A new perspective for the Quintic B-spline collocation method via the Lie-Trotter splitting algorithm to solitary wave solutions of the GEW equation

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Abstract

A hybrid method utilizing the collocation technique with B-splines and Lie-Trotter splitting algorithm applied for 3 model problems which include a single solitary wave, two solitary wave interaction, and a Maxwellian initial condition is designed for getting the approximate solutions for the generalized equal width (GEW) equation. Initially, the considered problem has been split into 2 sub-equations as linear $U_t = \hat{A}(U)$ and nonlinear $U_t = \hat{B}(U)$ in the terms of time. After, numerical schemes have been constructed for these sub-equations utilizing the finite element method (FEM) together with quintic B-splines. Lie-Trotter splitting technique $\hat{A}o\hat{B}$ has been used to generate approximate solutions of the main equation. The stability analysis of acquired numerical schemes has been examined by the Von-Neumann method. Also, the error norms L_2 and L_∞ with mass, energy, and momentum conservation constants I_1, I_2, and I_3, respectively are calculated to illustrate how perfect solutions this new algorithm applied to the problem generates and the ones produced are compared with those in the literature. These new results exhibit that the algorithm presented in this paper is more accurate and successful, and easily applicable to other non-linear partial differential equations (PDEs) as the present equation.

1. INTRODUCTION

The current study plans to explore the (GEW) equation in the following form with a strategy of obtaining approximate solutions superior according to other studies

$$U_t - \mu U_{xxt} + \epsilon U^p U_x = 0, \tag{1.1}$$

in which U stands for the amplitude of the surface of the water, ϵ and μ are non-negative parameters also p is a non-negative integer. Physical boundary conditions need to be $U \to 0$ when $|x| \to \infty$ and here the initial-boundary conditions for Eq. (1.1) with acceptable physically reasonable constraints are given as below:

$$U(x_L, t) = U(x_R, t) = 0, \tag{1.2}$$
$$U_x(x_L, t) = U_x(x_R, t) = 0,$$
$$U(x, 0) = g(x), \quad x_L \le x \le x_R.$$

It should also be noted that U is the negative of the electrostatic potential in plasma applications. Thus, the solitary wave results given in Eq. (1.1) help find many physical phenomena that are nonlinear and have weak dispersion waves. Also, it includes dispersion waves, examples of which are ion-acoustic and magneto-hydrodynamic waves in plasma, phonon packets in nonlinear crystals, and nonlinear transverse waves in shallow water [29]. One wants to figure out the physical mechanism of phenomena in nature defined by non-linear PDEs. Therefore, the analytical solution for non-linear PDEs needs to be investigated. Especially, Traveling wave solutions of nonlinear evolution equations (NLEEs), a special class of PDEs, play a significant role in the work of PDE models commonly seen in

natural events, in applied sciences. For instance, the wave events observed in sciences including physics, chemistry, and biology. The best-known models of such phenomena in nature, which include traveling waves, can be given as examples the nonlinear Korteweg-de Vries (KdV) equation introduced by Benjamin et al. [8], regularized long wave (RLW) equation presented by Peregrine [42] and equal width (EW) equation utilized by Morrison et al. [40], etc. The (GEW) equation based on the EW equation is relevant to the generalized regularized long wave (GRLW) equation and the generalized Korteweg-de Vries (GKdV) equation. These equations that have solitary solutions, pulse-like Raslan [44], are non-linear wave equations with $(p+1)$th nonlinearity. The EW equation, which is a one-dimensional PDE likened by the RLW and KdV equations [3], is obtained for $p = 1$ in Eq. (1.1) and Eq. (1.1) for $p = 2$ submits the MEW equation used in modeling non-linear events. One can look at the references [12, 14, 18, 19, 35, 47, 50] and [4, 15, 16, 22, 33, 36, 45, 51] for studies on these equation. Moreover, the reader can refer to works in references [2, 5–7, 10, 11, 13, 21, 28, 34, 37, 39] and [23, 25, 38]. Such equations mentioned above, such as the GEW equation have been extensively analyzed and many important computer algorithms are coded to investigate such models. In this study, the Matlab 2019b computer program with a memory 20GB and 64 bit has been used.

When a wide literature review is made, it has not been found as many studies for the GEW equation as for the other nonlinear (PDE) equations. The analytical solitary wave results of the generalized EW and EW-Burges equation were acquired by Hamdi et al. [24]. The GEW equation was worked collocation method with the quadratic and cubic B-spline, respectively by Evans and Raslan [17] and Raslan [44]. Roshan [46] investigated with Petrov-Galerkin (FEM) utilizing a linear hat one as the test function and a quadratic B-splines as the trial function. [28, 30–32] and [52] obtained approximate results of the GEW equation utilizing Galerkin, collocation, subdomain, Petrov-Galerkin's (FEMs) with the help of B-splines, and also [9] disputed the existence and uniqueness of the problem. This mentioned equation has been solved approximately with the usage of meshless method together with global collocation with normal types of radial basis functions (RBFs) by Panahipour [41]. Abbaszadeh et al. [1] used the local meshless collocation method. Taghizadeh et al. [49] constructed with the homogeneous balance method for the exact traveling wave solutions of the GEW equation. İnan and Bahadır [27] submitted a fully implicit finite difference method.

This study is organized part by part as follows: First of all, a comprehensive literature review is made. Then, a brief information for the splitting algorithm is submitted. To start the integration of the GEWE, the main equation is split to have the couple equations then these are applied to the Lie-Trotter splitting algorithm utilizing the collocation method with quintic B-splines. Then these partial differential equations are discretized in time so that the coupled system of ordinary differential equations are obtained in terms of parameters described for the approximation of the unknown of the (PDEs) using quintic B-splines. Then fully integration of the GEWE is managed using the Crank-Nicolson time integrator. It has been reached coupled recursive algebraic system and mentioned in the manuscript that the unknown variables of the algebraic system is found by using Lie Trotter splitting algorithm. To measure the sturdiness and effectiveness of the technique used in the present study, 3 numerical test problems are taken and the solutions obtained from these problems are compared with those in other methods. Finally, a brief conclusion is given about the benefits of the algorithm attempted for the study and the data it produces.

2. The Splitting Method

The simplest splitting method, which degrades the solution of the Cauchy problem to the sequential solution of two sub-problems, is the Lie-Trotter Splitting method. Let us now consider the Cauchy problem given below

$$U'(t) = \Lambda U(t), \quad U(0) = U_0, t \geq 0, \tag{2.1}$$

in which Λ operator can be discretized such that $\Lambda = \hat{A} + \hat{B}$ as in [48] and (2.1) turns into the following form

$$U'(t) = (\hat{A} + \hat{B})U(t), \quad U(0) = U_0, t \geq 0.$$

Here, $U_0 \in X$ is a vector which is found to be utilizing the initial condition, $u(x, t)$ is solution vector, $\Lambda, \hat{A}$, and $\hat{B}$ are operators in X Banach space. In the Lie-Trotter schemes, the first sub-problem with operator $\hat{A}$ is solved utilizing the original initial condition given with the problem. Then, the solutions obtained with the operator $\hat{A}$ are used as the initial condition for the solution of the second sub-problem given with the operator $\hat{B}$ and given as the solution of

the main problem in the first time step. Thus, approximate results at the next time levels are obtained similarly to those at the initial time level. The algorithm of the Lie-Trotter method, with $t_0 = 0$ and $t_N = T$,

$$\frac{dU^*(t)}{dt} = \hat{A}U^*(t), \quad U^*(t_n) = U_n^0 \quad , \quad t \in [t_n, t_{n+1}],$$

$$\frac{dU^{**}(t)}{dt} = \hat{B}U^{**}(t), \quad U^{**}(t_n) = U^*(t_{n+1}) \quad , \quad t \in [t_n, t_{n+1}].$$

Here, U_n^0 is the original initial condition given in (2.1), Δt is the time step, $\Delta t = t_{n+1} - t_n, n = 0, 1, ..., N - 1$. Thus, the required results are found with $U(t_{n+1}) = u^{**}(t_{n+1})$. This approach $(\hat{A} - \hat{B})$ is called as the splitting one. The formal solution (2.1) can be given as follows

$$U(t_{n+1}) = e^{\Lambda \Delta t} U(t_n) = e^{(\hat{A}+\hat{B})\Delta t} U(t_n). \tag{2.2}$$

Taylor series expansion for this solution may be specified as

$$U(t_{n+1}) = e^{\Delta t(\hat{A}+\hat{B})} U(t_n) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\hat{A}(u(t))\frac{\partial}{\partial U} + \hat{B}(u(t))\frac{\partial}{\partial U} \right)^k U(t_n).$$

If the sum of operators $\hat{A}$ and $\hat{B}$ is calculated instead of Λ, a new approach to Equation (2.2) is obtained as follows

$$U(t_{n+1}) = e^{\hat{A}\Delta t} e^{\hat{B}\Delta t} U(t_n). \tag{2.3}$$

Using Eq. (2.3) instead of (2.2) produces an error. This is a local splitting error and it is given as follows

$$\begin{aligned} Te &= \frac{1}{\Delta t}(e^{\Delta t(\hat{A}+\hat{B})} - e^{\Delta t \hat{B}} e^{\Delta t \hat{A}})U(t_n) \\ &= \frac{1}{\Delta t}[\frac{\Delta t^2}{2}(\hat{A}\hat{B} - \hat{B}\hat{A})U(t_n) + O(\Delta t^3)] \\ &= \frac{1}{\Delta t}[\hat{A}, \hat{B})U(t_n) + O(\Delta t^2)], \end{aligned}$$

here, $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is first order approximation as a commutator of two non-commuting operators. Thus, the approximation (2.3) is a first order approximation unless the operators $\hat{A}$ and $\hat{B}$ are commutative. If the operators are commutative, there is no splitting error and the method results in a precise solution [26].

It can be noted that a new splitting technique can be obtained as $(\hat{B} - \hat{A})$ if the $\hat{A}$ and $\hat{B}$ operators are swapped, and the solutions are acquired by applying similar operations.

3. Implementation to the constructed numerical schemes of the method

First of all, the solution domain has been split into finite elements to apply the numerical approach to the equation and it is limited to the closed interval $[x_L, x_R]$. This interval is divided into finite elements uniformly of length h by the nodal x_j such that $x_L = x_0 \leq x_1 \leq ... \leq x_N = x_R, h = x_{j+1} - x_j = \dfrac{x_R - x_L}{N}$ for $j = 0(1)N - 1$. The set of quintic B-splines $\varphi_{-2}(x), \varphi_{-1}(x), \ldots, \varphi_{N+2}(x)$ composed a base at the nodes x_j for $j = -2(1)N + 2$ on the solution region $[x_L, x_R]$ may be stated as in [43]

$$\varphi_j(x) = \frac{1}{h^5} \begin{cases} p_0 = (x - x_{j-3})^5, & x \in [x_{j-3}, x_{j-2}], \\ p_1 = p_0 - 6(x - x_{j-2})^5, & x \in [x_{j-2}, x_{j-1}], \\ p_2 = p_1 - 6(x - x_{j-2})^5 + 15(x - x_{j-1})^5, & x \in [x_{j-1}, x_j], \\ p_3 = p_2 - 6(x - x_{j-2})^5 - 20(x - x_j)^5, & x \in [x_j, x_{j+1}], \\ p_4 = p_3 - 6(x - x_{j-2})^5 + 15(x - x_{j+1})^5, & x \in [x_{j+1}, x_{j+2}] \\ p_5 = p_4 - 6(x - x_{j-2})^5 - 6(x - x_{j+2})^5, & x \in [x_{j+2}, x_{mj3}], \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

The approximate solution $U_N(x,t)$ can be represented in terms of the quintic B-splines as follows

$$U_N(x,t) = \sum_{j=-2}^{N+2} \varphi_j(x)\delta_j(t), \tag{3.2}$$

in which the unknown time parameters $\delta_j(t)$ are found out using the boundary conditions and collocation requirements. Quintic B-splines (3.1) on $[0,h]$ in terms of ζ on a typical element $[x_j, x_{j+1}]$ by the local coordinate transformation defined as $h = x - x_j$ for $0 \leq \zeta \leq h$ can be presented in the form given below

$$\begin{aligned}
\varphi_{j-2} &= 1 - 5\zeta + 10\zeta^2 - 10\zeta^3 + 5\zeta^4 - \zeta^5, \\
\varphi_{j-1} &= 26 - 50\zeta + 20\zeta^2 + 20\zeta^3 - 20\zeta^4 + 5\zeta^5, \\
\varphi_j &= 66 - 60\zeta^2 + 30\zeta^4 - 10\zeta^5, \\
\varphi_{j+1} &= 26 + 50\zeta + 20\zeta^2 - 20\zeta^3 - 20\zeta^4 + 10\zeta^5, \\
\varphi_{j+2} &= 1 + 5\zeta + 10\zeta^2 + 10\zeta^3 + 5\zeta^4 - 5\zeta^5, \\
\varphi_{j+3} &= \zeta^5.
\end{aligned} \tag{3.3}$$

All of the quintic basis are null outside of $\phi_{j-2}, \varphi_{j-1}, \varphi_j, \varphi_{j+1}, \varphi_{j+2}$ and φ_{j+3}. The nodal values U_j, U_j', U_j'' utilizing Eqs. (3.2) and (3.3) are submitted in terms of the parameter $\delta_j(t)$ by

$$\begin{aligned}
U_j &= \delta_{m-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2}, \\
U_j' &= \frac{5}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}), \\
U_j'' &= \frac{20}{h^2}(\delta_{j-2} + 2\delta_{j-1} - 6\delta_j + 2\delta)_{j+1} + \delta_{j+2}),
\end{aligned} \tag{3.4}$$

and the variation of U with the interval $[x_j, x_{j+1}]$ is given as follows

$$U = \sum_{j=-2}^{N+2} \varphi_j \delta_j. \tag{3.5}$$

It is time to split the GEW equation as given below

$$U_t - \mu U_{xxt} = 0, \tag{3.6}$$

$$U_t - \mu U_{xxt} + \epsilon U^p U_x = 0. \tag{3.7}$$

Due to the implementation of the collocation method in the aforementioned work, and the collocation points with the knots are defined and the Eqs. (3.6) and (3.7) are used to appraise U_j and its position derivatives given in (3.4), and they are substituted in the Eqs. (3.6) and (3.7). Thus, the following ordinary differential equation systems are acquired.

$$\dot{\delta}_{j-2} + 26\dot{\delta}_{j-1} + 66\dot{\delta}_j + 26\dot{\delta}_{j+1} + \dot{\delta}_{j+2} - \frac{20\mu}{h^2}(\dot{\delta}_{j-2} + 2\dot{\delta}_{j-1} - 6\dot{\delta}_j + 2\dot{\delta}_{j+1} + \dot{\delta}_{j+2}) = 0, \tag{3.8}$$

$$\begin{aligned}
\dot{\delta}_{j-2} + 26\dot{\delta}_{j-1} + 66\dot{\delta}_j + 26\dot{\delta}_{j+1} + \dot{\delta}_{j+2} - \frac{20\mu}{h^2}(\dot{\delta}_{j-2} + 2\dot{\delta}_{j-1} - 6\dot{\delta}_j + 2\dot{\delta}_{j+1} + \dot{\delta}_{j+2}) \\
+ \frac{5z_j}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}) = 0,
\end{aligned} \tag{3.9}$$

in which first order derivative according to time t is presented with symbol $''.''$ and z_j is taken as

$$z_j = \epsilon(\delta_{j-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2})^p,$$

for linearization operation. When $\dfrac{\delta_j^{n+1} + \delta_j^n}{2}$ instead of the parameter δ_j and $\dfrac{\delta_j^{n+1} - \delta_j^n}{\Delta t}$ instead of the parameter $\dot{\delta}_j$ in Eqs. (3.8) and (3.9) are written, equation systems submitted in the following are acquired

$$k_1\delta_{m-2}^{n+1} + k_2\delta_{m-1}^{n+1} + k_3\delta_m^{n+1} + k_4\delta_{m+1}^{n+1} + k_5\delta_{m+2}^{n+1} = k_5\delta_{m-2}^n + k_4\delta_{m-1}^n + k_3\delta_m^n + k_2\delta_{m+1}^n + k_1\delta_{m+2}^n, \tag{3.10}$$

$$l_1\delta_{m-2}^{n+1} + l_2\delta_{m-1}^{n+1} + l_3\delta_m^{n+1} + l_4\delta_{m+1}^{n+1} + l_5\delta_{m+2}^{n+1} = l_5\delta_{m-2}^n + l_4\delta_{m-1}^n + l_3\delta_m^n + l_2\delta_{m+1}^n + l_1\delta_{m+2}^n, \tag{3.11}$$

in which k_i, $l_i (i = 1(1)5)$, and z_j are $z_j = \epsilon U^p$

$$k_1 = 1 - \frac{20\mu}{h^2}, \quad k_2 = 26 - \frac{40\mu}{h^2}, \quad k_3 = 66 + \frac{120\mu}{h^2}, \quad k_4 = 26 - \frac{40\mu}{h^2}, \quad k_5 = 1 - \frac{20\mu}{h^2},$$

$$l_1 = 1 - \frac{20\mu}{h^2} - \frac{5z_j\Delta t}{2h}, \quad l_2 = 26 - \frac{40\mu}{h^2} - \frac{25z_j\Delta t}{h}, \quad l_3 = 66 + \frac{120\mu}{h^2},$$

$$l_4 = 26 - \frac{40\mu}{h^2} + \frac{25z_j\Delta t}{h}, \quad l_5 = 1 - \frac{20\mu}{h^2} + \frac{5z_j\Delta t}{2h}.$$

The systems submitted in (3.10) and (3.11) which have the generalized rows occur from unknown time-parameters δ_j in $(N + 5)$ quantity and equations $(N + 1)$. Since the virtual parameters δ_{-2}, δ_{-1}, δ_{N+1}, and δ_{N+2} in each system are not in the solution region and they are annihilated first to obtain unique solution of each system. Because of this, U and $U^{'}$ in Eq. (3.4) and the boundary conditions $U(x_L, t) = U(x_R, t) = 0$ and $U_x(x_L, t) = U_x(x_R, t) = 0$ must be used. Therefore, matrix system $(N + 1)$ x $(N + 1)$ for systems (3.10) and (3.11) are acquired. The closed form of these two matrix systems can be written as

$$A_1\delta^{n+1} = A_1^T\delta^n,$$

$$B_1\lambda^{n+1} = B_1^T\lambda^n,$$

with unknown time-dependent parameters $\delta^T = [\delta_0\delta_1...\delta_N]$ and $\lambda^T = [\lambda_0\lambda_1...\lambda_N]$ that need to be found and the coefficient matrices A_1 and B_1.

$$A_1 = \begin{bmatrix} \bar{k}_3 & \bar{k}_4 & \bar{k}_5 & & & & & & \\ \bar{k}_2 & \bar{k}_3 & \bar{k}_4 & k_5 & & & & & \\ k_1 & k_2 & k_3 & k_4 & k_5 & & & & \\ & & & & \ddots & & & & \\ & & & & & k_1 & k_2 & k_3 & k_4 & k_5 \\ & & & 1 & & & k_1 & \bar{\bar{k}}_2 & \bar{\bar{k}}_3 & \bar{\bar{k}}_4 \\ & & & & & & & \bar{\bar{k}}_1 & \bar{\bar{k}}_2 & \bar{\bar{k}}_3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \bar{l}_3 & \bar{l}_4 & \bar{l}_5 & & & & & & \\ \bar{l}_2 & \bar{l}_3 & \bar{l}_4 & l_5 & & & & & \\ l_1 & l_2 & l_3 & l_4 & l_5 & & & & \\ & & & & \ddots & & & & \\ & & & & & l_1 & l_2 & l_3 & l_4 & l_5 \\ & & & 1 & & & l_1 & \bar{\bar{l}}_2 & \bar{\bar{l}}_3 & \bar{\bar{l}}_4 \\ & & & & & & & \bar{\bar{l}}_1 & \bar{\bar{l}}_2 & \bar{\bar{l}}_3 \end{bmatrix},$$

$$\bar{k}_3 = \frac{165}{4}k_1 - \frac{33}{8}k_2 + k_3, \quad \bar{k}_4 = \frac{65}{2}k_1 - \frac{9}{4}k_2 + k_4, \quad \bar{k}_5 = \frac{9}{4}k_1 - \frac{1}{8}k_2 + k_5,$$

$$\bar{k}_2 = -\frac{33}{8}k_1 + k_2, \quad \bar{k}_3 = -\frac{9}{4}k_1 + k_3, \quad \bar{k}_4 = -\frac{1}{8}k_1 + k_4,$$

$$\bar{\bar{k}}_2 = -\frac{1}{8}k_5 + k_2, \quad \bar{\bar{k}}_3 = -\frac{9}{4}k_5 + k_3, \quad \bar{\bar{k}}_4 = -\frac{33}{8}k_5 + k_4,$$

$$\bar{\bar{k}}_1 = \frac{9}{4}k_5 - \frac{1}{8}k_4 + k_1, \quad \bar{\bar{k}}_2 = \frac{65}{2}k_5 - \frac{9}{4}k_4 + k_2, \quad \bar{\bar{k}}_3 = \frac{165}{4}k_5 - \frac{33}{8}k_4 + k_3,$$

$$\bar{l}_3 = \frac{165}{4}l_1 - \frac{33}{8}l_2 + l_3, \quad \bar{l}_4 = \frac{65}{2}l_1 - \frac{9}{4}l_2 + l_4, \quad \bar{l}_5 = \frac{9}{4}l_1 - \frac{1}{8}l_2 + l_5,$$

$$\bar{l}_2 = -\frac{33}{8}l_1 + l_2, \quad \bar{l}_3 = -\frac{9}{4}l_1 + l_3, \quad \bar{l}_4 = -\frac{1}{8}l_1 + l_4,$$

$$\bar{\bar{l}}_2 = -\frac{1}{8}l_5 + l_2, \quad \bar{\bar{l}}_3 = -\frac{9}{4}l_5 + l_3, \quad \bar{\bar{l}}_4 = -\frac{33}{8}l_5 + l_4,$$

$$\bar{\bar{l}}_1 = \frac{9}{4}l_5 - \frac{1}{8}l_4 + l_1, \quad \bar{\bar{l}}_2 = \frac{65}{2}l_5 - \frac{9}{4}l_4 + l_2, \quad \bar{\bar{l}}_3 = \frac{165}{4}l_5 - \frac{33}{8}l_4 + l_3.$$

Solutions of these systems are realized by Lie-Trotter splitting technique. The nonlinear term z_j in Eq. (3.11) are applied 3 or 5 times inner iteration given by the formula $(\delta^*)^n = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ throughout run of the computer code to provide that the numerical solutions are pretty close to the analytical ones.

4. THE INITIAL VECTOR δ_j^0

In order to start the solution process of the systems (3.10) and (3.11), the initial vector δ_j^0 must be determined. This vector will be calculated using the initial condition $U(x_j, 0) = U_N(x_j, 0) = g_0(x_j), j = 0(1)N$ and first and second order derivatives at the boundaries given in the main problem. At time $t = t_0 = 0$, approximation (3.2) can be rewritten as follows, with δ_j^0 being the parameters to be determined

$$U_N(x, 0) = \sum_{j=-2}^{N+2} \varphi_j(x)\delta_j^0(t), \tag{4.1}$$

and the following algebraic equations are obtained

$$U_m = \delta_{j-2}^0 + 26\delta_{j-1}^0 + 66\delta_j^0 + 26\delta_{j+1}^0 + \delta_{j+2}^0, j = 0(1)N,$$
$$U_0 = \delta_{-2}^0 + 26\delta_{-1}^0 + 66\delta_0^0 + 26\delta_1^0 + \delta_2^0,$$
$$U_1 = \delta_{-1}^0 + 26\delta_0^0 + 66\delta_1^0 + 26\delta_2^0 + \delta_3^0,$$
$$\vdots$$
$$\tag{4.2}$$
$$U_{N-1} = \delta_{N-3}^0 + 26\delta_{N-2}^0 + 66\delta_{N-1}^0 + 26\delta_N^0 + \delta_{N+1}^0,$$
$$U_N = \delta_{N-2}^0 + 26\delta_{N-1}^0 + 66\delta_N^0 + 26\delta_{N+1}^0 + \delta_{N+2}^0.$$

As it can be seen, the above system is the matrix system in the form of $(N + 1) \times (N + 5)$. For linear independent solutions of this system, the number of unknowns and equations have to be equalized. In order to this, the boundary

conditions $U_x(x_L, t) = U_x(x_R, t) = 0$ and $U_{xx}(x_L, t) = U_{xx}(x_R, t) = 0$ are utilized as below

$$\delta^0_{m-2} + 26\delta^0_{m-1} + 66\delta^0_m + 26\delta^0_{m+1} + \delta^0_{m+2} = g_0(x_m), m = 0(1)N$$

$$-\delta^0_{-2} - 10\delta^0_{-1} + 10\delta^0_1 + \delta^0_2 = g'_0(x_L),$$

$$\delta^0_{-2} + 2\delta^0_{-1} - 6\delta^0_0 + 2\delta^0_1 + \delta^0_2 = g''_0(x_L), \tag{4.3}$$

$$\delta^0_{N-2} + 2\delta^0_{N-1} - 6\delta^0_N + 2\delta^0_{N+1} + \delta^0_{N+2} = g''_0(x_R),$$

$$-\delta^0_{N-2} - 10\delta^0_{N-1} + 10\delta^0_{N+1} + \delta^0_{N+2} = g'_0(x_R).$$

If the equalities in the system (4.2) are used, $(N+1) \times (N+1)$ dimensional matrix equation that may be happened with variant of Thomas algorithm for the initial vector δ^0 is gained as follows

$$\begin{bmatrix} 54 & 60 & 6 & & & & & \\ 25.25 & 67.5 & 26.25 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & \\ & & & \ddots & & & & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & 1 & 26.25 & 67.5 & 25.25 \\ & & & & 6 & 60 & 54 \end{bmatrix} \begin{bmatrix} \delta^0_0 \\ \delta^0_1 \\ \delta^0_2 \\ . \\ . \\ . \\ \delta^0_{N-2} \\ \delta^0_{N-1} \\ \delta^0_N \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ . \\ . \\ . \\ U_{N-2} \\ U_{N-1} \\ U_N \end{bmatrix}.$$

Calculation of such matrices with existing symbolic programming languages is quite easy and practical. These characteristics of the presented schemes reflect their reliable and robust properties.

5. STABILITY ANALYSIS

The stability analysis of the two numerical schemes (3.10) and (3.11) obtained by splitting the GEW equation is investigated by the Von Neumann method by applying the quintic B-spline collocation method. Firstly, the Fourier modes, $\delta^n_j = \varrho^n_1 e^{ij\gamma h}$ in approximation (3.10) and $\Psi^n_j = \varrho^n_2 e^{ij\gamma h}$ in approximation (3.11), are substituted. Here, where γ is the mode number and h is element size. For the system (3.11), the moment the ϵU^p in the term $\epsilon U^p U_x$ is linearized, z_j be going to act as a local constant and thus the von Neumann method becomes applicable for the stability of the system (3.11). Using the Euler formula $e^{i\Phi} = cos\Phi + isin\Phi$, growth factors ϱ_1 and ϱ_2 presented as follows are obtained

$$\varrho_1 = \frac{A_1 - iB_1}{A_1 + iB_1}, \quad \varrho_2 = \frac{A_1 - iC_1}{A_1 + iC_1}, \tag{5.1}$$

and for the expressions $k_1, k_2, ..., k_9, k_{10}$ and $l_1, l_2, ..., l_9, l_{10}$ in section 3

$$A_1 = 2a_1 - \frac{40\mu}{h^2}a_2, \quad B_1 = 0, \quad C_1 = \frac{5z_j\Delta t}{h}a_3,$$

$$a_1 = cos2\gamma h + 26cos\gamma h + 33, \quad a_2 = cos2\gamma h + 2cos\gamma h - 3, \quad a_3 = sin2\gamma h + 10sin\gamma h,$$

$|\varrho_1| = |\varrho_2| = 1$ from Eq. (5.1) and hence $|\varrho_1|.|\varrho_2| = 1$. Because the conditions $|\varrho_1| \leq 1$, and $|\varrho_2| \leq 1$ are satisfied, it can be clearly expressed that the systems (3.10) and (3.11) with Lie Trotter-Splitting technique are unconditionally stable.

6. NUMERICAL EXPERIMENTS AND DISCUSSION

The GEW equation, which is considered only with homogeneous boundary conditions are investigated and error norms are calculated for them

$$L_2 = ||U - U_N||_2 = \sqrt{h \sum_{j=0}^{N}(U - U_N)^2},$$

and

$$L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|.$$

The exact solution of the GEW equation with the initial-boundary value problem is given as in [52]

$$U(x,t) = \left(\frac{c(p+1)(p+2)}{2\epsilon} sech^2\left[\frac{p}{2\sqrt{\mu}}(x - ct - x_0)\right]\right)^{1/p},$$

This solution is a solitary wave with initially x_0 centered, amplitude $\dfrac{c(p+1)(p+2)}{2\epsilon}$, wave velocity c, and width $\dfrac{p}{2\sqrt{\mu}}$.
The preservation quantities of the GEW equation submitted as mass, momentum and energy are presented as follows [17, 44, 52]

$$I_1 = \int_{x_L}^{x_R} U(x,t)dx,$$

$$I_2 = \int_{x_L}^{x_R} [U^2(x,t) + \mu U_x^2(x,t)]dx,$$

$$I_3 = \int_{x_L}^{x_R} [U^{(p+2)}(x,t)]dx.$$

After calculating the motion of solitary waves, the values I_1, I_2 and I_3 are examined to evaluate the performance of the approximate algorithm used for the calculation.

6.1. **Example I: A single solitary wave movement.** This section will deal with the solutions produced by applying the numerical schemes presented in the study, in tables with five parameter sets for various values p, c and $amplitude = \left(\dfrac{c(p+1)(p+2)}{2\epsilon}\right)^{1/p}$. Throughout the study, the parameters $h = 0.1, \Delta = 0.2, \epsilon = 3, \mu = 1, x_0 = 30$ over the solution area $[0, 80]$ are selected for all of these five sets as in the studies in the tables. All numerical calculations are made from 0 to 20 time levels in five time increments.

Implementation 1.1 In the first implementation, $c = 1/32$ and $p = 2$ are taken and hence the solitary wave amplitude is 0.25. The error norms L_2 and L_∞, the three invariant values I_1, I_2, I_3 are computed using the current algorithm and the results produced are shown in Table 1. This table shows that all invariants are fixed and the error norm results are as small as desired.

Implementation 1.2 In the second implementation, $c = 1/2$ and $p = 2$ are taken and hence the solitary wave amplitude is 1. The error norms L_2 and L_∞, the three invariant values I_1, I_2, I_3 are calculated using the current algorithm and the results produced are exhibited in Table 2. It can be clearly viewed from Table 2 that the invariants I_1, I_2 are almost the same, I_3 constant and the error norms are quite small.

Implementation 1.3 Taking the parameters $c = 0.001, p = 3$, the solitary wave amplitude becomes 0.15. The error norms L_2 and L_∞, the three invariant values I_1, I_2, I_3 are calculated using the current algorithm and the results generated are demonstrated in Table 3. It can be clearly observed from this table that the invariants are perfectly preserved and the error norms are small as intended.

Implementation 1.4 If the parameters $c = 0.3, p = 3$ are selected, the solitary wave amplitude will be 1. The error norms L_2 and L_∞, the three invariant values I_1, I_2, I_3 are computed using the present algorithm. The newly obtained results generated have been displayed in Table 4 and this table shows that the the invariants I_1, I_2 are almost the same, I_3 constant and the error norm results are fit for purpose.

Implementation 1.5 For values of $p = 4$ and $c = 1$, the solitary wave amplitude becomes 1. The error norms L_2 and L_∞, the three invariant values I_1, I_2, I_3 are computed via the present algorithm and he results generated are displayed in Table 5. One may easily observe from this table that as it is hoped, the invariants I_1, I_2 are almost the same, I_3 constant and the error norm values are quite small.

Implementation 1.6 In this application, the values of the error norms and invariants for numerical results studied by Galerkin, Collocation , Petrov- Galerkin finite element methods by [17, 28, 31, 44, 46, 52] with the help of different

TABLE 1. The invariants and error norms of the single solitary wave with $h = 0.1, \Delta t = 0.2$, $\epsilon = 3, \mu = 1$ on the region $[0, 80]$ with the amplitudes 0.25 for $p = 2$.

method		time	I_1	I_2	I_3	$L_2 x 10^5$	$L_\infty x 10^5$
Present	$p = 2$	0	0.78539816	0.16666667	0.00520833	0.000000	0.000000
		5	0.78539816	0.16666667	0.00520833	0.030210	0.023478
		10	0.78539816	0.16666667	0.00520833	0.060360	0.048785
		15	0.78539816	0.16666667	0.00520833	0.090395	0.075056
		20	0.78539816	0.16666667	0.00520833	0.120270	0.101365

TABLE 2. The invariants and error norms of the single solitary wave with $h = 0.1, \Delta t = 0.2$, $\epsilon = 3, \mu = 1$ on the region $[0, 80]$ with the amplitudes 1 for $p = 2$.

method		time	I_1	I_2	I_3	L_2	L_∞
Present	$p = 2$	0	3.14159265	2.66666666	1.33333333	0.00000000	0.00000000
		5	3.14156224	2.66662603	1.33333333	0.00438756	0.00289477
		10	3.14153186	2.66658546	1.33333333	0.00855926	0.00540459
		15	3.14150149	2.66654489	1.33333333	0.01269506	0.00792951
		20	3.14147112	2.66650433	1.33333333	0.01685700	0.01047330

TABLE 3. The invariants and error norms of the single solitary wave with $h = 0.1, \Delta t = 0.2$, $\epsilon = 3, \mu = 1$ on the region $[0, 80]$ with the amplitudes 0.15 for $p = 3$.

method		time	I_1	I_2	I_3	$L_2 x 10^7$	$L_\infty x 10^7$
Present	$p = 3$	0	0.4189163	0.0549808	0.0000733	0.00000	0.00000
		5	0.4189163	0.0549808	0.0000733	0.04111	0.05222
		10	0.4189163	0.0549808	0.0000733	0.08222	0.01051
		15	0.4189163	0.0549808	0.0000733	0.01233	0.01585
		20	0.4189163	0.0549808	0.0000733	0.01644	0.02126

B-splines are compared with those of the algorithm presented in the present study at $t = 20$ and are displayed in Table 6. For $p = 2, 3$ and 4, it may be observed that the error norms produced by the numerical schemes in the study are extremely small than all studies in the table, as desired. It has been found that the newly found invariant values are in good harmony with those of the previous ones.

Implementation 1.7 The solitary amplitude wave becomes 1 with the parameters c=1/2 for p=2, c=0.3 for p=3, and c=1 for p=4. Table 7 reports the values of invariants and error norms for various time levels. for single solitary wave this last table submits a comparison of the values of the error norms and invariants with those previously worked with the Galerkin, Collocation methods. From this table, besides expressing that the three conservation properties remain almost the same as time progresses and are consistent with their peers, it can be said that their norms are satisfactorily quite small. Figure 1 plots a simulation of a single solitary wave at times $t = 0, 10, 20$. with $c = 0.3$ for values of $p = 3, 4$. This figure shows that as the solitary wave moves to the right, its velocity hardly changes and also one may clearly visualize that the wave maintains its amplitude and shape as time progresses.

TABLE 4. The invariants and also error norms of the single solitary wave with $\Delta t = 0.2, h = 0.1,$ $\mu = 1, \epsilon = 3$ on the region $[0, 80]$ with the amplitudes 1 for $p = 3$.

method		time	I_1	I_2	I_3	L_2	L_∞
Present	$p = 3$	0	2.80436421	2.46391383	0.98556555	0.00000000	0.00000000
		5	2.80433498	2.46387067	0.98556555	0.00205514	0.00145533
		10	2.80430576	2.46382753	0.98556555	0.00409713	0.00278287
		15	2.80427655	2.46378441	0.98556555	0.00614498	0.00413034
		20	2.80424735	2.46374129	0.98556555	0.00821984	0.00550017

TABLE 5. The invariants and also error norms of the single solitary wave with $\Delta t = 0.2, h = 0.1,$ $\mu = 1, \epsilon = 3$ on the region $[0, 80]$ with the amplitudes 1 for $p = 4$.

method		time	I_1	I_2	I_3	L_2	L_∞
Present	$p = 4$	0	2.62205755	2.35619437	0.78539816	0.00000000000	0.00000000000
		5	2.62203468	2.35615854	0.78539816	0.00107282684	0.00079859296
		10	2.62201182	2.35612272	0.78539816	0.00216193142	0.00154443834
		15	2.62198896	2.35608691	0.78539816	0.00326704600	0.00230677859
		20	2.62196610	2.35605110	0.78539816	0.00439458001	0.00308793394

(a)   (b)
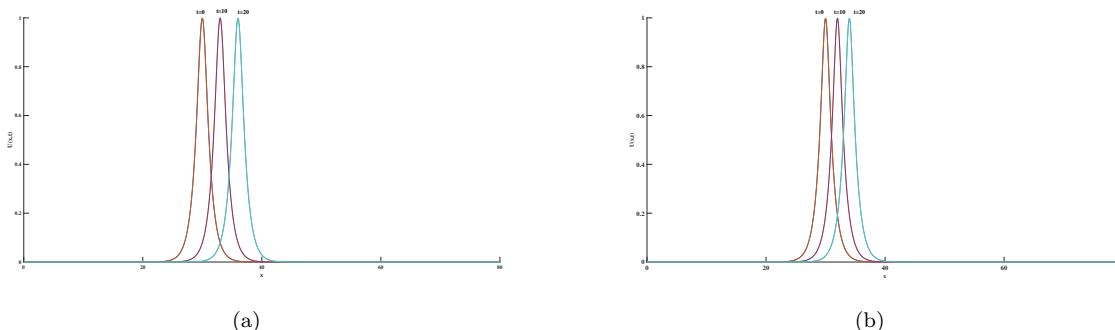
FIGURE 1. Graph for $c = 0.3, x_0 = 30, [0, 80]$ with $p = 3, 4$ at $t = 0, 10, 20$ of single solitary wave.

6.2. **Example II: Two solitary wave Interactions.** This section is going to examine two solitary wave interactions with the parameter values $\epsilon = 3, \mu = 1, \Delta t = 0.025, h = 0.1$ over the region $[0, 80]$ throughout the study. For this, the following initial condition is used

$$U(x, 0) = \sum_{i=1}^{2} \left(\frac{c_i(p+1)(p+2)}{2\epsilon} sech^2 \left[\frac{p}{2\sqrt{\mu}} (x - x_i) \right] \right)^{1/p},$$

The condition given at the initial time above models the propagation of two waves which have different amplitudes, one at x_1 and the other at x_2 in which x_i and c_i (i=1,2) are arbitrary positive constants. Here, three parameter sets in tables are considered for various values p, c_i and these are presented in three applications.

Implementation 1.1 In the first application, invariants for $p = 2$ with those of the parameters $c_1 = 0.5, c_2 = 0.125$ are calculated and the amounts found are compared with the Petrov Galerkin method [46]. Numerical calculations are made from 0 to 20 time levels in five time increments. The compared results are reported in Table 8. One may obviously see that they do not change at all during the simulation and are very compatible with the reference being

TABLE 6. The invariants and error norms of the single solitary wave with $\Delta t = 0.2, h = 0.1, \mu = 1, \epsilon = 3$ on the region $[0, 80]$ with the amplitudes $0.25, 0.15$ and 1 for $p = 2, 3$ and 4.

method		time	I_1	I_2	I_3	$L_2 x 10^3$	$L_\infty x 10^3$
Present	$p = 2$	0	0.78539816	0.16666667	0.00520833	0.00000000	0.00000000
		5	0.78539816	0.16666667	0.00520833	0.00030210	0.00023478
		10	0.78539816	0.16666667	0.00520833	0.00060360	0.00048785
		15	0.78539816	0.16666667	0.00520833	0.00090395	0.00075056
		20	0.78539816	0.16666667	0.00520833	0.00120270	0.00101365
[17]		20	0.78528640	0.16658180	0.00520600	0.15695390	0.20214760
[28]		20	0.78539680	0.16666630	0.00520830	0.07833789	0.04448503
[44]		20	0.78466760	0.16643400	0.00519380	0.19588780	0.17443300
[46]		20	0.78539800	0.16666900	0.00520829	0.00250172	0.00275164
[31]		20	0.78539650	0.16666630	0.00520830	0.00127758	0.00068872
[52]		20	0.78539	0.16666	0.00520	0.00135	0.00091
Present	$p = 3$	0	0.4189163	0.0549808	0.0000733	0.00000000	0.00000000
		5	0.4189163	0.0549808	0.0000733	0.00000411	0.00000522
		10	0.4189163	0.0549808	0.0000733	0.0000082	0.00000105
		15	0.4189163	0.0549808	0.0000733	0.00000123	0.00000158
		20	0.4189163	0.0549808	0.0000733	0.00000164	0.00000212
[17]		20	–	–	–	–	–
[28]		20	0.41891540	0.05498050	0.00007330	0.00282488	0.00183291
[44]		20	0.65908330	0.05938137	0.00006871	0.51496770	0.32060590
[46]		20	0.41891600	0.05497830	0.00007330	0.00006407	0.00008206
[31]		20	0.41891540	0.05498070	0.00007330	0.00000633	0.00000345
[52]		20	0.41891	0.05498	0.000073	0.000019	0.000023
Present	$p = 4$	0	2.62205755	2.35619437	0.78539816	0.00000000	0.00000000
		5	2.62203468	2.35615854	0.78539816	1.07282684	0.79859296
		10	2.62201182	2.35612272	0.78539816	2.16193142	1.54443834
		15	2.62198896	2.35608691	0.78539816	3.26704600	2.30677859
		20	2.62196610	2.35605110	0.78539816	4.39458001	3.08793394
[17]		20	–	–	0.00520600	–	–
[28]		20	2.63278330	2.37300320	0.8023383	8.90617000	8.21991000
[44]		20	–	–	–	–	–
[46]		20	2.62206000	2.35615000	0.78534400	2.30499000	1.88285000
[31]		20	2.62192110	2.35590930	0.78511300	3.41485000	2.49360000
[52]		20	2.62192	2.35593	0.78513	3.39086	2.47031

compared.

Implementation 1.2 In the second implementation, invariants for $p = 3$ with the parameters $c_1 = 0.3, c_2 = 0.0325$ are computed and Numerical ones are made from 0 to 100 time levels in ten time increments. Calculated values are compared with the Petrov Galerkin method [46] in Table 9. As requested, it can be seen to be in agreement with the reference presented in the table. Figure 2 plots a simulation of two solitary waves generated at various time levels.

Implementation 1.3 In the last application, for $p = 4$, the invariants are calculated by choosing the parameters $c_1 = 0.2, c_2 = 1/80$ and their values are compared with the Petrov Galerkin method [46] and displayed in Table 10. This application shows that as a result of the numerical results and their comparisons obtained with the proposed algorithm, the invariants are well preserved. Figure 3 plots a simulation of two solitary waves generated at various time levels. For Figures 2 and 3, the wave with the smaller amplitude at the beginning is located to the right of the

TABLE 7. Comprasions of The invariants and error norms of the single solitary wave with $\Delta t = 0.2, h = 0.1, \mu = 1, \epsilon = 3$ on the region $[0, 80]$ with the amplitude 1 for $p = 2, 3$ and 4. .

method		time	I_1	I_2	I_3	L_2	L_∞
Present	$p = 2$	0	3.14159265	2.66666666	1.33333333	0.0000000	0.0000000
		5	3.14156224	2.66662603	1.33333333	0.0043875	0.0028947
		10	3.14153186	2.66658546	1.33333333	0.0085592	0.0054045
		15	3.14150149	2.66654489	1.33333333	0.0126950	0.0079295
		20	3.14147112	2.66650433	1.33333333	0.0168570	0.0104733
[28]		20	3.1589605	2.6902580	1.3570299	0.0380303	0.0262900
[32]		20	3.1251634	2.6447698	1.3115241	0.0544214	0.0360834
[52]1		20	3.1253043	2.6445829	1.3113394	0.0513210	0.0341675
[52]2		20	3.1416722	2.6669051	1.3335718	0.0167509	0.0102639
Present	$p = 3$	0	2.80436421	2.46391383	0.98556555	0.0000000	0.0000000
		5	2.80433498	2.46387067	0.98556555	0.0020551	0.0014553
		10	2.80430576	2.46382753	0.98556555	0.0040971	0.0027828
		15	2.80427655	2.46378441	0.98556555	0.0061449	0.0041303
		20	2.80424735	2.46374129	0.98556555	0.0082198	0.0055001
[28]		20	2.8187398	2.4852249	0.0070200	0.0165563	0.0137045
[32]		20	2.7911063	2.4444001	0.9661645	0.0460246	0.0333820
[52]1		20	2.8043570	2.4639086	0.9855602	0.0080147	0.0053823
[52]2		20	2.8042943	2.4637495	0.9854011	0.0070855	0.0048047
Present	$p = 4$	0	2.62205755	2.35619437	0.78539816	0.0000000	0.0000000
		5	2.62203468	2.35615854	0.78539816	1.0728268	0.7985929
		10	2.62201182	2.35612272	0.78539816	2.1619314	1.5444383
		15	2.62198896	2.35608691	0.78539816	3.2670460	2.3067785
		20	2.62196610	2.35605110	0.78539816	4.3945800	3.0879339
[28]		20	2.63278330	2.37300320	0.8023383	0.0089061	0.0082199
[32]		20	2.6122055	2.3408135	0.7701119	0.0375343	0.0287549
[52]1		20	2.6220508	2.3561901	0.7853939	0.0042169	0.0029795
[52]2		20	2.6219284	2.3559327	0.7851364	0.0033908	0.0024703

(a) $t = 10$ (b) $t = 50$ (c) $t = 70$ (d) $t = 100$
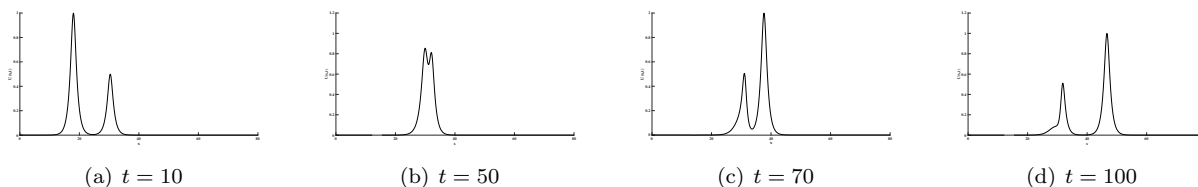
FIGURE 2. $p = 3$ of two solitary wave interactions.

wave with the larger amplitude. Later on, the interaction between the waves begins and then they overlap with each other. Then, with the time $t = 50$, those waves initialy start taking their original shapes.

6.3. **Example III: Evolution process of solitons.** This example will handle the Eq. (1.1) together with the Maxwellian condition presented at the initial time [17, 44, 46]

$$U(x, 0) = exp(-x^2), \quad [-20, 20].$$

TABLE 8. Calculated invariants of two solitary waves for $p = 2$ with values $x_1 = 15, x_2 = 30, c_1 = 0.5, c_2 = 0.125, \epsilon = 3, \Delta t = 0.025, h = 0.1$ over $[0, 80]$ at various times.

	method			[46]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	4.71239	3.33333	1.41666	4.71239	3.33324	1.14166
10	4.71239	3.33333	1.41666	4.71239	3.33324	1.14166
20	4.71239	3.33333	1.41666	4.71239	3.33324	1.14166
30	4.71239	3.33333	1.41666	4.71239	3.33324	1.14166
40	4.71239	3.33333	1.41666	4.71239	3.33333	1.14166
50	4.71239	3.33333	1.41666	4.71239	3.33338	1.14166
60	4.71239	3.33333	1.41666	4.71239	3.33333	1.14166

TABLE 9. Computed two solitary wave invariants for $p = 3$ with values $x_1 = 15, x_2 = 30, c_1 = 0.3, c_2 = 0.0325, \epsilon = 3, \Delta t = 0.025, h = 0.1$ over $[0, 80]$ at various times.

	method			[46]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	4.20655	3.07989	1.01636	4.20655	3.97977	1.01634
10	4.20655	3.07989	1.01636	4.20655	2.07986	1.01634
20	4.20655	3.07989	1.01636	4.20655	3.07982	1.01634
30	4.20655	3.07989	1.01636	4.20655	3.07980	1.01634
40	4.20655	3.07989	1.01636	4.20655	3.07986	1.01634
50	4.20655	3.07989	1.01636	4.20655	3.07981	1.01633
60	4.20655	3.07989	1.01636	4.20655	3.07987	1.01633
70	4.20655	3.07989	1.01636	4.20655	3.07976	1.01634
80	4.20655	3.07989	1.01636	4.20655	3.07991	1.01633
90	4.20655	3.07989	1.01636	4.20655	3.07974	1.01633
100	4.20655	3.07989	1.01636	4.20655	3.07972	1.01634

TABLE 10. Computed two solitary wave invariants for $p = 4$ with values $x_1 = 15, x_2 = 30, c_1 = 0.2, c_2 = 1/80, \epsilon = 3, \Delta t = 0.025, h = 0.1$ over $[0, 80]$ at various times.

	method			[46]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	3.93309	2.94524	0.797671	3.93309	2.94512	0.797614
20	3.93309	2.94524	0.797671	3.93309	2.94517	0.797611
40	3.93309	2.94524	0.797671	3.93309	2.94515	0.797612
60	3.93309	2.94524	0.797671	3.93309	2.94505	0.797622
80	3.93309	2.94524	0.797671	3.93309	2.94506	0.797613
100	3.93309	2.94524	0.797671	3.93309	2.94508	0.797611
120	3.93309	2.94524	0.797671	3.93308	2.94511	0.797611

It is taken the parameter values $\epsilon = 3, h = 0.05, \Delta t = 0.01$ on the domain $[0, 80]$ throughout the study for $p = 2, 3, 4$ with different values of $\mu = 0.01, 0.025, 0.05$ and $\mu = 0.1$. For numerical simulations, the code has been run till $t = 12$. The results generated are compared with the Petrov–Galerkin and collocation techniques and reported in Table 11.

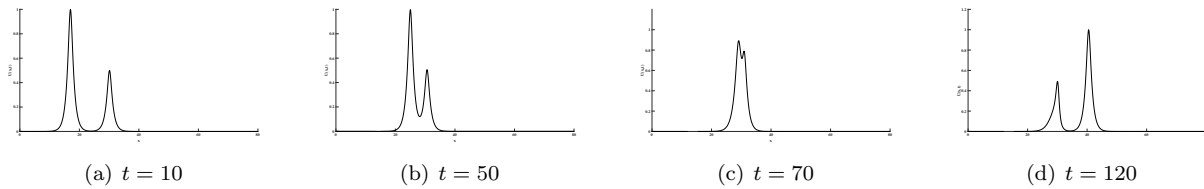(a) $t = 10$ (b) $t = 50$ (c) $t = 70$ (d) $t = 120$

FIGURE 3. $p = 4$ of two solitary wave interactions.

This table concludes that the proposed algorithm yields results consistent with [17, 44, 46]. Figures 4 and 5 indicate the evolution of the Maxwell initial condition to solitary waves for $p = 3, 4$ with different values of μ at $t = 12$. One can observe from this figure that the smaller solitary wave located to the left of the larger one. Figures 4 and 5 depict only one stable soliton with (a) for $p = 3$ and 4 when $\mu = 0.1$, and two stable solitary waves with (b) when $\mu = 0.05$ for $p = 3, 4$. Also, these figures depicts three and five stable solitary waves with (c) and (d) for $p = 3, 4$ when $\mu = 0.025$ and $\mu = 0.01$, respectively. From Figure 4 and 5, one may coclude that the number of stable solitary waves increases as the value μ decreases.

TABLE 11. The invariant values of Maxwellian initial condition with $\Delta t = 0.01, h = 0.05$ for different values of μ.

	p	2			3			4		
		I_1			I_2	I_3	I_1	I_2	I_3	
	test	test	test		1.7724	1.2658	0.7926	1.7724	1.2658	0.7236
	4	1.7724	1.2655	0.8868	1.7721	1.2649	0.7928	1.7720	1.2648	0.7265
I1	8	1.7723	1.2653	0.8863	1.7717	1.2641	0.7907	1.7712	1.2636	0.7242
	12	1.7721	1.2651	0.8858	1.7711	1.2626	0.7862	1.7693	1.2582	0.7051
[52]	12	1.7739	1.2711	0.9123	1.7813	1.2901	0.8664	1.8156	1.3901	1.2707
	0	1.7724	1.2846	0.8862	1.7724	1.2846	0.7927	1.7724	1.2846	0.7236
	4	1.7724	1.2846	0.8864	1.7724	1.2845	0.7929	1.7724	1.2846	0.7243
0.025	8	1.7724	1.2846	0.8864	1.7724	1.2845	0.7929	1.7724	1.2845	0.7238
	12	1.7724	1.2846	0.8864	1.7724	1.2845	0.7928	1.7724	1.2844	0.7237
[52]	12	1.7725	1.2846	0.8881	1.7730	1.2837	0.7946	1.7791	1.3056	0.8198
	0	1.7724	1.3159	0.8862	1.7724	1.3159	0.7926	1.7724	1.3159	0.7236
	4	1.7724	1.3159	0.8863	1.7724	1.3159	0.7927	1.7724	1.3159	0.7237
0.05	8	1.7724	1.3159	0.8863	1.7724	1.3159	0.7927	1.7724	1.3159	0.7237
	12	1.7724	1.3159	0.8863	1.7724	1.3159	0.7928	1.7724	1.3159	0.7237
[52]	12	1.7724	1.3159	0.8864	1.7725	1.3160	0.7940	1.7735	1.3188	0.7345
	0	1.7724	1.3786	0.8862	1.77245	1.87997	0.88623	1.7724	1.3786	0.7236
	4	1.7724	1.3786	0.8862	1.77245	1.87997	0.88623	1.7724	1.3786	0.7236
0.1	8	1.7724	1.3786	0.8862	1.77245	1.87997	0.88623	1.7724	1.3786	0.7236
	12	1.7724	1.3786	0.8862	1.77245	1.87997	0.88623	1.7724	1.3786	0.7236
[52]	12	1.7724	1.3786	0.8862	1.7724	1.3786	0.7928	1.7725	1.3786	0.7243

(a) $\mu = 0.1$ (b) $\mu = 0.05$ (c) $\mu = 0.025$ (d) $\mu = 0.01$
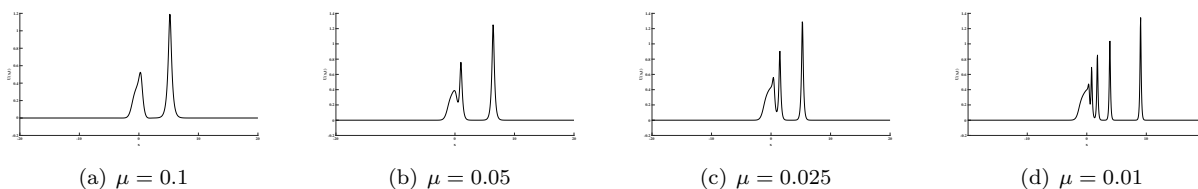
FIGURE 4. Maxwellian initial condition $p = 3$ at $t = 12$.

(a) $\mu = 0.1$ (b) $\mu = 0.05$ (c) $\mu = 0.025$ (d) $\mu = 0.01$
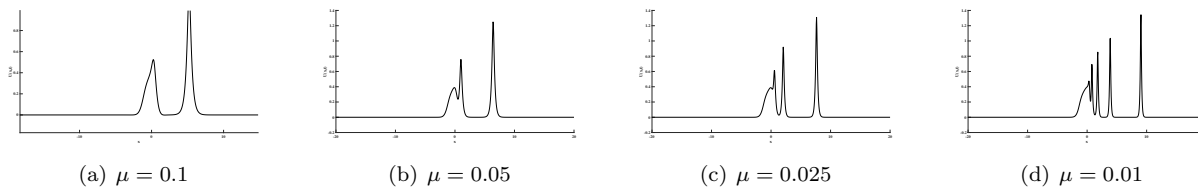
FIGURE 5. Maxwellian initial condition $p = 4$ at $t = 12$.

7. CONCLUSION

The Lie-Trotter splitting techniques together with the quintic B-spline collocation method have been used for approximate solutions of the (GEW) equation. Three model problems have been applied to the proposed algorithm and a comparison is carried out with the values of Petrov Galerkin, Galerkin, and collocation methods by computing the error norms L_2 and L_∞ and invariant values I_1, I_2 and I_3 representing mass, energy, and momentum to measure its accuracy, practicality and usefulness. For the invariants and error norms calculated with the method applied in this study, it can be said that the invariants are well preserved and the error norms are satisfactorily small according to Petrov Galerkin, Galerkin, some collocation methods and this situation indicates the success of the proposed algorithm. Additionally, numerical algorithms are shown to be unconditionally stable. Therefore, readers may be encouraged to utilize the present method about the other nonlinear models stated by PDEs in applied sciences such as engineering, physics.

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