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The complex SEE transform technique in difference equations and differential difference equations

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Abstract

Differential equations are used to represent different scientific problems are handled efficiently by integral transformations, where integral transforms represent an easy and effective tool for solving many problems in the mentioned fields. This work utilizes the integral transform of the Complex SEE integral transformation to provide an efficient solution method for the difference and differential-difference equations by benefiting from the properties of this complex transform to solve some problems related to difference and differential-difference equations. The 3D, contour and 2D surfaces, as well as the related density plot surfaces of some acquired data, are used to draw the physical aspect of the obtained findings. The proposed approach offers an efficient and rapid solution for addressing the inherent complexity of differential-difference problems with initial conditions.

Keywords. The complex SEE transform, Inverse of complex SEE transform, Differential equations, Difference equations, Differential-difference equations.

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1. Introduction

Difference equations are very much analogous to differential equations. Difference equations are more elementary, but differential equations are more familiar. Differential equations are a standard part of the curriculum for science and engineering students, but difference equations are not. And not without reason: differential equations are a prerequisite for future courses but difference equations are not. But if a curriculum were to include difference equations and differential equations, it might make sense to teach the former first because the topic is more tangible. Difference equations play a key role in the mathematical modeling of a wide range of real-world events in neural variability, bioscience and control theory, in determining the anticipated time for random synaptic action potential production in nerve cells, studying on the human pupil light reflex, mathematical biology, diverse physiological process models, and disease controllability [6, 11, 12].

The study of differential-difference equations has grown rapidly over the past few decades. It is not just a theoretical interest that propels the study of this type of problems, but also the fact that many real-world physical phenomena appear in various fields can be modeled by them. Such as optimal bridge designing, control theory, the process of heat conduction, the chemical diffusion process, underground water flow, the fluid flow in the porous medium and thermal explosion. Studies reveal that, in some cases, differential-difference equations are often the most scientifically the rational choice for mathematical modeling of diverse biological and physical processes.

In the last few decades, numerous researchers have investigated (analytically and numerically) the differential equations for difference equations and differential-difference equations. Differential-difference equations reducible to difference and q-difference equations was studied by Romanenko [30]. Ahmed et al. presented a wavelet collocation

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method for efficiently solving singularly perturbed differential-difference equations and one-parameter singularly perturbed differential equations taking into account the singular perturbations inherent in control systems [2]. Gegen and Hu [3] proposed three complex differential-difference equations related to the integrable differential-difference equations. A non-asymptotic solution was established for the differential-difference equation governing the time evolution of a birth-death process [28]. The rational integrable differential-difference equations were proposed from a different discrete spectral problem by Zhao and Amin and they proved the Liouville integrability of rational integrable differential-difference equations by deriving its bi-Hamiltonian structures [31]. A geometrical index for the group S^n and its applications in periodic solutions of differential and differential-difference equations was studied in [10].

The developed exact analytical solutions to the steady state thermal analysis of the extended surfaces are not explicitly expressed. Alternatively, most of the nonlinear models have been solved numerically. However, the recent developments in nonlinear analysis have used various approximate analytical method for symbolic solutions of nonlinear equations. The developments of symbolic solutions to the problems often lead to the applications of series solution techniques. Unfortunately, these techniques require high mathematical skills, and the eventual solutions give the series of solutions with huge expressions and a large number of terms which are not appealing to learners and readers. Also, applying such series of solution techniques to transient nonlinear problems make the series solutions so huge for convenient especially, for practical applications. Moreover, such analysis comes with complex mathematical analysis involving high computational cost and time.

However, the solutions of the numerical methods are numeric in nature and the methods have inherent high computational cost and time. Therefore, the need and the obvious advantages of developing highly accurate symbolic analytical solutions to the nonlinear problems cannot be overemphasized. In addition, the obvious advantages of generating exact analytical solutions to the nonlinear problems are important. There have been few attempts where exact analytical solutions were developed for the nonlinear models, but the developed solutions are implicit in nature which add to the computational efforts, time and costs. Consequently, over the years, recourse has been made to approximate analytical, semi-analytical, semi-numerical, and numerical methods for the thermal analysis of the passive devices as shown in the reviewed studies. By applying the proper limits and providing the integration of the product between two functions, one of them is a fixed function for the specified integration and is called the kernel function, usually denoted by $k(t,y)$ with the other function $F(y)$. The result is called integral transformation, resulting in a new function denoted as $f(t)$. The integral transformation can be represented mathematically considering y as a parameter by the following formula:

$$I\left\{f\left(t\right)\right\}=\int_{t=0}^{\infty}k\left(t,y\right)f\left(t\right)dt=f\left(y\right). \tag{1.1}$$

Mansour et al [18] offered a new complex integral transform, the complex Sadaq-Emad-Emann (SSE) transform to solve the differential equation and the properties of this transform were investigated. The solution to linear higher order ordinary differential equation was found by using the new general complex integral transform [26]. Another fundamental change in particular SEE change was applied to address straight normal deferential conditions with consistent coefficients and SEE change of incomplete derivative was inferred [19]. A new integral transformation named Emad-Sara Transform has been proposed to solve linear ordinary differential equations with constant coefficients [8]. The complex SEE integral transform was going to be used to find the exact solution for the first order ordinary differential equations of population growth and decay problems, some growth and decay problems [20]. The complex SEE integral transform of Bessel's functions was used to solve the Schrödinger equation, Laplace equation, heat equation, wave equation and Helmholtz equation in cylindrical or spherical [21]. A new integral transform called the Kuffi-Abbas-Jawad integral transform has been proposed and applied to solve ordinary differential equations [1].

The Sadiq-Emad-Emann transform was determined the solution of difference and differential-difference equations, with neurotrophic versions of difference and differential difference equations [25]. The defined integral transform is considered efficient and widely used to find the exact solution to many scientific and engineering problems [7]. The exact solution of differential and differential-difference [22–24] has been found through the usage of the SEE integral transformation by benefiting from the properties of this complex transformation and applying it to problems.

Authors of [17] explained the generalized fifth-order KdV like equation with prime number $p = 3$ via a generalized bilinear differential operator. The N-lump was investigated to the variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation [16]. Applications of $tan(\phi/2)$-expansion method for the Biswas-Milovic equation [15], the Gerdjikov-Ivanov model [14], the Kundu-Eckhaus equation [13] and the fifth-order integrable equations [9] were studied. Lump solutions were analyzed to the fractional generalized CBS-BK equation [32] and the (3+1)-D Burger system [5]. The approximations of one-dimensional hyperbolic equation with non-local integral conditions were constructed by reduced differential transform method [27].

This paper is being drafted in the following way: In section 2, we introduce a fundamental definitions. Section 3, examines the properties and theorems of complex SEE integral transform. Some applicability of the Complex SEE integral transformation in solving difference equations and differential-difference equations are provided to demonstrate the efficacy and applicability of the proposed scheme in section 4. Additionally, this section contrasts the numerical outcomes obtained by the proposed approach with existing techniques. Finally, the paper concludes in section 5.

2. Fundamental Definitions

The main concept of the Complex SEE integral transform includes the following cases in order:

Definition 2.1. [25] The relationship between two functions, $y(t)$ with another function, $y(t{-}\alpha)$, is called the difference equation, noticing that α is a constant.

Definition 2.2. [25] If multiple derivatives of the same function, let it be $y(t)$, exist in the same difference equation. The resulting function is called a differential-difference equation.

Definition 2.3. [18] On the interval $[0,\infty)$, the integrable function $f(t)$, for the parameter v , the complex number i and $n\in\mathbb{Z}$. The Complex SEE integral transform $T\,(iv)$ to the function $f\,(t)$ can be defined by the following formula:

$$S^c\,\{f\,(t)\} = T\,(iv) = \frac{1}{v^n}\int_0^\infty e^{-ivt} f\,(t)\,dt.$$

As long as there is an integral for some parameter v.

3. Properties and Theorems of Complex SEE Integral Transform

This section includes the most important properties for the Complex SEE integral transform that are eminent for this work.

Property 3.1. *Linearity of Complex SEE transform[18]*

$$S^C\,\{Af\,(t) \pm Bg\,(t)\} = AS^c\,\{f\,(t)\} \pm BS^c\,\{g\,(t)\}\,,$$

where A and B are arbitrary constants.

Property 3.2. *Shifting of Complex SEE Transform [18]*

$$S^c\left\{e^{at} f(t)\right\} = \frac{(v+ia)^n}{v^n} T\,(v+ia)\,,$$

3.1. The Complex SEE Transform of SomeFundamental Functions [18, 23].

I: $S^c\,\{k\} = \frac{-ik}{v^{n+1}}$, where k is a constant.

$$S^c\,\{t\} = \frac{-1}{v^{n+2}}. \tag{3.1}$$

II: In general: $S^c\,\{t^m\} = \frac{(-1)^m (i)^{m-1}\ m!}{v^{n+m+1}}$, m is a positive integer number.

III: $S^c\left\{e^{ab}\right\} = \frac{-1}{v^n}\left[\frac{a}{a^2+v^2} + \frac{iv}{a^2+v^2}\right]$, in which a is a known constant.

IV: $S^c\,\{\sin\,(at)\ \} = \frac{a}{v^n (v^2-a^2)}$, in which a is a known constant.

V: $S^c\,\{\cos\,(at)\ \} = \frac{-iv}{v^n (v^2-a^2)}$, in which a is a known constant.

VI: $S^c\left\{\sinh\left(at\right)\ \right\}=\frac{-a}{v^n\left(v^2+a^2\right)}$, in which a is a known constant.
VII: $S^c\left\{\cosh\left(at\right)\ \right\}=\frac{-iv}{v^n\left(v^2+a^2\right)}$, in which a is a known constant.

3.2. **Some of Inverse "Complex SEE Transform" Functions** [23]**.** I: $S^{C^{-1}}\left\{\frac{-ik}{v^{n+1}}\right\}=k$, where k is a known constant.

$$S^{C^{-1}}\left\{\frac{\left(-1\right)^m\left(i\right)^{m-1}\ m!}{v^{n+m+1}}\right\}=t^m\ ,\quad m\in\mathbb{Z}^+. \tag{3.2}$$

$$S^{C^{-1}}\left\{\frac{-1}{v^n}\left[\frac{c}{c^2+v^2}+\frac{iv}{c^2+v^2}\right]\right\}=e^{ct},\quad c\equiv\ \text{constant}. \tag{3.3}$$

$$S^{C^{-1}}\left\{\frac{1}{v^n\left(v^2-c^2\right)}\right\}=\frac{1}{c}\sin\left(ct\right),\quad c\equiv\ \text{constant}. \tag{3.4}$$

$$S^{C^{-1}}\left\{\frac{-iv}{v^n\left(v^2-c^2\right)}\right\}=\cos\left(ct\right),\quad c\equiv\ \text{constant}. \tag{3.5}$$

$$S^{C^{-1}}\left\{\frac{-1}{v^n\left(v^2+c^2\right)}\right\}=\frac{1}{c}\sinh\left(ct\right),\quad c\equiv\ \text{constant}. \tag{3.6}$$

$$S^{c^{-1}}\left\{\frac{-iv}{v^n\left(v^2+c^2\right)}\right\}=\cosh\left(ct\right),\quad c\equiv\ \text{constant}. \tag{3.7}$$

3.3. **The Complex SEE Integral Transform of Derivatives** [18]**.** This section explains the application of the Complex SEE to the derivatives: and time shifting property.

Theorem 3.3. *[18, 22] Let $f(t)$ be a defined function of $t>1$, and its derivatives $f^{'}(t), f^{''}(t), f^{'''}(t), ..., f^{(m)}(t)$ exists, then:*

$$S^c\left\{f^{(m)}(t)\right\}=\frac{1}{v^n}\left[-f^{(m-1)}(0)-ivf^{(m-2)}(0)-(iv)^2f^{(m-3)}(0)\cdots-(iv)^{m-1}f(0)\right]+(iv)^mT\left(iv\right).$$

Special case:

$$S^c\left\{f^{'}\left(t\right)\right\}=\frac{-f(0)}{v^n}+ivT\left(iv\right), \tag{3.8}$$

$$S^c\left\{f^{''}\left(t\right)\right\}=-\frac{f^{'}\left(0\right)}{v^n}-\frac{if\left(0\right)}{v^{n-1}}-v^2T\left(iv\right). \tag{3.9}$$

3.4. **Time Shifting Property.** If $S^c\{f(t)\}=T(iv)$, then $S^c\left\{f\left(t-t_0\right)\right\}=e^{-ivt_0}T\left(iv\right)$, since $f(t)=0,\forall t<0$, hence we have $f\left(t-t_0\right)=0$ for all $t<0$.

Proof.

$$S^c\left\{f\left(t-t_0\right)\right\}=\frac{1}{v^n}\int_0^{\infty}\ e^{-ivt}\ f\left(t-t_0\right)\ dt=\frac{1}{v^n}\left(\int_0^{t_0}\ e^{-ivt}f\left(t-t_0\right)\ dt+\int_{t_0}^{\infty}\ e^{-ivt}f\left(t-t_0\right)\ dt\right).$$

Because in the close interval $\left[t_0,0\right], f\left(t-t_0\right)=0$, it is possible to suppose the variable $s=t-t_0\Rightarrow t=s+t_0$.

When $t|_{t_0}^{\infty}$ then $s|_0^{\infty}$ and $dt{=}ds$, so we substitute the above in the last integral, the results are:

$$\begin{aligned} s^c\{f(t-t_0)\} &= \frac{1}{v^n}\int_{s=0}^{\infty} e^{-iv(s+t_0)} f(s)\,ds \\ &= e^{-ivt_0}\frac{1}{v^n}\int_{s=0}^{\infty} e^{-ivs} f(s)\,ds \\ &= e^{-ivt_0}\,T(iv). \end{aligned}$$

□

4. Applications

The applicability of the Complex SEE integral transformation in solving difference equations and differential-difference equations is presented in this section.

4.1. **Application on Difference Equation.** Solve the following difference equation

$$6y(t) - 5y(t-1) + y(t-2) = t \text{ if } y(t) = 0 \text{ for all } t < 0.$$

The Complex SEE integral transform is applied on both sides of above equation

$$6S^c\{y(t)\} - 5S^c\{y(t-1)\} + S^c\{y(t-2)\} = S^c\{t\}.$$

The time shifting property is applied on the above equation

$$6T(iv) - 5e^{-iv}T(iv) + e^{-2iv}T(iv) = \frac{-1}{v^{n+2}}.$$

Then:

$$T(iv) = \frac{-1}{v^{n+2}\left(6-5e^{-iv}+e^{-2iv}\right)},$$

$$T(iv) = \frac{-1}{v^{n+2}\left(e^{-iv}-2\right)\left(e^{-iv}-3\right)},$$

$$T(iv) = \frac{-1}{v^{n+2}}\left[\frac{1}{(2-e^{-iv})} - \frac{1}{(3-e^{-iv})}\right],$$

$$T(iv) = \frac{-1}{v^{n+2}}\left[\frac{1}{2\left(1-\frac{e^{-iv}}{2}\right)} - \frac{1}{3\left(1-\frac{e^{-iv}}{3}\right)}\right],$$

$$T(iv) = -\frac{1}{v^{n+2}}\left[\frac{1}{2}\left(1+\frac{e^{-iv}}{2}+\frac{e^{-2iv}}{(2)^2}+\frac{e^{-3iv}}{(2)^3}+\frac{e^{-4iv}}{(2)^4}+\cdots\right) - \frac{1}{3}\left(1+\frac{e^{-iv}}{3}+\frac{e^{-2iv}}{(3)^2}+\frac{e^{-3iv}}{(3)^3}+\cdots\right)\right],$$

$$T(iv) = \frac{-1}{6v^{n+2}} + \sum_{j=1}^{\infty}\left[\frac{1}{(2)^{j+1}} - \frac{1}{(3)^{j+1}}\right]\frac{\left(-e^{-ivj}\right)}{v^{2+n}}.$$

Taking the inverse of the Complex SEE integral transform of the above equation gives the required exact solution

$$\begin{aligned} S^{c^{-1}}\{T(iv)\} &= \frac{1}{6}S^{c^{-1}}\left\{\frac{-1}{v^{n+2}}\right\} + S^{c^{-1}}\left\{\sum_{j=1}^{\infty}\left[\frac{1}{(2)^{j+1}} - \frac{1}{(3)^{j+1}}\right]\frac{\left(-e^{-ivj}\right)}{v^{2+n}}\right\} \\ &= \frac{1}{6}(t) + \sum_{j=1}^{[t]}\left[\frac{1}{(2)^{j+1}} - \frac{1}{(3)^{j+1}}\right](t-j). \end{aligned}$$

Then the exact solution is

$$y(t) = \frac{t}{6} + \sum_{j=1}^{[t]} \left[\frac{1}{(2)^{j+1}} - \frac{1}{(3)^{j+1}} \right] (t-j), \tag{4.1}$$

where $[t]$ is the greatest integer, $[t] \leq t$.

4.2. **Application on Differential-Difference Equation.**

Example 4.1. *Find the exact solution of differential-difference equation*

$$y^{'}(t) + y(t-1) = t \text{ if } y(0) = 0.$$

Applying the Complex SEE integral transform on both sides of above equation obtaining

$$S^c\left\{y^{'}(t)\right\} + S^c\{y(t-1)\} = S^c\{t\}.$$

Now, applying the Complex SEE integral transform of derivatives and time shifting property to give

$$(iv)\,T(iv) - \frac{1}{v^n} y(0) + e^{-iv} T(iv) = \frac{-1}{v^{n+2}},$$

$$T(iv) = \frac{-1}{v^{n+2}\left(iv - e^{-iv}\right)},$$

$$T(iv) = \frac{-1}{v^{n+2}(iv)\left[1 - \frac{e^{-iv}}{iv}\right]},$$

$$T(iv) = \frac{-1}{v^{n+3}}\left[1 - \frac{e^{-iv}}{iv} + \frac{e^{-2iv}}{(iv)^2} - \frac{e^{-3iv}}{(iv)^3} + \cdots\right],$$

$$T(iv) = \frac{-1}{i^3 v^{n+3}} \sum_{j=0}^{\infty} \frac{e^{-ivj}}{(iv)^j}, \quad i^2 = -1,$$

$$T(iv) = \frac{1}{v^n} \sum_{j=0}^{\infty} \frac{e^{-ivj}}{(iv)^{j+3}}.$$

Taking the inverse of the Complex SEE integral transform of the above equation gives the required exact solution of the application.

$$S^{c^{-1}}\{T(iv)\} = S^{c^{-1}}\left\{\frac{1}{v^n}\sum_{j=0}^{\infty}\frac{e^{-ivj}}{(iv)^{j+3}}\right\} = \begin{cases} \frac{(t-j)^{j+2}}{\{j+2)!}, & t \geq j \\ 0, & \text{otherwise} \end{cases}$$

$$S^c\{T(iv)\} = \tfrac{1}{6}(t) + \textstyle\sum_{j=1}^{[t]} \left[\frac{1}{(2)^{j+1}} - \frac{1}{(3)^{j+1}}\right](t-j).$$

Then

$$y(t) = \frac{t}{6} + \sum_{j=1}^{[t]} \left[\frac{1}{(2)^{j+1}} - \frac{1}{(3)^{j+1}} \right] (t-j), \tag{4.2}$$

where $[t] \in \mathbb{Z}$, $[t] \leq t$.

Example 4.2. *Consider the second problem of differential-difference equation in the following:*

$$y'(t) - y(t-1) = 1, \quad t > 1, \quad y(t) = 1, \quad 0 \leq t \leq 1, \tag{4.3}$$

such that we have

$$\int_0^{\infty} y(t)\, e^{-vt} dt = \frac{v + e^{-v}}{v\left(v - e^{-v}\right)}, \tag{4.4}$$

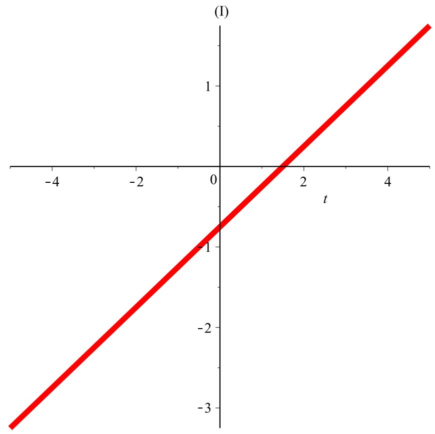

FIGURE 1. Exact solution (4.2).

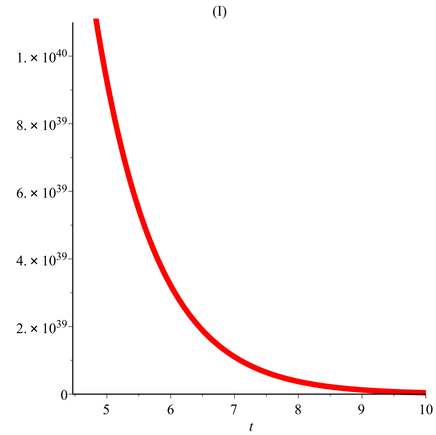

FIGURE 2. Exact solution (4.5).

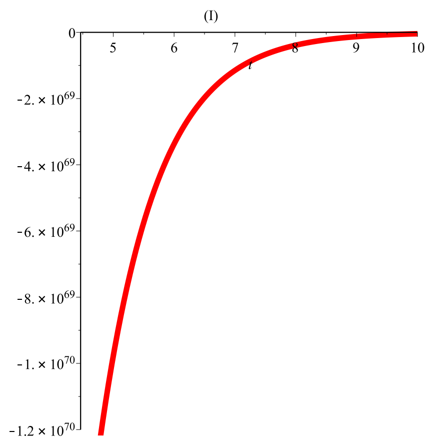

FIGURE 3. Exact solution (4.7).

by using the mentioned issues in Example 1 we can get to the final solutions in the below

$$y(t) = -1 + 2\sum_{j=0}^{N} \frac{(t-j)^j}{j!}, \qquad N \le t < N+1, \quad N = 0,1,2,.... \tag{4.5}$$

Example 4.3. *Consider the third problem of differential-difference equation in the following*

$$y'(t) = 2y(t-1), \quad t > 1, \quad y(t) = t, \quad 0 \le t \le 1, \tag{4.6}$$

by applying the Complex SEE integral transform of derivatives and time shifting property and taking the inverse of the Complex SEE integral transform of the above equation gives the required exact solution as below

$$y(t) = \frac{1}{2} + \sum_{j=0}^{N} \left\{ \frac{2^j (t-j)^{j+1}}{j+1!} - \frac{2^{j-1}(t-j)^j}{j!} \right\}, \qquad N \le t < N+1, \quad N = 0,1,2,.... \tag{4.7}$$

Example 4.4. *Consider the following linear third-order multiplicatively advanced differential equation*

$$y^{(3)}(t) = p^3 y(pt), \tag{4.8}$$

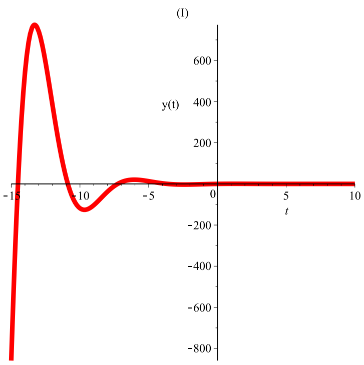

FIGURE 4. Exact solution (4.12) with $p = 1$.

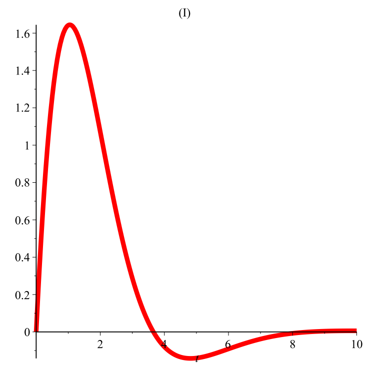

FIGURE 5. Exact solution (4.13) with $p = 1.2$.

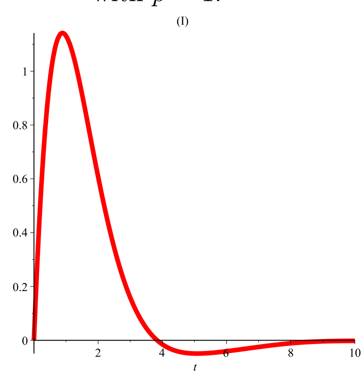

FIGURE 6. Approximate solution (4.13) with $p = 1.5$.

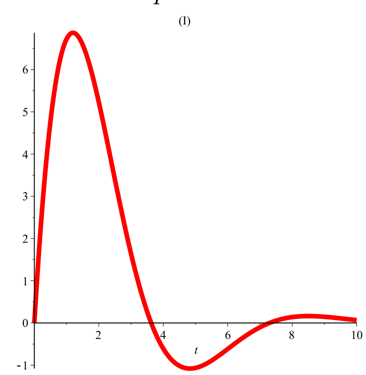

FIGURE 7. Approximate solution (4.13) with $p = 1.01$.

for $p > 1$, on the interval $t \in [0, 8)$, satisfying the initial conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -p. \tag{4.9}$$

For small $p > 1$, as $p \to 1^+$, Eqs. (4.8) and (4.9) can be considered to be a perturbation of the classical analogue, which is the ODE

$$Y^{(3)}(t) = Y(t), \tag{4.10}$$

with initial conditions

$$Y(0) = 0, \quad Y'(0) = 1, \quad Y''(0) = -1, \tag{4.11}$$

obtained by setting $p = 1$ in (4.8) and (4.9). One can check directly that (4.10) and (4.11) is solved uniquely by

$$Y(t) = 2\exp(-t/2)\sin(\sqrt{3}t/2)/\sqrt{3}. \tag{4.12}$$

Now, utilizing techniques mirroring those of Theorem 3.2 of [29], a particular solution to (4.8) will be as

$$\widetilde{y}(t) = \sum_{l=-\infty}^{\infty} \frac{\exp(-p^l t/2)\sin(\sqrt{3}p^l t/2)}{p^{l(l-1)/(2/3)}}, \tag{4.13}$$

for $t \geq 0$. Note that the expression in Eq. (4.13) does not have the alternation $(-1)^k$ and this will allow a sharp bound on $\widetilde{y}(t)$ for all $t \geq 0$, independent of $p > 1$. The first derivative of $\widetilde{y}(t)$ is seen to be

$$\widetilde{y}'(t) = \sum_{l=-\infty}^{\infty} \frac{p^l \exp(-p^l t/2)\ \left[(-1/2)\sin(\sqrt{3}p^l t/2) + (\sqrt{3}/2)\cos(\sqrt{3}p^l t/2)\right]}{p^{l(l-1)/(2/3)}} \tag{4.14}$$

$$= \sum_{l=-\infty}^{\infty} \frac{\exp(-p^l t/2)\ \sin(\sqrt{3}p^l t/2 + 2\pi/3)}{p^{l(l-1-2/3)/(2/3)}}.$$

Using this identity implicitly, we get

$$\widetilde{y}''(t) = \sum_{l=-\infty}^{\infty} \frac{p^l\ \exp(-p^l t/2)\ \sin(\sqrt{3}p^l t/2 + 4\pi/3)}{p^{l(l-1-2/3)/(2/3)}} \tag{4.15}$$

$$= \sum_{l=-\infty}^{\infty} \frac{\exp(-p^l t/2)\ \sin(\sqrt{3}p^l t/2 + 4\pi/3)}{p^{l(l-1-4/3)/(2/3)}},$$

and finally we verify

$$\widetilde{y}'''(t) = \sum_{l=-\infty}^{\infty} \frac{\exp(-p^l t/2)\ \sin(\sqrt{3}p^l t/2 + 6\pi/3)}{p^{l(l-1-6/3)/(2/3)}} \tag{4.16}$$

$$= \sum_{l=-\infty}^{\infty} \frac{\exp(-p^{l-1}(pt)/2)\ \sin(\sqrt{3}p^{l-1}pt/2)}{p^{[(l-1+1)((l-1)-1)]/(2/3)}}$$

$$= \sum_{k=-\infty}^{\infty} \frac{\exp(-p^k(pt)/2)\ \sin(\sqrt{3}p^k pt/2)}{p^{[(k+1)(k-1)]/(2/3)}}$$

$$= p^3 \sum_{k=-\infty}^{\infty} \frac{\exp(-p^k(pt)/2)\ \sin(\sqrt{3}p^k pt/2)}{p^{k(k-1)/(2/3)}} = p^3\, \widetilde{y}(pt).$$

From (4.13)-(4.14), one sees that

$$\widetilde{y}(0) = \sum_{l=-\infty}^{\infty} \frac{\sin(0)}{p^{l(l-1)/(2/3)}} = 0, \tag{4.17}$$

$$\widetilde{y}'(0) = \sum_{l=-\infty}^{\infty} \frac{\sin(2\pi/3)}{p^{l(l-5/3)/(2/3)}} = \frac{\sqrt{3}}{2} \sum_{l=-\infty}^{\infty} \frac{p^l}{(p^3)^{l(l-1)/2}} = \frac{\sqrt{3}}{2} F(p^3;p), \tag{4.18}$$

$$\widetilde{y}''(0) = \sum_{l=-\infty}^{\infty} \frac{\sin(4\pi/3)}{p^{l(l-7/3)/(2/3)}} = -\frac{\sqrt{3}}{2} \sum_{l=-\infty}^{\infty} \frac{(p^2)^l}{(p^3)^{l(l-1)/2}} = -\frac{\sqrt{3}}{2} F(p^3;p^2), \tag{4.19}$$

where

$$F(p^3;p^r) = \sum_{l=-\infty}^{\infty} \frac{(p^r)^l}{(p^3)^{l(l-1)/2}}, \tag{4.20}$$

by normalizing $\widetilde{y}(t)$ by $\widetilde{y}'(0) = \frac{\sqrt{3}}{2}F(p^3;p)$, then we can obtain

$$y(t) = \widetilde{y}(t)/\widetilde{y}'(0), \tag{4.21}$$

one sees that $y(t)$ now satisfies Eq. (4.9) along with the initial conditions (4.10). Also, the last initial condition follows from the finding that

$$y''(0) = \widetilde{y}''(0)/\widetilde{y}'(0) = \frac{-\frac{\sqrt{3}}{2}F(p^3;p^2)}{\frac{\sqrt{3}}{2}F(p^3;p)} = \frac{-F(p^3;p^2)}{F(p^3;p)} = -p, \tag{4.22}$$

and also $F(p^3;p)$ have the following properties

$$\frac{F(p^3;p^2)}{F(p^3;p)} = p = \frac{F(p^3;-p^2)}{F(p^3;-p)}. \tag{4.23}$$

For easily one can write

$$F(p^3;p^2) = F(p^3;p^3(1/p)) = p^3(1/p)\,F(p^3;1/p) = p\left[pF(p^3;1/p)\right] = p[F(p^3;p)], \tag{4.24}$$

such that one have

$$F(p;\lambda) \equiv \sum_{l=-\infty}^{\infty} \frac{\lambda^l}{p^{l(l-1)/2}} = \chi_p(1+\lambda)\prod_{l=1}^{\infty}\left(1+\frac{\lambda}{p^l}\right)\left(1+\frac{1}{\lambda p^l}\right), \tag{4.25}$$

in which

$$\chi_p = \prod_{l=1}^{\infty}\left(1-\frac{1}{p^l}\right). \tag{4.26}$$

We see

$$\begin{aligned} F(p^3;1) &= \chi_{p^3}\prod_{l=0}^{\infty}\left[\left(1+\frac{1}{p^{3l}}\right)\left(1+\frac{1}{p^{3(l+1)}}\right)\right] \\ &= \chi_{p^3}\left[\prod_{l=0}^{\infty}\left(1+\frac{1}{p^{3l}}\right)\right]\left(1+\frac{1}{p^3}\right)\left[\prod_{l=0}^{\infty}\left(1+\frac{1}{p^{6+3l}}\right)\right], \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} F(p^3;p) &= \chi_{p^3}\prod_{l=0}^{\infty}\left[\left(1+\frac{p}{p^{3l}}\right)\left(1+\frac{1}{pp^{3(l+1)}}\right)\right] \\ &= \chi_{p^3}\left[\prod_{l=0}^{\infty}\left(1+\frac{p}{p^{3l}}\right)\right]\left[\prod_{l=0}^{\infty}\left(1+\frac{1}{p^{4+3l}}\right)\right]. \end{aligned} \tag{4.28}$$

Example 4.5. *Let us first consider the linear system of differential-difference with variable coefficients as [4]:*

$$\begin{cases} 2tf_1'(t+1) - 3f_1(t+1) + f_2(t+1) = 0, \\ 2tf_2'(t+1) + f_1(t+1) - 3f_2(t+1) = 0, \end{cases} \tag{4.29}$$

with the below conditions

$$f_1(2) = 1, \quad f_2(2) = -1, \tag{4.30}$$

by applying the Complex SEE integral transform of derivatives and time shifting property and taking the inverse of the Complex SEE integral transform of the above equation gives the required exact solution as below

$$f_1(t) = 1 + t^2 - 2t, \quad f_2(t) = 2t - 1 - t^2. \tag{4.31}$$

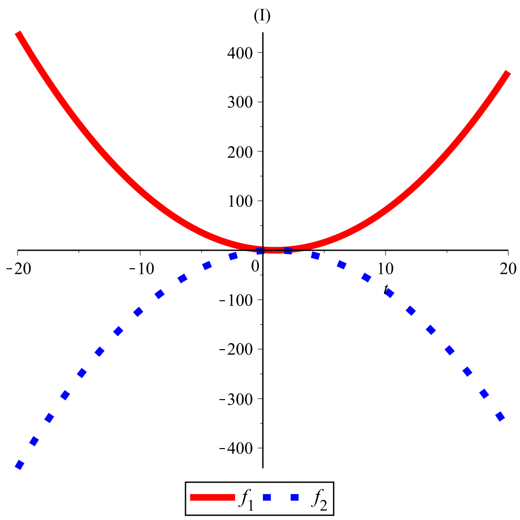

FIGURE 8. Exact solution $f_1(t) and f_2(t)$ (4.31).

Example 4.6. *Let us second consider the linear system of differential-difference with variable coefficients [4]:*

$$\begin{cases} f_1'(t) + f_2'(t) + f_2(t) = t - \exp(-t), \\ f_1'(t) + 4f_2'(t) + f_1(t) = 1 + 2\exp(-t), \quad 0 \le t \le 1, \end{cases} \tag{4.32}$$

with the below conditions

$$f_1(0) = 1, \quad f_2(0) = 0, \tag{4.33}$$

by applying the Complex SEE integral transform of derivatives and time shifting property and taking the inverse of the Complex SEE integral transform of the above equation gives the required exact solution as below

$$\begin{cases} ivT_1(iv) - \frac{1}{v^n} f_1(0) + ivT_2(iv) - \frac{1}{v^n} f_2(0) + T_2(iv) = -\frac{1}{v^{n+2}} + \frac{1}{v^n}\left[\frac{-1+iv}{1+v^2}\right], \\ ivT_1(iv) - \frac{1}{v^n} f_1(0) + 4ivT_2(iv) - \frac{1}{v^n} f_2(0) + T_1(iv) = -\frac{1}{v^{n+1}} + \frac{2}{v^n}\left[\frac{-1+iv}{1+v^2}\right], \end{cases} \tag{4.34}$$

by simplifying the above system taking the inverse of the Complex SEE integral transform of the above equation gives the required exact solutions as below

$$\begin{aligned} f_1(t) = \sum_{i=0}^{\infty} f_{i1}(t) &= 1 - 2t + 0.66666783t^2 - 0.18518544t^3 \\ &+ (0.43211635e-1)t^4 - (0.84431064e-2)t^5 + (0.14109473e-2)t^6 \\ &- (0.22259609e-3)t^7 + (0.45879299e-4)t^8 \\ &- (0.12963993e-4)t^9 + (0.23318594e-5)t^{10}, \end{aligned} \tag{4.35}$$

$$\begin{aligned} f_2(t) = \sum_{i=0}^{\infty} f_{i2}(t) &= t - 0.16666785t^2 + (0.74074385e-1)t^3 \\ &- (0.20064678e-1)t^4 + (0.4122386e-2)t^5 - (0.71348994e-3)t^6 \\ &+ (0.131827769e-3)t^7 - (0.41893269e-4)t^8 \\ &+ (0.1603362e-4)t^9 - (0.320804353e-5)t^{10}, \\ &\ldots \end{aligned} \tag{4.36}$$

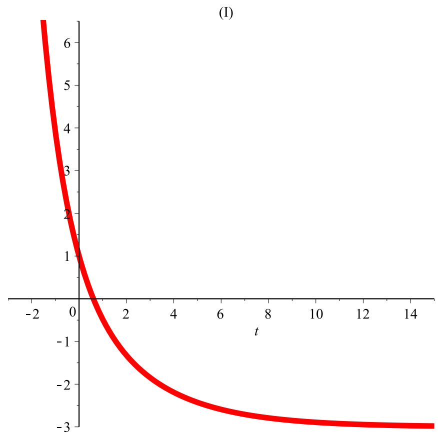

FIGURE 9. Exact solution $f_1(t)$ (4.37).

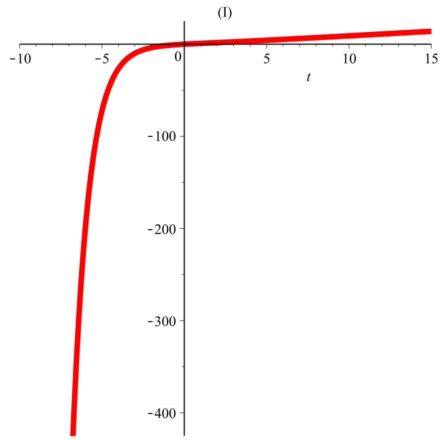

FIGURE 10. Exact solution $f_2(t)$ (4.37).

and also, for the system the exact solutions will be as below

$$\begin{aligned} f_1(t) &= \exp(-t) + 3\exp(-t/3) - 3, \\ f_2(t) &= -\frac{1}{2}\exp(-t) + \frac{3}{2}\exp(-t/3) - 1 + t. \end{aligned} \tag{4.37}$$

5. Discussion and Results

For the graphical analysis, we observe that outcomes $y(t)$ represent exponential form function. In particular, we depict the dynamics of (4.2) in Figure 1 and (4.5) in Figure 2. Also, the solution of differential-difference is depicted the dynamics of (4.7) in Figure 3. In Figure 4, behavior of (4.12) is presented with $p = 1$. Plot of exact solution (4.13) for $p = 1.2$ is presented in Figure 5. Moreover, behavior of exact solution (4.13) for $p = 1.5$ is analyzed in Figure 6. In addition, Plot of exact solution (4.13) for $p = 1.01$ is shown in Figure 7. In Figure 8, the exact solution of (4.31) is plotted as well for $f_1(t)$ and $f_2(t)$. In Figures 9 and 10, the exact solutions of $f_1(t)$ and $f_2(t)$ are obtained for the system of (4.32) are shown. Our research is highlighted in this section by its novelty and accomplishments. By combining our results with those of formerly reported articles presented, we can reach our destination.

6. Conclusion

This paper represents the proof of the Complex SEE integral transform's capability to solve problems using simple techniques. However, an essential property of the Complex SEE integral transform is proved first. Then, the exact solution to the difference and differential-difference equations has been solved. Numerical applications prove the ability of the transform to handle the difference and differential-difference equations. We then think that these results will help for conducting future research in various areas of physics such as mathematical physics, nonlinear mechanics and other applied fields and so on. The method used here can be applied to other nonlinear partial differential equations.

References

[1] E. S. Abbas, E. A. Kuffi, and A. A. Jawad, *New integral "Kuffi-Abbas-Jawad" KAJ transform and its application on ordinary differential equations*, Journal of Interdisciplinary Mathematics, *25* (2022), 1427-1433.

[2] S. Ahmed, S. Jahan, K. J. Ansari, K. Shah, and T. Abdeljawad, *Wavelets collocation method for singularly perturbed differential-difference equations arising in control system*, Results in Applied Math., *21* (2024), 100415.

[3] H. Gegen and X. Hu, *Three complex differential-difference equations*, Partial Differential Equations in Applied Mathematics 5, (2022), 100300.

[4] E. Gökmen and M. Sezer, *Taylor collocation method for systems of high-order linear differential-difference equations with variable coefficients*, Ain Shams Engineering Journal, *4* (2013), 117-125.

[5] Y. Gu, S. Malmir, J. Manafian, O. A. Ilhan, A. A. Alizadeh, and A. J. Othman, *Variety interaction between k-lump and k-kink solutions for the (3+1)-D Burger system by bilinear analysis*, Results Phys., *43* (2022), 106032.

[6] H. Khan, M. Alipour, R. A. Khan, H. Tajadodi, and A. Khan, *On approximate solution of fractional order logistic equations by operational matrices of bernstein polynomials*, J. Math. Comput. Sci., *14* (2015), 222-232.

[7] E. A. Kuffi and E. A. Mansour, *On Hewit and Story Method for Construction Liapunov Function of Differential Algebraic Equations*, Journal of College of Education, Al Mustansiriyah University, 2015.

[8] E. Kuffi and S. F. Maktoof, *"Emad-Falih Transform" a new integral transform*, Journal of Interdisciplinary Mathematics, *24* (2021), 2381-2390.

[9] M. Lakestani, J. Manafian, A. R. Najafizadeh, and M. Partohaghighi, *Some new soliton solutions for the nonlinear the fifth-order integrable equations*, Comput. Meth. Diff. Equ., *10*(2) (2022), 445-460.

[10] L. Li and W. Ge, *A geometrical index for the group S^n and its applications in periodic solutions of differential and differential-difference equations*, Journal of Mathematical Analysis and Applications, *530*(1) (2024), 127665.

[11] A. Longtin and J. G. Milton, *Complex oscillations in the human pupil light reflex with mixed and delayed feedback*, Math Biosci, 90 (1988), pp. 183-199.

[12] M. C. Mackey and L. Glass, *Oscillation and chaos in physiological control systems*, Science, *197*(4300) (1977), 287-289.

[13] J. Manafian and M. Lakestani, *Abundant soliton solutions for the Kundu-Eckhaus equation via tan($\phi(\xi)$)-expansion method*, Optik, *127*(14) (2016), 5543-5551.

[14] J. Manafian and M. Lakestani, *Optical soliton solutions for the Gerdjikov-Ivanov model via tan($\phi/2$)-expansion method*, Optik, *127*(20) (2016), 9603-9620.

[15] J. Manafian and M. Lakestani, *Application of tan($\phi/2$)-expansion method for solving the Biswas-Milovic equation for Kerr law nonlinearity*, Optik, *127*(4) (2016), 2040-2054.

[16] J. Manafian and M. Lakestani, *N-lump and interaction solutions of localized waves to the (2+ 1)- dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation*, J. Geom. Phys., *150* (2020), 103598.

[17] J. Manafian, L. A. Dawood, and M. Lakestani, *New solutions to a generalized fifth-order KdV like equation with prime number $p = 3$ via a generalized bilinear differential operator*, Partial Differ. Equ. Appl. Math., *9* (2024), 100600.

[18] E. A. Mansour, S. Mehdi, and E. A. Kuffi, *The new integral transform and its applications*, International Journal of Nonlinear Analysis and Applications, *12* (2021) 849-856.

[19] E. A. Mansour, E. A.Kuffi, and S. A. Mehdi, *The new integral transform "SEE transform" and its applications*, Periodicals of engineering and natural sciences, *9* (2021), 1016-1029.

[20] E. A. Mansour, E. A. Kuffi, and S. A. Mehdi, *Solving Population Growth and Decay Problems Using Complex SEE Transform*, 2021 7th International Conference on Contemporary Information Technology and Mathematics (ICCITM), (2021), 243-246.

[21] E. A. Mansour, E. A. Kuffi, and S. A. Mehd, *Complex SEE Transform of Bessel's Functions*, Journal of Physics: Conference Series, *1999*(1) (2021), 012154, IOP Publishing.

[22] E. A. Mansour, S. A. Mehdi, and A. K. Emad, *Application of New Transform "Complex SEE Transform" to Partial Differential Equations*, Journal of Physics: Conference Series, IOP Publishing, *1999*(1) (2021), 012155.

[23] E. A. Mansour, S. Mehdi, and E. A. Kuffi, *On the complex SEE change and systems of ordinary differential equations*,International Journal of Nonlinear Analysis and Applications, *12* (2021) ,1477-1483.

[24] E. A. Mansour, E. A. Kuffi, and S. A. Mehdi, *The Solution of Faltung Type Volterra Integro-Differential Equation of First Kind using Complex SEE Transform*,Journal of College of Education Al Mustansiriyah University, *23* 2022.

[25] E. A. Mansour and E. A. Kuffi, *SEE Transform in Difference Equations and Differential-Difference Equations Compared With Neutrosophic Difference Equations*, International Journal of Neutrosophic Science (IJNS), *22* (2023), 36-43.

[26] S. A. Mehdi, E. A. Kuffi, and J. A. Jasim, *Solving ordinary differential equations using a new general complex integral transform*, Journal of interdisciplinary mathematics, *25* (2022), 1919-1932.

[27] S. R. Moosavi, N, Taghizadeh, and J. Manafian, *Analytical approximations of one-dimensional hyperbolic equation with non-local integral conditions by reduced differential transform method*, Comput. Meth. Diff. Equ., *8*(3) 2020, 537-552.

[28] S. J. Park and M. Y. Choi, *Counting statistics based on the analytic solutions of the differential-difference equation for birth-death processes*, Chaos Solitons Fractals, *173* (2023), 113679.

[29] D. W. Pravica, N. Randriampiry, and M. J. Spurr, *Solutions of a class of multiplicatively advanced differential equations*, C.R. Acad. Sci. Paris Ser. I, *356* 2018, 776-817.

[30] E. Yu. Romanenko, *Differential-difference equations reducible to difference and q-difference equations*, Comput. Math. Appl., *42*(3-5) (2001), 615-626.

[31] Q. Zhao and M. A. Amin, *Explicit solutions of rational integrable differential-difference equations*,Partial Differential Equations in Applied Mathematics 5, 2022, 100338.

[32] M. Zhang, X. Xie, J. Manafian, O. A. Ilhan, and G. Singh, *Characteristics of the new multiple rogue wave solutions to the fractional generalized CBS-BK equation*, J. Adv. Res., *38* (2022), 131-142.