\varepsilon_1+\frac{\partial f_i}{\partial u_2}.\varepsilon_2+...+\frac{\partial f_i}{\partial u_N}.\varepsilon_N]_{u_i=V_i}=0. \tag{2.9}$$

Thus, the nonlinear equation system turns into a linear equation system consisting of N unknowns $\varepsilon_1,\varepsilon_2,...,\varepsilon_N$. The method for obtaining a linear equation system from the non-linear equation system with the Taylor series method given above is called the Newtonian type linearization method. This process can be continued until the desired degree of accuracy is found.

3. Finite-Difference Scheme Obtained by Newtonian Type Linearization Method for the Time-Fractional NWS Equation

In this section, it will be given how to find the finite difference scheme obtained by Newtonian type linearization method for Equation (1.1). Using the difference approximations of the time-fractional derivative and the second-order derivative

$$\begin{gathered}\frac{\partial^\alpha u}{\partial t^\alpha}\approx\frac{1}{\Delta t^\alpha}\sum_{j=0}^{k+1}\omega_j^\alpha u_m^{k-j+1},\\ \frac{\partial^2 u}{\partial x^2}=\frac{u_{m+1}^k-2u_m^k+u_{m-1}^k}{\Delta x^2},\end{gathered} \tag{3.1}$$

and substituting in Equation (1.1), difference scheme is obtained like

$$\frac{1}{\Delta t^\alpha}\sum_{j=0}^{k+1}\omega_j^\alpha u_m^{k-j+1}=\frac{u_{m+1}^k-2u_m^k+u_{m-1}^k}{\Delta x^2}+u_m^{k+1}-(u_m^{k+1})^p, \tag{3.2}$$

by multiplying with Δt^α both sides of Equation (3.2), the scheme is obtained as

$$\sum_{j=0}^{k+1}\omega_j^\alpha u_m^{k-j+1}=\frac{\Delta t^\alpha}{\Delta x^2}.(u_{m+1}^k-2u_m^k+u_{m-1}^k)+\Delta t^\alpha u_m^{k+1}-\Delta t^\alpha(u_m^{k+1})^p. \tag{3.3}$$

By rearranging Equation (3.3), the finite-difference scheme can also be expressed as follows.

$$\begin{cases}u_m^{k+1}-\Delta t^\alpha u_m^{k+1}+\Delta t^\alpha(u_m^{k+1})^p-\frac{\Delta t^\alpha}{\Delta x^2}.(u_{m+1}^k-2u_m^k+u_{m-1}^k)+\sum_{j=1}^{k+1}\omega_j^\alpha u_m^{k-j+1}=0,\\ u_m^{k+1}=V_m^k+\varepsilon_m^k, m=1,2,...M-1, k=0,1,...,K-1.\end{cases} \tag{3.4}$$

Here V_m^k is known values and ε_m^k are unknown values. By extending to the Taylor series up to the first order derivative around $u_m^{k+1}=V_m^k$ in the Equation (3.4) and writing the equation in terms of the dependent variable, the difference scheme can be written as

$$\begin{cases}\varepsilon_m^{k+1}=\frac{-V_m^k+\Delta t^\alpha V_m^k-\Delta t^\alpha(V_m^k)^p-\sum_{j=1}^{k+1}\omega_j^\alpha u_m^{k-j+1}+\frac{\Delta t^\alpha}{\Delta x^2}(u_{m+1}^k-2u_m^k+u_{m-1}^k)}{1-\Delta t^\alpha+p\Delta t^\alpha(V_m^k)^{p-1}},\\ V_m^k=u_m^k,\\ u_m^{k+1}=V_m^k+\varepsilon_m^k, \qquad m=1,2,...M-1, \quad k=0,1,...,K-1.\end{cases} \tag{3.5}$$

By rewriting u_m^{k+1} in Equation (3.5), the following scheme is found.

$$\begin{cases}u_m^{k+1}=\frac{(p-1)\Delta t^\alpha(u_m^k)^p-\sum_{j=1}^{k+1}\omega_j^\alpha u_m^{k-j+1}+\frac{\Delta t^\alpha}{\Delta x^2}(u_{m+1}^k-2u_m^k+u_{m-1}^k)}{1-\Delta t^\alpha+p\Delta t^\alpha(u_m^k)^{p-1}},\\ u_m^0=\xi(x_m), u_0^k=\psi(t_k), u_M^k=\eta(t_k),\\ m=1,2,...M-1, \quad k=0,1,...,K-1, \quad 0<\alpha\le 1.\end{cases} \tag{3.6}$$

This founded scheme is called finite-difference scheme (NTLM scheme) obtained by Newtonian type linearization method.

4. Stability and Convergence Criteria of Newtonian Type Linearization Method for Time-Fractional NWS Equation

In this section, stability and convergence criteria of Newtonian type linearization method for time-fractional NWS equation are given.

4.1. Courant-Friedrichs-Lewy (CFL) Condition for Finite-Difference Methods.

The Courant-Friedrichs-Lewy (CFL) condition is used to obtain stable and convergent numerical schemes for fluid dynamics equations, parabolic and hyperbolic partial differential equations or systems to provide more consistent and accurate solutions numerically. The CFL condition expresses the existence of an upper bound on the local time step of the solution element given in explicit numerical schemes for second-order parabolic equations. Namely,

$$C_{CFL} = \sigma \frac{\Delta t}{\Delta x^2} \leq C_{max}, \tag{4.1}$$

Here, σ is the coefficient appearing in the equations, such as wave velocity, velocity magnitude. C_{CFL} is a dimensionless number called the Courant number. C_{max} is a number that can take different values in implicit or explicit schemes. It is usually taken as $C_{max} = 1$ in explicit schemes. [3, 4]

Theorem 4.1. *CFL condition; It is a necessary condition for the convergence of numerical approximations of linear, non-linear, variable coefficient, given by initial-boundary value problems or formed according to any norm, etc. partial differential equations. [18]*

Theorem 4.2. *Let the finite-difference approximation be a consistent numerical approximation for the well-positioned linear initial-boundary value problem. The CFL condition is a necessary condition for the stability of the finite-difference approximation. [18]*

The CFL condition is a necessary condition for numerical finite-difference schemes to be stable and convergent. However, it is not a sufficient condition. This is also the disadvantage of the CFL condition.

4.2. Investigation of Convergence of Newtonian Type Linearization Method for Time-Fractional NWS Equation.

Remark 4.3. The inequality $px^{p+1} \leq (p-1)x^p$ is satisfied for $p \geq 2, p \in \mathbb{N}, 0 \leq x \leq \frac{1}{2}$.[2]

Proof. Since $0 \leq x \leq \frac{1}{2}$, it can be written the inequality

$$x \leq \frac{1}{2} \leq \frac{2}{3} \leq \ldots \leq \frac{p-1}{p}. \tag{4.2}$$

By rearranging in the (4.2), it be found the following inequality

$$x \leq \frac{p-1}{p} \Rightarrow px \leq (p-1) \Rightarrow px^{p+1} \leq (p-1)x^p. \tag{4.3}$$

□

For $p \geq 2, p \in \mathbb{N}, 0 \leq x \leq \frac{1}{2}$, the inequality $\frac{1}{1+(p-1)x^p} \leq \frac{1}{1+px^{p+1}}$ is satisfied. For $0 < \alpha \leq 1$, inequality $\left|\omega_j^\alpha\right| \leq \frac{1}{j}$ is satisfied.[17] By rearranging u_m^{k+1} and taking $R = \frac{\Delta t^\alpha}{\Delta x^2}$ in Equation (3.6), the difference scheme can be written as

$$u_m^{k+1} = \frac{(p-1)\Delta t^\alpha (u_m^k)^p - \sum_{j=2}^{k+1} \omega_j^\alpha u_m^{k-j+1} + (\alpha - 2R)u_m^k + R(u_{m+1}^k + u_{m-1}^k)}{1 - \Delta t^\alpha + p\Delta t^\alpha (u_m^k)^{p-1}}. \tag{4.4}$$

First, we will examine the positivity conditions of Equation (4.4). For this, let's assume $0 \leq u_m^k \leq 1$.It is sufficient to satisfy the condition $\alpha - 2R \geq 0$ for u_m^{k+1} to be positive. Namely,

$$\alpha - 2R \geq 0 \Leftrightarrow R = \frac{\Delta t^\alpha}{\Delta x^2} \leq \frac{\alpha}{2}, 0 < \alpha \leq 1. \tag{4.5}$$

Now, let's examine the bounded conditions of Equation (4.4). For this, it is necessary to show that $0 \le u_m^{k+1} \le 1$ while $0 \le u_m^k \le 1$.In the first stage of boundedness, the condition $(p-1)\Delta t^\alpha (u_m^k)^p < 1 - \Delta t^\alpha$ for $2 \le p \in \mathbb{N}$ must be satisfied. Namely,

$$(p-1)\Delta t^\alpha (u_m^k)^p < 1 - \Delta t^\alpha \Leftrightarrow \Delta t^\alpha \le \frac{1}{1+(p-1)(u_m^k)^p}. \tag{4.6}$$

Because of the (4.6), satisfying the condition $(p-1)\Delta t^\alpha (u_m^k)^p < 1 - \Delta t^\alpha$ is equivalent to satisfying the condition $\Delta t^\alpha \le \frac{1}{1+(p-1)(u_m^k)^p}$. In the second stage, let's prove by induction that $\frac{\alpha}{2} \le \frac{1}{1+(p-1)(u_m^k)^p}$ when conditions $2 \le p \in \mathbb{N}, R = \frac{\Delta t^\alpha}{\Delta x^2} \le \frac{\alpha}{2}, 0 < \alpha \le 1, u_m^k \le \frac{1}{2}$ are satisfied. By using the inequalities $(u_m^k)^2 < 1$ and $0 < \alpha \le 1$ we get following inequality.

$$0 < \alpha + \alpha (u_m^k)^2 < 2. \tag{4.7}$$

By rearranging in (4.7), $\frac{\alpha}{2} < \frac{1}{1+(u_m^k)^2}$ is obtained. This relation is provided for $p = 2$. Let the given inequality be true for $p = r$. That is, let the inequality $\frac{\alpha}{2} < \frac{1}{1+(r-1)(u_m^k)^r}$ be satisfied. Let us show that the given inequality is also true for $p = r+1$. By using Corollary 1, we get the following inequality.

$$\frac{1}{1+(r-1)(u_m^k)^r} \le \frac{1}{1+(r)(u_m^k)^{r+1}}. \tag{4.8}$$

So, the inequality $\frac{\alpha}{2} < \frac{1}{1+(r-1)(u_m^k)^r} \le \frac{1}{1+(r)(u_m^k)^{r+1}}$ is found. So, it is also true for $p = r+1$. When Δx is chosen small enough, $\frac{\Delta t^\alpha}{\Delta x^2} \le \frac{\alpha}{2} \Rightarrow \Delta t^\alpha \le \frac{\alpha}{2}\Delta x^2 \le \frac{\alpha}{2}, 0 < \alpha \le 1$ can be written. Thus, we get following inequality.

$$\Delta t^\alpha \le \frac{\alpha}{2} < \frac{1}{1+(p-1)(u_m^k)^p} \le \frac{1}{1+(p)(u_m^k)^{p+1}}. \tag{4.9}$$

Because of the (4.6), inequality $(p-1)\Delta t^\alpha (u_m^k)^p < 1 - \Delta t^\alpha$ is satisfied for $2 \le p \in \mathbb{N}$. Let's use abbreviation $B(\alpha) = (\alpha - 2R)u_m^k + R(u_{m+1}^k + u_{m-1}^k) - \sum_{j=2}^{k+1} \omega_j^\alpha u_m^{k-j+1}$ in Equation (4.4). Taking $\alpha = 1, R = 0.5$, and $u_{m+1}^k = u_{m-1}^k = 0$ it is found $B(\alpha) = 0$ for $k \ge 1$. Thus, we get following equality.

$$\begin{aligned} |B(\alpha)| &= \left| (\alpha - 2R)u_m^k + R(u_{m+1}^k + u_{m-1}^k) - \textstyle\sum_{j=2}^{k+1} \omega_j^\alpha u_m^{k-j+1} \right| \\ &\le R + \tfrac{\alpha - 2R}{2} + \tfrac{1}{2} \textstyle\sum_{j=2}^{k+1} \left|\omega_j^\alpha\right| \le R + \tfrac{\alpha-2R}{2} + \tfrac{k}{4} = \tfrac{\alpha}{2} + \tfrac{k}{4}. \end{aligned} \tag{4.10}$$

Since $B(\alpha) \le \frac{\alpha}{2} + \frac{k}{4}$ and $0 < \alpha \le 1, 0 \le B(\alpha) < 1$ is found.Because of $0 \le B(\alpha) < 1$, it can written following inequality

$$1 - \Delta t^\alpha + p\Delta t^\alpha (u_m^k)^{p-1} \le 1 - \Delta t^\alpha + p\Delta t^\alpha (u_m^k)^{p-1} + B(\alpha). \tag{4.11}$$

By using the (4.11) and after various editing, we gets following inequality

$$\frac{(p-1)\Delta t^\alpha (u_m^k)^p + B(\alpha)}{1 - \Delta t^\alpha + p\Delta t^\alpha (u_m^k)^{p-1} + B(\alpha)} \le \frac{(p-1)\Delta t^\alpha (u_m^k)^p + B(\alpha)}{1 - \Delta t^\alpha + p\Delta t^\alpha (u_m^k)^{p-1}}. \tag{4.12}$$

In the last step, let us show that $(p-1)\Delta t^\alpha (u_m^k)^p + B(\alpha) \le 1 - \Delta t^\alpha$ for $\forall B(\alpha)$ when the conditions for $R = \frac{\Delta t^\alpha}{\Delta x^2} \le \frac{\alpha}{2}, 0 < \alpha \le 1$ and $u_m^k \le \frac{1}{2}$ are satisfied. In order to prove by the method of contradiction, let's assume inequality $(p-1)\Delta t^\alpha (u_m^k)^p + B(\alpha) > 1 - \Delta t^\alpha$.Then, Since this inequality will be satisfied for $\forall B(\alpha)$ it must also provide for $B(\alpha) = 0$. Then we can be written inequality $(p-1)\Delta t^\alpha (u_m^k)^p > 1 - \Delta t^\alpha$ After arranging, it is found inequality $\Delta t^\alpha > \frac{1}{1+(p-1)(u_m^k)^p}$.This inequality contradicts with the (4.6). Then, it can be written following inequality

$$(p-1)\Delta t^\alpha (u_m^k)^p + B(\alpha) \le 1 - \Delta t^\alpha \le 1 - \Delta t^\alpha + p\Delta t^\alpha (u_m^k)^{p-1}. \tag{4.13}$$

Finally, taking into account the (4.12), the following inequality for the boundedness is obtained.

$$\frac{(p-1)\Delta t^\alpha (u_m^k)^p + B(\alpha)}{1 - \Delta t^\alpha + p\Delta t^\alpha (u_m^k)^{p-1} + B(\alpha)} \le \frac{(p-1)\Delta t^\alpha (u_m^k)^p + B(\alpha)}{1 - \Delta t^\alpha + p\Delta t^\alpha (u_m^k)^{p-1}} = u_m^{k+1} \le 1. \tag{4.14}$$

Table 1. Comparison of numerical solutions for $p = 2$ in Example 5.1.

x_m	*Exact*	$\alpha = 1$	$\alpha = \frac{e}{3}$	$\alpha = 0.9$
0	*0.4858916457*	*0.4858916457*	*0.4858916457*	*0.4858916457*
0.1	*0.4738528457*	*0.4738856794*	*0.4732948535*	*0.4732294208*
0.2	*0.4617820225*	*0.4618702204*	*0.4608292604*	*0.4607140288*
0.3	*0.4496921627*	*0.4498464941*	*0.4484921509*	*0.4483423340*
0.4	*0.4375965071*	*0.4378162487*	*0.4362816189*	*0.4361120206*
0.5	*0.4255085007*	*0.4257817550*	*0.4241965144*	*0.4240215394*
0.6	*0.4134417432*	*0.4137458037*	*0.4122364145*	*0.4120700663*
0.7	*0.4014099287*	*0.4017117059*	*0.4004015839*	*0.4002574629*
0.8	*0.3894267963*	*0.3896832894*	*0.3886929661*	*0.3885842596*
0.9	*0.3775060651*	*0.3776648965*	*0.3771121514*	*0.3770516284*
1.0	*0.3656613806*	*0.3656613806*	*0.3656613806*	*0.3656613806*

Consequently, Thus, u_m^{k+1} is both positive and bounded when the conditions for $R = \frac{\Delta t^\alpha}{\Delta x^2} \leq \frac{\alpha}{2}$ and $u_m^k \leq \frac{1}{2}$ are satisfied. Under these conditions, Equation (3.6) is consistent. In this case, the CFL condition is satisfied when the conditions are met $R = \frac{\Delta t^\alpha}{\Delta x^2} \leq \frac{\alpha}{2}$ and $u_m^k \leq \frac{1}{2}$. According to Theorems 4.1 and 4.2, the NTLM scheme for time-fractional NWS equation is both stable and convergent.

5. Applications

In this section, it has used the Newtonian type linearization method to get numerical solutions of the time-fractional NWS equation. Difference schemes for different nonlinear cases of time fractional NWS equation have been generated and the results have been evaluated with the help of tables and graphs. In this section, it has used the Newtonian type linearization method to get numerical solutions of the time-fractional NWS equation. Difference schemes for different nonlinear cases of time fractional NWS equation have been generated and the results have been evaluated with the help of tables and graphs. All numerical results are obtained by taking $t = 1, \Delta x = \frac{1}{10}, \Delta t = \frac{1}{1000}, M = 10, K = 1000$.

Example 5.1. *We consider the following initial-boundary value problem for $k = 1, c = 1, d = 1, p = 2$ in the Equation (1.1):*

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u - u^2, \\ u(x,0) = \xi(x) = \frac{1}{(1+e^{\frac{x}{\sqrt{6}}})^2}, \\ u(0,t) = \psi(t) = \frac{1}{(1+e^{\frac{-5t}{6}})^2}, u(1,t) = \eta(t) = \frac{1}{(1+e^{\frac{1}{\sqrt{6}}-\frac{5t}{6}})^2}. \end{cases} \tag{5.1}$$

The exact solution of Equation (3.5) introduced by Zulfiqar et al. [1] is as follows:

$$u(x,t) = \frac{1}{(1+e^{\frac{x}{\sqrt{6}}-\frac{5t}{6}})^2}. \tag{5.2}$$

By taking $p = 2$ in Equation (3.6), the NTLM scheme for the time-fractional NWS equation is found as follows.

$$\begin{cases} u_m^{k+1} = \frac{\Delta t^\alpha (u_m^k)^2 - \sum_{j=1}^{k+1} \omega_j^\alpha u_m^{k-j+1} + \frac{\Delta t^\alpha}{\Delta x^2}(u_{m+1}^k - 2u_m^k + u_{m-1}^k)}{1 - \Delta t^\alpha + 2\Delta t^\alpha (u_m^k)}, \\ u_m^0 = \xi(x_m), \quad u_0^k = \psi(t_k), \quad u_M^k = \eta(t_k), \\ m = 1, 2, ...M-1, \quad k = 0, 1, ..., K-1, \quad 0 < \alpha \leq 1. \end{cases} \tag{5.3}$$

TABLE 2. Comparison of numerical solutions for $p = 2$ in Example 5.1.

x_m	*Exact*	$\alpha = \frac{\phi}{2}$	$\alpha = 0.8$	$\alpha = \frac{\pi}{4}$
0	0.4858916457	0.4858916457	0.4858916457	0.4858916457
0.1	0.4738528457	0.4715459108	0.4712808496	0.4708106577
0.2	0.4617820225	0.4577496019	0.4572828984	0.4564186095
0.3	0.4496921627	0.4444893013	0.4438827815	0.4428103019
0.4	0.4375965071	0.4317525740	0.4310665139	0.4297932698
0.5	0.4255085007	0.4195275125	0.4188205660	0.4175725667
0.6	0.4134417432	0.4078023849	0.4071314533	0.4058857803
0.7	0.4014099287	0.3965653812	0.3959854221	0.3949615516
0.8	0.3894267963	0.3858044667	0.3853682780	0.3845596719
0.9	0.3775060651	0.3755073216	0.3752653166	0.3748373601
1.0	0.3656613806	0.3656613806	0.3656613806	0.3656613806

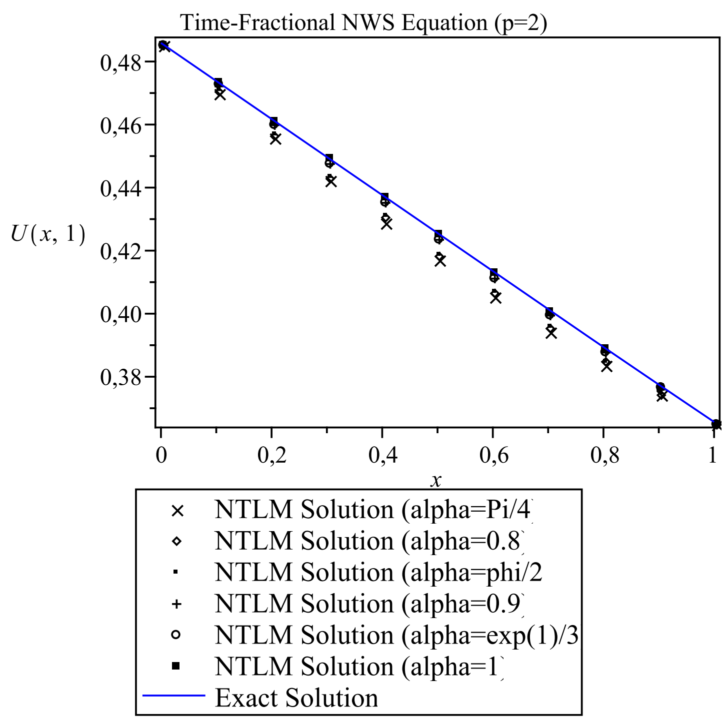

FIGURE 1. Comparison of numerical solutions for $p = 2$ in Example 5.1.

Example 5.2. *We consider the following initial-boundary value problem for* $k = 1, c = 1, d = 1, p = 3$ *in the Equation* (1.1)*:*

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u - u^3, \\ u(x,0) = \xi(x) = \frac{-1+e^{\frac{x}{\sqrt{2}}}}{1+e^{\frac{x}{\sqrt{2}}}}, \\ u(0,t) = \psi(t) = 0, \\ u(1,t) = \eta(t) = \frac{e^{\frac{1}{\sqrt{2}}} - e^{\frac{-1}{\sqrt{2}}}}{e^{\frac{1}{\sqrt{2}}} + e^{\frac{-1}{\sqrt{2}}} + 2e^{\frac{-3t}{2}}}. \end{cases} \tag{5.4}$$

TABLE 3. Comparison of numerical solutions for $p = 3$ in Example 5.2.

x_m	*Exact*	$\alpha = 1$	$\alpha = \frac{e}{3}$	$\alpha = 0.9$
0	0	0	0	0
0.1	0.0577413647	0.0577882904	0.0569782605	0.0569134305
0.2	0.1150659662	0.1151554103	0.1135832318	0.1134579061
0.3	0.1715694731	0.1716930394	0.1694550404	0.1692777848
0.4	0.2268716326	0.2270177201	0.2242595036	0.2240429213
0.5	0.2806264256	0.2807812879	0.2776984746	0.2774589298
0.6	0.3325301859	0.3326791432	0.3295178285	0.3292751227
0.7	0.3823273369	0.3824560078	0.3795129312	0.3792899601
0.8	0.4298136205	0.4299090461	0.4275316295	0.4273540568
0.9	0.4748369009	0.4748884530	0.4734749448	0.4733709277
1.0	0.5172957920	0.5172957920	0.5172957920	0.5172957920

TABLE 4. Comparison of numerical solutions for $p = 3$ in Example 5.2.

x_m	*Exact*	$\alpha = \frac{\phi}{2}$	$\alpha = 0.8$	$\alpha = \frac{\pi}{4}$
0	0	0	0	0
0.1	0.0577413647	0.0559271748	0.0558317182	0.0563405527
0.2	0.1150659662	0.1115523753	0.1113679363	0.1098178257
0.3	0.1715694731	0.1665850368	0.1663243766	0.1676337250
0.4	0.2268716326	0.2207565039	0.2204383200	0.2179078096
0.5	0.2806264256	0.2738290204	0.2734774814	0.2750366871
0.6	0.3325301859	0.3256029055	0.3252471240	0.3226690413
0.7	0.3823273369	0.3759218266	0.3755953236	0.3767776209
0.8	0.4298136205	0.4246762159	0.4244164399	0.4227665491
0.9	0.4748369009	0.4718049831	0.4716529318	0.4720570996
1.0	0.5172957920	0.5172957920	0.5172957920	0.5172957920

The exact solution of (4.11) *introduced by Zulfiqar et al.* [1] *is as follows:*

$$u(x,t) = \frac{e^{\frac{x}{\sqrt{2}}} - e^{\frac{-x}{\sqrt{2}}}}{e^{\frac{x}{\sqrt{2}}} + e^{\frac{-x}{\sqrt{2}}} + 2e^{\frac{-3t}{2}}}. \tag{5.5}$$

By taking $p = 3$ *in Equation* (3.6)*, the NTLM scheme for time-fractional NWS equation is found as follows.*

$$\begin{cases} u_m^{k+1} = \frac{2\Delta t^\alpha (u_m^k)^3 - \sum_{j=1}^{k+1} \omega_j^\alpha u_m^{k-j+1} + \frac{\Delta t^\alpha}{\Delta x^2}(u_{m+1}^k - 2u_m^k + u_{m-1}^k)}{1 - \Delta t^\alpha + 3\Delta t^\alpha (u_m^k)^2}, \\ u_m^0 = \xi(x_m), u_0^k = \psi(t_k), u_M^k = \eta(t_k), \\ m = 1, 2, \ldots M-1, \quad k = 0, 1, \ldots, K-1, \quad 0 < \alpha \leq 1. \end{cases} \tag{5.6}$$

Example 5.3. *We consider the following initial-boundary value problem for* $k = 1, c = 1, d = 1, p = 4$ *in the Equation* 1.1*:*

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u - u^4, \\ u(x,0) = \xi(x) = \frac{1}{(1+e^{\frac{3x}{\sqrt{10}}})^{\frac{2}{3}}}, \\ u(0,t) = \psi(t) = (\frac{1}{2}tanh(\frac{21t}{20}) + \frac{1}{2})^{\frac{2}{3}}, \\ u(1,t) = \eta(t) = (\frac{1}{2}tanh(\frac{-3}{2\sqrt{10}} + \frac{21t}{20}) + \frac{1}{2})^{\frac{2}{3}}. \end{cases} \tag{5.7}$$

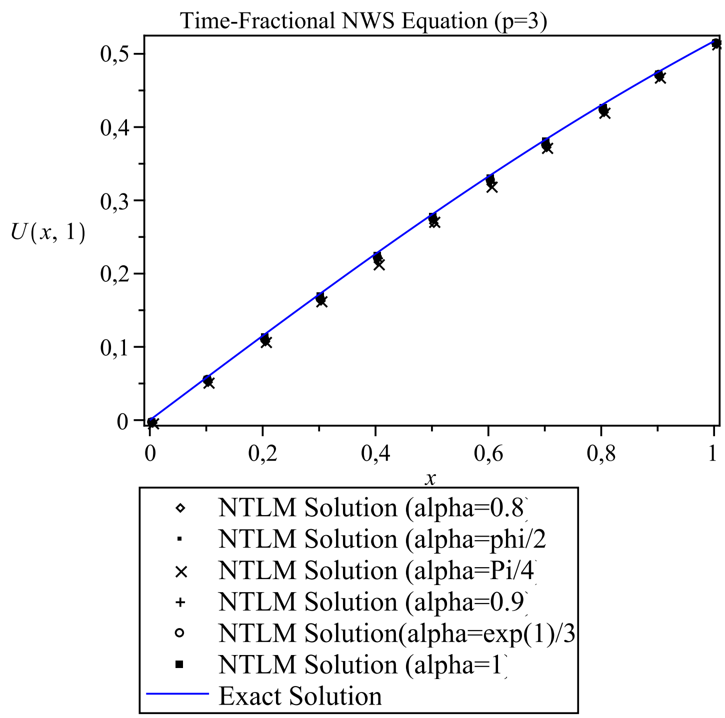

FIGURE 2. Comparison of numerical solutions for $p = 3$ in Example 5.2.

The exact solution of Equation (5.7) *introduced by Zulfiqar et al. [1] is as follows:*

$$u(x,t) = (\frac{1}{2}tanh(\frac{-3}{2\sqrt{10}}(x - \frac{7t}{\sqrt{10}}) + \frac{1}{2})^{\frac{2}{3}}. \tag{5.8}$$

By taking $p = 4$ *in Equation* (3.6)*, the NTLM scheme for time-fractional NWS equation is found as follows.*

$$\begin{cases} u_m^{k+1} = \frac{3\Delta t^\alpha (u_m^k)^4 - \sum_{j=1}^{k+1} \omega_j^\alpha u_m^{k-j+1} + \frac{\Delta t^\alpha}{\Delta x^2}(u_{m+1}^k - 2u_m^k + u_{m-1}^k)}{1 - \Delta t^\alpha + 4\Delta t^\alpha (u_m^k)^3}, \\ u_m^0 = \xi(x_m), \quad u_0^k = \psi(t_k), \quad u_M^k = \eta(t_k), \\ m = 1, 2, ...M-1, \quad k = 0, 1, ..., K-1, \quad 0 < \alpha \leq 1. \end{cases} \tag{5.9}$$

TABLE 5. Comparison of numerical solutions for $p = 4$ in Example 5.3.

x_m	*Exact*	$\alpha = 1$	$\alpha = \frac{e}{3}$	$\alpha = 0.9$
0	*0.9258778028*	*0.9258778028*	*0.9258778028*	*0.9258778028*
0.1	*0.9192365524*	*0.9192296090*	*0.9155227292*	*0.9152839901*
0.2	*0.9120702733*	*0.9120581185*	*0.9055380919*	*0.9051186898*
0.3	*0.9043500324*	*0.9043342929*	*0.8958487145*	*0.8953034979*
0.4	*0.8960475153*	*0.8960297316*	*0.8863943913*	*0.8857759632*
0.5	*0.8871354786*	*0.8871171274*	*0.8771274929*	*0.8764869797*
0.6	*0.8775882553*	*0.8775707682*	*0.8680111087*	*0.8673987582*
0.7	*0.8673823111*	*0.8673670925*	*0.8590176680*	*0.8584833228*
0.8	*0.8564968414*	*0.8564852873*	*0.8501279723*	*0.8497214461*
0.9	*0.8449144011*	*0.8449079120*	*0.8413305837*	*0.8411019802*
1.0	*0.8326215520*	*0.8326215520*	*0.8326215520*	*0.8326215520*

TABLE 6. Comparison of numerical solutions for $p = 4$ in Example 5.3.

x_m	*Exact*	$\alpha = \frac{\phi}{2}$	$\alpha = 0.8$	$\alpha = \frac{\pi}{4}$
0	*0.9258778028*	*0.9258778028*	*0.9258778028*	*0.9258778028*
0.1	*0.9192365524*	*0.9118251063*	*0.9114953379*	*0.9109675807*
0.2	*0.9120702733*	*0.8990562619*	*0.8984798127*	*0.8975553992*
0.3	*0.9043500324*	*0.8874374516*	*0.8866911487*	*0.8854984459*
0.4	*0.8960475153*	*0.8768673746*	*0.8760236246*	*0.8746716383*
0.5	*0.8871354786*	*0.8672710195*	*0.8663992705*	*0.8650072299*
0.6	*0.8775882553*	*0.8585950018*	*0.8577629124*	*0.8564302124*
0.7	*0.8673823111*	*0.8508041508*	*0.8500785847*	*0.8489202150*
0.8	*0.8564968414*	*0.8438791776*	*0.8433270704*	*0.8424428886*
0.9	*0.8449144011*	*0.8378152697*	*0.8375044519*	*0.8370080196*
1.0	*0.8326215520*	*0.8326215520*	*0.8326215520*	*0.8326215520*

6. Results and Discussion

The time-fractional NWS equations given by the initial-boundary problem are solved numerically using the Newtonian type linearization finite method associated with the Grunwald-Letnikov approach. In addition, finite-difference schemes obtaining Newtonian type linearization method are solved in Maple. The discrete schemes are explicit. Also, $t = 1, \Delta x = \frac{1}{10}, \Delta t = \frac{1}{1000}, M = 10, K = 1000, \alpha = \frac{\pi}{4}, 0.8, \frac{\phi}{2}, 0.9, \frac{e}{3}, 1$ is taken for all examples in this study. Where, ϕ is the golden ratio. Furthermore, the order of fractional derivative is chosen as irrational and rational numbers to illustrate the performance of methodology. Table 1-6 show finite-difference solutions using the Newtonian type linearization schemes with different fractional order for p=2 in Example 5.1, p=3 in Example 5.2, p=4 in Example 5.3 two by two respectively.

7. Conclusions

Finite-difference solutions of the time-fractional Newell-Whitehead-Segel equations have been obtained with the help of Newtonian type linearization method. When the results obtained from these figures and tables are interpreted, as the α value approaches 1, finite-difference solutions for time-fractional NWS equations are highly compatible with exact solutions. Newtonian type linearization methods are highly effective and reliable methods to numerically solve the time-fractional NWS equation for $p = 2, 3$, and 4. For irrational order of fractional derivative, results to obtain from Newtonian type finite-difference schemes are compatible with other and effective with the programming. It

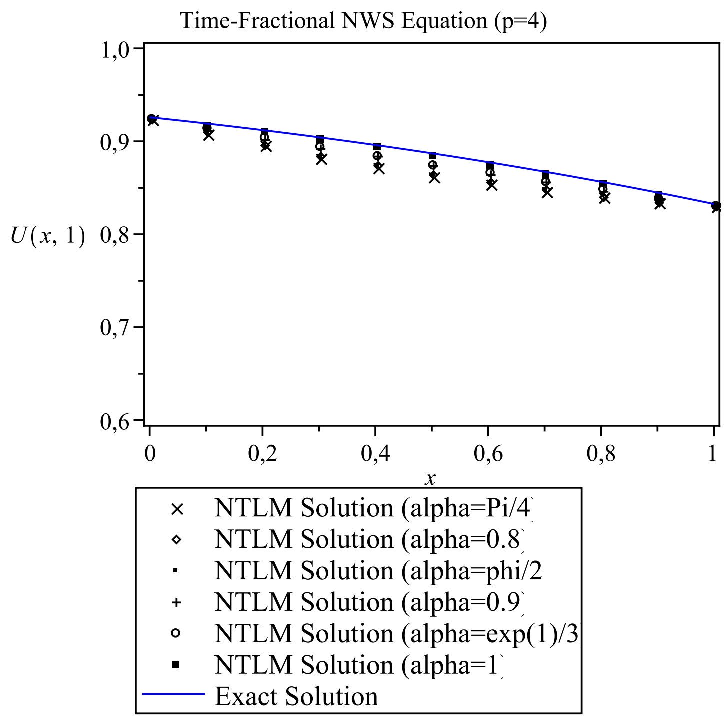

FIGURE 3. Comparison of numerical solutions for $p = 4$ in Example 5.3.

is concluded that the Newtonian type linearization method is very powerful and efficient techniques for solving the time-fractional Newell-Whitehead-Segel equation even if irrational order. The stability and convergence criteria of the Newtonian type linearization method were determined for the time-fractional NWS equation. The time and spatial step size in all examples are taken according to these stability criteria. Considering the stability criteria in all examples solved with the Newtonian type linearization method proposed for the time-fractional NWS equation, it is seen that the numerical solutions are in harmony with the exact solution and numerical solution for $\alpha = 1$ coincides with the exact solution from the figures. Consequently, it is concluded that the numerical solutions found by the Newtonian type linearization method proposed for the time-fractional NWS equation can be used to determine the dynamic motion of Rayleigh-Bénard convection near the bifurcation point of the binary fluid combinations heated from below. In addition, by taking larger p values, stability properties can be sampled and the study can be moved to a more general form by examining the spatial-fractional NWS equation.

This study is obtained from Emre Aydin's Master Thesis titled "Numerical solutions and stability properties of time fractional Newell-Whitehead-Segel equations" dated March 5, 2021. The supervisor of the thesis is Assist Prof. Dr. Inci Cilingir Sungu [2].

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