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Eigenvalue intervals of parameters for iterative systems of nonlinear Hadamard fractional boundary value problems

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Abstract

This study uses a classic fixed point theorem of cone type in a Banach space to identify the eigenvalue intervals of parameters for which an iterative system of a Hadamard fractional boundary value problem has at least one positive solution. To the best of our knowledge, no attempt has been made to obtain such results for Hadamard-type problems in the literature. We provided an example to illustrate the feasibility of our findings in order to show how effective they are.

Keywords. Hadamard fractional derivative, Boundary value problem; Kernel, Fixed-point theorems, Positive solution.
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1. Introduction

The area of fractional differential equations (FDEs) has expanded over the past several years as a result of its applicability in numerous real-world situations in the disciplines of physics, thermodynamics, economics, modeling, and possibly other fields as well [17, 18]. Many real-world problems benefit significantly from using fractional calculus (FC) in mathematical modeling. However, we are still in the early stages of implementing this powerful tool in a variety of research domains. At this point, differential calculus expanded its scope to include the dynamics of the complex real world, and new theories began to be put into effect and assessed on real data [31]. The nonlocal nature of the FC facilitates a precise description of a wide range of materials and processes with characteristics related to memory and heredity [16, 23]. There are numerous applications in a variety of scientific disciplines, including biomathematics [11], random processes [28], viscoelasticity [29], non-Newtonian fluid mechanics [3], and characterization of anomalous diffusion [30].

The theories of an iterative system of Hadamard FDEs (HFDEs) under diverse boundary conditions (BCs) are not sufficiently established. The main focus of research on FDEs, as per the literature, is on Riemann–Liouville or Caputo derivatives. The literature on FDEs of the Hadamard-type is not enriched yet. Contrary to the sort of derivatives noted above, the Hadamard derivative, which originally appeared in 1892, has a logarithmic function with any exponent as the integral's Kernel [12]. For a detailed description of Hadamard derivative and integral, see [2, 5, 6, 14, 15]. In literature, different investigations on Hadamard fractional order boundary value problems (HFBVPs) have appeared to explain the existence, uniqueness of solutions, positive solutions under suitable conditions, see Thiramanus et al. [32], Pei et al. [33], Tariboon et al. [34], Wang et al. [35], Zhang et al. [38, 40].

In [36], Yang investigated the existence of at least one positive solution for the coupled system of HFDEs. Zhai et al. [37], in their research article, have established the existence results for the FBVP. By means of a fixed point theorem (FPT), Zhai et al. [39] investigated the existence, uniqueness of solutions for a system of HFDE with integral BCs. They achieved to arrive at a unique conclusion for the problem that rely on twin parameters. Ding et al. [8]

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studied the existence of positive solutions for a system of HFDEs with semipositone nonlinearities via the fixed-point index and nonnegative matrices. Abbas et al. [1] in their publication, briefly discussed the random solutions to a coupled system of Hilfer–HFDEs with finite delay. Matar et al. [22] addressed the uniqueness, stability in the sense of Ulam to a system of nonlinear Langevin equations including Caputo-Hadamard derivative with nonperiodic BCs. BVPs are widely employed in a variety of sectors, including telecom devices, chemical compounds, motor vehicles, and pharmaceuticals. Positive solutions seem to be beneficial in these operations, see [13, 20, 24–27].

In contrast to the aforementioned approaches, it has the advantage of being able to include integral and multi-point fractional BCs via Guo–Krasnosel'skii FPT of cone compression and expansion of norm kind (see [10, 19]) to the considered FDE. As a result, we are able to determine the eigenvalue intervals of parameters in a Banach space for which there are positive solutions on the appropriate cone. In order to locate suitable fixed points for the newly indicated operator, we create the Kernel for the related linear FBVP explicitly and compute its bounds in a better way. In the current work, the two primary strategies for achieving the required results are the fixed point technique and the bootstrapping argument. The main attraction of this article lies in the fact that it is the first to study the novel iterative systems of HFDE with fractional integral BCs:

$$\left.\begin{aligned} {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}}_{1^+}\mathtt{u}_1(\mathtt{z})+\lambda_1\mathtt{p}_1(\mathtt{z})\mathtt{g}_1\big(\mathtt{u}_2(\mathtt{z})\big)&=0,\\ {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}}_{1^+}\mathtt{u}_2(\mathtt{z})+\lambda_2\mathtt{p}_2(\mathtt{z})\mathtt{g}_2\big(\mathtt{u}_3(\mathtt{z})\big)&=0,\\ &\cdots\\ {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}}_{1^+}\mathtt{u}_\ell(\mathtt{z})+\lambda_\ell\mathtt{p}_\ell(\mathtt{z})\mathtt{g}_\ell\big(\mathtt{u}_{\ell+1}(\mathtt{z})\big)&=0,\\ \mathtt{u}_{\ell+1}(\mathtt{z})=\mathtt{u}_1(\mathtt{z}),\ \mathtt{z}\in(1,e),& \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} &\mathtt{u}_1^{(\mathtt{j})}(1)=0,\ \mathtt{j}=0,1,\cdots,\ell-2,\ {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}-1}_{1^+}\mathtt{u}_1(e)-\int_1^e \text{ß}(\mathtt{z})\mathtt{u}_1(\mathtt{z})\frac{d\mathtt{z}}{\mathtt{z}}=\sum_{\mathtt{i}=1}^{\mathtt{p}}\mho_{\mathtt{i}}\ {}^{\mathtt{H}}\mathtt{I}^{\gamma_{\mathtt{i}}}_{1^+}\mathtt{u}_1(\sigma),\\ &\mathtt{u}_2^{(\mathtt{j})}(1)=0,\ \mathtt{j}=0,1,\cdots,\ell-2,\ {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}-1}_{1^+}\mathtt{u}_2(e)-\int_1^e \text{ß}(\mathtt{z})\mathtt{u}_2(\mathtt{z})\frac{d\mathtt{z}}{\mathtt{z}}=\sum_{\mathtt{i}=1}^{\mathtt{p}}\mho_{\mathtt{i}}\ {}^{\mathtt{H}}\mathtt{I}^{\gamma_{\mathtt{i}}}_{1^+}\mathtt{u}_2(\sigma),\\ &\qquad\qquad\qquad\qquad\cdots\\ &\mathtt{u}_\ell^{(\mathtt{j})}(1)=0,\ \mathtt{j}=0,1,\cdots,\ell-2,\ {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}-1}_{1^+}\mathtt{u}_\ell(e)-\int_1^e \text{ß}(\mathtt{z})\mathtt{u}_\ell(\mathtt{z})\frac{d\mathtt{z}}{\mathtt{z}}=\sum_{\mathtt{i}=1}^{\mathtt{p}}\mho_{\mathtt{i}}\ {}^{\mathtt{H}}\mathtt{I}^{\gamma_{\mathtt{i}}}_{1^+}\mathtt{u}_\ell(\sigma), \end{aligned}\right\} \tag{1.2}$$

where $\mathtt{q}\in(\ell-1,\ell],\ \ell\in\mathbb{N}$ for $\ell\geq 3, \sigma\in(1,e), \gamma_i\in[1,\mathtt{q}-1], \mho_{\mathtt{i}}\geq 0,\ \mathtt{i}=1,2,\cdots,\mathtt{p},\ {}^{\mathtt{H}}\mathtt{D}^{*}_{1^+},\ {}^{\mathtt{H}}\mathtt{I}^{\star}_{1^+}$ are the Hadamard derivative and integral respectively.

Throughout the entire work, we propose a few hypotheses:

- (H_1) $\mathtt{p}_{\mathtt{k}}:[1,e]\to\mathbb{R}^+$ is continuous and $\mathtt{p}_{\mathtt{k}}$ does not vanish identically on any closed subinterval of $[1,e]$, for $\mathtt{k}=1,2,\cdots,\ell$,
- (H_2) $\mathtt{g}_{\mathtt{k}}:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, for $\mathtt{k}=1,2,\cdots,\ell$,
- (H_3) $\text{ß}:[1,e]\to(0,\infty)$ is continuous,
- (H_4) each of $\mathtt{g}_{\mathtt{k}0}=\lim\limits_{\mathtt{x}\to 1^+}\dfrac{\mathtt{g}_{\mathtt{k}}(\mathtt{x})}{\mathtt{x}}$ and $\mathtt{g}_{\mathtt{k}\infty}=\lim\limits_{\mathtt{x}\to\infty}\dfrac{\mathtt{g}_{\mathtt{k}}(\mathtt{x})}{\mathtt{x}}$, for $1\leq\mathtt{k}\leq\ell$, exists as positive real numbers.

The rest of the paper is organized as follows. In section 2, we generate the Kernel and its bounds. In section 3, we address the key theorems related to the main problem. In section 4, an example is coined in support of validity of the results obtained in the previous sections.

2. Preliminaries, Kernel and Bounds

Definition 2.1. [16] The Hadamard fractional integral of order $\mathtt{q}\in\mathbb{R}^+$ of the function $\mathtt{h}(\mathtt{z})$ is defined as

$${}^{\mathtt{H}}\mathtt{I}^{\mathtt{q}}_{1}\mathtt{h}(\mathtt{z})=\frac{1}{\Gamma(\mathtt{q})}\int_1^{\mathtt{z}}\left(\ln\frac{\mathtt{z}}{\mathtt{y}}\right)^{\mathtt{q}-1}\mathtt{h}(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}},\quad \mathtt{z}\in[1,e].$$

Definition 2.2. [16] The Hadamard derivative of order $q \in (\ell-1, \ell]$, $\ell \in \mathbb{Z}^+$ of the function $h(z)$ is defined as

$$ {}^{H}D_1^{q}h(z) = \frac{1}{\Gamma(\ell - q)}\left(z\frac{d}{dz}\right)^{\ell}\int_1^z \left(\ln\frac{z}{y}\right)^{\ell-q+1} h(y)\frac{dy}{y}, \quad z \in [1, e]. $$

Lemma 2.3. *[16] If* $a, q, \varpi > 0$, *then*

$$ (1) \quad \left({}^{H}D_a^{q}\left(\ln\frac{z}{a}\right)^{\varpi-1}\right)(y) = \frac{\Gamma(\varpi)}{\Gamma(\varpi - q)}\left(\ln\frac{y}{a}\right)^{\varpi-q-1}, $$

$$ (2) \quad \left({}^{H}I_a^{q}\left(\ln\frac{z}{a}\right)^{\varpi-1}\right)(y) = \frac{\Gamma(\varpi)}{\Gamma(\varpi + q)}\left(\ln\frac{y}{a}\right)^{\varpi+q-1}, $$

$$ (3) \quad \left({}^{H}D_a^{q}\left(\ln\frac{z}{a}\right)^{q-k}\right)(y) = 0, \quad k = 1, 2, \cdots, [q]+1. $$

Denote:

- $\Upsilon = \Gamma(q) - \sum_{i=1}^{p} \frac{\mho_i \Gamma(q)}{\Gamma(q+\gamma_i)}(\ln\sigma)^{q+\gamma_i-1}$,
- $\Upsilon_1 = \Upsilon - \int_1^e \text{ß}(z)(\ln z)^{q-1}\frac{dz}{z}$

Lemma 2.4. *Let* $h(z) \in C\big([1, e], \mathbb{R}\big)$. *Then the FBVP*

$$ {}^{H}D_{1^+}^{q}u_1(z) + h(z) = 0, \; z \in (1, e), \tag{2.1} $$

$$ \left.\begin{aligned} & u_1^{(j)}(1) = 0, \qquad 0 \le j \le \ell - 2, \\ & {}^{H}D_{1^+}^{q-1}u_1(e) - \int_1^e \text{ß}(z)u_1(z)\frac{dz}{z} = \sum_{i=1}^{p}\mho_i \; {}^{H}I_{1^+}^{\gamma_i}u_1(\sigma), \end{aligned}\right\} \tag{2.2} $$

has a unique solution, $u_1(z) = \int_1^e G(z, y)h(y)\frac{dy}{y}$, *where*

$$ G(z, y) = G_1(z, y) + G_2(z, y), \tag{2.3} $$

and

$$ G_1(z, y) = \aleph(z, y) + \sum_{i=1}^{p}\frac{\mho_i(\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)}\aleph^*(\sigma, y), $$

$$ G_2(z, y) = \frac{(\ln z)^{q-1}}{\Upsilon_1}\int_1^e G_1(z, y)\text{ß}(z)\frac{dz}{z}, $$

$$ \aleph(z, y) = \begin{cases} \frac{(\ln z)^{q-1}}{\Gamma(q)} - \frac{1}{\Gamma(q)}\left(\ln\frac{z}{y}\right)^{q-1}, & 1 \le y \le z \le e, \\ \frac{(\ln z)^{q-1}}{\Gamma(q)}, & 1 \le z \le y \le e, \end{cases} $$

$$ \aleph^*(\sigma, y) = \begin{cases} (\ln\sigma)^{q+\gamma_i-1} - \left(\ln\frac{\sigma}{y}\right)^{q+\gamma_i-1}, & y \le \sigma, \\ (\ln\sigma)^{q+\gamma_i-1}, & \sigma \le y. \end{cases} $$

Proof. Let $u_1(z) \in C^{[q]+1}[1, e]$ be a solution of FBVP (2.1)-(2.2) and is uniquely expressed as

$$ u_1(z) = \sum_{k=1}^{\ell}\mathfrak{c}_k\big(\ln z\big)^{q-k} - \int_1^z \frac{1}{\Gamma(q)}\left(\ln\frac{z}{y}\right)^{q-1}h(y)\frac{dy}{y}. \tag{2.4} $$

From $u_1^{(j)}(1) = 0,\ 0 \le j \le \ell - 2$, we get $\mathfrak{c}_\ell = \mathfrak{c}_{\ell-1} = \cdots = \mathfrak{c}_2 = 0$. Therefore

$$u_1(z) = \mathfrak{c}_1 \big(\ln z\big)^{q-1} - \int_1^z \frac{1}{\Gamma(q)} \left(\ln \frac{z}{y}\right)^{q-1} h(y) \frac{dy}{y}.$$

By using Lemma 2.3, we obtain

$$^{H}D_{1^+}^{q-1} u_1(z) = \mathfrak{c}_1 \Gamma(q) - \int_1^z h(y) \frac{dy}{y}.$$

Then

$$^{H}D_{1^+}^{q-1} u_1(e) - \int_1^e \beta(z) u_1(z) \frac{dz}{z} = \sum_{i=1}^{p} \mho_i \ ^{H}I_{1^+}^{\gamma_i} u_1(\sigma)$$

implies that

$$\mathfrak{c}_1 = \frac{1}{\Upsilon} \left[\int_1^e \beta(z) u_1(z) \frac{dz}{z} + \int_1^e h(y) \frac{dy}{y} - \sum_{i=1}^{p} \frac{\mho_i}{\Gamma(q+\gamma_i)} \int_1^\sigma \left(\ln \frac{\sigma}{y}\right)^{q+\gamma_i-1} h(y) \frac{dy}{y} \right].$$

Hence the unique solution of FBVP (2.1)-(2.2) is

$$u_1(z) = \begin{cases} \dfrac{(\ln z)^{q-1}}{\Upsilon} \left[\displaystyle\int_1^e \beta(z) u_1(z) \frac{dz}{z} + \int_1^e h(y) \frac{dy}{y} - \sum_{i=1}^{p} \frac{\mho_i}{\Gamma(q+\gamma_i)} \times \right. \\ \left. \displaystyle\int_1^\sigma \left(\ln \frac{\sigma}{y}\right)^{q+\gamma_i-1} h(y) \frac{dy}{y} \right] - \int_1^z \frac{1}{\Gamma(q)} \left(\ln \frac{z}{y}\right)^{q-1} h(y) \frac{dy}{y} \end{cases}$$

$$= \begin{cases} \dfrac{(\ln z)^{q-1}}{\Upsilon} \displaystyle\int_1^e \beta(z) u_1(z) \frac{dz}{z} + \frac{1}{\Upsilon} \left[\frac{\Upsilon + (\Gamma(q) - \Upsilon)}{\Gamma(q)} \right] (\ln z)^{q-1} \int_1^e h(y) \frac{dy}{y} \\ \displaystyle - \sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon \Gamma(q+\gamma_i)} \int_1^\sigma \left(\ln \frac{\sigma}{y}\right)^{q+\gamma_i-1} h(y) \frac{dy}{y} \\ \displaystyle - \int_1^z \frac{1}{\Gamma(q)} \left(\ln \frac{z}{y}\right)^{q-1} h(y) \frac{dy}{y} \end{cases}$$

$$
= \begin{cases} \dfrac{(\ln z)^{q-1}}{\Upsilon} \displaystyle\int_1^e ß(z)u_1(z)\frac{dz}{z} + \frac{(\ln z)^{q-1}}{\Gamma(q)} \int_1^e h(y)\frac{dy}{y} \\ + \dfrac{1}{\Upsilon}\left[\dfrac{\Gamma(q)-\Upsilon}{\Gamma(q)}\right](\ln z)^{q-1} \displaystyle\int_1^e h(y)\frac{dy}{y} \\ - \displaystyle\sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)} \int_1^{\sigma} \left(\ln \frac{\sigma}{y}\right)^{q+\gamma_i-1} h(y)\frac{dy}{y} \\ - \displaystyle\int_1^z \frac{1}{\Gamma(q)}\left(\ln\frac{z}{y}\right)^{q-1} h(y)\frac{dy}{y} \end{cases}
$$

$$
= \begin{cases} \dfrac{(\ln z)^{q-1}}{\Upsilon} \displaystyle\int_1^e ß(z)u_1(z)\frac{dz}{z} + \frac{(\ln z)^{q-1}}{\Gamma(q)} \int_1^e h(y)\frac{dy}{y} \\ + \displaystyle\sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)} \int_1^{\sigma} \left(\ln \sigma\right)^{q+\gamma_i-1} h(y)\frac{dy}{y} \\ - \displaystyle\sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)} \int_1^{\sigma} \left(\ln \frac{\sigma}{y}\right)^{q+\gamma_i-1} h(y)\frac{dy}{y} \\ - \displaystyle\int_1^z \frac{1}{\Gamma(q)}\left(\ln\frac{z}{y}\right)^{q-1} h(y)\frac{dy}{y} \end{cases}
$$

$$
= \begin{cases} \displaystyle\int_1^e \aleph(z,y)h(y)\frac{dy}{y} + \sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)} \int_1^e \aleph^*(\sigma,y)h(y)\frac{dy}{y} \\ + \dfrac{(\ln z)^{q-1}}{\Upsilon}\left[\displaystyle\int_1^e ß(z)\int_1^e G_1(z,y)h(y)\frac{dy}{y}\frac{dz}{z}\right. \\ \left. + \displaystyle\int_1^e ß(z)\int_1^e \frac{(\ln z)^{q-1}}{\Upsilon_1}\int_1^e G_1(z,y)ß(z)\frac{dz}{z}h(y)\frac{dy}{y}\frac{dz}{z}\right] \end{cases}
$$

$$
= \begin{cases} \displaystyle\int_1^e \left[\aleph(z,y) + \sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)} \aleph^*(\sigma,y)\right] h(y)\frac{dy}{y} \\ + \dfrac{(\ln z)^{q-1}}{\Upsilon}\cdot\dfrac{\Upsilon}{\Upsilon_1}\displaystyle\int_1^e ß(z)\int_1^e G_1(z,y)h(y)\frac{dy}{y}\frac{dz}{z} \end{cases}
$$

$$
= \int_1^e \left[G_1(z,y) + G_2(z,y)\right]h(y)\frac{dy}{y}
$$

$$
= \int_1^e G(z,y)h(y)\frac{dy}{y},
$$

where $G(z,y)$ is given in (2.3). The proof is completed. □

Lemma 2.5. *The Kernel* $G(z,y)$ *given in* (2.3) *is nonnegative, for all* $z, y \in [1, e]$.

Proof. The Kernel $G(z,y)$ is given in (2.3). Let $1 \le z \le y \le e$. Then:

$$
\aleph(z,y) = \frac{(\ln z)^{q-1}}{\Gamma(q)} \ge 0.
$$

Let $1 \le y \le z \le e$. Then:

$$
\begin{aligned} \aleph(z,y) &= \frac{1}{\Gamma(q)}\left[(\ln z)^{q-1} - \left(\ln\frac{z}{y}\right)^{q-1}\right] \\ &\ge \frac{(\ln z)^{q-1}}{\Gamma(q)}\left[1 - (1 - \ln y)^{q-1}\right] \ge 0. \end{aligned}
$$

On the other hand, let $1 \le \sigma \le y \le e$. Then:

$$
\aleph^*(\sigma,y) = (\ln\sigma)^{q+\gamma_i-1} \ge 0.
$$

Let $1 \leq \mathtt{y} \leq \sigma \leq e$. Then:

$$\begin{aligned}\aleph^*(\sigma, \mathtt{y}) &= (\ln \sigma)^{\mathtt{q}+\gamma_i-1} - \left(\ln \frac{\sigma}{\mathtt{y}}\right)^{\mathtt{q}+\gamma_i-1} \\ &\geq \frac{(\ln \sigma)^{\mathtt{q}+\gamma_i-1}}{\Gamma(\mathtt{q})}\left[1-(1-\ln \mathtt{y})^{\mathtt{q}+\gamma_i-1}\right] \geq 0.\end{aligned}$$

Hence $\mathtt{G}(\mathtt{z}, \mathtt{y}) \geq 0$. □

Lemma 2.6. *The Kernel* $\aleph(\mathtt{z}, \mathtt{y})$ *has the properties:*

(1) $\aleph(\mathtt{z}, \mathtt{y}) \leq \aleph(e, \mathtt{y}), \ \forall \ \mathtt{z}, \mathtt{y} \in [1, e]$,

(2) $\aleph(\mathtt{z}, \mathtt{y}) \geq \left(\frac{1}{4}\right)^{\mathtt{q}-1} \aleph(e, \mathtt{y}), \ \forall \mathtt{z} \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right], \mathtt{y} \in (1, e)$.

Proof. We prove (1). Let $1 \leq \mathtt{z} \leq \mathtt{y} \leq e$. Then:

$$\frac{\partial \aleph}{\partial \mathtt{z}} = \frac{(\ln \mathtt{z})^{\mathtt{q}-2}}{\Gamma(\mathtt{q}-1)} \geq 0.$$

Let $1 \leq \mathtt{y} \leq \mathtt{z} \leq e$. Then:

$$\begin{aligned}\frac{\partial \aleph}{\partial \mathtt{z}} &= \frac{1}{\Gamma(\mathtt{q}-1)}\left[\frac{1}{\mathtt{z}}(\ln \mathtt{z})^{\mathtt{q}-2} - \frac{1}{\mathtt{z}}\left(\ln \frac{\mathtt{z}}{\mathtt{y}}\right)^{\mathtt{q}-2}\right] \\ &\geq \frac{(\ln \mathtt{z})^{\mathtt{q}-2}}{\Gamma(\mathtt{q}-1)}\left[\frac{1-(1-\ln \mathtt{y})^{\mathtt{q}-2}}{\mathtt{z}}\right] \geq 0.\end{aligned}$$

Hence the inequality (1). We establish the inequality (2). Let $1 \leq \mathtt{z} \leq \mathtt{y} \leq e$ and $\mathtt{z} \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]$. Then:

$$\frac{\aleph(\mathtt{z}, \mathtt{y})}{\aleph(e, \mathtt{y})} = (\ln \mathtt{z})^{\mathtt{q}-1} \geq \left(\frac{1}{4}\right)^{\mathtt{q}-1}.$$

Let $1 \leq \mathtt{y} \leq \mathtt{z} \leq e$and $\mathtt{z} \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]$. Then:

$$\begin{aligned}\frac{\aleph(\mathtt{z}, \mathtt{y})}{\aleph(e, \mathtt{y})} &= \frac{(\ln \mathtt{z})^{\mathtt{q}-1} - \left(\ln \frac{\mathtt{z}}{\mathtt{y}}\right)^{\mathtt{q}-1}}{1-(1-\ln \mathtt{y})^{\mathtt{q}-1}} \\ &\geq \frac{(\ln \mathtt{z})^{\mathtt{q}-1} - (\ln \mathtt{z} - \ln \mathtt{y} \ln \mathtt{z})^{\mathtt{q}-1}}{1-(1-\ln \mathtt{y})^{\mathtt{q}-1}} \\ &= \frac{(\ln \mathtt{z})^{\mathtt{q}-1}\left[1-(1-\ln \mathtt{y})^{\mathtt{q}-1}\right]}{1-(1-\ln \mathtt{y})^{\mathtt{q}-1}} \\ &= (\ln \mathtt{z})^{\mathtt{q}-1} \geq \left(\frac{1}{4}\right)^{\mathtt{q}-1}.\end{aligned}$$

Hence $\aleph(\mathtt{z}, \mathtt{y}) \geq \left(\frac{1}{4}\right)^{\mathtt{q}-1} \aleph(e, \mathtt{y}), \quad \forall (\mathtt{z}, \mathtt{y}) \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right] \times (1, e)$. □

Lemma 2.7. *The Kernel* $\mathtt{G}(\mathtt{z}, \mathtt{y})$ *has the properties:*

(1) $\mathtt{G}(\mathtt{z}, \mathtt{y}) \leq \varpi \varphi(\mathtt{y}), \ \forall \ \mathtt{z}, \mathtt{y} \in [1, e]$,

(2) $\min\limits_{\mathtt{z} \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]} \mathtt{G}(\mathtt{z}, \mathtt{y}) \geq \left(\frac{1}{4}\right)^{2\mathtt{q}-2} \varpi \varphi(\mathtt{y}), \ \forall \ \mathtt{z}, \mathtt{y} \in [1, e]$,

where $\varphi(y) = \aleph(e, y) + \sum_{i=1}^{p} \frac{\mho_i}{\Upsilon\Gamma(q+\gamma_i)} \aleph^*(\sigma, y), \ \varpi = 1 + \frac{1}{\Upsilon_1}\int_1^e \text{ß}(z)\frac{dz}{z}.$

Proof. To show (1). For this we have:

$$\begin{aligned}
G(z, y) &= G_1(z, y) + G_2(z, y) \\
&= \aleph(z, y) + \sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)} \aleph^*(\sigma, y) + \frac{(\ln z)^{q-1}}{\Upsilon_1} \int_1^e G_1(z, y)\text{ß}(z)\frac{dz}{z} \\
&\leq \begin{cases} \aleph(e, y) + \sum_{i=1}^{p} \frac{\mho_i}{\Upsilon\Gamma(q+\gamma_i)} \aleph^*(\sigma, y) + \\ \frac{1}{\Upsilon_1}\int_1^e \left[\aleph(e, y) + \sum_{i=1}^{p} \frac{\mho_i}{\Upsilon\Gamma(q+\gamma_i)} \aleph^*(\sigma, y)\right]\text{ß}(z)\frac{dz}{z} \end{cases} \\
&= \varpi\varphi(y).
\end{aligned}$$

Hence the inequality (1). We establish the inequality (2). For $z, y \in [1, e]$, we have:

$$\begin{aligned}
\min_{z\in\left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]} G(z, y) &= \min_{z\in\left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]} \begin{cases} \aleph(z, y) + \sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)} \aleph^*(\sigma, y) + \\ \frac{(\ln z)^{q-1}}{\Upsilon_1} \int_1^e G_1(z, y)\text{ß}(z)\frac{dz}{z} \end{cases} \\
&\geq \begin{cases} \left(\frac{1}{4}\right)^{q-1} \left[\aleph(e, y) + \sum_{i=1}^{p} \frac{\mho_i}{\Upsilon\Gamma(q+\gamma_i)} \aleph^*(\sigma, y)\right] + \\ \left(\frac{1}{4}\right)^{q-1} \frac{1}{\Upsilon_1}\int_1^e \left[\aleph(z, y) + \sum_{i=1}^{p} \frac{\mho_i (\ln z)^{q-1}}{\Upsilon\Gamma(q+\gamma_i)} \times \right. \\ \left. \aleph^*(\sigma, y)\right]\text{ß}(z)\frac{dz}{z} \end{cases} \\
&\geq \begin{cases} \left(\frac{1}{4}\right)^{q-1} \varphi(y) + \left(\frac{1}{4}\right)^{2q-2} \frac{1}{\Upsilon_1}\int_1^e \left[\aleph(e, y) + \right. \\ \left. \sum_{i=1}^{p} \frac{\mho_i}{\Upsilon\Gamma(q+\gamma_i)} \aleph^*(\sigma, y)\right]\text{ß}(z)\frac{dz}{z} \end{cases} \\
&\geq \left(\frac{1}{4}\right)^{q-1} \varphi(y) + \left(\frac{1}{4}\right)^{2q-2} \frac{1}{\Upsilon_1}\int_1^e \varphi(y)\text{ß}(z)\frac{dz}{z} \\
&\geq \left(\frac{1}{4}\right)^{2q-2} \varphi(y)\left[1 + \frac{1}{\Upsilon_1}\int_1^e \text{ß}(z)\frac{dz}{z}\right].
\end{aligned}$$

Hence $\min_{z\in\left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]} G(z, y) \geq \left(\frac{1}{4}\right)^{2q-2} \varpi\varphi(y), \ \forall \ z, y \in [1, e].$ □

3. Main results

From [4, 9, 21], it can be seen that an ℓ-tuple $\Big(\mathtt{u}_1(\mathtt{z}),\mathtt{u}_2(\mathtt{z}),\cdots,\mathtt{u}_\ell(\mathtt{z})\Big)$ is a solution of the FBVP (1.1)-(1.2) if and only if $\mathtt{u}_\mathtt{k}(\mathtt{z})\in\mathtt{C}^{[\mathtt{q}]+1}[1,e]$ satisfies

$$\mathtt{u}_1(\mathtt{z})=\left\{\begin{array}{l}\lambda_1\displaystyle\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2\int_1^e \mathtt{G}(\mathtt{y}_1,\mathtt{y}_2)\mathtt{p}_2(\mathtt{y}_2)\cdots\\ \mathtt{g}_{\ell-1}\Bigg(\lambda_\ell\displaystyle\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\Big(\mathtt{u}_1(\mathtt{y}_\ell)\Big)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\frac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\frac{d\mathtt{y}_1}{\mathtt{y}_1},\end{array}\right.$$

and

$$\left\{\begin{array}{l}\mathtt{u}_2(\mathtt{z})=\lambda_2\displaystyle\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y})\mathtt{p}_2(\mathtt{y})\mathtt{g}_2\big(\mathtt{u}_3(\mathtt{y})\big)\frac{d\mathtt{y}}{\mathtt{y}},\\ \mathtt{u}_3(\mathtt{z})=\lambda_3\displaystyle\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y})\mathtt{p}_3(\mathtt{y})\mathtt{g}_3\big(\mathtt{u}_4(\mathtt{y})\big)\frac{d\mathtt{y}}{\mathtt{y}},\\ \qquad\cdots\\ \mathtt{u}_\ell(\mathtt{z})=\lambda_\ell\displaystyle\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y})\mathtt{p}_\ell(\mathtt{y})\mathtt{g}_\ell\big(\mathtt{u}_{\ell+1}(\mathtt{y})\big)\frac{d\mathtt{y}}{\mathtt{y}},\end{array}\right.$$

where $\mathtt{u}_{\ell+1}(\mathtt{z})=\mathtt{u}_1(\mathtt{z}),\ 1<\mathtt{z}<e$. By a positive solution of the FBVP (1.1)-(1.2), we mean $\big(\mathtt{u}_1(\mathtt{z}),\mathtt{u}_2(\mathtt{z}),\cdots,$ $\mathtt{u}_\ell(\mathtt{z})\big)\in\Big(\mathtt{C}^{[\mathtt{q}]+1}[1,e]\Big)^\ell$ which satisfying the FDE (1.1) and BCs (1.2) with $\mathtt{u}_\mathtt{k}(\mathtt{z})>0,\mathtt{k}=1,2,\cdots,\ell\ \forall\ \mathtt{z}\in[1,e]$.

Let $\mathtt{B}=\mathtt{C}([1,e],\mathbb{R})$ be the Banach space endowed with the norm

$$\|\mathtt{x}\|=\max_{\mathtt{z}\in[1,e]}|\mathtt{x}(\mathtt{z})|$$

and $\mathtt{P}\subset\mathtt{B}$ be a cone defined as

$$\mathtt{P}=\left\{\mathtt{x}\in\mathtt{B}\ :\ \mathtt{x}(\mathtt{z})\geq 0 \text{ on } [1,e] \text{ and } \min_{\mathtt{z}\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]}\mathtt{x}(\mathtt{z})\geq\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\|\mathtt{x}\|\right\}.$$

Define an integral operator $\mathtt{T}:\mathtt{P}\to\mathtt{B}$, for $\mathtt{u}_1\in\mathtt{P}$, by

$$\left.\begin{array}{r}\mathtt{T}\mathtt{u}_1(\mathtt{z})=\lambda_1\displaystyle\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2\int_1^e \mathtt{G}(\mathtt{y}_1,\mathtt{y}_2)\\ \mathtt{p}_2(\mathtt{y}_2)\cdots\mathtt{g}_{\ell-1}\Bigg(\lambda_\ell\displaystyle\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\\ \mathtt{g}_\ell\Big(\mathtt{u}_1(\mathtt{y}_\ell)\Big)\dfrac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\dfrac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\dfrac{d\mathtt{y}_1}{\mathtt{y}_1}.\end{array}\right\}\tag{3.1}$$

Notice from (H_3) and Lemma 2.5 that, for $\mathtt{u}_1\in\mathtt{P}$, $\mathtt{T}\mathtt{u}_1(\mathtt{z})\geq 0$ on $[1,e]$. In addition, we have

$$\mathtt{T}\mathtt{u}_1(\mathtt{z})\leq\varpi\left\{\begin{array}{l}\lambda_1\displaystyle\int_1^e \varphi(\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2\int_1^e \varphi(\mathtt{y}_2)\mathtt{p}_2(\mathtt{y}_2)\cdots\\ \mathtt{g}_{\ell-1}\Bigg(\lambda_\ell\displaystyle\int_1^e \varphi(\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\Big(\mathtt{u}_1(\mathtt{y}_\ell)\Big)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\frac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\frac{d\mathtt{y}_1}{\mathtt{y}_1}\end{array}\right.$$

so that

$$
\left.\begin{array}{r}
\|\mathtt{Tu}_1\| \leq \varpi\lambda_1 \displaystyle\int_1^e \varphi(\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2 \int_1^e \varphi(\mathtt{y}_2)\mathtt{p}_2(\mathtt{y}_2)\cdots \\
\mathtt{g}_{\ell-1}\Bigg(\lambda_\ell \displaystyle\int_1^e \varphi(\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\Big(\mathtt{u}_1(\mathtt{y}_\ell)\Big) \\
\dfrac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\dfrac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\dfrac{d\mathtt{y}_1}{\mathtt{y}_1}.
\end{array}\right\} \tag{3.2}
$$

If $\mathtt{u}_1 \in \mathtt{P}$, from Lemma 2.6 and (3.2), we deduce that

$$
\begin{aligned}
\min_{\mathtt{z}\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]} \mathtt{Tu}_1(\mathtt{z}) &= \left\{\begin{array}{l}
\displaystyle\min_{\mathtt{z}\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]} \lambda_1 \int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2 \int_1^e \mathtt{G}(\mathtt{y}_1,\mathtt{y}_2) \\
\mathtt{p}_2(\mathtt{y}_2)\cdots\mathtt{g}_{\ell-1}\Bigg(\lambda_\ell \displaystyle\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\Big(\mathtt{u}_1(\mathtt{y}_\ell)\Big) \\
\dfrac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\dfrac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\dfrac{d\mathtt{y}_1}{\mathtt{y}_1}
\end{array}\right. \\
&\geq \left\{\begin{array}{l}
\left(\dfrac{1}{4}\right)^{2\mathtt{q}-2}\varpi\lambda_1 \displaystyle\int_1^e \varphi(\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2 \int_1^e \varphi(\mathtt{y}_2)\times \\
\mathtt{p}_2(\mathtt{y}_2)\cdots\mathtt{g}_{\ell-1}\Bigg(\lambda_\ell \displaystyle\int_1^e \varphi(\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\Big(\mathtt{u}_1(\mathtt{y}_\ell)\Big) \\
\dfrac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\dfrac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\dfrac{d\mathtt{y}_1}{\mathtt{y}_1}
\end{array}\right. \\
&\geq \left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\|\mathtt{Tu}_1\|.
\end{aligned}
$$

Therefore $\displaystyle\min_{\mathtt{z}\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]} \mathtt{Tu}_1(\mathtt{z}) \geq \left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\|\mathtt{Tu}_1\|$. Hence $\mathtt{Tu}_1 \in \mathtt{P}$ and so $\mathtt{T}:\mathtt{P}\to\mathtt{P}$. Further the operator $\mathtt{T}$ is a completely continuous by an application of the Arzela–Ascoli Theorem [7].

3.1. **Notations.** We introduce:

$$
\Psi_1 = \max\left\{\begin{array}{l}
\left[\left(\dfrac{1}{4}\right)^{2\mathtt{q}-2}\varpi \displaystyle\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(\mathtt{y})\mathtt{p}_1(\mathtt{y})\dfrac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{1\infty}\right]^{-1}, \\
\left[\left(\dfrac{1}{4}\right)^{2\mathtt{q}-2}\varpi \displaystyle\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(\mathtt{y})\mathtt{p}_2(\mathtt{y})\dfrac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{2\infty}\right]^{-1}, \\
\cdots \\
\left[\left(\dfrac{1}{4}\right)^{2\mathtt{q}-2}\varpi \displaystyle\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(\mathtt{y})\mathtt{p}_\ell(\mathtt{y})\dfrac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{\ell\infty}\right]^{-1}
\end{array}\right\},
$$

and

$$\Psi_2 = \min\left\{\begin{array}{c}\left[\varpi\int_1^e \varphi(\mathtt{y})\mathtt{p}_1(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{10}\right]^{-1},\\ \left[\varpi\int_1^e \varphi(\mathtt{y})\mathtt{p}_2(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{20}\right]^{-1},\\ \cdots \\ \left[\varpi\int_1^e \varphi(\mathtt{y})\mathtt{p}_\ell(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{\ell 0}\right]^{-1}\end{array}\right\}.$$

Theorem 3.1. *Suppose* (H_1)-(H_4) *hold. Then for each* $\lambda_{\mathtt{k}}, \mathtt{k} = 1, 2, \cdots, \ell$ *satisfying*

$$\Psi_1 < \lambda_{\mathtt{k}} < \Psi_2,\ \mathtt{k} = 1, 2, \cdots, \ell, \tag{3.3}$$

there exists an ℓ*-tuple* $(\mathtt{u}_1, \mathtt{u}_2, \cdots, \mathtt{u}_\ell)$ *satisfying the FBVP* (1.1)-(1.2) *s.t.* $\mathtt{u}_{\mathtt{k}}(\mathtt{z}) > 0$, $\mathtt{k} = 1, 2, \cdots, \ell$ *on* $(1, e)$.

Proof. Let $\lambda_{\mathtt{k}}$, $\mathtt{k} = 1, 2, \cdots, \ell$ be given as in (3.3). Now let $\epsilon > 0$ be chosen s.t.

$$\max\left\{\begin{array}{c}\left[\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(\mathtt{y})\mathtt{p}_1(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}(\mathtt{g}_{1\infty}-\epsilon)\right]^{-1},\\ \left[\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(\mathtt{y})\mathtt{p}_2(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}(\mathtt{g}_{2\infty}-\epsilon)\right]^{-1},\\ \cdots \\ \left[\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(\mathtt{y})\mathtt{p}_\ell(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}(\mathtt{g}_{\ell\infty}-\epsilon)\right]^{-1}\end{array}\right\} \le \min\left\{\begin{array}{c}\lambda_1,\\ \lambda_2,\\ \cdot\\ \cdot\\ \cdot\\ \lambda_\ell\end{array}\right\},$$

and

$$\max\left\{\begin{array}{c}\lambda_1,\\ \lambda_2,\\ \cdot\\ \cdot\\ \cdot\\ \lambda_\ell\end{array}\right\} \le \min\left\{\begin{array}{c}\left[\varpi\int_1^e \varphi(\mathtt{y})\mathtt{p}_1(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}(\mathtt{g}_{10}+\epsilon)\right]^{-1},\\ \left[\varpi\int_1^e \varphi(\mathtt{y})\mathtt{p}_2(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}(\mathtt{g}_{20}+\epsilon)\right]^{-1},\\ \cdots \\ \left[\varpi\int_1^e \varphi(\mathtt{y})\mathtt{p}_\ell(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}(\mathtt{g}_{\ell 0}+\epsilon)\right]^{-1}\end{array}\right\}.$$

Now from the definitions of $\mathtt{g}_{\mathtt{k}0}$, $\mathtt{k} = 1, 2, \cdots, \ell$, there exists an $\ell_1 > 0$ s.t., for each $1 \le \mathtt{k} \le \ell$, $\mathtt{g}_{\mathtt{k}}(\mathtt{x}) \le (\mathtt{g}_{\mathtt{k}0} + \epsilon)\mathtt{x}$, $1 < \mathtt{x} \le \ell_1$.

Let $\mathtt{u}_1 \in \mathtt{P}$ with $\|\mathtt{u}_1\| = \ell_1$. By Lemma 2.6 and the choice of ϵ, for $1 \le \mathtt{y}_{\ell-1} \le e$,

$$\begin{aligned}&\lambda_\ell\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(\mathtt{y}_\ell)\big)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\\ &\quad\le \lambda_\ell\int_1^e \varpi\varphi(\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\big(\mathtt{g}_{\ell 0}+\epsilon\big)\mathtt{u}_1(\mathtt{y}_\ell)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\\ &\quad\le \lambda_\ell\int_1^e \varpi\varphi(\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\big(\mathtt{g}_{\ell 0}+\epsilon\big)\|\mathtt{u}_1\|\\ &\quad\le \|\mathtt{u}_1\| = \ell_1.\end{aligned}$$

It follows from Lemma 2.6, in the same way, for $1 \le \mathtt{y}_{\ell-2} \le e$:

$$\lambda_{\ell-1}\int_1^e \mathtt{G}(\mathtt{y}_{\ell-2},\mathtt{y}_{\ell-1})\mathtt{p}_{\ell-1}(\mathtt{y}_{\ell-1})\mathtt{g}_{\ell-1}\left(\lambda_\ell\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\,\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell(\mathtt{u}_1(\mathtt{y}_\ell))\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\right)\frac{d\mathtt{y}_{\ell-1}}{\mathtt{y}_{\ell-1}}$$
$$\le \lambda_{\ell-1}\int_1^e \varpi\varphi(\mathtt{y}_{\ell-1})\mathtt{p}_{\ell-1}(\mathtt{y}_{\ell-1})\frac{d\mathtt{y}_{\ell-1}}{\mathtt{y}_{\ell-1}}(\mathtt{g}_{\overline{\ell-1}0}+\epsilon)\|\mathtt{u}_1\|$$
$$\le \|\mathtt{u}_1\| = \ell_1.$$

Continuing with this bootstrapping argument, for $1 \le \mathtt{z} \le e$:

$$\left.\begin{aligned}&\lambda_1\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2\int_1^e \mathtt{G}(\mathtt{y}_1,\mathtt{y}_2)\mathtt{p}_2(\mathtt{y}_2)\cdots\\ &\quad \mathtt{g}_{\ell-1}\Bigg(\lambda_\ell\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell(\mathtt{u}_1(\mathtt{y}_\ell))\\ &\quad\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\frac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\frac{d\mathtt{y}_1}{\mathtt{y}_1}\end{aligned}\right\} \le \ell_1,$$

so that, for $1 \le \mathtt{z} \le e$, $\mathtt{Tu}_1(\mathtt{z}) \le \ell_1$. Hence $\|\mathtt{Tu}_1\| \le \ell_1 = \|\mathtt{u}_1\|$. If we set $\mathtt{E}_1 = \{\mathtt{x} \in \mathtt{B} : \|\mathtt{x}\| < \ell_1\}$, then

$$\|\mathtt{Tu}_1\| \le \|\mathtt{u}_1\|, \text{ for } \mathtt{u}_1 \in \mathtt{P} \cap \partial\mathtt{E}_1. \tag{3.4}$$

From the definition of $\mathtt{g}_{\mathtt{k}\infty}$, $\mathtt{k} = 1, 2, \cdots, \ell$, there exists $\widehat{\ell}_2 > 0$ s.t., for each $1 \le \mathtt{k} \le \ell$, $\mathtt{g}_\mathtt{k}(\mathtt{x}) \ge (\mathtt{g}_{\mathtt{k}\infty} - \epsilon)\mathtt{x}$, $\mathtt{x} \ge \widehat{\ell}_2$. Choose $\ell_2 = \max\left\{2\ell_1, \left(\frac{1}{4}\right)^{2-2\mathtt{q}}\frac{\widehat{\ell}_2}{\varpi}\right\}$. Let $\mathtt{u}_1 \in \mathtt{P}$ and $\|\mathtt{u}_1\| = \ell_2$. Then

$$\min_{\mathtt{z}\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]} \mathtt{u}_1(\mathtt{z}) \ge \left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\|\mathtt{u}_1\| \ge \widehat{\ell}_2.$$

Based on Lemma 2.6 and choice of ϵ, for $1 \le \mathtt{y}_{\ell-1} \le e$, we have:

$$\begin{aligned}&\lambda_\ell\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell(\mathtt{u}_1(\mathtt{y}_\ell))\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\\ &\ge\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\lambda_\ell\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)(\mathtt{g}_{\ell\infty}-\epsilon)\mathtt{u}_1(\mathtt{y}_\ell)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\\ &\ge\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\lambda_\ell\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}(\mathtt{g}_{\ell\infty}-\epsilon)\|\mathtt{u}_1\|\\ &\ge\|\mathtt{u}_1\| = \ell_2.\end{aligned}$$

It stems in the same way from Lemma 2.6 and choice of ϵ, for $1 \le \mathtt{y}_{\ell-2} \le e$:

$$\left.\begin{aligned}&\lambda_{\ell-1}\int_1^e \mathtt{G}(\mathtt{y}_{\ell-2},\mathtt{y}_{\ell-1})\mathtt{p}_{\ell-1}(\mathtt{y}_{\ell-1})\mathtt{g}_{\ell-1}\Bigg(\lambda_\ell\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\\ &\quad\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell(\mathtt{u}_1(\mathtt{y}_\ell))\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\frac{d\mathtt{y}_{\ell-1}}{\mathtt{y}_{\ell-1}}\end{aligned}\right\} \ge \left\{\begin{aligned}&\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\lambda_{\ell-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y}_{\ell-1})\times\\ &\mathtt{p}_{\ell-1}(\mathtt{y}_{\ell-1})\frac{d\mathtt{y}_{\ell-1}}{\mathtt{y}_{\ell-1}}(\mathtt{g}_{\overline{\ell-1}\infty}-\epsilon)\|\mathtt{u}_1\|\end{aligned}\right.$$
$$\ge \|\mathtt{u}_1\| = \ell_2.$$

By bootstrapping argument, we discover:

$$\left.\begin{array}{r}\lambda_1 \displaystyle\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2 \int_1^e \mathtt{G}_1(\mathtt{y}_1,\mathtt{y}_2)\mathtt{p}_2(\mathtt{y}_2)\cdots \\ \mathtt{g}_{\ell-1}\Bigg(\lambda_\ell \displaystyle\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(\mathtt{y}_\ell)\big) \\ \dfrac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\dfrac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\dfrac{d\mathtt{y}_1}{\mathtt{y}_1}\end{array}\right\}\geq \ell_2,$$

so that $\mathtt{Tu}_1(\mathtt{z}) \geq \ell_2 = \|\mathtt{u}_1\|$. Hence $\|\mathtt{Tu}_1\| \geq \|\mathtt{u}_1\|$. So if we set $\mathtt{E}_2 = \big\{\mathtt{x}\in \mathtt{B} : \|\mathtt{x}\| < \ell_2\big\}$, then

$$\|\mathtt{Tu}_1\| \geq \|\mathtt{u}_1\|, \text{ for } \mathtt{u}_1 \in \mathtt{P}\cap\partial\mathtt{E}_2. \tag{3.5}$$

By utilizing (3.4), (3.5) and Guo–Krasnosel'skii FPT (see [10, 19]), we conclude that $\mathtt{T}$ has a fixed point $\mathtt{u}_1 \in \mathtt{P}\cap(\overline{\mathtt{E}}_2\backslash\mathtt{E}_1)$. Setting $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, we obtain a positive solution $\big(\mathtt{u}_1,\mathtt{u}_2,\cdots,\mathtt{u}_\ell\big)$ of the FBVP (1.1)–(1.2) iteratively indicated by:

$$\mathtt{u}_\mathtt{k}(\mathtt{z}) = \lambda_\mathtt{k}\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y})\mathtt{p}_\mathtt{k}(\mathtt{y})\mathtt{g}_\mathtt{k}\big(\mathtt{u}_{\mathtt{k}+1}(\mathtt{y})\big)\frac{d\mathtt{y}}{\mathtt{y}}, \quad \mathtt{k} = \ell,\ell-1,\cdots,1.$$

□

3.2. Notations.

$$\Psi_3 = \max\left\{\begin{array}{c}\left[\left(\dfrac{1}{4}\right)^{2\mathtt{q}-2}\varpi\displaystyle\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y})\mathtt{p}_1(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{10}\right]^{-1}, \\ \left[\left(\dfrac{1}{4}\right)^{2\mathtt{q}-2}\varpi\displaystyle\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y})\mathtt{p}_2(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{20}\right]^{-1}, \\ \cdots \\ \left[\left(\dfrac{1}{4}\right)^{2\mathtt{q}-2}\varpi\displaystyle\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y})\mathtt{p}_\ell(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{\ell 0}\right]^{-1}\end{array}\right\}$$

and

$$\Psi_4 = \min\left\{\begin{array}{c}\left[\varpi\displaystyle\int_1^e\varphi(\mathtt{y})\mathtt{p}_1(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{1\infty}\right]^{-1}, \\ \left[\varpi\displaystyle\int_1^e\varphi(\mathtt{y})\mathtt{p}_2(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{2\infty}\right]^{-1}, \\ \cdots \\ \left[\varpi\displaystyle\int_1^e\varphi(\mathtt{y})\mathtt{p}_\ell(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{\ell\infty}\right]^{-1}\end{array}\right\}.$$

Theorem 3.2. *Suppose* (H_1)-(H_4) *hold, then for each* $\lambda_\mathtt{k}, \mathtt{k} = 1,2,\cdots,\ell$ *satisfying*

$$\Psi_3 < \lambda_\mathtt{k} < \Psi_4,\ \mathtt{k} = 1,2,\cdots,\ell, \tag{3.6}$$

there exists an ℓ*-tuple* $\big(\mathtt{u}_1,\mathtt{u}_2,\cdots,\mathtt{u}_\ell\big)$ *satisfying the FBVP* (1.1)-(1.2) *s.t.* $\mathtt{u}_\mathtt{k}(\mathtt{z}) > 0$, $\mathtt{k} = 1,2,\cdots,\ell$ *on* $(1,e)$.

Proof. Let λ_k, $k = 1, 2, \cdots, \ell$ be given as in (3.6). Now let $\epsilon > 0$ be chosen s.t.

$$\max \left\{ \begin{array}{c} \left[\left(\frac{1}{4}\right)^{2q-2} \varpi \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(y) p_1(y) \frac{dy}{y} (g_{10} - \epsilon) \right]^{-1}, \\ \left[\left(\frac{1}{4}\right)^{2q-2} \varpi \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(y) p_2(y) \frac{dy}{y} (g_{20} - \epsilon) \right]^{-1}, \\ \cdots \\ \left[\left(\frac{1}{4}\right)^{2q-2} \varpi \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(y) p_\ell(y) \frac{dy}{y} (g_{\ell 0} - \epsilon) \right]^{-1} \end{array} \right\} \le \min \left\{ \begin{array}{c} \lambda_1, \\ \lambda_2, \\ \cdot \\ \cdot \\ \cdot \\ \lambda_\ell \end{array} \right\},$$

and

$$\max \left\{ \begin{array}{c} \lambda_1, \\ \lambda_2, \\ \cdot \\ \cdot \\ \cdot \\ \lambda_\ell \end{array} \right\} \le \min \left\{ \begin{array}{c} \left[\varpi \int_1^e \varphi(y) p_1(y) \frac{dy}{y} (g_{1\infty} + \epsilon) \right]^{-1}, \\ \left[\varpi \int_1^e \varphi(y) p_2(y) \frac{dy}{y} (g_{2\infty} + \epsilon) \right]^{-1}, \\ \cdots \\ \left[\varpi \int_1^e \varphi(y) p_\ell(y) \frac{dy}{y} (g_{\ell\infty} + \epsilon) \right]^{-1} \end{array} \right\}.$$

From the definitions of g_{k0}, $1 \le k \le \ell$ there exists $\widehat{\ell}_3 > 0$ s.t., for each $1 \le k \le \ell$,

$$g_k(x) \ge \big(g_{k0} - \epsilon\big)x, \ \ 1 < x \le \widehat{\ell}_3.$$

According to the definitions of g_{k0}, it follows that $g_{k0}(1) = 0, \ 1 \le k \le \ell$ and so there exist $1 < \Theta_\ell < \Theta_{\ell-1} < \cdots < \Theta_2 < \widehat{\ell}_3$ s.t.

$$\left. \begin{array}{r} \lambda_k g_k(z) \le \dfrac{\Theta_{k-1}}{\int_1^e \varphi(y_\ell) p_k(y) dy}, \ z \in \big[1, \Theta_k\big], \quad 3 \le k \le \ell, \ \text{and} \\ \lambda_2 g_2(z) \le \dfrac{\widehat{\ell}_3}{\int_1^e \varphi(y_\ell) p_2(y) dy}, \ z \in \big[1, \Theta_2\big]. \end{array} \right\}$$

Let $u_1 \in P$ with $\|u_1\| = \Theta_\ell$. Then:

$$\begin{aligned} &\lambda_\ell \int_1^e G(y_{\ell-1}, y_\ell) p_\ell(y_\ell) g_\ell\big(u_1(y_\ell)\big) \frac{dy_\ell}{y_\ell} \\ &\qquad \le \lambda_\ell \int_1^e \varphi(y_\ell) p_\ell(y_\ell) g_\ell\big(u_1(y_\ell)\big) \frac{dy_\ell}{y_\ell} \\ &\qquad \le \frac{\int_1^e \varphi(y_\ell) p_\ell(y_\ell) \Theta_{\ell-1} \frac{dy_\ell}{y_\ell}}{\int_1^e \varphi(y_\ell) p_\ell(y_\ell) \frac{dy_\ell}{y_\ell}} \\ &\qquad \le \Theta_{\ell-1}. \end{aligned}$$

Utilizing this bootstrapping technique, it implies that

$$\left. \begin{array}{r} \lambda_2 \int_1^e G(y_1, y_2) p_2(y_2) g_2\Big(\lambda_3 \int_1^e G(y_2, y_3) p_3(y_3) \cdots \\ g_{\ell-1}\Big(\lambda_\ell \int_1^e G(y_{\ell-1}, y_\ell) p_\ell(y_\ell) g_\ell\big(u_1(y_\ell)\big) \\ \frac{dy_\ell}{y_\ell}\Big) \cdots \frac{dy_2}{y_2}\Big) \frac{dy_1}{y_1} \end{array} \right\} \le \widehat{\ell}_3.$$

Then

$$\begin{aligned}
\mathtt{T}\mathtt{u}_1(\mathtt{z}) &= \begin{cases} \lambda_1 \displaystyle\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2 \int_1^e \mathtt{G}(\mathtt{y}_1,\mathtt{y}_2)\mathtt{p}_2(\mathtt{y}_2)\cdots \\ \mathtt{g}_{\ell-1}\Bigg(\lambda_\ell \displaystyle\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(\mathtt{y}_\ell)\big)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\frac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\frac{d\mathtt{y}_1}{\mathtt{y}_1} \end{cases} \\
&\geq \left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\lambda_1 \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \varphi(\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\big(\mathtt{g}_{10}-\epsilon\big)\|\mathtt{u}_1\|\frac{d\mathtt{y}_1}{\mathtt{y}_1} \\
&\geq \|\mathtt{u}_1\|.
\end{aligned}$$

So $\|\mathtt{T}\mathtt{u}_1\| \geq \|\mathtt{u}_1\|$. If we set $\mathtt{E}_3 = \Big\{\mathtt{x}\in\mathtt{B} : \|\mathtt{x}\| < \Theta_\ell\Big\}$, then

$$\|\mathtt{T}\mathtt{u}_1\| \geq \|\mathtt{u}_1\|, \text{ for } \mathtt{u}_1 \in \mathtt{P}\cap\partial\mathtt{E}_3. \tag{3.7}$$

Since each $\mathtt{g}_{\mathtt{k}\infty}$ is taken to be a positive real number, it follows that $\mathtt{g}_\mathtt{k}$, $1\leq\mathtt{k}\leq\ell$ is unbounded at ∞. For each $1\leq\mathtt{k}\leq\ell$, set

$$\mathtt{g}_\mathtt{k}^*(\mathtt{x}) = \sup_{\mathtt{y}\in[1,\mathtt{x}]}\mathtt{g}_\mathtt{k}(\mathtt{y}).$$

By definition of $\mathtt{g}_{\mathtt{k}\infty}$, $1\leq\mathtt{k}\leq\ell$, there exists $\widehat{\ell}_4$ s.t., for each $1\leq\mathtt{k}\leq\ell$,

$$\mathtt{g}_\mathtt{k}^*(\mathtt{x}) \leq \big(\mathtt{g}_{\mathtt{k}\infty}+\epsilon\big)\mathtt{x},\ \ \mathtt{x}\geq\widehat{\ell}_4.$$

It follows that there exists $\ell_4 = \max\Big\{2\widehat{\ell}_3,\widehat{\ell}_4\Big\}$ s.t., for each $1\leq\mathtt{k}\leq\ell$,

$$\mathtt{g}_\mathtt{k}^*(\mathtt{x}) \leq \mathtt{g}_\mathtt{k}^*\big(\ell_4\big),\ \ 1<\mathtt{x}\leq\ell_4.$$

Choose $\mathtt{u}_1\in\mathtt{P}$ with $\|\mathtt{u}_1\| = \ell_4$. Then, by using bootstrapping argument, we have:

$$\begin{aligned}
\mathtt{T}\mathtt{u}_1(\mathtt{z}) &= \begin{cases} \lambda_1 \displaystyle\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1\Bigg(\lambda_2 \int_1^e \mathtt{G}(\mathtt{y}_1,\mathtt{y}_2)\mathtt{p}_2(\mathtt{y}_2)\cdots \\ \mathtt{g}_{\ell-1}\Bigg(\lambda_\ell \displaystyle\int_1^e \mathtt{G}(\mathtt{y}_{\ell-1},\mathtt{y}_\ell)\mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(\mathtt{y}_\ell)\big)\frac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\frac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\frac{d\mathtt{y}_1}{\mathtt{y}_1} \end{cases} \\
&\leq \begin{cases} \lambda_1 \displaystyle\int_1^e \varphi(\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1^*\Bigg(\lambda_2 \int_1^e \varphi(\mathtt{y}_2)\mathtt{p}_2(\mathtt{y}_2)\cdots\mathtt{g}_{\ell-1}\Bigg(\lambda_\ell \int_1^e \varphi(\mathtt{y}_\ell) \\ \mathtt{p}_\ell(\mathtt{y}_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(\mathtt{y}_\ell)\big)\dfrac{d\mathtt{y}_\ell}{\mathtt{y}_\ell}\Bigg)\cdots\dfrac{d\mathtt{y}_2}{\mathtt{y}_2}\Bigg)\dfrac{d\mathtt{y}_1}{\mathtt{y}_1} \end{cases} \\
&\leq \lambda_1 \int_1^e \varphi(\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\mathtt{g}_1^*\big(\ell_4\big)\frac{d\mathtt{y}_1}{\mathtt{y}_1} \\
&\leq \lambda_1 \int_1^e \varphi(\mathtt{y}_1)\mathtt{p}_1(\mathtt{y}_1)\frac{d\mathtt{y}_1}{\mathtt{y}_1}\big(\mathtt{g}_{1\infty}+\epsilon\big)\ell_4 \\
&\leq \ell_4 = \|\mathtt{u}_1\|.
\end{aligned}$$

Thus $\|\mathtt{T}\mathtt{u}_1\| \leq \|\mathtt{u}_1\|$. So, if we let $\mathtt{E}_4 = \Big\{\mathtt{x}\in\mathtt{B} : \|\mathtt{x}\| < \ell_4\Big\}$, then

$$\|\mathtt{T}\mathtt{u}_1\| \leq \|\mathtt{u}_1\|, \text{ for } \mathtt{u}_1 \in \mathtt{P}\cap\partial\mathtt{E}_4. \tag{3.8}$$

By utilizing (3.7), (3.8) and Guo–Krasnosel'skii FPT (see [10, 19]), we get that $\mathtt{T}$ has a fixed point $\mathtt{u}_1 \in \mathtt{P}\cap(\overline{\mathtt{E}}_4\backslash\mathtt{E}_3)$, which in turn with $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$ yields an ℓ-tuple $\big(\mathtt{u}_1,\mathtt{u}_2,\cdots,\mathtt{u}_\ell\big)$ satisfying the FBVP (1.1)-(1.2) for the chosen values of $\lambda_\mathtt{k}$, $\mathtt{k}=1,2,\cdots,\ell$. □

4. Example

Let $\sigma = 2, n = 3, \mathtt{q} = 2.5, \gamma_i = 1.2, \mathtt{p} = 1$. Consider the FBVP for $\mathtt{z} \in (1, e)$:

$$\left.\begin{aligned} {}^{\mathtt{H}}\mathtt{D}^{2.5}_{1^+}\mathtt{u}_1(\mathtt{z}) + \lambda_1\mathtt{p}_1(\mathtt{z})\mathtt{g}_1\big(\mathtt{u}_2(\mathtt{z})\big) = 0,\\ {}^{\mathtt{H}}\mathtt{D}^{2.5}_{1^+}\mathtt{u}_2(\mathtt{z}) + \lambda_2\mathtt{p}_2(\mathtt{z})\mathtt{g}_2\big(\mathtt{u}_3(\mathtt{z})\big) = 0,\\ {}^{\mathtt{H}}\mathtt{D}^{2.5}_{1^+}\mathtt{u}_3(\mathtt{z}) + \lambda_3\mathtt{p}_3(\mathtt{z})\mathtt{g}_3\big(\mathtt{u}_1(\mathtt{z})\big) = 0, \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \mathtt{u}_1(1) = 0, \quad \mathtt{u}_1'(1) = 0, \quad {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}-1}_{1^+}\mathtt{u}_1(e) - \int_1^e \text{ß}(\mathtt{z})\mathtt{u}_1(\mathtt{z})\frac{d\mathtt{z}}{\mathtt{z}} = \sum_{\mathtt{i}=1}^{\mathtt{p}} \frac{1}{2}\ {}^{\mathtt{H}}\mathtt{I}^{1.2}_{1^+}\mathtt{u}_1(2),\\ \mathtt{u}_2(1) = 0, \quad \mathtt{u}_2'(1) = 0, \quad {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}-1}_{1^+}\mathtt{u}_1(e) - \int_1^e \text{ß}(\mathtt{z})\mathtt{u}_1(\mathtt{z})\frac{d\mathtt{z}}{\mathtt{z}} = \sum_{\mathtt{i}=1}^{\mathtt{p}} \frac{1}{2}\ {}^{\mathtt{H}}\mathtt{I}^{1.2}_{1^+}\mathtt{u}_1(2),\\ \mathtt{u}_3(1) = 0, \quad \mathtt{u}_3'(1) = 0, \quad {}^{\mathtt{H}}\mathtt{D}^{\mathtt{q}-1}_{1^+}\mathtt{u}_1(e) - \int_1^e \text{ß}(\mathtt{z})\mathtt{u}_1(\mathtt{z})\frac{d\mathtt{z}}{\mathtt{z}} = \sum_{\mathtt{i}=1}^{\mathtt{p}} \frac{1}{2}\ {}^{\mathtt{H}}\mathtt{I}^{1.2}_{1^+}\mathtt{u}_1(2), \end{aligned}\right\} \tag{4.2}$$

where

$$\begin{cases} \text{ß}(\mathtt{z}) = \mathtt{z}, \mathtt{p}_1(\mathtt{z}) = \mathtt{p}_2(\mathtt{z}) = \mathtt{p}_3(\mathtt{z}) = \mathtt{z},\\ \mathtt{g}_1(\mathtt{u}) = \mathtt{u}\left(\dfrac{1}{20} - 0.049e^{-\mathtt{u}}\right),\\ \mathtt{g}_2(\mathtt{u}) = \dfrac{\mathtt{u}}{20} - 0.049\sin\mathtt{u},\\ \mathtt{g}_3(\mathtt{u}) = \dfrac{\mathtt{u}^2 + 15\mathtt{u}}{15(5\times 10^3 + \mathtt{u})}. \end{cases}$$

In view of the data given, we get $\Upsilon \approx 1.237371600, \Upsilon_1 \approx 0.4025348954, \mathtt{g}_{10} \approx 0.001, \mathtt{g}_{1\infty} \approx 0.05, \mathtt{g}_{20} \approx 0.001, \mathtt{g}_{2\infty} \approx 0.05, \mathtt{g}_{30} \approx 0.2\times 10^{-3}, \mathtt{g}_{3\infty} \approx 0.07$,

$$\begin{aligned} \Psi_1 &= \max\left\{\begin{aligned} &\left[\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y})\mathtt{p}_1(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{1\infty}\right]^{-1},\\ &\left[\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y})\mathtt{p}_2(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{2\infty}\right]^{-1},\\ &\left[\left(\frac{1}{4}\right)^{2\mathtt{q}-2}\varpi\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}}\varphi(\mathtt{y})\mathtt{p}_3(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{3\infty}\right]^{-1} \end{aligned}\right\}\\ &= \max\Big\{550.0030423, 550.0030423, 412.5022817\Big\}\\ &\approx 550.0030423, \end{aligned}$$

and

$$\begin{aligned} \Psi_2 &= \min\left\{\left[\varpi\int_1^e\varphi(\mathtt{y})\mathtt{p}_1(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{10}\right]^{-1}, \left[\varpi\int_1^e\varphi(\mathtt{y})\mathtt{p}_2(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{20}\right]^{-1}, \left[\varpi\int_1^e\varphi(\mathtt{y})\mathtt{p}_3(\mathtt{y})\frac{d\mathtt{y}}{\mathtt{y}}\mathtt{g}_{30}\right]^{-1}\right\}\\ &= \min\big\{1632.763452, 1632.763452, 6834.572496\big\}\\ &\approx 1632.763452. \end{aligned}$$

Then all the conditions of Theorem 3.1 are fulfilled. Therefore, by Theorem 3.1, we get an optimal eigenvalue interval $550.0030423 < \lambda_{\mathtt{k}} < 1632.763452$, for $\mathtt{k} = 1, 2, 3$ in which the FBVP (4.1)-(4.2) has at least one positive solution.

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