Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 12, No. 4, 2024, pp. 749-762
DOI:10.22034/cmde.2024.57112.2390

Solitary waves with two new nonlocal boussinesq types equations using a couple of integration schemes

Islam Samir[1], Ahmed H. Arnous[2], Ahmed M. Elsherbeny[1], Mohammad Mirzazadeh[3,*], Mir Sajjad Hashemi[4], and Mostafa Eslami[5,*]

[1]Department of Physics and Mathematics Engineering, Faculty of Engineering, Ain Shams University-11517, Cairo, Egypt.

[2]Ain Shams University-11517, Cairo, Egypt.

[3]Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, Rudsar-Vajargah, Iran.

[4]Department of Computer Engineering, Biruni University, 34010 Istanbul, Turkey.

[5]Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.

Abstract

The Boussinesq equation and its related types are able to provide a significant explanation for a variety of different physical processes that are relevant to plasma physics, ocean engineering, and fluid flow. Within the framework of shallow water waves, the aim of this research is to find solutions for solitary waves using newly developed nonlocal models of Boussinesq's equations. The extraction of bright and dark solitary wave solutions along with bright–dark hybrid solitary wave solutions is accomplished through the implementation of two integration algorithms. The general projective Riccati equations method and the enhanced Kudryashov technique are the ones that have been implemented as techniques. The enhanced Kudryashov method combines the benefits of both the original Kudryashov method and the newly developed Kudryashov method, which may generate bright, dark, and singular solitons. The Projective Riccati structure is determined by two functions that provide distinct types of hybrid solitons. The solutions get increasingly diverse as these functions are combined. The techniques that were applied are straightforward and efficient enough to provide an approximation of the solutions discovered in the research. Furthermore, these techniques can be utilized to solve various kinds of nonlinear partial differential equations in mathematical physics and engineering. In addition, plots of the selected solutions in three dimensions, two dimensions, and contour form are provided.

Keywords. Nonlocal, Boussinesq, Water waves, Kudryashov, Riccati.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Wave and nonlinear evolution equations, also known as NLEEs, include first- or second-order derivatives concerning the time variable are occupied a significant role in various scientific fields, including water waves, nonlinear optics, plasma physics, electronic devices, fluid mechanics, and so on [2, 20–26, 31, 33]. In recent years, there has been a surge in interest in these equations due to the complexity associated with finding a solution to NLPDEs. In recent years, various computational tools and direct algebraic approaches have gradually emerged due to the progression of scientific research and the ongoing development of computational computer software [1, 3–5, 7, 10, 12, 13, 15, 32]. Coastal engineers need to understand the nonlinear and shallow-water wave characteristics in order to assess the impact of shallow-water waves on nonlinear forms in coastal and ocean engineering. Because the world will need energy to do things in the future, people worldwide are studying how to design wave energy devices that can get energy from natural sources like waves. Aside from analyzing different environmental and design conditions, it is important to use different analytical methods to analyze nonlinear wave loads on floating structures. This enables us to gain insight

Received: 14 June 2023; Accepted: 27 March 2024.
∗ Corresponding author. Email: mostafa.eslami@umz.ac.ir, mirzazadehs2@guilan.ac.ir.

into the interactions between nonlinear waves and structures. In significant work, Boussinesq successfully derived an equation that describes the propagation of water waves within a rectangular channel [6]. The equation under consideration is an important feature in the field of water wave theory, specifically in the modeling of nonlinear dispersive long waves characterized by small amplitudes [11].

Boussinesq demonstrated that a delicate interaction between the steepening effect of the nonlinearity and the flattening effect of the dispersion preserves the shape of a wave. He derived the equation to model shallow water wave action read as

$$u_{tt} = \left(u - \alpha u^2 + \beta u_{xx}\right)_{xx}. \tag{1.1}$$

Eq. (1.2) holds significant importance not only in the field of fluid dynamics but also in various other physical phenomena. It has been observed to accurately describe nonlinear lattice waves in the continuum limit [30], the propagation of ion-sound waves in a uniform isotropic plasma [19], and the dynamics of the anharmonic lattice in the Fermi-Pasta-Ulam (FPU) problem. These diverse applications highlight the versatility and wide-ranging implications of Eq. (1.2) in different areas of physics.

Hence, various Boussinesq models have been created to examine the interaction between nonlinear waves and floating objects in various water depths. These models can be applied to marine operations and wave energy conversion [6, 8, 16–18]. It characterizes the motion of water with tiny amplitudes and long waves. Boussinesq-type equations combine dispersion, time dependence, and weak nonlinearity and have emerged as essential equations for predicting wave transformations in coastal locations. In coastal and ocean engineering, the Boussinesq equation is commonly employed. Some of the applications of this equation in ocean engineering are tsunami wave simulations and mathematical modeling of tidal swings. The equation can also be found in other domains and applications [9, 27–29]. Bang-Qing Li et al. [14] developed two new types of nonlocal Boussinesq equations as follows:

The first nonlocal Boussinesq type equation is considered as

$$\begin{cases} u_t = v_x, \\ v_t = \alpha u_x - v_x - (u^2)_x + \left(\frac{v^2 + {u_x}^2}{u - \frac{\alpha}{2}}\right)_x. \end{cases} \tag{1.2}$$

The second nonlocal Boussinesq type equation is considered as

$$\begin{cases} u_t = v_x, \\ v_t = \alpha u_x - v_x - u_{xxx} - (u^2)_x + \left(\frac{v^2 + uv + {u_x}^2}{u - \frac{\alpha}{2}}\right)_x. \end{cases} \tag{1.3}$$

In water waves, $u = u(x,t)$ and $v = v(x,t)$ represent two wave envelopes in two direction at different layers. The following sections are structured: In section 2, we will go over the core procedure of the enhanced Kudryashov's (EK) technique and the general projective Riccati equations (GPRES) algorithm. Section 3 and section 4 implement the two integration techniques to two nonlocal models of Boussinesqs equations. Finally, the conclusion of the present investigation is reported in section 4.

2. Methods Summary

Exact solutions of nonlinear differential equations refer to solutions that can be expressed in a closed-form mathematical expression, rather than as a numerical approximation. These solutions are often sought after as they provide deeper insight into the underlying physical or mathematical systems being described by the differential equation. There are various methods for finding exact solutions of nonlinear differential equations, including analytical techniques such as separation of variables, reduction of order, and substitution methods, as well as numerical methods like the shooting method and finite difference method. The choice of method depends on the specific form of the differential equation and the desired level of accuracy. Exact solutions can provide valuable information about the behavior of the system, including its stability, periodicity, and bifurcation points.

Let us now proceed to provide a detailed explanation of these methods.

2.1. **EK Procedure.** The EK technique is a method for solving nonlinear differential equations. The technique starts by transforming the nonlinear differential equation into a polynomial of unknown functions and their derivatives, along with unknown parameters. The polynomial is then simplified by collecting terms with the same power and equating them to zero. This results in an over-determined system of algebraic equations that can be solved using software such as Mathematica to determine the unknown parameters and functions. By balancing the highest-order derivatives and nonlinear terms in the original differential equation, the EK technique can determine the value of a positive integer parameter that controls the degree of the polynomial. The resulting solutions of the differential equation are then expressed in terms of elementary functions. The EK method combines the benefits of the original Kudryashov method, which can obtain dark and singular solitons, and the newly developed Kudryashov method, which can give bright and dark solitons. Our proposed techniques can generate both bright and dark solitons and singular solitons. The EK technique has been shown to be effective in solving a wide range of nonlinear differential equations and has been widely used in various fields of science and engineering.
Taking into consideration the following form of the NLEE.

$$G(y, y_x, y_t, y_{xt}, y_{xx}, ...) = 0. \tag{2.1}$$

The central procedure of the EK method involves a polynomial, G, that is defined based on the variable y, as well as independent variables of time and space. The unknown function, $y = y(x,t)$, is represented by y.
By employing the transformation:

$$y(x,t) = Y(\varpi),\ \varpi = \mu(x - \upsilon t), \tag{2.2}$$

where μ and υ are variables to be determined later, the Eq. (2.1) is thus transformed to the NLODE.

$$P(Y, -\mu\upsilon Y', \mu Y', \mu^2 Y'', ...) = 0. \tag{2.3}$$

Step–1: Considering that the solution to (2.3) may be stated as [3]

$$Y(\varpi) = \lambda_0 + \sum_{l=1}^{N} \sum_{i+j=l} \lambda_{ij} Q^i(\varpi) R^j(\varpi), \tag{2.4}$$

the constants $\lambda_0, \lambda_{ij}(i,j = 0,1,...,N)$ need to be found and the functions $Q(\varpi)$ and $R(\varpi)$ must satisfy the following ODEs:

$$Q'(\varpi) = Q(\varpi)(\eta Q(\varpi) - 1), \tag{2.5}$$

$$R'(\varpi)^2 = R(\varpi)^2(1 - \chi R(\varpi)^2). \tag{2.6}$$

The solutions to Equations (2.5) and (2.6) can be found by using these formulas:

$$Q(\varpi) = \frac{1}{\eta + be^{\varpi}}, \tag{2.7}$$

$$R(\varpi) = \frac{4a}{4a^2 e^{\varpi} + \chi e^{-\varpi}}, \tag{2.8}$$

where a, b, η and χ are arbitrary constants.
Step–2: To find the positive integer value N in Eq. (2.4), the highest order derivatives and the nonlinear term in Eq. (2.3) must be balanced.
Step–3: By substituting Equations (2.4), (2.6), and (2.5) into Eq. (2.3), a polynomial expression is obtained that involves the functions $R(\varpi)$, $R'(\varpi)$, and $Q(\varpi)$. By equating all terms of equal power in this polynomial to zero, an over-determined system of algebraic equations is created. This system can be solved using Mathematica to determine the unknown parameters $k, v, a, b, \eta, \chi, \lambda_0, \lambda_{ij}(i,j = 0,1,...,N)$. This results in the solutions of Eq. (2.1).

2.2. **GPREs Procedure.** Moreover, the GPREs technique is a method for solving nonlinear differential equations. It involves transforming the original nonlinear equation into a set of algebraic equations by using a projective change of variables. The transformed equation is then solved using Mathematica to determine the unknown parameters. The solutions obtained from the transformed equation are then used to find the solutions of the original nonlinear equation. The Projective Riccati structure is determined by two functions that provide distinct and unique types of hybrid solitons which give it m. The solutions get increasingly diverse as these functions are combined. The GPREs technique has been used to solve a wide range of nonlinear differential equations, including those in mathematical physics, engineering, and other fields, and has been found to be a reliable and effective method for finding solutions.

Step–1: Assuming That Eq. (2.1) has solution of the form [15]

$$Y(\varpi) = \alpha_0 + \sum_{i=1}^{N} \alpha_i \phi^i(\varpi) + \beta_i \phi^{i-1}(\varpi)\psi(\varpi), \tag{2.9}$$

the constants α_0, α_i, and β_i (for $i = 0, 1, ..., N$) need to be determined, and the functions $\psi(\varpi)$ and $\phi(\varpi)$ must fulfill the following system of ODEs:

$$\begin{aligned} \psi'(\varpi) &= \sigma + \varepsilon\psi^2(\varpi) - \delta\phi(\varpi), \\ \phi'(\varpi) &= \varepsilon\phi(\varpi)\psi(\varpi), \end{aligned} \tag{2.10}$$

where

$$\psi^2(\varpi) = -\varepsilon\left(\sigma - 2\delta\phi(\varpi) + \frac{\delta^2 + \vartheta}{\sigma}\phi^2(\varpi)\right), \tag{2.11}$$

In Eq. (2.10), σ is a positive constant and δ is an arbitrary constant. The solutions to this equation are presented below:

Case-1: $\varepsilon = \vartheta = -1$

$$\phi(\varpi) = \frac{\sigma sech\,[\sqrt{\sigma}\varpi]}{\delta sech\,[\sqrt{\sigma}\varpi] + 1}, \quad \psi(\varpi) = \frac{\sqrt{\sigma}\tanh[\sqrt{\sigma}\varpi]}{\delta sech\,[\sqrt{\sigma}\varpi] + 1}, \tag{2.12}$$

Case-2: $\varepsilon = -\vartheta = 1$

$$\phi(\varpi) = \frac{\sigma csch\,[\sqrt{\sigma}\varpi]}{\delta csch\,[\sqrt{\sigma}\varpi] + 1}, \quad \psi(\varpi) = \frac{\sqrt{\sigma}\coth[\sqrt{\sigma}\varpi]}{\delta csch\,[\sqrt{\sigma}\varpi] + 1}, \tag{2.13}$$

Case-3: $\varepsilon = -\vartheta = -1$

$$\phi(\varpi) = \frac{\sigma \sec[\sqrt{\sigma}\varpi]}{\delta \sec[\sqrt{\sigma}\varpi] + 1}, \quad \psi(\varpi) = \frac{\sqrt{\sigma}\tan[\sqrt{\sigma}\varpi]}{\delta \sec[\sqrt{\sigma}\varpi] + 1}, \tag{2.14}$$

Case-4: $\varepsilon = \vartheta = 1$

$$\phi(\varpi) = \frac{\sigma \csc[\sqrt{\sigma}\varpi]}{\delta \csc[\sqrt{\sigma}\varpi] + 1}, \quad \psi(\varpi) = -\frac{\sqrt{\sigma}\cot[\sqrt{\sigma}\varpi]}{\delta \csc[\sqrt{\sigma}\varpi] + 1}. \tag{2.15}$$

Step–2: Balance the highest order derivatives and the nonlinear term in Eq. (2.3) to get the positive integer value N in Eq. (2.9).

Step–3: By substituting Eqs. (2.9), (2.10), and (2.11) into Eq. (2.3), a polynomial expression is obtained that involves the functions $\phi(\varpi)$ and $\psi(\varpi)$. By equating all terms of equal power in this polynomial to zero, an over-determined system of algebraic equations is created. This system can be solved using Mathematica to determine the unknown parameters $k, v, \sigma, \delta, \alpha_0, \alpha_i$ and β_i (for $i = 0, 1, ..., N$). This results in the solutions of Eq. (2.1).

3. First Nonlocal Boussinesq's Type Equation

We start by introducing two transformations to Eq. (1.2) as follows:

$$u = \phi_x, \quad v = \phi_t. \tag{3.1}$$

$$\phi_{tt} = \alpha\phi_{xx} - \phi_{xt} - ((\phi_x)^2)_x + \left(\frac{{\phi_t}^2 + {\phi_{xx}}^2}{\phi_x - \frac{\alpha}{2}}\right)_x. \tag{3.2}$$

We then present a travelling wave transformation to (3.2) as

$$\phi(x,t) = U(\eta), \quad \eta = k(x - \nu t), \tag{3.3}$$

$$k\left(\frac{3\alpha}{2} + \nu\right)U'^2 - \frac{1}{2}\alpha(\alpha + (1-\nu)\nu)U' + k^3U''^2 - k^2U'^3 = 0. \tag{3.4}$$

Replace U' by V

$$k\left(\frac{3\alpha}{2} + \nu\right)V^2 - \frac{1}{2}\alpha(\alpha + (1-\nu)\nu)V + k^3V'^2 - k^2V^3 = 0. \tag{3.5}$$

3.1. Application of EK Procedure.

The balance between V'^2 and V^3 gives $N = 2$. As a results

$$V(\varpi) = \lambda_0 + \lambda_{01}R(\varpi) + \lambda_{10}Q(\varpi) + \lambda_{11}Q(\varpi)R(\varpi) + \lambda_{02}R(\varpi)^2 + \lambda_{20}Q(\varpi)^2. \tag{3.6}$$

By substituting Equations (3.6), (2.6), and (2.5) into Eq.(3.5), a polynomial expression is obtained that involves the functions $Q(\varpi), R(\varpi)$ and $R'(\varpi)$. The following results can be obtained by solving the over-determined system of algebraic equations generated by this polynomial using Mathematica:

Result (1):

$$\lambda_0 = \lambda_{01} = \lambda_{02} = 0,\ \lambda_{10} = \pm 2\eta\sqrt{2(3\alpha + 2\nu)},\ \lambda_{20} = -\eta\lambda_{10},\ k = \mp\sqrt{-\frac{1}{2}(3\alpha + 2\nu)}. \tag{3.7}$$

Plugging Eq. (3.7) along with Eq. (2.7) into Eq. (3.6) provides

$$u(x,t) = -2b\sqrt{-(3\alpha+2\nu)^2}\left\{\frac{\eta\ b\ \exp\left[\mp\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\eta + b\ \exp\left[\mp\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right)^2}\right\}, \tag{3.8}$$

$$v(x,t) = 2b\sqrt{-(3\alpha+2\nu)^2}\nu\left\{\frac{\eta\ b\ \exp\left[\mp\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\eta + b\ \exp\left[\mp\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right)^2}\right\}. \tag{3.9}$$

When $\eta = \pm\, b$, Eq. (3.8) and Eq. (3.9) give complex solutions

$$u(x,t) = -\frac{1}{2}\sqrt{-(3\alpha+2\nu)^2}\ sech^2\left[\sqrt{-\frac{1}{8}(3\alpha+2\nu)}(x-\nu t)\right], \tag{3.10}$$

$$v(x,t) = \frac{\nu}{2}\sqrt{-(3\alpha+2\nu)^2}\ sech^2\left[\sqrt{-\frac{1}{8}(3\alpha+2\nu)}(x-\nu t)\right], \tag{3.11}$$

and

$$u(x,t) = \frac{1}{2}\sqrt{-(3\alpha+2\nu)^2}\ csch^2\left[\sqrt{-\frac{1}{8}(3\alpha+2\nu)}(x-\nu t)\right], \tag{3.12}$$

$$v(x,t) = -\frac{\nu}{2}\sqrt{-(3\alpha+2\nu)^2}\ csch^2\left[\sqrt{-\frac{1}{8}(3\alpha+2\nu)}(x-\nu t)\right]. \tag{3.13}$$

Result (2):

$$\lambda_0 = \mp\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}},\ \lambda_{02} = \pm\chi\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}},$$
$$k = \mp\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}},\ \lambda_{01} = \lambda_{10} = \ \lambda_{20} = 0,. \tag{3.14}$$

Plugging Eq. (3.14) along with Eq. (2.8) into Eq. (3.6) provides

$$u(x,t) = \frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{4(2\alpha-1)}\times$$
$$\left\{1-\frac{16a^2\chi}{\left(4a^2\exp\left[\mp\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}}(x-\nu t)\right]+\chi\exp\left[\pm\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}}(x-\nu t)\right]\right)^2}\right\}, \tag{3.15}$$

$$v(x,t) = -\frac{(6\alpha^2-3\alpha+(4\alpha-2)\nu)\nu}{4(2\alpha-1)}\times$$
$$\left\{1-\frac{16a^2\chi}{\left(4a^2\exp\left[\mp\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}}(x-\nu t)\right]+\chi\exp\left[\pm\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}}(x-\nu t)\right]\right)^2}\right\}, \tag{3.16}$$

when $\chi = \pm 4a^2$, Eqs. (3.15) and (3.16) give solitary wave solutions with $(6\alpha^2-3\alpha+(4\alpha-2)\nu)(2\alpha-1) > 0$.

$$u(x,t) = \frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{4(2\alpha-1)}\ \tanh^2\left[\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}}(x-\nu t)\right], \tag{3.17}$$

$$v(x,t) = -\frac{(6\alpha^2-3\alpha+(4\alpha-2)\nu)\nu}{4(2\alpha-1)}\ \tanh^2\left[\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}}(x-\nu t)\right], \tag{3.18}$$

and singular soliton solutions

$$u(x,t) = \frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{4(2\alpha-1)}\ \coth^2\left[\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}}(x-\nu t)\right], \tag{3.19}$$

$$v(x,t) = -\frac{(6\alpha^2-3\alpha+(4\alpha-2)\nu)\nu}{4(2\alpha-1)}\ \coth^2\left[\frac{1}{4}\sqrt{\frac{6\alpha^2-3\alpha+(4\alpha-2)\nu}{2\alpha-1}}(x-\nu t)\right]. \tag{3.20}$$

3.2. **Application of GPREs Procedure.** The balance between V'^2 and V^3 in Eq. (3.5) yields $N=2$. As a result, we obtain:

$$V(\varpi) = \alpha_0 + \alpha_1\phi(\varpi) + \beta_1\psi(\varpi) + \alpha_2\phi^2(\varpi) + \beta_2\phi(\varpi)\psi(\varpi), \tag{3.21}$$

with the aid of Step-4 which is presented in the preceding section, the following results are obtained

<u>Case-1</u>: $\varepsilon = \vartheta = -1$

$$\alpha_0 = 0,\ \alpha_1 = -2\delta k,\ \alpha_2 = -\frac{4\left(\delta^2-1\right)k^3}{3\alpha+2\nu},\ \beta_1 = 0,\ \beta_2 = 2k^2\sqrt{-\frac{2\left(\delta^2-1\right)}{3\alpha+2\nu}},\ \sigma = -\frac{3\alpha+2\nu}{2k^2}. \tag{3.22}$$

So then, consider the following solutions:

$$u(x,t) = \left\{\delta + \operatorname{sech}\left[\sqrt{-\frac{1}{2}(2\nu+3\alpha)}(x-\nu t)\right] - \sqrt{\delta^2-1}\ \tanh\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right\}$$
$$\times\frac{(3\alpha+2\nu)\ \operatorname{sech}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\delta\ \operatorname{sech}\left[\sqrt{-\frac{1}{2}(2\nu+3\alpha)}(x-\nu t)\right]+1\right)^2}, \tag{3.23}$$

$$v(x,t) = \left\{\delta + \operatorname{sech}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right] - \sqrt{\delta^2-1}\ \tanh\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right\}$$
$$\times\frac{-\nu\ (3\alpha+2\nu)\ \operatorname{sech}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\delta\ \operatorname{sech}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+1\right)^2}. \tag{3.24}$$

<u>Case-2</u>: $\varepsilon = -\vartheta = 1$

$$\alpha_0 = 0,\ \alpha_1 = -2\delta k,\ \alpha_2 = -\frac{4\left(\delta^2+1\right)k^3}{3\alpha+2\nu},\ \beta_1 = 0,\ \beta_2 = 2k^2\sqrt{-\frac{2\left(\delta^2+1\right)}{3\alpha+2\nu}},\ \sigma = -\frac{3\alpha+2\nu}{2k^2}. \tag{3.25}$$

Then, the following solutions are obtained

$$u(x,t) = \left\{-\delta + \operatorname{csch}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right] + \sqrt{\delta^2+1}\ \coth\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right\}$$
$$\times\frac{-(3\alpha+2\nu)\ \operatorname{csch}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\delta\ \operatorname{csch}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+1\right)^2}, \tag{3.26}$$

$$v(x,t) = \left\{-\delta + \operatorname{csch}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right] + \sqrt{\delta^2+1}\ \coth\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right\}$$
$$\times\frac{\nu\ (3\alpha+2\nu)\ \operatorname{csch}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\delta\ \operatorname{csch}\left[\sqrt{-\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+1\right)^2}. \tag{3.27}$$

<u>Case-3</u>: $\varepsilon = -\vartheta = -1$

$$\alpha_0 = 0,\ \alpha_1 = 2\delta k,\ \alpha_2 = -\frac{4\left(\delta^2-1\right)k^3}{3\alpha+2\nu},\ \beta_1 = 0,\ \beta_2 = 2k^2\sqrt{-\frac{2\left(\delta^2-1\right)}{2\nu+3\alpha}},\ \sigma = \frac{3\alpha+2\nu}{2k^2}. \tag{3.28}$$

Then, the following solutions are obtained

$$u(x,t) = \left\{\delta + \sec\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right] + \sqrt{1-\delta^2}\ \tan\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right\}$$
$$\times\frac{(3\alpha+2\nu)\ \sec\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\delta\ \sec\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+1\right)^2}, \tag{3.29}$$

$$v(x,t)=\left\{\delta+\sec\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+\sqrt{1-\delta^2}\ \tan\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right\}$$
$$\times\frac{-\nu\ (3\alpha+2\nu)\ \sec\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\delta\ \sec\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+1\right)^2}. \tag{3.30}$$

Case-4: $\varepsilon=\vartheta=1$

$$\alpha_0=0,\ \alpha_1=2\delta k,\ \alpha_2=-\frac{4\left(\delta^2+1\right)k^3}{3\alpha+2\nu},\ \beta_1=0,\ \beta_2=-2k^2\sqrt{-\frac{2\left(\delta^2+1\right)}{3\alpha+2\nu}},\ \sigma=\frac{3\alpha+2\nu}{2k^2}. \tag{3.31}$$

Then, the following solutions are raised

$$u(x,t)=\left\{\delta-\csc\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+\sqrt{-\delta^2-1}\ \cot\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right\}$$
$$\times\frac{(3\alpha+2\nu)\ \csc\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\delta\ \csc\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+1\right)^2}, \tag{3.32}$$

$$v(x,t)=\left\{\delta-\csc\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+\sqrt{-\delta^2-1}\ \cot\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]\right\}$$
$$\times\frac{-\nu\ (3\alpha+2\nu)\ \csc\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]}{\left(\delta\ \csc\left[\sqrt{\frac{1}{2}(3\alpha+2\nu)}(x-\nu t)\right]+1\right)^2}. \tag{3.33}$$

4. Second Nonlocal Boussinesq's Type Equation

We begin by presenting two transformations to Eq. (1.3) as follows

$$u=\phi_x,\quad v=\phi_t, \tag{4.1}$$

$$\phi_{tt}=\alpha\phi_{xx}-\phi_{xt}-\phi_{xxxx}-((\phi_x)^2)_x+\left(\frac{{\phi_t}^2+\phi_x\phi_t+{\phi_{xx}}^2}{\phi_x-\frac{\alpha}{2}}\right)_x. \tag{4.2}$$

We then present a travelling wave transformation to (4.2) as

$$\phi(x,t)=U(\varpi),\quad \varpi=k(x-\nu t), \tag{4.3}$$

$$\frac{3k\alpha}{2}U'^2-k^3U'U'''-\frac{1}{2}\alpha(\alpha+(1-\nu)\nu)U'+\frac{\alpha k^2}{2}U'''+k^3U''^2-k^2U'^3=0, \tag{4.4}$$

Replace U' by V

$$\frac{3k\alpha}{2}V^2-k^3VV''-\frac{1}{2}\alpha(\alpha+(1-\nu)\nu)V+\frac{\alpha k^2}{2}V''+k^3V'^2-k^2V^3=0. \tag{4.5}$$

4.1. **Application of EK Procedure.** The balance between VV'' and V^3 in Eq. (4.5) gives $N = 2$. Consequently, we reach

$$V(\varpi) = \lambda_0 + \lambda_{01}R(\varpi) + \lambda_{10}Q(\varpi) + \lambda_{11}Q(\varpi)R(\varpi) + \lambda_{02}R(\varpi)^2 + \lambda_{20}Q(\varpi)^2, \tag{4.6}$$

By inserting Eq. (4.6) into Eq. (4.5), along with Equations (2.6) and (2.5), a polynomial is obtained that consists of $Q(\varpi)$, $R(\varpi)$, and $R'(\varpi)$. An over-determined system of algebraic equations is generated by this polynomial, which can be solved using Mathematica to obtain the following results:

$$\lambda_0 = 0,\ \lambda_{02} = \pm\chi\sqrt{\alpha-\nu^2+\nu},\ k = \pm\frac{1}{2}\sqrt{\alpha-\nu^2+\nu},\ \lambda_{01} = \lambda_{10} = \ \lambda_{20} = \lambda_{11} = 0. \tag{4.7}$$

Plugging Eq. (4.7) along with Eq. (2.8) into Eq. (4.6) provides

$$u(x,t) = \frac{\alpha-\nu^2+\nu}{2}\left\{\frac{16a^2\chi}{\left(4a^2\exp\left[\pm\frac{1}{2}\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]+\chi\exp\left[\mp\frac{1}{2}\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]\right)^2}\right\}, \tag{4.8}$$

$$v(x,t) = -\frac{(\alpha-\nu^2+\nu)\nu}{2}\left\{\frac{16a^2\chi}{\left(4a^2\exp\left[\pm\frac{1}{2}\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]+\chi\exp\left[\mp\frac{1}{2}\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]\right)^2}\right\}. \tag{4.9}$$

When $\chi = \pm 4a^2$, Eq. (4.8) and Eq. (4.9) give solitary wave solutions for $\alpha-\nu^2+\nu > 0$

$$u(x,t) = \frac{\alpha-\nu^2+\nu}{2}\ sech^2\left[\frac{1}{2}\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right], \tag{4.10}$$

$$v(x,t) = -\frac{(\alpha-\nu^2+\nu)\nu}{2}\ sech^2\left[\frac{1}{2}\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right], \tag{4.11}$$

and singular soliton solutions for $\alpha-\nu^2+\nu > 0$

$$u(x,t) = \frac{\alpha-\nu^2+\nu}{2}\ csch^2\left[\frac{1}{2}\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right], \tag{4.12}$$

$$v(x,t) = -\frac{(\alpha-\nu^2+\nu)\nu}{2}\ csch^2\left[\frac{1}{2}\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]. \tag{4.13}$$

4.2. **Application of GPRES Procedure.** Balancing V'^2 with V^3 in Eq. (4.5) gives $N = 2$. Consequently, we reach

$$V(\varpi) = \alpha_0 + \alpha_1\phi(\varpi) + \beta_1\psi(\varpi) + \alpha_2\phi^2(\varpi) + \beta_2\phi(\varpi)\psi(\varpi), \tag{4.14}$$

With the aid of Step-4 which is presented in the preceding section, the following results are obtained

<u>Case-1</u>: For $\varepsilon = -1$ and $\vartheta = -1$

$$\alpha_0 = 0,\ \alpha_1 = \delta k,\ \alpha_2 = \frac{\left(\delta^2-1\right)k^3}{-\alpha+\nu^2-\nu},\ \beta_1 = 0,\ \beta_2 = k^2\sqrt{\frac{1-\delta^2}{-\alpha+\nu^2-\nu}},\ \sigma = \frac{\alpha-\nu^2+\nu}{k^2}. \tag{4.15}$$

Then, we have the following solutions

$$u(x,t) = \left\{\delta + \operatorname{sech}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right] + \sqrt{\delta^2-1}\ \tanh\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]\right\} \times \frac{-(-\alpha+\nu^2-\nu)\ \operatorname{sech}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]}{\left(\delta\ \operatorname{sech}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]+1\right)^2}, \tag{4.16}$$

$$v(x,t) = \left\{\delta + \operatorname{sech}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right] + \sqrt{\delta^2-1}\ \tanh\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]\right\} \times \frac{\nu(-\alpha+\nu^2-\nu)\ \operatorname{sech}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]}{\left(\delta\ \operatorname{sech}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]+1\right)^2}, \tag{4.17}$$

Case-2: For $\varepsilon = -1$ and $\vartheta = 1$

$$\alpha_0 = 0,\ \alpha_1 = \delta k,\ \alpha_2 = \frac{\left(\delta^2+1\right)k^3}{-\alpha+\nu^2-\nu},\ \beta_1 = 0,\ \beta_2 = -k^2\sqrt{\frac{-1-\delta^2}{-\alpha+\nu^2-\nu}},\ \sigma = \frac{\alpha-\nu^2+\nu}{k^2}. \tag{4.18}$$

Then, the following solutions are obtained

$$u(x,t) = \left\{-\delta + \operatorname{csch}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right] + \sqrt{\delta^2+1}\ \coth\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]\right\} \times \frac{(-\alpha+\nu^2-\nu)\ \operatorname{csch}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]}{\left(\delta\ \operatorname{csch}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]+1\right)^2}, \tag{4.19}$$

$$v(x,t) = \left\{-\delta + \operatorname{csch}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right] + \sqrt{\delta^2+1}\ \coth\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]\right\} \times \frac{-\nu(-\alpha+\nu^2-\nu)\ \operatorname{csch}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]}{\left(\delta\ \operatorname{csch}\left[\sqrt{\alpha-\nu^2+\nu}(x-\nu t)\right]+1\right)^2}. \tag{4.20}$$

Case-3: For $\varepsilon = 1$ and $\vartheta = -1$

$$\alpha_0 = 0,\ \alpha_1 = -\delta k,\ \alpha_2 = \frac{\left(\delta^2-1\right)k^3}{-\alpha+\nu^2-\nu},\ \beta_1 = 0,\ \beta_2 = k^2\sqrt{\frac{1-\delta^2}{-\alpha+\nu^2-\nu}},\ \sigma = -\frac{\alpha-\nu^2+\nu}{k^2}. \tag{4.21}$$

Then, the following solutions are obtained

$$u(x,t) = \left\{-\delta - \sec\left[\sqrt{-\alpha+\nu^2-\nu}(x-\nu t)\right] + \sqrt{1-\delta^2}\ \tan\left[\sqrt{-\alpha+\nu^2-\nu}(x-\nu t)\right]\right\} \times \frac{(-\alpha+\nu^2-\nu)\ \sec\left[\sqrt{-\alpha+\nu^2-\nu}(x-\nu t)\right]}{\left(\delta\ \sec\left[\sqrt{-\alpha+\nu^2-\nu}(x-\nu t)\right]+1\right)^2}, \tag{4.22}$$

$$v(x,t) = \left\{-\delta - \sec\left[\sqrt{-\alpha+\nu^2-\nu}(x-\nu t)\right] + \sqrt{1-\delta^2}\ \tan\left[\sqrt{-\alpha+\nu^2-\nu}(x-\nu t)\right]\right\} \times \frac{-\nu(-\alpha+\nu^2-\nu)\ \sec\left[\sqrt{-\alpha+\nu^2-\nu}(x-\nu t)\right]}{\left(\delta\ \sec\left[\sqrt{-\alpha+\nu^2-\nu}(x-\nu t)\right]+1\right)^2}. \tag{4.23}$$

Case-4: For $\varepsilon = 1$ and $\vartheta = 1$

$$\alpha_0 = 0,\ \alpha_1 = -\delta k,\ \alpha_2 = \frac{\left(\delta^2 - 1\right) k^3}{-\alpha + \nu^2 - \nu},\ \beta_1 = 0,\ \beta_2 = k^2 \sqrt{\frac{1 - \delta^2}{-\alpha + \nu^2 - \nu}},\ \sigma = -\frac{\alpha - \nu^2 + \nu}{k^2}. \tag{4.24}$$

Then, the following solutions are raised

$$u(x,t) = \left\{ -\delta + \csc\left[\sqrt{-\alpha + \nu^2 - \nu}(x - \nu t)\right] + \sqrt{-\delta^2 - 1}\ \cot\left[\sqrt{-\alpha + \nu^2 - \nu}(x - \nu t)\right] \right\}$$
$$\times \frac{(-\alpha + \nu^2 - \nu)\ \csc\left[\sqrt{-\alpha + \nu^2 - \nu}(x - \nu t)\right]}{\left(\delta\ \csc\left[\sqrt{-\alpha + \nu^2 - \nu}(x - \nu t)\right] + 1\right)^2}, \tag{4.25}$$

$$v(x,t) = \left\{ -\delta + \csc\left[\sqrt{-\alpha + \nu^2 - \nu}(x - \nu t)\right] + \sqrt{-\delta^2 - 1}\ \cot\left[\sqrt{-\alpha + \nu^2 - \nu}(x - \nu t)\right] \right\}$$
$$\times \frac{-\nu(-\alpha + \nu^2 - \nu)\ \csc\left[\sqrt{-\alpha + \nu^2 - \nu}(x - \nu t)\right]}{\left(\delta\ \csc\left[\sqrt{-\alpha + \nu^2 - \nu}(x - \nu t)\right] + 1\right)^2}. \tag{4.26}$$

5. Physical Interpretation

In the realm of understanding solitons in real life, an understanding of their dynamics is of greatest significance. A crucial aspect in this process is the analysis of graphs depicting specific solitons, as they hold a significant role in the overall interpretation. This study focuses on the soliton vector u and its corresponding counterpart v, as demonstrated by Equations (5)(6). Significantly, u is defined as dark, while v is categorized as bright, with both vectors displaying equal directions of movement. This study aims to examine the interconnected properties of mutual limitations seen in water wave solitons across various strata. Through analyzing the interaction between these solitons, our objective is to elucidate their intricate dynamics and comprehend the fundamental mechanisms that dictate their behavior. We conduct thorough analysis and experiments to investigate the consequences of these mutual limitations, offering vital insights into the characteristics of water wave solitons and their possible uses in other industries. Figure 1 displays the wave profile for solution (3.17) and (3.18). The 3D graphs (a) and (c) depict the structural properties of $u(x,t)$ and $v(x,t)$, respectively, while the contours (b) and (d) aid in characterizing their propagation. Figure 2 displays the wave profile for solution (4.10) and (4.11). The 3D graphs (a) and (c) depict the structural properties of $u(x,t)$ and $v(x,t)$, respectively, while the contours (b) and (d) aid in characterizing their propagation. The water soliton layers show conjugate features of mutual constraints, as depicted in the illustration of the included figures.

6. Conclusions

This study paper discussed the investigation of solitary waves, dark solitons, and bright-dark hybrid solitary waves within the framework of water waves. The study addresses two nonlocal Boussinesq's type equations that were proposed in 2022 [14]. The study seeks to expand our comprehension of the dynamics and behavior of water waves by conducting a comprehensive investigation of these equations. This will provide insights into the complex nature of solitary wave events. Overall, this research study has effectively utilized two integration strategies, specifically the augmented Kudryashov's methodology and the general projective Riccati equations algorithm, to achieve the intended outcomes. These strategies have demonstrated efficacy in retrieving the required data for the investigation.

The water soliton layers show conjugate features of mutual constraints, as depicted in the illustration of the included figures. The findings of the study are highly novel, suggesting significant potential for future research in this field. In the future, some aspects will be considered, such as the integrability and multiple soliton solutions. Also, some effects on nonlinear partial differential equations recently studied may be considered for this model. Also, the two implemented methods will be employed for many nonlinear evolution equations that model some nonlinear phenomena.

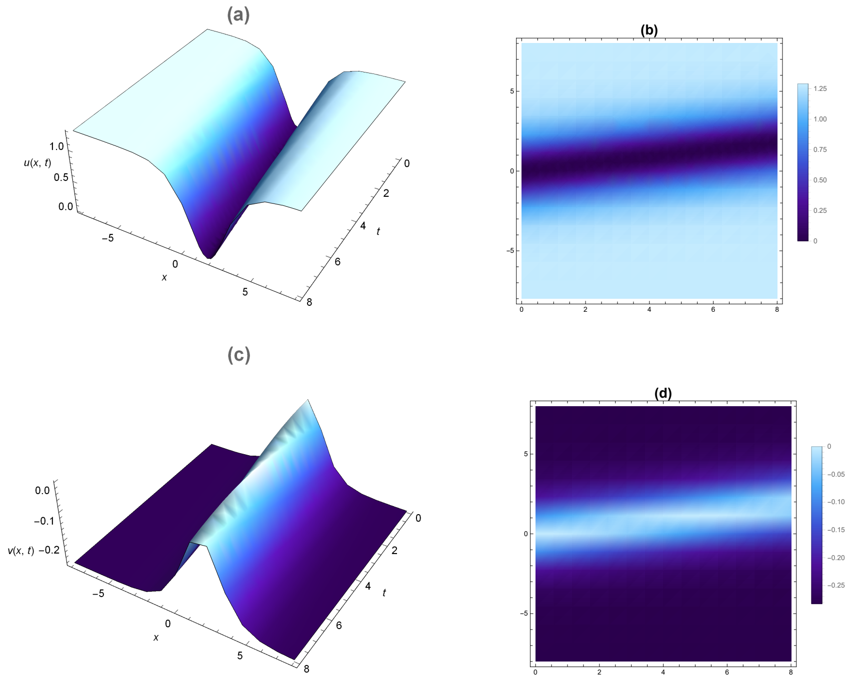

FIGURE 1. Wave profile for solution (3.17) and (3.18).

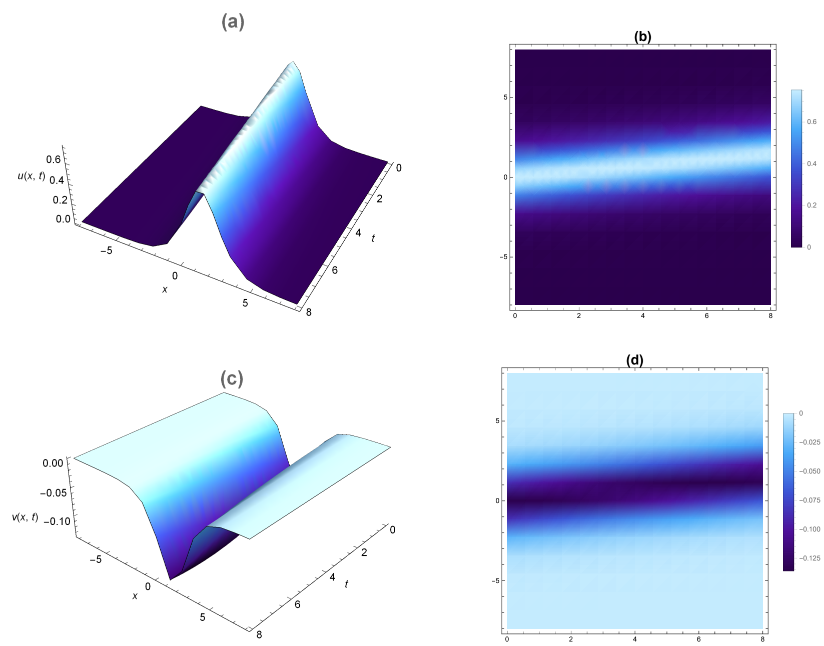

FIGURE 2. Wave profile for solution (4.10) and (4.11).

Declarations

Ethical Approval: Not applicable.

Competing interests: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors' contributions: All authors contributed equally to this work.

Funding: No funding received for this paper.

Availability of data and materials: No applicable for this paper.

References

[1] G. Akram, M. Sadaf, S. Arshed, and Fizza Sameen, *Bright, dark, kink, singular and periodic soliton solutions of LakshmananPorsezianDaniel model by generalized projective Riccati equations method*, Optik, *241* (2021), 167051.

[2] A. H. Arnous, M. Ekici, S. P. Moshokoa, M. Zaka Ullah, A. Biswas, and M. Belic. *Solitons in Nonlinear Directional Couplers with Optical Metamaterials by Trial Function Scheme.* Acta Physica Polonica A, *132*(4) (2017), 13991410.

[3] A. H. Arnous, *Optical solitons with Biswas–Milovic equation in magneto-optic waveguide having Kudryashov's law of refractive index*, Optik, *247* (2021), 167987.

[4] A. H. Arnous, Q. Zhou, A. Biswas, P. Guggilla, S. Khan, Y. Yıldırım, A. S. Alshomrani, and H. M. Alshehri, *Optical solitons in fiber Bragg gratings with cubic-quartic dispersive reectivity by enhanced Kudryashov's approach*, Physics Letters A, *422* (2022), 127797.

[5] A. H. Arnous, A. Biswas, Y. Yıldırım, Q. Zhou, W. Liu, A. S. Alshomrani, and H. M. Alshehri, *Cubic– quartic optical soliton perturbation with complex Ginzburg–Landau equation by the enhanced Kudryashov's method*, Chaos, Solitons & Fractals, *155* (2022), 111748.

[6] J. V. Bolussinesq, *Theorie des ondes et des remous qui se propagent le long dun canal rectangulaire horizontal, en communiquant au liquide contene dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pures Appl, *17*(9) (1872), 55-108.

[7] Y. M. Chu, M. Inc, M. S. Hashemi, and S. Eshaghi, *Analytical treatment of regularized Prabhakar fractional differential equations by invariant subspaces*, Computational and Applied Mathematics, *41*(6) (2022), 271.

[8] M. T. Darvishi, M. Najafi, and A. M. Wazwaz, *Soliton solutions for Boussinesq-like equations with spatio-temporal dispersion*, Ocean Eng, *130* (2017) 228240.

[9] M. M. A. El-Sheikh, A. R. Seadawy, H. M. Ahmed, A. H. Arnous, and W. B. Rabie, *Dispersive and propagation of shallow water waves as a higher order nonlinear Boussinesq-like dynamical wave equations*, Physica A: Statistical Mechanics and its Applications, *537* (2020), 122662.

[10] M. S. Hashemi, *A novel approach to find exact solutions of fractional evolution equations with non-singular kernel derivative*, Chaos, Solitons & Fractals, *152* (2021), 111367.

[11] R. S. Johnson, *A modern introduction to the mathematical theory of water waves. Cambridge Texts in Applied Mathematics*, Cambridge University Press, Cambridge, 1997.

[12] N. A. Kudryashov, *One method for finding exact solutions of nonlinear differential equations*, Communications in Nonlinear Science and Numerical Simulation, *17*(6) (2012), 2248-2253.

[13] N. A. Kudryashov, *Method for finding highly dispersive optical solitons of nonlinear differential equations*, Optik, *206* (2020), 163550.

[14] B-Q. Li, A. M. Wazwaz, and Y. L. Ma, *Two new types of nonlocal Boussinesq equations in water waves: Bright and dark soliton solutions*, Chinese Journal of Physics, *77* (2022), 1782-1788.

[15] B. Li and Y. Chen, *Nonlinear Partial Differential Equations Solved by Projective Riccati Equations Ansatz*, Z. Naturforsch, *58*(a) (2003), 511-519.

[16] P. A. Madsen, H. B. Bingham, H. Liu, *A new Boussinesq method for fully nonlinear waves from shallow to deep water*, J. Fluid Mech., *462* (2002), 130.

[17] W. X. Ma and J. S. He, *A second Wronskian formulation of the Boussinesq equation*, Nonlinear Anal, *70* (2009), 42454258.

[18] Y. L. Ma and B-Q. Li, *Bifurcation solitons and breathers for the nonlocal Boussinesq equations*, Appl. Math. Lett, *124* (2022), 107677.
[19] V. G. Makhankov, *Dynamics of classical solitons (in nonintegrable systems)*, Phys. Rep., *35* (1978), 1128.
[20] T. Mathanaranjan, *An effective technique for the conformable space-time fractional cubic-quartic nonlinear Schrodinger equation with different laws of nonlinearity*, Computational Methods for Differential Equations, *10*(3) (2022), 701–715.
[21] T. Mathanaranjan, Dipankar Kumar, Hadi Rezazadeh, and Lanre Akinyemi, *Optical solitons in metamaterials with third and fourth order dispersions*, Optical and Quantum Electronics, *54*(5) (2022), 271.
[22] T. Mathanaranjan and Dayalini Vijayakumar, *New soliton solutions in nano-fibers with space-time fractional derivatives*, Fractals, *30*(7) (2022), 2250141.
[23] T. Mathanaranjan, *Optical solitons and stability analysis for the new (3+ 1)-dimensional nonlinear Schrödinger equation*, Journal of Nonlinear Optical Physics & Materials, *32*(2) (2023), 2350016.
[24] T. Mathanaranjan, *New Jacobi elliptic solutions and other solutions in optical metamaterials having higher-order dispersion and its stability analysis*, International Journal of Applied and Computational Mathematics, *9*(5) (2023), 66.
[25] T. Mathanaranjan, M. S. Hashemi, H. Rezazadeh, L. Akinyemi, and A. Bekir, *Chirped optical solitons and stability analysis of the nonlinear Schrödinger equation with nonlinear chromatic dispersion*, Communications in Theoretical Physics, *75*(8) (2023), 085005.
[26] T. Mathanaranjan, *Optical soliton, linear stability analysis and conservation laws via multipliers to the integrable Kuralay equation*, Optik, *290* (2023), 171266.
[27] S. C. Mohapatra and C. Guedes Soares, *Comparing solutions of the coupled Boussinesq equations in shallow water*, In Maritime Technology and Engineering; Guedes Soares, C., Santos, T.A., Eds.; Taylor & Francis Group: London, UK, (2015), 947954,
[28] S. C. Mohapatra, R. B. Fonseca, and C. Guedes Soares, *A comparison between analytical and numerical simulations of solutions of the coupled Boussinesq equations*, In Maritime Technology and Engineering; Guedes Soares, C., Santos, T.A., Eds.; Taylor & Francis Group: London, UK, (2016), 11751180.
[29] S. C. Mohapatra, R. B. Fonseca, and C. Guedes Soares, *Comparison of analytical and numerical simulations of long nonlinear internal waves in shallow water*, J. Coast. Res., *34* (2018), 928938.
[30] M. Toda, *Studies of a non-linear lattice*, Phys. Rep, *18*(C) (1975), 1123.
[31] J. Vega-Guzman, M. F. Mahmood, Qin Zhou, Houria Triki, Ahmed H. Arnous, Anjan Biswas, Seithuti P. Moshokoa, and Milivoj Belic. *Solitons in Nonlinear Directional Couplers with Optical Metamaterials*, Nonlinear Dynamics, *87*(1) (2016), 42758.
[32] N. M. Yamurlu and A. S. Karaka, *A novel perspective for simulations of the MEW equation by trigonometric cubic B-spline collocation method based on Rubin-Graves type linearization*, Computational Methods for Differential Equations, *10*(4) (2022), 1046-1058.
[33] Yun-Hui Zhao, T. Mathanaranjan, Hadi Rezazadeh, Lanre Akinyemi, and Mustafa Inc, *New solitary wave solutions and stability analysis for the generalized (3+ 1)-dimensional nonlinear wave equation in liquid with gas bubbles*, Results in Physics, *43* (2022), 106083.