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A Chebyshev pseudo-spectral based approach for solving Troesch's problem with convergence analysis

Mehrdad Ghaznavi* **and Mohammad Hadi Noori Skandari**

Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran.

Abstract

In this article, the Chebyshev pseudo-spectral (CPS) method is presented for solving Troesch's problem, which is a singular, highly sensitive, and nonlinear boundary problem and occurs in an consideration of the confinement of a plasma column by radiation pressure. Here, a continuous time optimization (CTO) problem corresponding to Troesch's problem is first proposed. Then, the Chebyshev pseudo-spectral method is used to convert the CTO problem to a discrete time optimization problem that its optimal solution can be find by nonlinear programming methods. The feasibility and convergence of the generated approximate solutions are analyzed. The proposed method is used to solve various kinds of Troesch's equation. The obtained results have been compared with approximate solutions resulted from well-known numerical methods. It can be confirmed that the numerical solutions resulted from this method are completely acceptable and accurate, compared with other techniques.

Keywords. Troesch's problem, Nonlinear programming, Chebyshev pseudo-spectral method, Continuous and discrete time optimization..
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1. Introduction

A boundary value problem (BVP) is a differential equation which has values assigned on the physical boundary of the given domain. Boundary value problems are applicable in different areas of science including: physics, mathematics, engineering and chemistry.
Troesch's problem [1, 13, 42, 48] is an special nonlinear two point BVP that occurs in an consideration of the restriction of a plasma column by radiation pressure [48]. This problem also occurs in the study of aspects of gas porous electrodes [22].
The standard Troesch's problem is given by

$$\begin{aligned} y''(t) &= nsinh(ny), \ 0 \leq t \leq 1, \\ y(0) &= 0, y(1) = 1, \end{aligned} \tag{1.1}$$

where n is a positive constant.
This problem contains a singularity at the upper boundary and meantime includes an analytic solution of the Cauchy problem equivalent to it. This problem has a closed form solution as follows [42]:

$$y(t) = \frac{2}{n} arcsinh\{\frac{y'(0)}{2} sc[nx, 1 - \frac{1}{4}(y'(0))^2]\}, \tag{1.2}$$

with $y'(0) = 2\sqrt{1-m}$, in which m is the solution of the following equation

$$\frac{sinh(\frac{n}{2})}{\sqrt{1-m}} = sc(n, m), \tag{1.3}$$

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* Corresponding author. Email: ghaznavi@shahroodut.ac.ir.

where $sc(n, m)$ is the Jacobian elliptic function determined by $sc(n, m) = tan(\phi)$, and m $(0 < m < 1)$ is a modulus of the Jacobian elliptic function. Here, ϕ, n and m satisfy the following integral:

$$n = \int_o^\phi \frac{1}{\sqrt{1 - msin^2\theta d\theta}}.$$

Since at the upper boundary there exists singularity, therefore numerical approximation methods have significant problems. A number of methods have been reported to obtain a closed form and approximate solution for Troesch's problem. Roberts and Shipman [42] applied the perturbation and continuation methods, simultaneously, to obtain a closed form solution for this problem. Troesch [47] utilized shooting method to attain a numerical approximate solution for this problem. Deeba et al. [7] solved this problem utilizing a numerical method based on the decomposition technique. Khuri [25] find an approximate solution for this problem using a modified decomposition technique and the Laplace transformation. Feng et al. [18] presented a numerical technique inspired by the modified homotopy perturbation method. Moreover, Khuri and Sayfy [26] executed the finite element technique utilizing the cubic B-spline collocation technique for solving Troesch's problem. Chang and Chang [4] utilized a novel approach based on the differential transformation for finding the solution of Troesch's problem. Also, Chang [6] solved this problem using the variational method and variable transformation. In [14] a modern version of finite difference approach for this problem was presented. Zarebnia and Sajjadian [50] employed sinc-Galerkin approach for solving this problem. Leal et al. [27] find an approximate solution using the homotopy perturbation method. Raja [41] applied the neural networks optimized with three meta-heuristic methods including active set (AS), particle swarm optimization (PSO), and PSO hybridized with AS (PSO-AS) algorithms to find a solution for this problem. Moreover, Saadatmandi and Abdolahi-Niasar [43] suggested a collocation method to obtain the approximate solution of Troesch's problem. To find more detailed information and numerical methods for solving Troesch's problem, we refer to Momani et al. [32], Bougoffa and Al-khadhi [2], Mirmoradi et al. [30], Chang [5], Nabati and Jalalvand [33], Zuniga et al. [51], Youssef and Baumann [49], Markov and Dragunov [29] and references therein.

Now, in this article, the Chebyshev pseudo-spectral (CPS) method will be applied to solve Troesch's problem. First, we propose a continuous time optimization (CTO) problem related to the main differential equation. Then, we convert the CTO problem to a discrete time optimization problem, using the Chebyshev-Gauss-Lobatto nodes. The feasibility and the convergence of the obtained solution are discussed. Numerical solutions show that the obtained results for different values of n, are extremely precise in comparison with those attained by the other numerical snd analytical approaches.

The rest of this paper is formed as follows: Section 2 is devoted to Chebyshev pseudo-spectral method. Feasibility and Convergence analysis are included in section 3. The accuracy and efficiency of the suggested CPS method is illustrated by numerical results in section 4. Eventually, in section 5 conclusions are stated.

2. Chebyshev pseudo-spectral method

In the literature, there are many numerical methods for solving differential equations and the Troesch's problem (see for example [10, 40]). Pseudo-spectral (PS) methods are a category of direct numerical methods for continuous-time problems (see [3, 8, 9, 11, 15–17, 19, 23, 28, 38, 46]). In [3], it was shown that PS methods provide a fast convergence for the approximation of the functions. In PS methods, we can prove some sufficient conditions related to the convergence as well as feasibility of the obtained approximate solution.

Among the PS methods, Chebyshev pseudo-spectral (CPS) method has a high degree of accuracy in approximations [21, 31, 34–37]. In CPS method, we first use an interpolating polynomial in Chebyshev-Gauss-Lobatto (CGL) nodes, that offers a fast computation in comparison with the other nodes. Then, we calculate the derivatives of interpolating polynomials in CGL nodes by a differentiation matrix. So, by CPS method, a nonlinear algebraic equation is attained related to the dynamical equation.

However, here, in order to find an approximation of the exact solution of Troesch's problem (1.1), we attain a nonlinear programming (NLP) problem. In CPS method, the state variable $y(.)$ is discretized, and values of the state in CGL nodes are unknowns and the ordinary differential equation (ODE) is represented according to these values. Moreover,

to attain a continuous approximation for the state variable, an interpolation procedure is utilized.
Related to differential Equation (1.1), we suggest a continuous-time optimization (CTO) problem as follows:

$$\begin{aligned} &min \quad J = y(0)^2 + (y(1)-1)^2 \\ &subject\ to \begin{cases} y''(t) = nsinh(ny), \\ 0 \le t \le 1. \end{cases} \end{aligned} \tag{2.1}$$

It is obvious that zero is the best objective value of the CTO problem (2.1). Therefore, by solving the CTO problem (2.1), we are going to find a feasible solution $y(t)$ with $y(0) =$ and $y(1) - 1 = 0$. Consequently, the obtained optimal solution of (2.1) satisfies all the equations of the Troesch's problem (1.1). Hence, finding optimal solution of CTO problem (2.1) is equivalent to obtain the solution of Troesch's problem (1.1). At follows, the CTO problem (2.1) is solved using the CPS method. Define $u(t) = y'(t), \quad 0 \le t \le 1$. Therefore, the following CTO problem is obtained from (2.1):

$$\begin{aligned} &min \quad J = y(0)^2 + (y(1)-1)^2 \\ &subject\ to \begin{cases} y'(t) = u(t), \\ u'(t) = nsinh(ny), \quad 0 \le t \le 1. \end{cases} \end{aligned} \tag{2.2}$$

The time transformation $t = \frac{1}{2}(\tau + 1)$ is applied, to change the interval to $[-1, 1]$. Define

$$\begin{cases} Y(\tau) = y(\frac{\tau+1}{2}) = y(t), \\ U(\tau) = u(\frac{\tau+1}{2}) = u(t). \end{cases} \tag{2.3}$$

Using the chain rule, we obtain $y'(t) = 2Y'(\tau)$ and $u'(t) = 2U'(\tau)$. Therefore, the CTO problem (2.2) is transformed to the following problem:

$$\begin{aligned} &min \quad J = Y^2(-1) + (Y(1)-1)^2 \\ &subject\ to \begin{cases} Y'(\tau) = \frac{1}{2}U(\tau), \\ U'(\tau) = \frac{1}{2}nsinh(nY(\tau)), \\ -1 \le \tau \le 1. \end{cases} \end{aligned} \tag{2.4}$$

The following CGL nodes on $[-1, 1]$ are utilized to obtain a discrete form of the CTO problem (2.4):

$$\tau_k = cos(\frac{N-k}{N}\pi), \quad k = 0, 1, 2, \dots, N, \tag{2.5}$$

where these nodes are the roots of the Chebyshev polynomials $T_j(\tau) = cos(jcos^{-1}(\tau)),\ j = 0, 1, \dots, N,\ \tau \in [-1, 1]$.
In CPS method, the next Lagrange interpolating polynomials are used for interpolating:

$$L_k(\tau) = \frac{2}{N\mu_k} \sum_{j=0}^{N} \frac{1}{\mu_j} T_j(\tau_k) T_j(\tau),\ \tau \in [-1, 1],\ k = 0, 1, \dots, N, \tag{2.6}$$

where $\mu_0 = \mu_N = 2$ and $\mu_k = 1,\ for\ k = 1, \dots, N-1$. It is obvious that $L_k(\tau_k) = 1,\ k = 0, 1, \dots, N$ and $L_k(\tau_j) = 0$, for all $k \ne j$.
The following CPS approximation is defined for the CTO problem (2.4):

$$Y(\tau) \approx \sum_{l=0}^{N} \bar{y}_l L_l(\tau),\ U(\tau) \approx \sum_{l=0}^{N} \bar{u}_l L_l(\tau), \tag{2.7}$$

where N is a given sufficiently big number and

$$Y(\tau_k) \approx \bar{y}_k, \quad U(\tau_k) \approx \bar{u}_k. \tag{2.8}$$

Moreover,

$$Y'(\tau_k) \approx \sum_{l=0}^{N} \bar{y}_l D_{lk},\ U'(\tau_k) \approx \sum_{l=0}^{N} \bar{u}_l D_{lk},\ k = 0, 1, \ldots, N, \tag{2.9}$$

where

$$D_{lk} = L_l'(\tau_k) = \begin{cases} \frac{\mu_k}{\mu_l}(-1)^{k+1}\frac{1}{\tau_k - \tau_l}, & if\ k \neq l, \\ -\frac{\tau_k}{2-2\tau_k^2}, & if\ 1 \leq k = l \leq N-1, \\ -\frac{(2N^2+1)}{6}, & if\ k = l = 0, \\ \frac{2N^2+1}{6}, & if\ k = l = N. \end{cases} \tag{2.10}$$

Utilizing relations (2.8) and (2.9), the following discrete time optimization (DTO) problem is a discrete approximation of the CTO problem (2.4):

$$\begin{aligned} & min \quad J_N = \bar{y}_0^2 + (\bar{y}_N - 1)^2 \\ & subject\ to \begin{cases} \sum\limits_{l=0}^{N} \bar{u}_l D_{lk} - \frac{nsinh(n\bar{y}_k)}{2} = 0, \\ \sum\limits_{l=0}^{N} \bar{y}_l D_{lk} - \frac{\bar{u}_k}{2} = 0, \\ k = 0, 1, 2, \ldots, N, \end{cases} \end{aligned} \tag{2.11}$$

where N is sufficiently large.
By finding the optimal solution of the DTO problem (2.11), a numerical approximation for the solution of Troesch's problem (1.1) as $y(t_k) \approx \bar{y}_k^*, k = 0, 1, 2, \ldots, N$ is obtained, where $t_k = \frac{\tau_k+1}{2}$ and $\tau_k, k = 0, 1, \ldots, N$ are the CGL nodes. Furthermore, a continuous approximation is obtained, using interpolation, as follows:

$$y^*(t) \approx \sum_{l=0}^{N} \bar{y}_l^* L_l(2t-1),\ 0 \leq t \leq 1. \tag{2.12}$$

Remark 2.1. We note that the DTO problem (2.11) can be solved by techniques of nonlinear programming.

3. Investigation of feasibility and convergence

In this section, we investigate the feasibility of the DTO problem (2.11). Also, we analyze the convergence of the approximation to the exact solution of Troesch's problem. it is demonstrated that we can relax the constraints of problem (2.11) to warranty feasibility. To this end, the following problem is suggested, in which the Polak's theory of consistent approximations [39] is applied for the relaxation:

$$\begin{aligned} & min \quad J_N = \bar{y}_0^2 + (\bar{y}_N - 1)^2 \\ & subject\ to \begin{cases} \left|\sum\limits_{l=0}^{N} \bar{u}_l D_{lk} - \frac{1}{2}nsinh(n\bar{y}_k)\right| \leq (N-1)^{\frac{3}{2}} - m, \\ \left|\sum\limits_{l=0}^{N} \bar{y}_l D_{lk} - \frac{1}{2}\bar{u}_k\right| \leq (N-1)^{\frac{3}{2}-m}, \end{cases} \end{aligned} \tag{3.1}$$

where $m \geq 2$ is specified. It is obvious that when $N \to \infty$, there is no differences between constraints of (2.11) and (3.1).

Let $W^{m,\infty}$ be the Sobolov space, which contains all functions $\eta : [-1, 1] \to \mathbb{R}$ with $\eta^{(j)} \in L^\infty,\ 0 \leq j \leq m$, with the norm:

$$\| \eta \| = \sum_{j=0}^{m} sup_{t \in [-1,1]} |\eta^{(j)}(t)|.$$

Lemma 3.1. *[3] Let $\eta \in W^{m,\infty}$ be given. Then, there exists a polynomial $p_N(t)$ of degree less than or equal to N, that*

$$|\eta(t) - p_N(t)| \leq cc_0 N^{-m}, \ -1 \leq t \leq 1 \tag{3.2}$$

in which c is a constant and $c_0 = \parallel \eta \parallel$.

Remark 3.2. Since $y(.)$, i.e. the solution of ODE (1.1) is continuously differentiable on $[0, 1]$, we can find a constant $k \in \mathbb{R}$ with

$$|y(t)| \leq k, \ t \in [0, 1].$$

Hence,

$$|\frac{d}{dy} sinh(ny)| = ncosh(ny) \leq ncosh(nk), \ y \in [-k, k].$$

Also, for function $f(y) = nsinh(ny), -k \leq y \leq k$ and for all $y_1, y_2 \in [-k, k]$, by mean value theorem, we have

$$|f(y_1) - f(y_2)| = |n||sinh(ny_1) - sinh(ny_2)| \leq |n|^2 cosh(nk)|y_1 - y_2|. \tag{3.3}$$

Remark 3.3. We note that since the solution $y(.)$ of BVP (1.1) is continuously differentiable on $[0, 1]$, so $y(.) \in W^{m,\infty}$.

Theorem 3.4. *Suppose that $(Y(.), U(.))$ is an optimal solution of (2.4), then there exists an integer $N_1 > 0$ such that problem (3.1) for any $N \geq N_1$, has a feasible solution $(\bar{y}_k, \bar{u}_k), \ k = 0, 1, \ldots, N$ such that*

$$\begin{cases} L(N-1)^{1-m} \geq |Y(\tau_k) - \bar{y}_k|, & k = 0, 1, \ldots, N, \\ L(N-1)^{1-m} \geq |U(\tau_k) - \bar{u}_k|, & k = 0, 1, \ldots, N, \end{cases} \tag{3.4}$$

where $L > 0$ is a constant and τ_k are the CGL nodes, $k = 0, 1, \ldots, N$.

Proof. From Lemma 3.1 and Remark 3.3, there are polynomials $p_1(.)$ and $p_2(.)$ from order $(N-1)$ and constants c_1 and c_2 that are independent of N such that

$$\begin{cases} c_1(N-1)^{1-m} \geq |Y'(\tau) - p_1(t)|, \\ c_2(N-1)^{1-m} \geq |U'(\tau) - p_2(t)|. \end{cases} \tag{3.5}$$

Define

$$\begin{cases} Y_N(\tau) = \int_{-1}^{\tau} p_1(x)dx + Y(-1), \\ U_N(\tau) = \int_{-1}^{\tau} p_2(x)dx + U(-1), \end{cases} \tag{3.6}$$

and $\bar{y}_k = Y_N(\tau_k)$ and $\bar{u}_k = U_N(\tau_k)$, for $k = 0, 1, 2, \ldots, N$. We demonstrate that $(\bar{y}_k, \bar{u}_k), k = 0, 1, 2, \ldots, N$ is a feasible solution for problem (3.1). From relations (3.5) and (3.6), it follows:

$$\begin{aligned} |Y(\tau) - Y_N(\tau)| &= |\int_{-1}^{\tau} (Y'(x) - p_1(x))dx| \leq \int_{-1}^{\tau} |Y'(x) - p_1(x)|dx \\ &\leq c_1(N-1)^{1-m} \int_{-1}^{\tau} dx = (\tau + 1)c_1(N-1)^{1-m} \leq 2c_1(N-1)^{1-m}, \ -1 \leq \tau \leq 1. \end{aligned} \tag{3.7}$$

By an analogous manner, we attain

$$|U(\tau) - U_N(\tau)| \leq 2c_2(N-1)^{1-m}. \tag{3.8}$$

Now, by (3.6), $Y_N(\tau)$ and $U_N(\tau)$ are polynomials with degree equal to N or less than it. But, derivative of any polynomial of degree equal to N or less than it, at the CGL nodes $\tau_0, \ldots, \tau_N$ equals to the value of the polynomial at the nodes multiplied by the matrix D, defined by (2.10). Hence, we attain

$$\sum_{l=0}^{N} \bar{y}_l D_{lk} = Y_N'(\tau_k), \ k = 0, 1, 2, \ldots, N, \ D = (D_{lk}),$$

and

$$\sum_{l=0}^{N} \bar{u}_l D_{lk} = U'_N(\tau_k), \ \ k = 0, 1, 2, \ldots, N.$$

Hence,

$$|\sum_{l=0}^{N} \bar{y}_l D_{lk} - \frac{1}{2}\bar{u}_k| \leq |Y'_N(\tau_k) - Y'(\tau_k)| + |Y'(\tau_k) - \frac{1}{2}\bar{u}_k|$$
$$= |p_1(\tau_k) - Y'(\tau_k)| + |\frac{1}{2}U(\tau_k) - \frac{1}{2}\bar{u}_k|$$
$$\leq c_1(N-1)^{1-m} + c_2(N-1)^{1-m} = (c_1 + c_2)(N-1)^{1-m}.$$

Also, by relations (3.3), (3.5) and (3.8), we have

$$|\sum_{l=1}^{N} \bar{u}_l D_{lk} - n sinh(n\bar{y}_k)| \leq |U'_N(\tau_k) - U'(\tau_k)| + |U'(\tau_k) - n sinh(n\bar{y}_k)|$$
$$= |p_2(\tau_k) - U'(\tau_k)| + |n sinh(nY(\tau_k)) - n sinh(n\bar{y}_k)|$$
$$\leq c_2(N-1)^{1-m} + n^2 cosh(nk)|Y(\tau_k) - \bar{y}_k|$$
$$\leq c_2(N-1)^{1-m} + c_1 n^2 cosh(nk)(N-1)^{1-m}.$$

By selecting N_1 such that $\max\{c_1 + c_2, c_2 + c_1 n^2 cosh(nk)\} \leq N^{\frac{1}{2}}$ for all $N \geq N_1$, we have:

$$|\sum_{i=0}^{N} \bar{y}_l D_{lk} - \frac{1}{2}\bar{u}_k| \leq (N-1)^{\frac{3}{2}-m}, \tag{3.9}$$

and

$$|\sum_{l=0}^{N} \bar{u}_l D_{lk} - n sinh(n\bar{y}_k)| \leq (N-1)^{\frac{3}{2}-m}. \tag{3.10}$$

By relations (3.9) and (3.10), we conclude that $(\bar{y}_k, \bar{u}_k), k = 0, 1, \ldots, N$ satisfy the constraints of DTO (3.1) and therefore the feasibility of this problem is proved. By setting $L = 2\max\{c_1, c_2\}$, we obtain relation (3.4) directly from (3.7) and (3.8). □

From Theorem 3.4 we conclude that the set of feasible points of problem (2.11) is nonempty and compact. Hence, the continuous objective function $J_N = \bar{y}_0^2 + (\bar{y}_N - 1)^2$ achieves the minimal solution.

Now, it is demonstrated that the sequence of solutions of DTO problem (3.1) converges to the optimal solution of CTO problem (2.4). The procedure concludes from [23, 24] and is according to the Polak's theory of consistent approximation [39]. Assume that optimal solution to problem (3.1) is as $(\bar{y}_k^*, \bar{u}_k^*), \ k = 0, 1, \ldots, N$. Define

$$\begin{cases} Y_N^*(t) = \sum_{k=0}^{N} \bar{y}_k^* L_k(t), & t \in [-1, 1] \\ U_N^*(t) = \sum_{k=0}^{N} \bar{u}_k^* L_k(t), & t \in [-1, 1], \end{cases}$$

where $L_k(.)$ is the Lagrange interpolating polynomial, for $k = 0, 1, \ldots, N$. Therefore, a sequence of direct solutions $\{(\bar{y}_k^*, \bar{u}_k^*), k = 0, 1, 2, \ldots, N\}_{N=N_1}^{\infty}$ and the sequence of interpolating functions $\{Y_N^*(.), U_N^*(.)\}_{N=N_1}^{\infty}$ are obtained.

Assumption I: We assume that the sequence $\{(\bar{y}_0, \bar{u}_0, Y_N'^*(.), U_N'^*(.))\}_{N=N_1}^{\infty}$ has a subsequence that uniformly converges to $\{(y_0^{\infty}, u_0^{\infty}, q_1(.), q_2(.))\}_{N=N_1}^{\infty}$, where $q_1(.)$ and $q_2(.)$ are continuous functions.

Theorem 3.5. *Assume that a sequence of optimal points of (3.1) is as* $\{(\bar{y}_k^*, \bar{u}_k^*), k = 0, 1, \ldots, N\}_{N=N_1}^{\infty}$ *and* $\{Y_N^*(.), U_N^*(.)\}_{N=N_1}^{\infty}$ *is their interpolating function sequence that satisfies Assumption* I*, then,*

$$\begin{cases} Y^*(\tau) = \int_{-1}^{\tau} q_1(t)dt + y_0^{\infty}, & -1 \le t \le 1 \\ U^*(\tau) = \int_{-1}^{\tau} q_2(t)dt + u_0^{\infty} & -1 \le t \le 1 \end{cases} \tag{3.11}$$

is an optimal point for problem (2.4).

Proof. From assumption I, we conclude that there exists a subsequence $\{Y_{N_i}^*{}'(.), U_{N_i}^*{}'(.)\}_{i=1}^{\infty}$ of sequence $\{Y_N^*{}'(.), U_N^*{}'(.)\}_{N=N_1}^{\infty}$ such that $lim_{i\to\infty} N_i = \infty$ and

$$\lim_{i\to\infty} (Y_{N_i}^*{}'(.), U_{N_i}^*{}'(.)) = (q_1(.), q_2(.)). \tag{3.12}$$

So, from (3.11) and (3.12) and Assumption I, we can conclude that

$$\begin{cases} \lim_{i\to\infty} Y_{N_i}^*(.) = Y^*(.) \\ \lim_{i\to\infty} U_{N_i}^*(.) = U^*(.). \end{cases}$$

Also, by investigating the objective function of DTO problem (3.1) and applying assumption I, it is easy to see that

$$\lim_{i\to\infty} (\bar{y}_0^{*2} + (\bar{y}_{N_i}^* - 1)^2) = Y^{*2}(-1) + (Y^*(1) - 1)^2.$$

Now, we first demonstrate that $(Y^*(.), U^*(.))$ is feasible for problem (2.4). Then, we show that $(Y^*(.), U^*(.))$ is optimal for CTO problem (2.4).

Step 1. We prove that $(Y^*(.), U^*(.))$ satisfies the differential equations of problem (2.4). With no loss of generality, suppose that the second differential equation is not satisfied at $(Y^*(.), U^*(.))$. Thus, there exists a time $\bar{t} \in [-1, 1]$ with

$$U'^*(\bar{t}) - \frac{1}{2} n sinh(nY^*(\bar{t})) \neq 0.$$

From density of CGL nodes τ_k in $[-1, 1]$, for $k = 0, 1, \ldots, N$ (see [20]), we conclude that a sequence k_{N_i} exists such that $0 < k_{N_i} < N_i$ and $lim_{i\to\infty} t_{k_{N_i}} = \bar{t}$. Thus,

$$U'^*(\bar{t}) - \frac{1}{2} n sinh(nY^*(\bar{t})) = lim_{i\to\infty} (U_{N_i}'^*(t_{k_{N_i}}) - \frac{1}{2} n sinh(nY_{N_i}^*(t_{k_{N_i}})) \neq 0. \tag{3.13}$$

On the other hand, by $lim_{i\to\infty}(N_i - 1)^{\frac{3}{2} - m} = 0$, and constraints of problem (3.1), we conclude

$$lim_{i\to\infty} (U_{N_i}'^*(t_{k_{N_i}}) - \frac{1}{2} n sinh(nY_{N_i}^*(t_{k_{N_i}})) = 0 \tag{3.14}$$

which contradicts (3.13). Hence $(Y^*(.), U^*(.))$ is a feasible point for (2.4).

Step 2. Since Troesch's problem has a solution, there exists a solution for the CTO problem (2.4) with the objective function equal to zero. Now, assume that $(Y^{**}(.), U^{**}(.))$ is an optimal solution for (2.4) with $Y^{**}(.) \in W^{m,\infty}, m \ge 2$. From Theorem 3.4, we conclude that there is a sequence of feasible points $\{\tilde{y}_k^N, \tilde{u}_k^N) : k = 0, 1, \ldots, N\}_{N=N_1}^{\infty}$ for CTO problem (2.4) that is convergent to $(Y^{**}(.), U^{**}(.))$, uniformly. Now, by optimality of $(Y^{**}(.), U^{**}(.))$ and $(\bar{y}_k^*, \bar{u}_k^*), k = 0, 1, \ldots, N$ we have:

$$\begin{aligned} 0 = (Y^{**}(-1))^2 + (Y^{**}(1) - 1)^2 \le (Y^*(-1))^2 + (Y^*(1) - 1)^2 \\ = lim_{i\to\infty}((Y_{N_i}^*(-1))^2 + (Y_{N_i}^*(1) - 1)^2) = \lim_{i\to\infty} (\bar{y}_0^{*^2} + (\bar{y}_{N_i}^* - 1)^2) \\ \le \lim_{N\to\infty} ((\tilde{y}_0^N)^2 + (\tilde{y}_N^N - 1)^2) = (Y^{**}(-1))^2 + (Y^{**}(1) - 1)^2 = 0 \end{aligned} \tag{3.15}$$

By (3.15), we have

$$Y^{*2}(-1) + (Y^*(1) - 1)^2 = 0$$

and therefore $(Y^*(.), U^*(.))$ is a feasible point with the objective value equal to zero that is optimal. □

4. Numerical illustrations

In this section, we apply the CPS method to particular instances of Troesch's problem, to show its accuracy and fast convergence.
At first, we consider Troesch's problem for $n = 0.5$ in Equation (1.1). We solve the NLP problem (2.11) for various values of N. The DTO problem (2.11) is solved in *MATLAB* software by *FMINCON* function. The pointwise approximate optimal solutions $\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N$ for $N = 9$ are given in Table 1. Also, the numerical point attained by the CPS method at the points $t = 0.1, 0.2, \ldots, 0.9$ for $n = 0.5$ are compared with the exact solution of Troesch's problem. We also compare the results with five other numerical procedures, namely, the decomposition technique given by Deeba et al. [7], the Laplace transformation procedure by Khuri [25], the modified homotopy perturbation method given by Feng et al. [18], the homotopy perturbation technique utilized by Leal et al. [27] and the sinc-Galerkin approach applied by Zarebnia and Sajjadian [50]. The results are given in Table 2. Moreover, in Table 3 we compare the absolute difference between the exact solution and the approximate ones, called the absolute error, of the proposed method with that of the aforementioned numerical methods. In Figure 1, the absolute error of the utilized method is compared with some other methods [25, 27, 50]. The graphical and numerical results demonstrate that the suggested approach is the most precise one, in comparison with other numerical techniques. Moreover, from Figure 2 we conclude that by increasing the number of Chebyshev nodes (N), the absolute error decreases, rapidly.

In the second case, the approximation is attained for Troesch's problem with $n = 1$. The pointwise approximate optimal solutions $\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N$ for $N = 10$ are given in Table 4. In Table 5, we compare the obtained approximate results for Troesch's problem for the case $n = 1$ with the exact ones and points attained by some other numerical approaches [7, 18, 27, 50]. In this table, we put the results of solving DTO problem (2.11) for $N = 10$ and $N = 16$. Moreover, in Table 6 the absolute error of the proposed method for $N = 10$ is compared with some other numerical techniques. Figure 3 shows the absolute error of our method compared with that of Zarebnia and Sajjadian [50]. Moreover, in Figure 4 absolute error for different number of nodes is shown. Numerical results demonstrate the validity, applicability and accuracy of the method compared with the other mentioned methods. Moreover, in Table 7 we present the values of the CPS solution at the endpoint $t = 0$ for $n = 0.5$ and $n = 1$. The numerical approximations, compared with the most accurate available results, affirm the proficiency and precision of the CPS method. Moreover, the solutions of Troesch's problem for some other values of n is depicted in Figure 5. Also, in Figure 6 we change the input domain to the interval $[0, 30]$ and find the solution of the Troesch's problem on this domain for different values on n including $n = 5, 10, 15$. The attained results confirm efficiency of the suggested approach for the new interval $[0, 30]$. Further, we run the proposed algorithm for different values of n including $n = 5, 10, 15$, and depict the results in Figure 7. Finally, we compared our results with those of references [45] and [44] for $n = 10$ in Table 8. As it can be seen in Figures 6 and 7, the proposed algorithm has the ability of solving Troesch's problem with a big n or a large span of the domain interval, efficiently. However, when n is big or the domain interval is large, in order to obtain acceptable results, it is necessary to increase N.

TABLE 1. Approximate solutions $y(\tau_k) \simeq \bar{y}_k, k = 0, 1, \ldots, N$ for $n = 0.5$ and $N = 9$.

k=0,1	k=2,3	k=4,5	k=6,7	k=8,9
0.00000000	0.112250798	0.399084001	0.736373459	0.967431003
0.028919804	0.240386265	0.570932754	0.874900442	1.00000000

TABLE 2. Approximate solution for Troesch's problem for n=0.5.

t	Exact solution	Deeba et al. [7]	Khuri [25]	Feng et al. [18]	Leal et al. [27]	*Zarebnia and sajjadian* [50]	*Our approach forN* $= 9$
0.1	0.0959443492923	0.0959477541	0.09594435202	0.0959395656	0.0959443155	0.095944347	0.095944349310930
0.2	0.1921287476603	0.1921352537	0.1921287539	0.1921193244	0.1921286848	0.192128740	0.192128747653106
0.3	0.2887944008934	0.2888034214	0.2887944107	0.2887806940	0.2887943176	0.288794409	0.288794400877506
0.4	0.3861848463623	0.3861955524	0.3861848612	0.3861675428	0.3861847539	0.386184841	0.386184846394200
0.5	0.4845471647449	0.4845585473	0.4845471832	0.4845274183	0.4845470753	0.484547165	0.484547164827194
0.6	0.5841332484456	0.5841442013	0.5841332650	0.5841127822	0.5841331729	0.584133254	0.584133248531741
0.7	0.6852011483018	0.6852105701	0.6852011675	0.6851822495	0.6852010943	0.685201142	0.685201148362007
0.8	0.7880165226496	0.7880234321	0.7880165463	0.788001867	0.7880164925	0.788016528	0.788016522690064
0.9	0.8928542161363	0.8928578710	0.8928542363	0.8928462193	0.8928542059	0.892854218	0.892854216158665

TABLE 3. Absolute error of solution of Troesch's problem for n=0.5, using different methods.

t	Deeba et al. [7]	Khuri [25]	Feng et al. [18]	Leal et al. [27]	*Zarebnia and Sajjadian* [50]	*Our approach forN* $= 9$
0.1	3.40×10^{-6}	2.72×10^{-9}	4.78×10^{-6}	3.38×10^{-8}	2.30×10^{-8}	1.09×10^{-11}
0.2	6.51×10^{-6}	6.20×10^{-9}	9.42×10^{-6}	6.29×10^{-8}	7.70×10^{-8}	4.69×10^{-11}
0.3	9.02×10^{-6}	9.80×10^{-9}	1.37×10^{-5}	8.33×10^{-8}	8.10×10^{-8}	2.25×10^{-11}
0.4	1.07×10^{-5}	1.48×10^{-8}	1.73×10^{-5}	9.25×10^{-8}	5.40×10^{-8}	0.50×10^{-11}
0.5	1.14×10^{-5}	1.85×10^{-8}	1.97×10^{-5}	8.89×10^{-8}	3.00×10^{-9}	1.27×10^{-10}
0.6	1.10×10^{-5}	1.66×10^{-8}	2.07×10^{-5}	7.75×10^{-8}	5.60×10^{-8}	1.31×10^{-10}
0.7	9.42×10^{-6}	1.92×10^{-8}	1.89×10^{-5}	5.40×10^{-8}	6.30×10^{-8}	9.42×10^{-11}
0.8	6.91×10^{-6}	2.36×10^{-8}	1.47×10^{-5}	3.02×10^{-8}	5.30×10^{-8}	0.99×10^{-11}
0.9	3.65×10^{-6}	2.02×10^{-8}	8.00×10^{-6}	1.02×10^{-8}	1.90×10^{-8}	5.87×10^{-11}

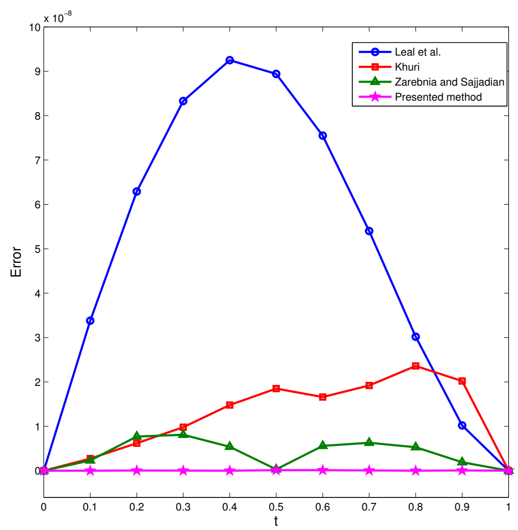

FIGURE 1. Error comparison of the CPS method with some other methods for $n = 0.5$.

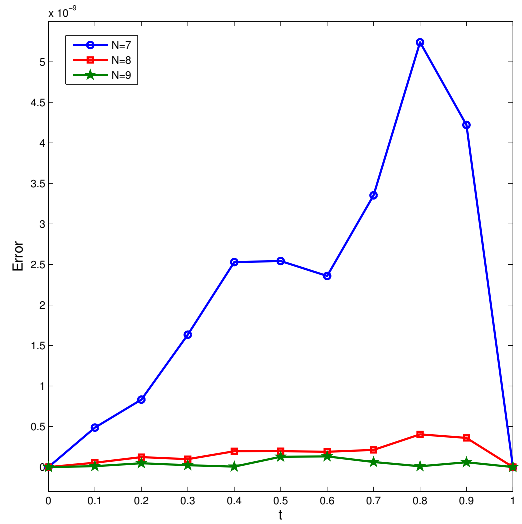

FIGURE 2. Error of Troesch's problem with different number of CGL nodes, for $n = 0.5$.

TABLE 4. Approximate solutions $y(\tau_k) \simeq \bar{y}_k, k = 0, 1, \ldots, N$ for $n = 1$ and $N = 10$.

k=0,1	k=2,3	k=4,5	k=6,7	k=8,9	$k = 10$
0.000000000000000	0.080832430522906	0.297879996209397	0.594225370814932	0.876936336964576	1.000000000000000
0.020685646429221	0.175440382486681	0.440599834126281	0.745638478215252	0.967509783495296	

TABLE 5. Approximate solution for Troesch's problem for n=1.

t	Exact solution	Deeba et al. [7]	Feng et al. [18]	Leal et al. [27]	*Zarebnia and Sajjadian* [50]	*Our approach for* $N = 10$	*Our approach for* $N = 16$
0.1	0.0846612565515	0.084248760	0.0843817004	0.0846607585	0.084661250	0.084661256233803	0.084661256551547
0.2	0.1701713581775	0.169430700	0.1696207644	0.1701704581	0.170171338	0.170171357775566	0.170171358177518
0.3	0.2573939080798	0.256414500	0.2565929224	0.2573927827	0.257393933	0.257393907199741	0.257393908079857
0.4	0.3472228551104	0.346085720	0.3462107378	0.3472217324	0.347222839	0.347222853916141	0.347222855110482
0.5	0.4405998351683	0.439401985	0.4394422743	0.4405989511	0.440599836	0.440599834126281	0.440599835168429
0.6	0.5385343980768	0.537365700	0.5373300622	0.5385339413	0.538534416	0.538534396340466	0.538534398076923
0.7	0.6421286091908	0.641083800	0.6410104651	0.6421286573	0.642128589	0.642128606902691	0.642128609190851
0.8	0.7526080940464	0.751788000	0.7517335467	0.7526085475	0.752608114	0.752608092252368	0.752608094046422
0.9	0.8713625197982	0.870908700	0.8708835371	0.8713630450	0.871362527	0.871362516502162	0.871362519798196

TABLE 6. Absolute error of solution of Troesch's problem for n=1, using different methods.

t	Deeba et al. [7]	Feng et al. [18]	Leal et al. [27]	*Zarebnia and Sajjadian* [50]	*Our approach for* $N = 10$
0.1	4.12×10^{-4}	2.76×10^{-4}	4.98×10^{-7}	6.55×10^{-9}	3.18×10^{-10}
0.2	7.00×10^{-4}	5.56×10^{-4}	9.00×10^{-7}	2.02×10^{-8}	4.02×10^{-10}
0.3	9.79×10^{-4}	8.10×10^{-3}	1.12×10^{-6}	2.50×10^{-8}	8.80×10^{-10}
0.4	1.14×10^{-3}	1.01×10^{-3}	1.12×10^{-6}	1.61×10^{-8}	1.19×10^{-9}
0.5	1.19×10^{-3}	1.16×10^{-3}	8.84×10^{-7}	0.83×10^{-9}	1.04×10^{-9}
0.6	1.16×10^{-3}	1.20×10^{-3}	4.57×10^{-7}	1.79×10^{-8}	1.74×10^{-9}
0.7	1.04×10^{-3}	1.19×10^{-3}	0.48×10^{-7}	2.10×10^{-8}	2.29×10^{-9}
0.8	8.20×10^{-4}	8.74×10^{-4}	4.53×10^{-7}	1.99×10^{-8}	1.79×10^{-9}
0.9	4.53×10^{-4}	4.79×10^{-4}	5.25×10^{-7}	7.20×10^{-9}	3.29×10^{-9}

Table 7. The approximate values for $y'(0)$.

n	N=10	N=12	N=14	N=16	N=18
n=0.5	0.959043795421242	0.959043795413151	0.959043795413204	0.959043795413238	0.959043795413266
n=1	0.845202682747389	0.845202685268858	0.845202685309271	0.845202685309938	0.845202685309953

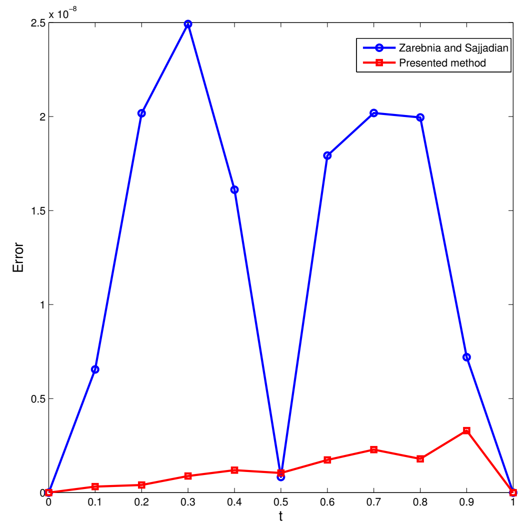

Figure 3. Error comparison of the CPS method with Zarebnia and Sajjadian method, for $n = 1$ and $N = 10$.

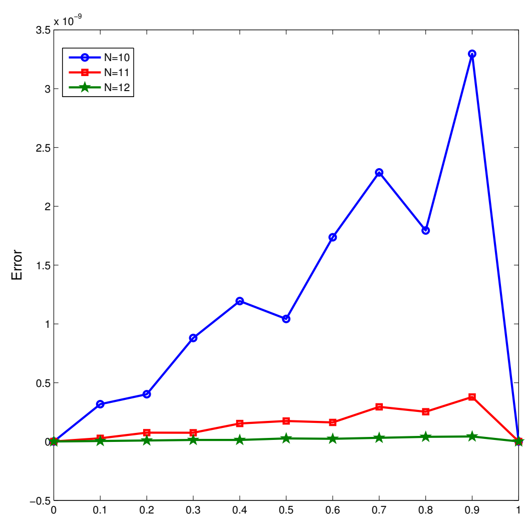

Figure 4. Error of Troesch's problem with different number of CGL nodes, for $n = 1$.

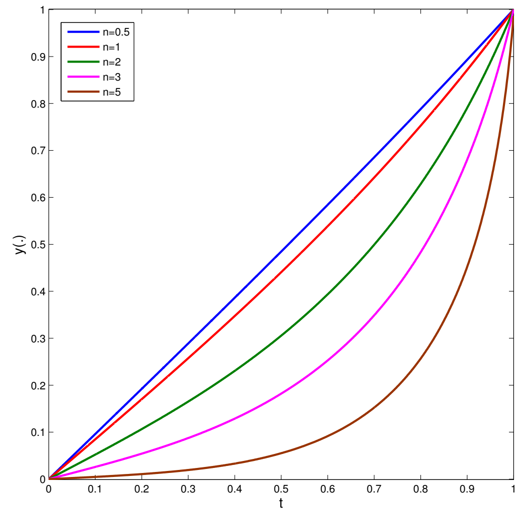

FIGURE 5. Solutions of Troesch's problem for different values of n.

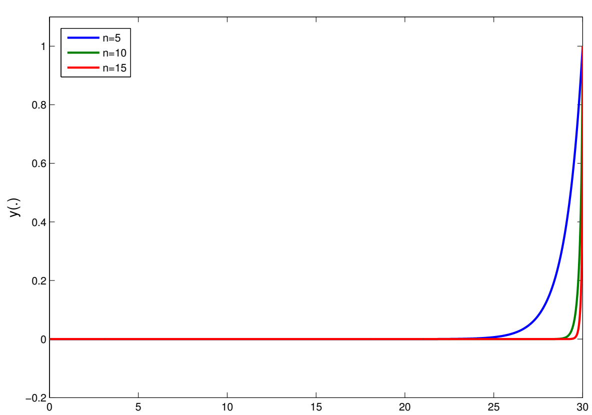

FIGURE 6. Solutions of Troesch's problem on time interval $[0, 30]$ and different values of n.

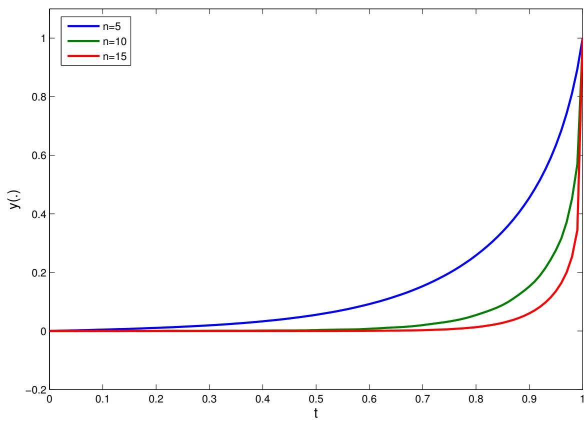

FIGURE 7. Solutions of Troesch's problem on time interval $[0, 1]$ and $n = 5, 10, 15$.

TABLE 8. Approximate solution of Troesch's problem for $n = 10$, using different methods.

t	Temimi and Kurkcu [45]	Scott [44]	Our approach
0.1	0.000042	0.000042	0.000045
0.2	0.000130	0.000130	0.000131
0.3	0.000359	0.000359	0.000359
0.4	0.000978	0.000978	0.000978
0.5	0.002659	0.002659	0.002659
0.6	0.007229	0.007229	0.007228
0.7	0.019664	0.019664	0.019662
0.8	0.053730	0.053730	0.053724
0.9	0.152114	0.152114	0.152095

5. Conclusions

In this paper, we demonstrated that the Chebyshev pseudo-spectral method is an applicable method for attaining an approximation of the exact solution of Troesch's problem, with high accuracy. The feasibility as well as convergence of the attained approximate solutions were analyzed. Moreover, we showed efficiency of the suggested method compared to other methods. We indicated that the CPS method, in comparison with other available methods, has very high accuracy.

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