Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-19
DOI:10.22034/cmde.2024.57676.2417

Direct and inverse problems of ROD equation using finite element method and a correction technique

Hanif Mirzaei[1,*], Kazem Ghanbari[1,2], Vahid Abbasnavaz[1], and Angelo Mingarelli[2]
[1]Faculty of Basic Sciences, Sahand University of Technology, Tabriz, Iran.
[2]School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada.

Abstract

The free vibrations of a rod are governed by a differential equation of the form $(a(x)y')' + \lambda a(x)y(x) = 0$, where $a(x)$ is the cross sectional area and λ is an eigenvalue parameter. Using the finite element method (FEM) we transform this equation to a generalized matrix eigenvalue problem of the form $(K - \Lambda M)u = 0$ and, for given $a(x)$, we correct the eigenvalues Λ of the matrix pair (K, M) to approximate the eigenvalues of the rod equation. The results show that with step size h the correction technique reduces the error from $O(h^2 i^4)$ to $O(h^2 i^2)$ for the i-th eigenvalue. We then solve the inverse spectral problem by imposing numerical algorithms that approximate the unknown coefficient $a(x)$ from the given spectral data. The cross section is obtained by solving a nonlinear system using Newton's method along with a regularization technique. Finally, we give numerical examples to illustrate the efficiency of the proposed algorithms.

Keywords. Rod equation, Eigenvalue, Finite element method, Direct problem, Inverse problem, Sturm-Liouville.
2010 Mathematics Subject Classification. 34B24, 34L16, 65F18, 74K10.

1. Introduction

The free longitudinal vibrations of a rod of length one is governed by the following differential equation:

$$(a(x)y')' + \lambda a(x)y(x) = 0, \ 0 < x < 1, \tag{1.1}$$

where $a(x)$ is the cross sectional area at a point x, λ is a parameter, and $y(x)$ is the displacement [17]. Normally, two boundary conditions at the end points $x = 0$ and $x = 1$ are considered for Equation (1.1), the most important ones being,

$$\begin{aligned} &free - free : y'(0) = 0, \quad y'(1) = 0, \\ &fixed - free : y(0) = 0, \quad y'(1) = 0, \\ &fixed - fixed : y(0) = 0, \quad y(1) = 0. \end{aligned} \tag{1.2}$$

When Equation (1.1) is considered together with one of the boundary conditions (1.2) it becomes an eigenvalue problem where λ is an eigenvalue, the corresponding non-trivial solution y being a corresponding eigenfunction [17]. It is known that the differential equation (1.1) with one set of the boundary conditions (1.2) has a sequence of nonnegative eigenvalues, say $\{\lambda_i\}_{i=1}^{\infty}$, where the λ_i are distinct, arranged in order of increasing magnitude, and $\lim_{i\to\infty} \lambda_i = \infty$. For more details see [17, 21]. The set of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ is called the *spectrum* of the problem (1.1)-(1.2). In the literature one usually finds three types of problems corresponding to (1.1), namely: (i) Direct problems, (ii) Inverse problems and (iii) Isospectral problems.

In direct problems, for a given $a(x)$, spectral data such as the eigenvalues, the eigenfunctions, and related asymptotic behaviour of either or both, are studied. It is clear that the rod equation is not explicitly solvable, in general. In practice, however, numerical methods such as finite differences, finite element, and other such finite dimensional

Received: 23 July 2023; Accepted: 27 March 2024.
∗ Corresponding author Email: h_mirzaei@sut.ac.ir.

methods are used to estimate the spectral data. Using such methods Equations (1.1) and (1.2) are then transformed to a matrix eigenvalue problem where the eigenvalues of the resulting matrix become approximations for the first N eigenvalues of the problem (1.1) and (1.2). Each of these methods can be used to approximate the eigenvalues of lower indices but for eigenvalues of higher indices they generally lead to poor numerical results. It is obvious that rod equation (1.1) is a special case of Sturm-Liouville equation. For the canonical form of the Sturm-Liouville equation, the finite difference method with step size h and finite element method with linear shape functions have error of order $O(h^2 i^4)$ for i-th eigenvalue [7, 37]. Other algorithms for solving direct problems can be found in [11, 22, 23, 26, 27, 30]. The asymptotic behaviour of the eigenvalues and eigenfunctions of Sturm-Liouville problems can be found in various references, e.g., [5, 17, 21, 31].

The corresponding inverse spectral problem is concerned with the construction of the unknown cross sectional area $a(x)$ using spectral data (such as eigenvalues, eigenfunctions, or both). By solving this inverse problem for a rod, we may design a rod with a prescribed structure and frequencies. The importance of finding the solution of such an inverse problem is thus realized. In general, the construction of a unique function, $a(x)$, requires two spectra corresponding to two different boundary conditions, e.g. fixed-fixed and fixed-free [6, 17, 20, 21]. In the case of a symmetric cross sectional area one can construct $a(x)$ uniquely using one spectrum, see [13, 17]. We refer the reader to [3, 4, 12, 21, 34, 35, 38, 42] and the references therein for information regarding inverse problems associated with a canonical Sturm-Liouville equation. By definition, *isospectral* rods are rods having the same spectrum. Such isospectral problems are studied in [18, 19, 28, 29, 32, 39]. Observe that the problem (1.1)-(1.2) with cross sections $a(x)$ and $ca(x)$ (where c is any positive constant) have the same set of eigenvalues. It follows that whenever $a(x)$ is a solution of the inverse spectral problem then, for any positive constant c, the function $ca(x)$ is also a solution. Therefore, in order to find a unique solution to inverse problem, we must assume some condition, for example, that $a(x)$ attain some specific value at a point of the interval $[0, 1]$, say $x = 1$. In other words, we will assume that:

$$\exists x^* \in [0,1],\ a(x^*) = 1. \tag{1.3}$$

Paine et al. [37] found a good approximation for the eigenvalues of the Sturm-Liouville equation $y'' + (\lambda - q(x))y = 0$ with fixed-fixed boundary conditions by correcting the eigenvalues obtained from the finite difference method. Correcting the eigenvalues of the finite element and Numerov methods for a canonical Sturm-Liouville equation are studied in [7, 14, 30]. The idea of correcting the eigenvalues in the finite difference and Numerov methods is also considered in solving inverse Sturm-Liouville problems with fixed-fixed and fixed-free boundary conditions, [8, 14, 15, 36]. The construction of the cross section $a(x)$ is studied in some papers: Morassi et al.[9, 33] constructed rods with given natural frequencies using the idea of quasi-isospectral rods. In [13] by correcting the eigenvalues of the finite difference method, the symmetric cross section $a(x)$ is constructed. In [20] the cross section $a(x)$ is constructed using one spectrum and a minimal mass condition.

To the best of our knowledge, the idea of using the FEM together with a correction has not been applied to solving direct and inverse problems of a rod equation. Therefore, first we discretize the rod equation by using the finite element method in order to obtain the corresponding matrix eigenvalue problem. In order to make good approximations for the eigenvalues of the rod equation we add a suitable correction term to the eigenvalues of the obtained matrix. Then we propose an algorithm based on a correction technique to solve direct and inverse (spectral) problems in the cases of symmetric and nonsymmetric functions, $a(x)$. Our results show that the FEM together with a correction technique can be applied successfully to solve direct and inverse problems of the type considered here.

2. Direct problem for rod equation

In this section, for a given cross section $a(x)$ we want to approximate the eigenvalues of the problem (1.1)-(1.2). To this end, we use the FEM to transform the Rod Equation (1.1) into a generalized matrix eigenvalue problem. Then, using a correction technique, we approximate the eigenvalues of the problem (1.1)-(1.2). Using the finite element method [1, 2, 10] with linear shape functions the Rod Equation (1.1) with fixed-free boundary condition can be discretized as a generalized matrix eigenvalue problem of the form,

$$(K_1 - \Lambda^1 M_1)\mathbf{v} = 0, \tag{2.1}$$

where K_1 is the stiffness matrix, M_1 is mass matrix,

$$K_1 = \frac{1}{h}\begin{pmatrix} a_1+a_2 & -a_2 & & & \\ -a_2 & a_2+a_3 & -a_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & a_{N-1}+a_N & -a_N \\ & & & -a_N & a_N \end{pmatrix}, \tag{2.2}$$

$$M_1 = \frac{h}{6}\begin{pmatrix} 2(a_1+a_2) & a_2 & & & \\ a_2 & 2(a_2+a_3) & a_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & 2(a_{N-1}+a_N) & a_N \\ & & & a_N & 2a_N \end{pmatrix}, \tag{2.3}$$

and where $h = \frac{1}{N}$, $x_i = ih$, $a_i = a(x_i - \frac{h}{2})$, Λ^1 is an eigenvalue of the corresponding discrete system, and $\mathbf{v} = [v_1, v_2, \ldots, v_n]^T$ is a corresponding eigenvector. For fixed-fixed boundary conditions, Equation (1.1) can be transformed into $(K_2 - \Lambda^2 M_2)\mathbf{u} = 0$ where K_2 and M_2 are tridiagonal matrices of order $N-1$, obtained by deleting the last row and column of matrices K_1 and M_1, respectively and $\mathbf{u} = [u_1, u_2, \ldots, u_{N-1}]^t$ is an eigenvector.

2.1. **A correction technique.** Let $\{\lambda_i(a(x))\}_{i=1}^{\infty}$ and $\{\mu_i(a(x))\}_{i=1}^{\infty}$ be the eigenvalues of Equation (1.1) with fixed-fixed and fixed-free boundary conditions, respectively. The eigenvalues $\Lambda_i^2(a(x))$ are approximations to the first $N-1$ eigenvalues of $\{\lambda_i(a(x))\}_{i=1}^{\infty}$ and $\Lambda_i^1(a(x))$ are approximations to the first N eigenvalues of $\{\mu_i(a(x))\}_{i=1}^{\infty}$. These eigenvalues are good approximations for the few first eigenvalues only but they lead to poor results for higher indices, see Tables 1-3.

Using the change of variable $y(x) = a(x)w(x)$, Equation (1.1) will be transformed to the Sturm-Liouville equation $w''(x) + (\lambda - q(x))w(x) = 0$, where $q(x) = \frac{a''(x)}{a(x)}$. Thus, using asymptotic formula of the eigenvalues of Sturm-Liouville problem [17] we have

$$\begin{cases} \lambda_i(a(x)) = \lambda_i(1) + C_1 + \alpha_i, \\ \mu_i(a(x)) = \mu_i(1) + C_2 + \beta_i, \end{cases} \tag{2.4}$$

where C_1 and C_2 are constants depending on the cross section $a(x)$ and $\lim_{i\to\infty} \alpha_i = 0$, $\lim_{i\to\infty} \beta_i = 0$.

Due to the asymptotic form (2.4) we note that the increasing error in approximations $\lambda_i(a(x)) \simeq \Lambda_i^2(a(x))$ and $\mu_i(a(x)) \simeq \Lambda_i^1(a(x))$ are results of existing error in approximations $\lambda_i(1) \simeq \Lambda_i^2(1)$ and $\mu_i(1) \simeq \Lambda_i^1(1)$. Thus, we expect a reduction of the error by adding correction terms $\epsilon_1(i,h) = \mu_i(1) - \Lambda_i^1(1)$ and $\epsilon_2(i,h) = \lambda_i(1) - \Lambda_i^2(1)$ to the eigenvalues $\Lambda_i^1(a(x))$ and $\Lambda_i^2(a(x))$, respectively. Using this correction idea, we correct the eigenvalues Λ_i^1 and Λ_i^2 to obtain good approximations for λ_i and μ_i as follows:

$$\begin{cases} \mu_i(a(x)) \simeq \tilde{\Lambda}_i^1 := \Lambda_i^1 + \epsilon_1(i,h), \ i = 1, 2, \ldots, N, \\ \lambda_i(a(x)) \simeq \tilde{\Lambda}_i^2 = \Lambda_i^2 + \epsilon_2(i,h), \ i = 1, 2, \ldots, N-1. \end{cases} \tag{2.5}$$

In order to calculate $\epsilon_1(i,h)$ and $\epsilon_2(i,h)$, and thus λ_i, μ_i we need to find the eigenvalues of the pair (K_1, M_1) and (K_2, M_2) corresponding to the uniform cross sectional area.

Lemma 2.1. [24] *The eigenvalues and eigenvectors of the tridiagonal matrix*

$$T = \begin{pmatrix} a-\alpha & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a-\beta \end{pmatrix}, \tag{2.6}$$

are given by

$$\lambda = a + 2\sqrt{bc}\cos\theta, \ \sin\theta \neq 0, \tag{2.7}$$

and the corresponding eigenvectors are,

$$\mathbf{u}_j = \frac{1}{\sin\theta} q^{j-1}\left[\sin j\theta + \frac{\alpha}{\sqrt{bc}}\sin(j-1)\theta\right], \ j = 1, 2, \ldots, n, \ q = \sqrt{\frac{b}{c}}, \tag{2.8}$$

where θ satisfies the equation,

$$\sin(n+1)\theta + \frac{\alpha+\beta}{\sqrt{bc}}\sin n\theta + \frac{\alpha\beta}{bc}\sin(n-1)\theta = 0. \tag{2.9}$$

Lemma 2.2. *The eigenvalues and the orthonormal eigenvectors of the matrix pair (K_2, M_2) corresponding to a uniform rod $(a(x) \equiv 1)$ with fixed-fixed boundary condition are*

$$\begin{aligned} \Lambda_k^{02} &= \tfrac{6}{h^2}\tfrac{1+\cos k\pi h}{2-\cos k\pi h}, \ k = 1, 2, \ldots, N-1, \\ u_{kj} &= \sqrt{6}\tfrac{\sin jk\pi h}{\sqrt{2+\cos k\pi h}}, \ k = 1, 2, \ldots, N-1. \end{aligned} \tag{2.10}$$

Proof. For a uniform rod, the system $(K_2 - \Lambda^{02}M_2)\mathbf{u} = 0$ can be written as follows:

$$\begin{pmatrix} \frac{2}{h} - \frac{2}{3}h\Lambda^{02} & -(\frac{1}{h} + \frac{h}{6}\Lambda^{02}) & & \\ -(\frac{1}{h} + \frac{h}{6}\Lambda^{02}) & & & \\ & \ddots & \ddots & \\ & & & -(\frac{1}{h} + \frac{h}{6}\Lambda^{02}) \\ & & -(\frac{1}{h} + \frac{h}{6}\Lambda^{02}) & \frac{2}{h} - \frac{2}{3}h\Lambda^{02} \end{pmatrix} \mathbf{u} = 0. \tag{2.11}$$

Comparing the coefficient matrix in (2.11) with the matrix T in (2.6), we have

$$a = c = -(\frac{1}{h} + \frac{h}{6}\Lambda^{02}), \ b = \frac{2}{h} - \frac{2}{3}h\Lambda^{02}, \ \alpha = \beta = 0. \tag{2.12}$$

Now, it is easy to see that zero is an eigenvalue of the coefficient matrix in system (2.11) with multiplicity $N-1$. Substituting (2.12) in (2.9) gives,

$$\theta_k = \frac{k\pi}{N}, \ k = 1, 2, \ldots, N-1. \tag{2.13}$$

Substituting this θ_k into (2.7) with $\lambda = 0$ we find that,

$$\Lambda_k^{02} = \frac{6}{h^2}\frac{1+\cos k\pi h}{2-\cos k\pi h}, \ k = 1, 2 \ldots, N-1. \tag{2.14}$$

Using the same θ_k, we obtain the components, u_{kj}, of $\mathbf{u}_j$ from (2.8), i.e.

$$u_{kj} = \frac{\sin jk\pi h}{\sin k\pi h}, \ k, j = 1, 2, \ldots, N-1. \tag{2.15}$$

The eigenvectors $\mathbf{u}_j$ whose components are (2.15) are orthogonal but not orthonormal with respect to M_2. Indeed, we have

$$\mathbf{u}_k^T M_2 \mathbf{u}_k = \frac{h}{6\sin^2 k\pi h}\left[2\sum_{i=1}^{N-2}\sin\frac{ik\pi}{N}\sin\frac{(i+1)k\pi}{N} + 4\sum_{i=1}^{N-1}\sin^2\frac{ik\pi}{N}\right]. \tag{2.16}$$

On the other hand,

$$\sum_{i=1}^{N-1}\sin^2\frac{ik\pi}{N} = \frac{1}{2}\sum_{i=1}^{N-1}(1-\cos\frac{2ik\pi}{N}) = \frac{1}{2}\left(N-1-\sum_{i=1}^{N-1}\cos\frac{2ik\pi}{N}\right). \tag{2.17}$$

Writing $\cos\frac{2ik\pi}{N}$ as an exponential function and doing some calculations we find

$$\sum_{i=1}^{N-1}\cos\frac{2ik\pi}{N}=-1, \tag{2.18}$$

so that,

$$\sum_{i=1}^{N-1}\sin^2\frac{ik\pi}{N}=\frac{N}{2}. \tag{2.19}$$

Use of a standard trigonometric identity gives,

$$\begin{aligned}\sum_{i=1}^{N-2}\sin\frac{ik\pi}{N}\sin\frac{(i+1)k\pi}{N} &= \frac{1}{2}\sum_{i=1}^{N-2}\left[\cos\frac{k\pi}{N}-\cos\frac{(2i+1)k\pi}{N}\right]\\ &= \frac{1}{2}\left[(N-2)\cos\frac{k\pi}{N}-\sum_{i=1}^{N-2}\cos\frac{(2i+1)k\pi}{N}\right].\end{aligned} \tag{2.20}$$

Using (2.18) we obtain the relation,

$$\sum_{i=1}^{2N+2}\cos\frac{2ik\pi}{2N+2}=0. \tag{2.21}$$

which is valid for all integer k, N. We can now rewrite (2.21) as follows

$$\sum_{i=1}^{N-1}\cos\frac{2(2i+1)k\pi}{2(N+1)}+\sum_{i=1}^{N+1}\cos\frac{2(2i)k\pi}{2(N+1)}+\cos\frac{2k\pi}{2N+2}+\cos\frac{2(2N+1)k\pi}{2N+2}=0.$$

Since $\cos\frac{2k\pi}{2N+2}=\cos\frac{2(2N+1)k\pi}{2N+2}$, there follows,

$$\sum_{i=1}^{N-1}\cos\frac{2(2i+1)k\pi}{2(N+1)}+\sum_{i=1}^{N+1}\cos\frac{2(2i)k\pi}{2(N+1)}=-2\cos\frac{k\pi}{N+1}.$$

From (2.21), we conclude that the second sum in the last equation is zero, hence

$$\sum_{i=1}^{N-2}\cos\frac{(2i+1)k\pi}{N}=-2\cos\frac{k\pi}{N}. \tag{2.22}$$

Combining (2.22) and (2.20) and substituting this result in (2.16), we find

$$\mathbf{u}_k^t M\mathbf{u}_k=\frac{1}{6\sin^2 k\pi h}(2+\cos\frac{k\pi}{N}). \tag{2.23}$$

Use of (2.23) and (2.15) completes the proof. □

Lemma 2.3. *The eigenvalues and the orthonormal eigenvectors of the matrix pair (K_1, M_1) corresponding to a uniform rod with fixed-free boundary conditions are given by,*

$$\begin{aligned}\Lambda_k^{01} &= \tfrac{6}{h^2}\tfrac{1+\cos(k-0.5)\pi h}{2-\cos(k-0.5)\pi h},\ \ k=1,2,\ldots,N,\\ v_{kj} &= \sqrt{6}\tfrac{\sin j(k-0.5)\pi h}{\sqrt{2+\cos(k-0.5)\pi h}},\ \ k=1,2,\ldots,N.\end{aligned} \tag{2.24}$$

Proof. The proof is similar to Lemma 2.2 and so is left to the reader. □

From Lemmas 2.2, 2.3 and the definitions of $\epsilon_1(i,h)$ and $\epsilon_2(i,h)$, we obtain

$$\begin{aligned} \epsilon_1(i,h) &= (i-0.5)^2\pi^2 - \tfrac{6}{h^2}\tfrac{1+\cos(i-0.5)\pi h}{2-\cos(i-0.5)\pi h},\ i=1,2,\dots,N, \\ \epsilon_2(i,h) &= i^2\pi^2 - \tfrac{6}{h^2}\tfrac{1+\cos i\pi h}{2-\cos i\pi h},\ i=1,2,\dots,N-1. \end{aligned} \tag{2.25}$$

Now, having the correction terms $\epsilon_1(i,h)$ and $\epsilon_2(i,h)$ we can approximate the eigenvalues $\lambda_i(a(x))$ and $\mu_i(a(x))$ by using (2.5).

2.2. **Numerical results for direct problem.** In this subsection, to show the efficiency of the corrected formulas (2.5) , we approximate the eigenvalues of (1.1) corresponding to the following cross sections

$$a_1(x) = 1 + 10e^{-25(x-0.5)^2},\ a_2(x) = 2+\cos(3\pi x),\ a_3(x) = 3-4x+2x^2.$$

To compare the results, the exact eigenvalues are computed using Matslise package [22]. The results of Tables 1-3 show that the errors $|\lambda_i - \Lambda_i^1|$ and $|\mu_i - \Lambda_i^2|$ increase rapidly with the index i. We observe that the corrected approximations $\tilde{\Lambda}_i^1$ and $\tilde{\Lambda}_i^2$ are efficient and much better than Λ_i^1 and Λ_i^2.

TABLE 1. Error of uncorrected and corrected eigenvalues for $a_1(x)$.

i	λ_i	$\lvert\lambda_i - \Lambda_i^2\rvert$	$\lvert\lambda_i - \tilde{\Lambda}_i^2\rvert$
1	2.3965	0.0019	0.0051
2	50.976	0.0332	0.0792
3	101.830	0.4086	0.1616
4	167.179	1.6075	0.1981
5	254.184	4.1327	0.2865
6	362.495	8.7564	0.4342
7	490.866	16.469	0.6144
8	638.924	28.422	0.8314
9	806.673	45.948	1.1002
10	994.157	70.580	1.4344
11	1201.39	104.05	1.8467
12	1428.37	148.28	2.3525
13	1675.09	204.2	2.9708
14	1941.56	205.37	3.7247
15	2227.77	367.26	4.6414
16	2533.72	476.815	5.7529

TABLE 2. Error of uncorrected and corrected eigenvalues for $a_2(x)$.

i	λ_i	$\lvert\lambda_i - \Lambda_i^2\rvert$	$\lvert\lambda_i - \tilde{\Lambda}_i^2\rvert$	μ_i	$\lvert\mu_i - \Lambda_i^1\rvert$	$\lvert\mu_i - \tilde{\Lambda}_i^1\rvert$
1	7.806	0.0086	0.0011	2.103	0.0005	0.00008
2	46.021	0.0758	0.0435	36.514	0.0080	0.0297
3	86.759	0.5483	0.0570	66.053	0.2469	0.0447
4	162.10	1.8059	0.1113	125.67	1.0255	0.0971
5	250.69	4.5271	0.1659	206.37	2.9391	0.1357
6	358.21	9.5214	0.2403	302.38	6.6841	0.1972
7	487.27	17.824	0.3246	420.69	13.192	0.2768
8	635.25	30.644	0.4390	559.06	23.590	0.3733
9	802.92	49.414	0.5856	716.66	39.190	0.5068
10	990.50	75.785	0.7612	894.28	61.54	0.6671
11	1197.74	111.59	0.9824	1091.68	92.39	0.8645
12	1424.72	158.89	1.2559	1308.77	133.67	1.1118
13	1671.46	219.92	1.5908	1545.63	187.55	1.4144
14	1937.93	297.04	2.0021	1802.23	256.32	1.7858
15	2224.14	392.746	2.5049	2078.57	342.42	2.2409
16	2530.09	509.51	3.1177	2374.65	448.34	2.7962

TABLE 3. Error of uncorrected and corrected eigenvalues for $a_3(x)$.

i	λ_i	$\lvert\lambda_i - \Lambda_i^2\rvert$	$\lvert\lambda_i - \tilde{\Lambda}_i^2\rvert$	μ_i	$\lvert\mu_i - \Lambda_i^1\rvert$	$\lvert\mu_i - \tilde{\Lambda}_i^1\rvert$
1	10.814	0.021	0.007	3.884	0.0004	0.002
2	40.491	0.426	0.025	23.239	0.129	0.014
3	89.838	2.238	0.059	62.698	1.064	0.040
4	158.92	7.198	0.115	121.91	4.187	0.084
5	247.75	17.802	0.202	200.87	11.61	0.154
6	356.31	37.326	0.333	299.56	26.21	0.261
7	158.92	69.798	0.529	121.91	51.67	0.421
8	247.75	119.82	0.818	200.87	92.316	0.660
9	356.31	192.121	1.239	299.56	152.91	1.009

The Computational order is defined by

$$Co_{\Lambda}^{h} = \frac{log\frac{|\Lambda_i^{h_1}-\lambda_i|}{|\Lambda_i^{h_2}-\lambda_i|}}{log\frac{h_1}{h_2}}, \; Co_{\Lambda}^{i} = \frac{log\frac{|\Lambda_i^{h}-\lambda_i|}{|\Lambda_{i+1}^{h}-\lambda_{i+1}|}}{log\frac{i}{i+1}},$$

where Co_{Λ}^{h} denotes the computational order with respect to the step size h and Co_{Λ}^{i} denotes the computational order with respect to the index i. The results of Tables 4-5 show that the correction technique reduces this error from $O(h^2i^4)$ to $O(h^2i^2)$. Similar results to Tables 4-5 can also be found for the cross sectional areas $a_1(x)$ and $a_3(x)$.

TABLE 4. The computational order of Λ_i^2 and $\tilde{\Lambda}_i^2$ with respect to h for $a_2(x)$ and $h = \frac{1}{10\times 2^n}$.

i	Without Correction			With Correction			
	$Co^h_{\Lambda_i^2}$	$Co^h_{\Lambda_i^2}$	$Co^h_{\Lambda_i^2}$	$Co^h_{\tilde{\Lambda}_i^2}$	$Co^h_{\tilde{\Lambda}_i^2}$	$Co^h_{\tilde{\Lambda}_i^2}$	$Co^h_{\tilde{\Lambda}_i^2}$
1	1.997	1.999	2.000	1.977	1.994	1.999	2.000
2	1.992	1.998	1.999	2.023	2.006	2.001	2.000
3	2.003	2.000	2.000	2.054	2.013	2.003	2.001
4	2.005	2.001	2.000	2.140	2.035	2.009	2.002
5	2.012	2.003	2.000	2.232	2.058	2.015	2.004
6	2.019	2.005	2.001	2.329	2.084	2.021	2.005
7	2.025	2.008	2.002	2.476	2.124	2.031	2.008
8	2.031	2.011	2.003	2.613	2.164	2.041	2.011
9	2.035	2.015	2.004	2.762	2.207	2.053	2.013
10		2.018	2.005		2.257	2.066	2.017
11		2.022	2.006		2.310	2.081	2.020
12		2.025	2.007		2.367	2.096	2.024
13		2.029	2.009		2.427	2.113	2.029
14		2.032	2.010		2.491	2.130	2.033
15		2.035	2.011		2.558	2.150	2.038
16		2.038	2.013		2.627	2.170	2.043
17		2.040	2.015		2.699	2.192	2.059
18		2.042	2.016		2.773	2.214	2.055
19		2.041	2.018		2.849	2.238	2.061

TABLE 5. The computational order of Λ_i^2 and $\tilde{\Lambda}_i^2$ with respect to index i for $a_2(x)$ and $h = \frac{1}{10\times 2^n}$.

i	Without Correction			With Correction		
	$Co^i_{\Lambda_i^2}$	$Co^i_{\Lambda_i^2}$	$Co^i_{\Lambda_i^2}$	$Co^i_{\tilde{\Lambda}_i^2}$	$Co^i_{\tilde{\Lambda}_i^2}$	$Co^i_{\tilde{\Lambda}_i^2}$
2	4.870	4.869	4.868	0.648	0.646	0.645
3	4.141	4.140	4.140	2.236	2.223	2.220
4	4.109	4.107	4.107	1.668	1.649	1.644
5	4.063	4.061	4.060	1.879	1.855	1.849
6	4.050	4.047	4.046	1.657	1.610	1.598
7	4.035	4.030	4.029	1.931	1.878	1.864
8	4.028	4.023	4.021	2.048	1.982	1.966
9	4.027	4.020	4.018	1.962	1.873	1.851
10	4.024	4.016	4.014	2.072	1.968	1.942
11	4.024	4.014	4.012	2.121	1.998	1.951
12	4.025	4.013	4.010	2.138	1.992	1.954
13	4.025	4.012	4.009	2.187	2.019	1.975
14	4.027	4.012	4.008	2.227	2.034	1.984
15	4.028	4.012	4.007	2.264	2.045	1.988
16	4.030	4.012	4.007	2.306	2.060	1.996
17	4.032	4.012	4.006	2.348	2.074	2.001
18	4.034	4.013	4.006	2.391	2.087	2.006

3. Inverse eigenvalue problems for the rod equation

In general, the construction of $a(x)$ from spectral data in analytic form is impossible. Therefore, in this section, by solving an inverse matrix eigenvalue problem by a method based on a correction technique, we construct the cross sectional area.

3.1. **Constructing a symmetric cross sectional area.** In this subsection, we want to construct $a(x)$ in the symmetric case (i.e., where $a(x) = a(1-x)$). It is well known that in this case one spectrum is enough to construct $a(x)$, uniquely. Thus, we consider the Equation (1.1) with fixed-fixed boundary condition and solve the following inverse eigenvalue problem:

Inverse Problem I. *Given a set of real, positive and distinct numbers $\{\lambda_i\}_{i=1}^n$, construct a symmetric cross sectional area $a(x)$ such that $\{\lambda_i\}_{i=1}^n$ are the first n eigenvalues of the problem (1.1) with fixed-fixed boundary conditions.*

NOTE: If n is large, the eigenvalue λ_n must satisfy the asymptotic relations (2.4) or, at the very least, lower/upper bounds for the eigenvalues for a Sturm-Liouville problem with fixed-fixed boundary conditions.

Let $N = 2n+2$. By the symmetric property of $a(x)$ we have $a_i = a_{N-i+1}$, $i = 1, 2, \ldots, n+1$. Using the given eigenvalues $\{\lambda_i\}_{i=1}^n$, we will construct the entries $\{a_i\}_{i=1}^{n+1}$ of the matrices K_2 and M_2. In order to strike a balance between the given data $\{\lambda_i\}_{i=1}^n$ and the unknowns $\{a_i\}_{i=1}^{n+1}$ taking into account the assumption (1.3), we will assume $a_{n+1} = 1$. The construction follows. First, by using the given eigenvalues $\{\lambda_i\}_{i=1}^n$ we compute the first n eigenvalues of the pair (K_2, M_2), i.e. $\{\Lambda_i^2\}_{i=1}^n$. Then by solving an inverse matrix eigenvalue problem as above we can construct the a_i and therefore the cross sectional area, $a(x)$. Using (2.5), we can approximate Λ_i^2 as follows:

$$\Lambda_i^2 \simeq \lambda_i - \epsilon_2(i,h), \; i = 1, 2, \ldots, n. \tag{3.1}$$

Thus, we try to construct $\{a_i\}_{i=1}^n$ in such a way that the eigenvalues $\{\Lambda_i^2\}_{i=1}^n$ given by (3.1) will be the eigenvalues of the pair (K_2, M_2). So, $\{a_i\}_{i=1}^n$ must be the solution of the following nonlinear system:

$$f_i(\mathbf{a}) = 0, \quad i = 1, 2, \cdots n, \tag{3.2}$$

where

$$f_i(\mathbf{a}) := \Lambda_i^2(\mathbf{a}) - \lambda_i(a(x)) + \epsilon_2(i,h), \quad i = 1, 2, \ldots n, \tag{3.3}$$

and $\mathbf{a} = [a_1, a_2, \cdots, a_n]$.

We solve the system (3.2) by using the modified Newton's method. The recursive formula for Newton's method is as follows:

$$\mathbf{a}_{k+1} = \mathbf{a}_k - [D\mathbf{f}]^{-1}\mathbf{f}(\mathbf{a}_k), \; k = 0, 1, \ldots, \quad \mathbf{a}_0 = [1, 1, \ldots, 1]. \tag{3.4}$$

where

$$D\mathbf{f}(i,j) = \left.\frac{\partial \Lambda_i^2(\mathbf{a})}{\partial a_j}\right|_{\mathbf{a}=\mathbf{a}_0},$$

is the Jacobian matrix corresponding to the nonlinear system (3.2).

The following lemma holds for the Jacobian matrix.

Lemma 3.1. *The entries of the Jacobian matrix $D\mathbf{f}$ are as follows:*

$$\begin{aligned} &D\mathbf{f}(i,1) := 2(\tfrac{1}{h} - \tfrac{\Lambda_i^{02}h}{3})u_{1i}^2, \; i = 1, 2, \ldots, n, \\ &D\mathbf{f}(i,j) := 2(\tfrac{1}{h} - \tfrac{\Lambda_i^{02}h}{3})(u_{j-1,i}^2 + u_{j,i}^2) - 4(\tfrac{1}{h} + \tfrac{\Lambda_i^{02}h}{6})u_{j-1,i}u_{j,i}, \\ &j = 2, 3, \ldots, n, \; i = 1, 2, \ldots, n, \end{aligned} \tag{3.5}$$

where u_{ij} and Λ_i^{02} are defined in Lemma 2.2 and u_{ij} is the j-th entries of the eigenvector corresponding to Λ_i^{02}.

Proof. From the matrix eigenvalue problem $(K_2 - \Lambda^2 M_2)u = 0$ we must have

$$K_2(\mathbf{a})u_i(\mathbf{a}) = \Lambda_i^2(\mathbf{a})M_2(\mathbf{a})u_i(\mathbf{a}),\ i = 1, 2, \ldots, n.$$

Differentiating both sides with respect to a_j we obtain

$$\frac{\partial K_2}{\partial a_j}u_i + K_2\frac{\partial u_i}{\partial a_j} = \frac{\partial \Lambda_i^2}{\partial a_j}M_2 u_i + \Lambda_i^2\frac{\partial M_2}{\partial a_j}u_i + \Lambda_i^2 M_2\frac{\partial u_i}{\partial a_j}.$$

Multiplying both sides of the last equation by u_i^t and using the orthonormal property of the eigenvectors, we find

$$\frac{\partial \Lambda_i^2}{\partial a_j} = u_i^t(\frac{\partial K_2}{\partial a_j} - \Lambda_i^2\frac{\partial M_2}{\partial a_j})u_i.$$

Computing $\frac{\partial K_2}{\partial a_j}$ and $\frac{\partial M_2}{\partial a_j}$ at $\mathbf{a} = \mathbf{a}_0$ and doing some matrix calculations, relations (3.5) are obtained. □

Remark 3.2. The matrix $D\mathbf{f}$ is a nonsingular constant matrix that is independent of the cross section $a(x)$. The condition number of $D\mathbf{f}$ is given in Table 6 for different values of n. For some large values of n, we may need to apply a regularization method for solving the nonlinear system (3.2). Here we apply a quasi-Newton's method by first defining

$$\mathbf{a}_{k+1} = \mathbf{a}_k - \alpha_k(G^tG + \sigma I)^{-1}G^t\mathbf{f}(\mathbf{a}_k), \quad G = D\mathbf{f}, \tag{3.6}$$

where α_k satisfy the Wolf conditions [16] and $\sigma > 0$ is a regularization parameter.

TABLE 6. Condition number of Jacobian matrix $D\mathbf{f}$.

n	4	8	16	32	64
$Cond(D\mathbf{f})$	25.749	105.284	419.481	$1.66e+3$	$6.63e+3$

Remark 3.3. Equation (1.1) is a special case of the general Sturm-Liouville equation $(p(x)y')' + (\lambda w(x) - q(x))y = 0$. It is proved in [43] that, in an appropriate Banach space, the eigenvalue λ is a differentiable function of $\frac{1}{p(x)}$, $w(x)$ and $q(x)$. Thus, the eigenvalue λ of Equation (1.1) is a differentiable function of $a(x)$.

Theorem 3.4. *Let $p \geq 1$ and $\|.\|_p$ denotes the L_p norm on interval $[0, 1]$, there exists a constant number $c_p > 0$ such that if $\|a(x) - 1\|_p < c_p$ and the sequence $\mathbf{a}_k$ obtained from (3.4) is positive, then the iteration (3.4) with initial point $\mathbf{a}_0$ converges to a solution of the system (3.2).*

Proof. Let $\|.\|$ be a norm on $\mathbb{R}^n$ and $S_r(\mathbf{a}_0) := \{\mathbf{a} : \|\mathbf{a} - \mathbf{a}_0\| < r\}$. The positivity of $\mathbf{a}_k$ implies that the matrices K_2 and M_2 are positive definite and the eigenvalues $\Lambda_i^2(\mathbf{a})$ are positive and simple, ([17], Chapter 3). Thus, $\mathbf{f}(\mathbf{a})$ is an analytic function of $\mathbf{a}$ [41]. Since $D\mathbf{f}$ is nonsingular, there exists a constant $K > 0$ such that $\|[D\mathbf{f}]^{-1}.(D\mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a}_0))\| \leq K\|\mathbf{a} - \mathbf{a}_0\|$. Suppose that $\eta = \|[D\mathbf{f}]^{-1}.\mathbf{f}(\mathbf{a}_0)\|$, $\rho = K\eta$ and $r_1 = \frac{1-\sqrt{1-2\rho}}{K}$. Now consider the variation of eigenvalues $\lambda_i(a(x))$ with respect to $a(x)$. By Remark 3.3, the eigenvalues $\lambda_i(a(x))$ are differentiable functions of $a(x)$. Also if $\|a(x) - 1\|_p = 0$, then we have,

$$f_i(\mathbf{a}_0) = \Lambda_i^{02}(\mathbf{a}_0) - \lambda_i(a(x)) + \epsilon_2(i, h) = 0,\ i = 1, 2, \ldots, n.$$

Thus, for all $p \geq 1$, there exists c_p, such that for all $a \in L_p[0, 1]$ satisfying $\|a(x) - 1\|_p \leq c_p$, we have $0 < \rho < \frac{1}{2}$. By the theory of the modified Newton's method [40], we conclude that all $\mathbf{a}_k$ lie in $S_{r_1}(\mathbf{a}_0)$ and the sequence $\{\mathbf{a}_k\}$ converges to a solution of $\mathbf{f}(\mathbf{a}) = 0$. □

Now we present the following algorithm for solving **Inverse Problem I**:

Algorithm 1:

1. Input the eigendata $\{\lambda_i\}_{i=1}^n$,

2. Compute the functions f_i from (3.3),
3. Compute the matrix $D\mathbf{f}$ from Lemma 3.1,
4. Compute the recursive Newton's sequence (3.4).

3.2. **Construction of nonsymmetric cross sectional area.** It is well-known that for a nonsymmetric cross sectional area, two sets of eigenvalues corresponding to two boundary conditions are required to determine $a(x)$ uniquely, [6, 17]. Now we consider Equation (1.1) with fixed-fixed and fixed-free boundary conditions and solve the following inverse problem:

Inverse Problem II. *Given two sets of real, distinct, positive numbers $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$, such that*

$$\mu_i < \lambda_i < \mu_{i+1},\ i = 1, 2, \dots, n-1,\ \mu_n < \lambda_n,$$

construct a nonsymmetric cross sectional area $a(x)$ such that $\{\mu_i\}_{i=1}^n$ are the first n eigenvalues of the problem (1.1) with fixed-free boundary condition and such that $\{\lambda_i\}_{i=1}^n$ are the first n eigenvalues of the problem (1.1)) with fixed-fixed boundary condition.

NOTE: If n is large, the eigenvalues λ_n and μ_n must satisfy the asymptotic relations (2.4) or, at the very least, lower/upper bounds for the eigenvalues for a Sturm-Liouville problem with fixed-fixed boundary conditions.

Let $N = 2n+1$. Using the finite element method, Equation (1.1) with fixed-free boundary conditions is transformed to the system $(K_1 - \Lambda^1 M_1)u = 0$. In the nonsymmetric case we are given $2n$ data and $2n+1$ unknown parameters $\{a_i\}_{i=1}^{2n+1}$. For a unique solution $a(x)$, we choose $a_{2n+1} = 1$. Using (2.5), we can approximate Λ_i^1 and Λ_i^2 as follows:

$$\begin{aligned} \Lambda_i^1 &\simeq \mu_i - \epsilon_1(i,h),\ i = 1, 2, \dots, n, \\ \Lambda_i^2 &\simeq \lambda_i - \epsilon_2(i,h),\ i = 1, 2, \dots, n. \end{aligned} \tag{3.7}$$

Now, we want to construct $\{a_i\}_{i=1}^{2n}$ such that Λ_i^1 and Λ_i^2, defined by (3.7), are the eigenvalues of (K_1, M_1) and (K_2, M_2), respectively. Thus, we must solve the following nonlinear system:

$$g_i(\mathbf{a}) = 0, \quad i = 1, 2, \dots, 2n, \tag{3.8}$$

where,

$$\begin{aligned} g_i(\mathbf{a}) &:= \Lambda_i^2(\mathbf{a}) - \lambda_i + \epsilon_2(i,h) \quad i = 1, 2, \dots, n, \\ g_{n+i}(\mathbf{a}) &:= \Lambda_i^1(\mathbf{a}) - \mu_i + \epsilon_1(i,h), \quad i = 1, 2, \dots, n, \end{aligned} \tag{3.9}$$

and $\mathbf{a} = [a_1, a_2, \cdots, a_{2n}]$.

As before, we solve the system (3.8) by using a modified Newton's method where the recursion formula is given by,

$$\mathbf{a}_{k+1} = \mathbf{a}_k - [D\mathbf{g}]^{-1}\mathbf{g}(\mathbf{a}_k),\ k = 0, 1, \dots, \quad \mathbf{a}_0 = [1, 1, \dots, 1]. \tag{3.10}$$

The Jacobian matrix of the system (3.8) is given by,

Lemma 3.5. *The entries of $D\mathbf{g}$ are as follows:*

$$D\mathbf{g}(i,1) = \left(\frac{1}{h} - \frac{\Lambda_i^{02} h}{3}\right) u_{i1}^2, \ i = 1, 2, \ldots, n,$$

$$D\mathbf{g}(n+i,1) = \left(\frac{1}{h} - \frac{\Lambda_i^{01} h}{3}\right) v_{i,1}^2, \ i = 1, \ldots, n,$$

$$D\mathbf{g}(i,j) = \left(\frac{1}{h} - \frac{\Lambda_i^{02} h}{3}\right) (u_{i,j-1}^2 + u_{i,j}^2) - 2(\frac{1}{h} + \frac{\Lambda_i^{02} h}{6}) u_{i,j-1} u_{ij},$$

$$i = 1, 2, \ldots, n, \ j = 2, \ldots, 2n$$

$$D\mathbf{g}(n+i,j) = \left(\frac{1}{h} - \frac{\Lambda_i^{01} h}{3}\right) (v_{i,j-1}^2 + v_{i,j}^2) - 2(\frac{1}{h} + \frac{\Lambda_i^{01} h}{6}) v_{i,j} v_{i,j-1},$$

$$i = 1, \ldots, n, \ j = 2, \ldots, 2n.$$

where Λ_i^{02}, u_{ij} are defined by (2.10) *and Λ_i^{01}, v_{ij} are defined by* (2.24).

Proof. The proof is similar to Lemma 3.1 and so is left to the reader. □

The convergence of the modified Newton iteration (3.10) can be proved in the same way as in Theorem 3.4. Now we can solve **Inverse problem II** using the following algorithm:

Algorithm 2

1. Input the eigendata $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$,
2. Compute the functions g_i from (3.9),
3. Compute the matrix $D\mathbf{g}$ from Lemma 3.5,
4. Compute the recursive Newton sequence (3.10).

3.3. **Numerical results for inverse problem.** In this subsection we present some numerical experiments for the solutions of **Inverse problems I** and **II** so as to display the efficiency of the proposed methods. In the following numerical examples we study the effect of the correction terms $\epsilon_1(i,h)$, and $\epsilon_2(i,h)$, in the nonlinear systems (3.2) and (3.8) . In Examples 3.6-3.9 the cross sectional area $a(x)$ without the correction terms is constructed by solving the nonlinear system $\Lambda_i^2(\mathbf{a}) - \lambda_i = 0$ for **Inverse problem I** and $\Lambda_i^2(\mathbf{a}) - \lambda_i = 0, \ \Lambda_i^1(\mathbf{a}) - \mu_i = 0$ for **Inverse problem II**. Numerical results (Figures 3.6-3.9, part (a)) suggest that Newton's method without the correction terms diverges. Our results show that using the FEM together with correction technique to solve the inverse problem of the rod equation is quite efficient.

The Computational order, denoted by "C-order", is defined by

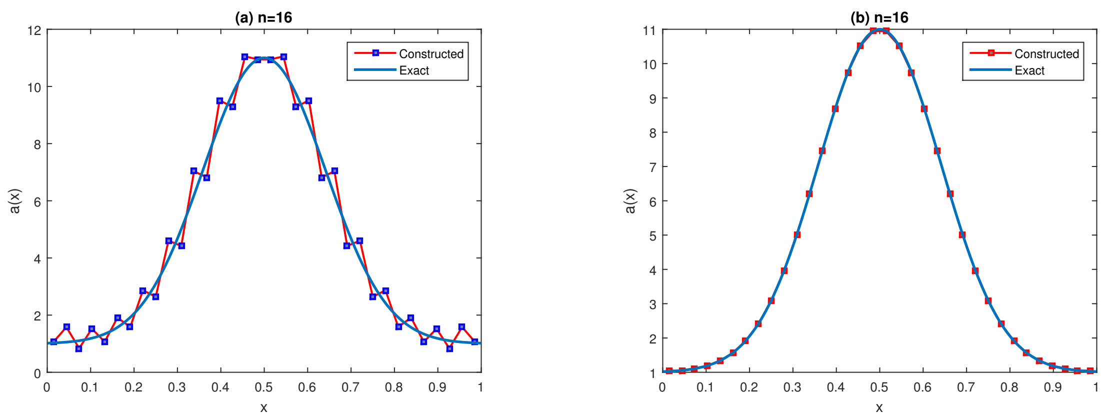
$$\text{C-order} = \frac{log(\frac{e_1}{e_2})}{log(\frac{h_1}{h_2})},$$

where $e_i = \|\mathbf{a} - \mathbf{a}_n\|_2$ denotes the L_2 error corresponding to h_i. From the results it can be seen that the numerical convergence rate is close to 1.

Example 3.6. Consider problem (1.1) with cross sectional area $a_1(x)$ which is a symmetric function. Thus, we can construct it by using **Algorithm 1**. Table 7 presents errors in finding $a_1(x)$ for various values of n. In Figure 3.6, the exact and computed cross sectional area are plotted. In Figure 3.6, the errors of the computed cross sectional area with and without the correction terms are plotted. The direct problem for $a_1(x)$ is solved in [13], but the inverse problem is not solved by the method of [13]. Here we construct $a_1(x)$ using regularized quasi-Newton's method (3.6).

TABLE 7. Error for $a_1(x)$ with different values of n.

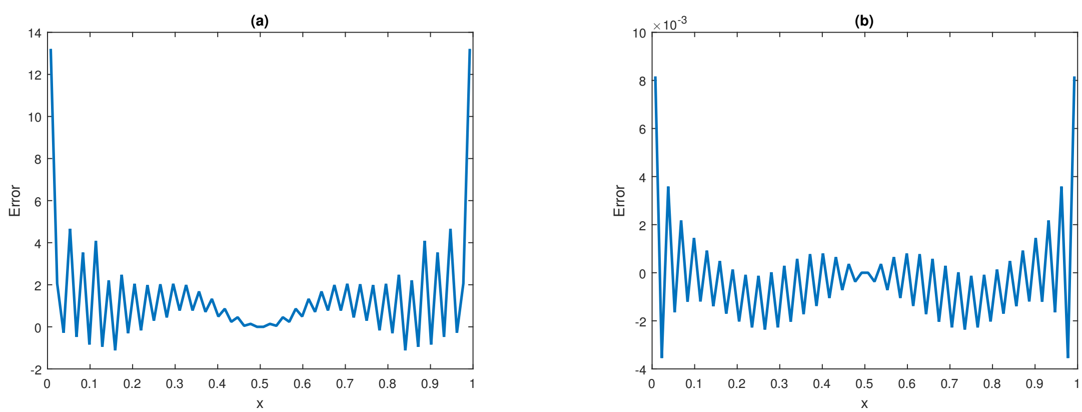
n	$\|\mathbf{a}-\mathbf{a}_n\|_2$	$\|\mathbf{a}-\mathbf{a}_n\|_\infty$	$\|\mathbf{f}\|_\infty$	C-order
4	0.1860	0.0826	$2.6e-13$	
8	0.0865	0.0382	$8.7e-12$	1.3025
16	0.0378	0.0176	$1.4e-10$	1.3017
32	0.0169	0.0082	$1.6e-10$	1.2136
64	0.0079	0.0039	$2.5e-10$	1.1218



(a) Indicates the result without correction terms.

(b) Indicates the results with correction terms.

FIGURE 1. Plot of $a_1(x)$ with $n = 16$.



(a) Indicates the errors without correction terms.

(b) Indicates the errors with correction terms.

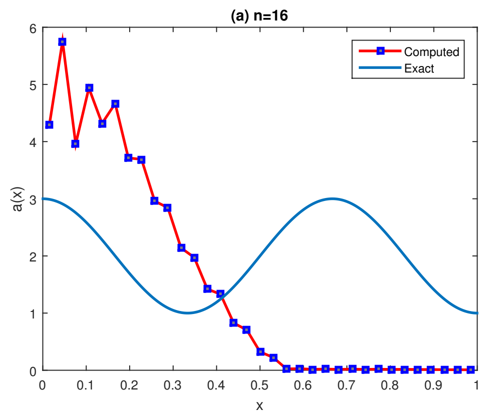
FIGURE 2. Error of $a_1(x)$ with $n = 32$.

Example 3.7. Consider problem (1.1) with cross sectional area $a_2(x)$. The function $a_2(x)$ is nonsymmetric so that we can construct it using **Algorithm 2**. The errors in approximating $a_2(x)$ for various values of n are presented in

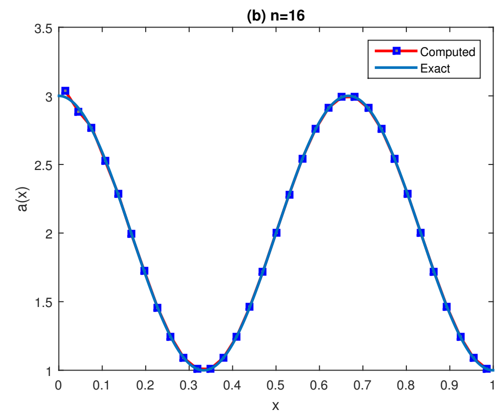
Table 8. The exact and computed cross sectional area are plotted in Figure 3. In Figure 3.7, the error with correction terms and without correction terms are plotted.

TABLE 8. Error for $a_2(x)$ with different values of n.

n	$\|\mathbf{a}-\mathbf{a}_n\|_2$	$\|\mathbf{a}-\mathbf{a}_n\|_\infty$	$\|\mathbf{g}\|_\infty$	C-order
4	0.1960	0.1770	$5.4e-13$	
8	0.1068	0.0906	$9.5e-13$	0.95
16	0.0549	0.0443	$2.04e-12$	1.00
32	0.0281	0.0217	$1.03e-10$	0.99

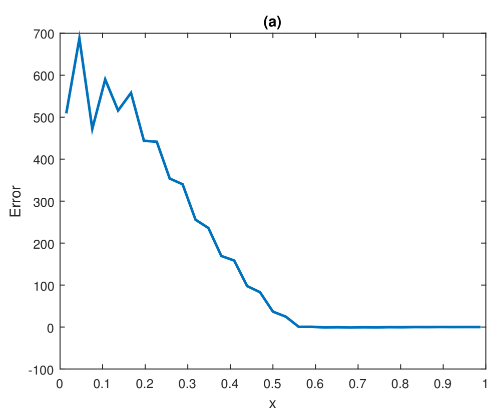


(a) Indicates the result without correction terms

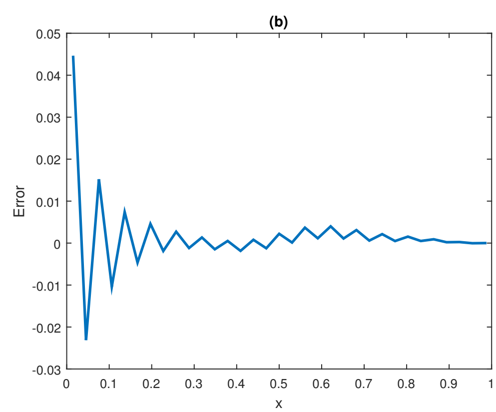


(b) Indicates the results with correction terms.

FIGURE 3. Plot of $a_2(x)$ with $n = 16$.



(a) Indicates the errors without correction terms.



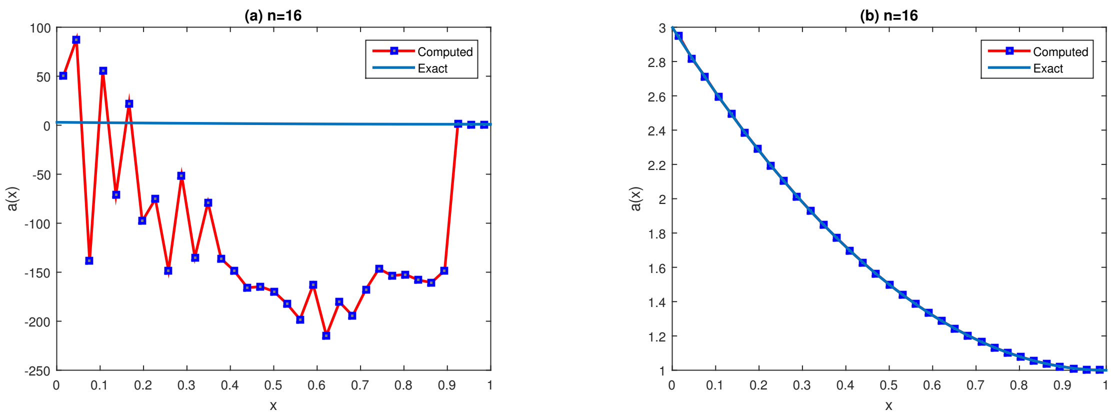
(b) Indicates the errors with correction terms.

FIGURE 4. Error of $a_2(x)$ with $n = 16$.

Example 3.8. Consider problem (1.1) with cross sectional area $a_3(x)$. The function $a_3(x)$ is nonsymmetric thus we can construct it using **Algorithm 2**. The errors of $a_3(x)$ for various values of n presented in Table 9. The exact and computed cross sectional area are plotted in Figure 5. The error of computed cross sectional area with correction terms and without correction terms are shown in Figure 3.8.

TABLE 9. Error for $a_3(x)$ with different values of n.

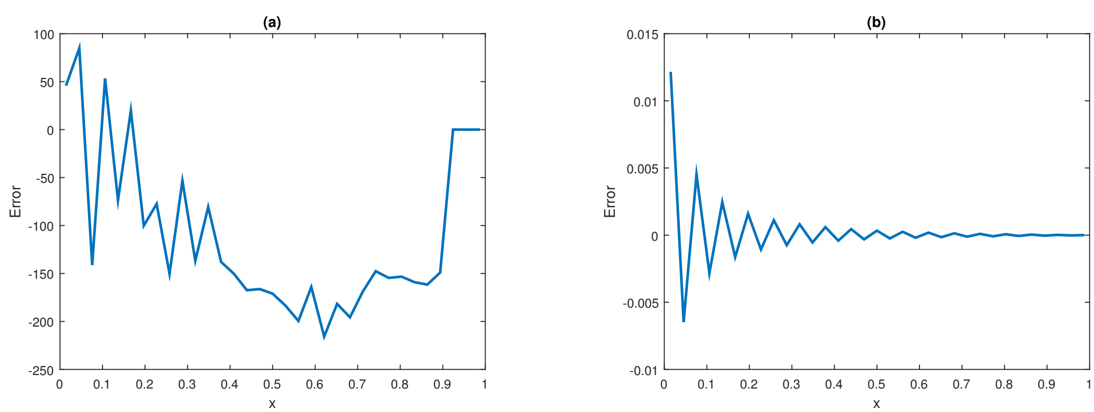
n	$\|\mathbf{a}-\mathbf{a}_n\|_2$	$\|\mathbf{a}-\mathbf{a}_n\|_\infty$	$\|\mathbf{g}\|_\infty$	C-order
4	0.0495	0.0436	$9.2e-14$	
8	0.028	0.0234	$1.2e-12$	0.89
16	0.0152	0.0121	$6.9e-12$	0.92
32	0.008	0.0061	$2.4e-11$	0.94



(a) Indicates the result without correction terms.

(b) Indicates the results with correction terms.

FIGURE 5. Plot of $a_3(x)$ with $n = 16$.



(a) Indicates the errors without correction terms.

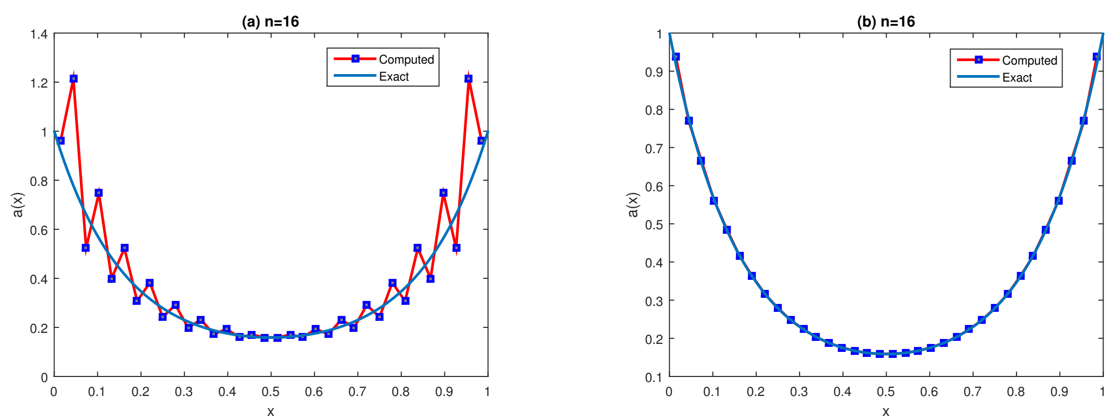
(b) Indicates the errors with correction terms.

FIGURE 6. Error of $a_3(x)$ with $n = 16$.

Example 3.9. Consider problem (1.1) with cross sectional area $a_4(x) = (\cosh(\pi x) - \tanh(\frac{\pi}{2})\sinh(\pi x))^2$. Construction of $a_4(x)$ was studied in [13]. This is a symmetric cross sectional area, thus we can construct it by using the eigenvalues $\{\lambda_i\}_{i=1}^n$ of problem (1.1) with fixed-fixed boundary condition. Table 10, presents error of $a_4(x)$ for various values of n. Also the results compared with the results of [13]. In Figure 3.9, the exact and computed cross sectional area are plotted for different values of n. In Figure 3.9, the error of computed cross sectional area with correction terms and without correction terms are plotted.

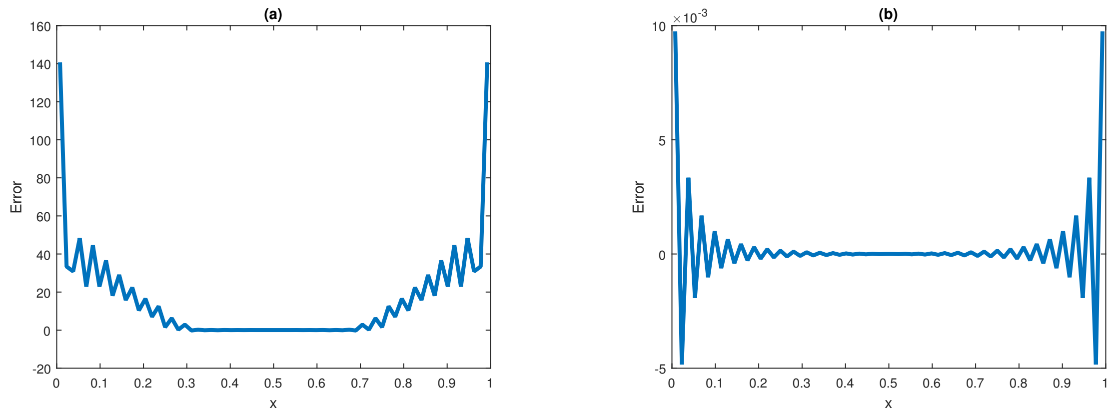
TABLE 10. Errors for $a_4(x)$ with different values of n.

n	$\|\mathbf{a}-\mathbf{a}_n\|_2$	$\|\mathbf{a}-\mathbf{a}_n\|_\infty$	$\|f\|_\infty$	C-order	$\|\mathbf{a}-\mathbf{a}_n\|_2$ [13]	$\|\mathbf{a}-\mathbf{a}_n\|_\infty$ [13]
4	0.0834	0.0572	$9.2e-14$		0.3532	0.2471
8	0.0523	0.0305	$3.4e-13$	0.79	0.2183	0.1468
16	0.0303	0.0234	$1.2e-12$	0.89	0.1315	0.0809
32	0.0167	0.0098	$4.7e-11$	0.90	0.0805	0.0426
64	0.0089	0.0050	$1.8e-10$	0.93		



(a) Indicates the errors without correction terms. (b) Indicates the errors without correction terms.

FIGURE 7. Results for $a_4(x)$ with different values of n.



(a) Indicates the errors without correction terms

(b) Indicates the errors with correction terms.

FIGURE 8. Errors for $a_4(x)$ with $n = 32$.

4. Conclusion

In this paper, the direct and inverse spectral problems of the rod equation are studied using the FEM along with a correction technique. Our main purpose is to examine the ability of the FEM together with the correction technique to improve the results in solving direct and inverse problems for Equation (1.1). We observed that the correction terms play an important role in estimating the eigenvalues and in constructing the unknown cross sectional area. Two algorithms based on the correction technique and a modified Newton's method for constructing symmetric and nonsymmetric cross sectional areas are proposed. The convergence of Newton's method is proved and computational orders are obtained. Numerical results demonstrate the efficiency of our algorithms in both symmetric and nonsymmetric cases. A weak point in this study is the large condition number of a related Jacobian matrix which makes Newton's method not work for some cross sections. Here we solved this problem by using a regularization method.

In future work we will extend the proposed method for Equation (1.1) to the case of general boundary conditions. Also, this method may be adapted to the study of a fourth order Euler-Bernoulli beam equation.

Declarations

Data Availability Statement: No data associated in the manuscript.
Funding. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.
Disclosure statement. The authors have no relevant financial or non-financial interests to disclose.

References

[1] M. O. Adewole, *Finite element solution of a class of parabolic integro-differential equations with non-homogeneous jump condition using FreeFEM++*, Computational Methods for Differential Equations, doi:10.22034/cmde.2023.50871.2114.

[2] S. Akbarpoor and A. H. Dabbaghian, *Approximate solution of inverse Nodal problem with conformable derivative*, Computational Methods for Differential Equations, doi:10.22034/cmde.2023.56851.2380.

[3] H. Altundağ and H. Taşeli, *Singular inverse Sturm-Liouville problems with Hermite pseudospectral methods*, Eur. Phys. J. Plus., *136* (2021), 1–13.

[4] H. Altundağ, C. Böckmann, and H. Taşeli, *Inverse Sturm-Liouville problems with pseudospectral methods*, International journal of computer mathematics., *92*(7) (2015), 1373–1384.

[5] R. Amirov and I. Adalar, *Eigenvalues of Sturm-Liouville operators and prime numbers*, Electronic journal of differential equations., (2017), 1–3.

[6] L. E. Andersson, *Inverse eigenvalue problems for a Sturm-Liouville equation in impedance form*, Inverse Probl., *4* (1988), 929–971.

[7] A. L. Andrew and J. W. Paine, *Correction of finite element estimates for Sturm-Liouville eigenvalues*, Numer. Math., *50* (1986), 205–215.

[8] A. L. Andrew,*Computing Sturm–Liouville potentials from two spectra*, Inverse problems., *22* (2006), 1–12.

[9] A. Bilotta, A. Morassi, and E. Turco, *Reconstructing blockages in a symmetric duct via quasi-isospectral horn operators* Journal of sound and vibration., *366* (2016), 149–172.

[10] Z. Chen, *Finite element method and their applications*, Springer-Verlag, Berlin., 2005.

[11] S. S. Ezhak and M. Y. Telnova, *Estimates for The first eigenvalue of the Sturm-Liouville problem with potentials in weighted spaces*, Journal of mathematical sciences., *244* (2020), 216-234.

[12] G. Freiling and V. A. Yurko, *Inverse Sturm-Liouville problems and their applications*, NOVA science publications, New York., 2001.

[13] Q. Gao, Z. Huang, and X. Cheng, *A finite difference method for an inverse Sturm-Liouville problem in impedance form*, Numer. Algor., *70* (2015), 669–690.

[14] Q. Gao, Q. Zhao, X. Zheng, and Y. Ling, *Convergence of Numerov's method for inverse Sturm-Liouville problems*, Applied mathematics and computation., *293* (2017), 1–17.

[15] Q. Gao, Q. Zhao and M. Chen, *On a modified Numerov's method for inverse Sturm-Liouville problems*, International journal of computer mathematics., *95*(2) (2018), 412–426.

[16] J. C. Gilbert, *On the realization of the Wolf conditions in reduced Quasi-Newton for equality constrained optimization*, SIAM J. Optim., *7*(3) (1997), 780–813.

[17] G. M. L. Gladwell, *Inverse problem in vibration*, Kluwer academic publishers, New York., 2004.

[18] G. M. L. Gladwell and A. Morassi, *On isospectral rods, horns and strings*, Inverse problems., *11* (1995), 533–554.

[19] G. M. L. Gladwell and A. Morassi, *A family of isospectral EulerBernoulli beams*, Inverse problems., *26* (2010), 1–12.

[20] K. Ghanbari, H. Mirzaei, and G. L. M. Gladwell, *Reconstructing of a rod using one spectrum and minimal mass condition*, Inverse problem in science and engineering., *22* (2014), 325–333.

[21] A. Kirsch, *An introduction to the mathematical theory of inverse problems*, Springer-Verlag, New York., 1996.

[22] V. Ledoux, M. V. Daele, and G. V. Berghe, *Matslise: A Matlab package for the numerical solution of Sturm-Liouville and Schrodinger equations*, ACM translations on mathematical software., *31* (2005), 532–554.

[23] C. S. Liu, *Computing the eigenvalues of the generalized Sturm-Liouville problems based on the Lie-group SL(2,R)*, Journal of computational and applied mathematics., *236*(17) (2012), 4547–4560.

[24] L. Losonczi, *Eigenvalues and eigenvactors of some tridiagonal matrices*, Acta Math.Hung., *60*(1-2) (1992), 309–323.

[25] G. T. Lubo and G. F. Duressa,*Linear B-spline finite element Method for solving delay reaction diffusion equation*, Computational Methods for Differential Equations., *11* (2023), 161–174.

[26] C. Magherini, *A corrected spectral method for Sturm-Liouville problems with unbounded potential at one endpoint*, Journal of computational and applied mathematics., *364* (2020), 1–15.

[27] H. Mirzaei, *Computing the eigenvalues of fourth order Sturm–Liouville problems with Lie Group method*, Iranian journal of numerical analysis and optimization., *7* (2017), 1–12.

[28] H. Mirzaei, *A family of isospectral fourth order Sturm–Liouville problems and equivalent beam equations*, Mathematical communications., *23* (2018), 15–27.

[29] H. Mirzaei, *Higher-order Sturm–Liouville problems with the same eigenvalues*, Turkish journal of mathematics., *44*(2) (2020), 409–417.

[30] H. Mirzaei, K. Ghanbari, and M. Emami, *Direct and Inverse Problems of String Equation by Numerov's Method*, Iranian Journal of Science., *47* (2023), 871–884.

[31] A. B. Mingarelli, *A note on Sturm-Liouville problems whose spectrum is the set of prime numbers*, Electronic journal of differential equations., *44*(123) (2011), 1–4.

[32] A. Morassi, *Quasi-isospectral vibrating systems*, Technical report, University of Udine., 2012.
[33] A. Morassi, *Constructing rods with given natural frequencies*, Mechanical systems and signal processing., *40* (2013), 288–300.
[34] S. Mosazadeh, *A new approach to uniqueness for inverse Sturm-Liouville problems on finite intervals*, Turkish journal of mathematics., *41* (2017), 1224–1234.
[35] A. S. Ozkan and A. Adalar, *Inverse nodal problems for Sturm-Liouville equation with nonlocal boundary conditions*, Journal of Mathematical Analysis and Applications., *520*(1) (2023), 126904.
[36] J. W. Paine, *A numerical method for the inverse Sturm-Liouville problem*, Siam J. Sci. Stat. Comput., *5* (1984), 149–156.
[37] J. W. Paine, F. R. Hoog, and R. S. Anderssen, *On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems*, Computing., *26* (1981), 123–139.
[38] M. Shahriari, A. A. Jodayree, and G. Teschl, *Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions* Journal of mathematical analysis and applications., *365* (2012), 19–29.
[39] C. W. Soh, *Isospectral Euler-Bernoulli beams via factorization and the Lie method*, Int. J. Non-Linear Mech., *44* (2009), 396-403.
[40] J. Stoer and R. Bulirsch, *Introduction to numerical analysis*, Third edition, Springer-Verlag, Wurzburg., 2002.
[41] J.G. Sun, *Multiple eigenvalue sensitivity analysis*, Linear algebra appl., *137–138* (1999), 183–211.
[42] X. Wu, P. Niu and G. Wei, *An inverse eigenvalue problem for a nonlocal Sturm-Liouville operator*, Journal of Mathematical Analysis and Applications., *494*(2) (2021), 124661.
[43] M. Zhang and K. Li, *Dependence of eigenvalues of Sturm-Liouville problems with eigenparameter dependent boundary conditions*, Applied mathematics and computation., *378* (2020), 1–14.