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Gradient estimates for a nonlinear equation under the almost Ricci soliton condition

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Abstract

In this paper, we study the gradient estimate for the positive solutions of the equation $\Delta u + au(\log u)^p + bu = f$ on an almost Ricci soliton (M^n, g, X, λ). In a special case, when $X = \nabla h$ for a smooth function h, we derive a gradient estimate for an almost gradient Ricci soliton.

Keywords. Gradient almost Ricci solitons, Gradient estimates, Almost Ricci solitons.
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1. Introduction

Let (M^n, g) be a complete Riemannian manifold with fixed base point $O \in M$. Consider the following lower bound on the Ricci curvature

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g \geq -\lambda g, \tag{1.1}$$

for a smooth function $\lambda : M \to \mathbb{R}$, and smooth vector field X, which satisfies

$$|X|(y) \leq \frac{K}{d(y,O)^\alpha}, \quad \forall y \in M. \tag{1.2}$$

Here $d(y, O)$ represents the distance from O to y, K is a positive constant, and $0 \leq \alpha < 1$. We say that a Riemannian manifold M is an almost Ricci soliton when equipped with (1.1), and the Ricci soliton when λ is a constant. An almost Ricci soliton (M, g, X, λ) is trivial, if it is a Ricci soliton, and a Ricci soliton is trivial when X is Killing. There is some newly published article about the characterization of almost Ricci solitons, and their isometries. We refer to the [3, 6–8, 14, 16] for more study.

In the pioneering work of Zhang and Zhu [18], they proposed the main conditions (1.1), (1.2) with constant λ, and also the volume non-collapsing condition when $\alpha \neq 0$:

$$Vol(B(x, 1)) \geq \rho, \tag{1.3}$$

for all $x \in M$ and some constant $\rho > 0$. They first studied volume comparison, then following the techniques in [5], they proved Sobolev inequalities on manifolds as follows:

Theorem 1.1 (Sobolev inequality)**.** *Assume that (1.1), (1.2), and (1.3) hold. Then there is a constant $r_0 = r_0(n, \lambda, K, \alpha, \rho)$ such that for any $f \in C_0^\infty(B(x, r))$, $r \leq r_0$, we have the following Sobolev inequalities:*

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}} dg \right)^{\frac{n-1}{n}} \leq C(n) r \oint_{B(x,r)} |\nabla f| dg, \tag{1.4}$$

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and

$$\left(\oint_{B(x,r)} |f|^{\frac{2n}{n-2}} dg \right)^{\frac{n-2}{n}} \leq C(n) r^2 \oint_{B(x,r)} |\nabla f|^2 dg.$$

Moreover, for the case that $X = \nabla f$ for some smooth function f, we get

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}} dg \right)^{\frac{n-1}{n}} \leq C(n) r \oint_{B(x,r)} |\nabla f| dg.$$

These results enabled authors to state local gradient estimates for elliptic and parabolic heat equations. Motivated by this work, in [1], we have studied volume comparison for an almost Ricci soliton:

Theorem 1.2 (Volume comparison)**.** *Assume that for an n-dimension Riemannian manifold (1.1) and (1.2) hold. Moreover, consider a positive constant N as an upper bound for λ. Suppose in addition that the volume non-collapsing condition (1.3) holds for positive constants $\rho > 0$, $K \geq 0$ and $0 \leq \alpha < 1$, then for any $0 < r_1 < r_2 \leq 1$, the volume ratio bound is as follows*

$$\frac{Vol(B(x, r_2))}{r_2^n} \leq e^{C(n,N,K,\alpha,\rho)[N(r_2^2 - r_1^2) + K(r_2 - r_1)^{1-\alpha}]} . \frac{Vol(B(x, r_1))}{r_1^n},$$

where $C = C(n, N, K, \alpha, \rho)$ is the constant depends on (n, N, K, α, ρ) and $B(x, r)$ is a ball centered at x with radius r. In particular, this result are true by considering the gradient soliton vector field $X = \nabla f$.

It is known that results such as volume comparison and gradient estimate are powerful tools in geometry. For example, see [2, 4, 10, 11, 15, 17]. As an important application, Li and Yau [12] deduced a Harnack inequality, and also they obtained upper and lower bounds for heat kernel under the Dirichlet and Neumann boundary conditions. Recently, Peng et al. [13], established Yau-type gradient estimates for the following equation on Riemannian manifolds

$$\Delta u + au(\log u)^p + bu = 0,$$

where $a, b \in \mathbb{R}$, and p is a rational number with $p = \frac{k_1}{2k_2 + 1} \geq 2$, where k_1 and k_2 are positive integer numbers. Lately, in [9] we studied gradient estimate on an almost Ricci soliton M for the solutions of

$$\Delta u = f + Y.u,$$

where Y is a smooth vector field.
In this paper, using the sufficient instrument like Sobolev inequality, volume comparison Theorem, and the same method as in [18], we want to obtain a gradient estimate for the smooth function u, which satisfies

$$\Delta u + au(\log u)^p + bu = f, \tag{1.5}$$

here a, b, and $p > 0$ are real constants, and $f : M \to \mathbb{R}$ is a smooth function.

2. Main results

In this section, we are going to state local gradient estimates for solutions of the nonlinear equation (1.5) for an almost Ricci soliton M^n. Note that all results in this section hold without the non-collapsing condition, if $\alpha = 0$. Here is our main result:

Theorem 2.1. *Suppose that for an almost Ricci soliton M^n, (1.1), (1.2), and (1.3) hold. For $q > \frac{n}{2}$, if u and f be smooth functions such that (1.5) holds with $0 \leq u \leq l_1$ and $|(\log u)^p| \leq l_2$, and $|\lambda| \leq l_3$ for constants l_1, l_2, l_3, then there exists a positive constant $r_0 = r_0(n, K, \alpha, \rho, l_1, l_2, l_3)$ such that for any $x \in M$ and $0 < r \leq r_0$ we have*

$$\sup_{B(x, \frac{1}{2} r)} |\nabla u|^2 \leq C(n, K, \alpha, \rho, l_1, l_2, l_3) \big[(\|f\|^*_{2q, B(x,r)})^2 + r^{-2} (\|u\|^*_{2, B(x,r)})^2 \big].$$

If, in the definition of Ricci soliton and almost Ricci soliton, the vector field is a gradient of a smooth function, then we derive gradient Ricci soliton and gradient almost Ricci soliton. By this definition, for a gradient almost Ricci soliton $(M, g, X = \nabla h, \lambda)$, the same computation as in Theorem 2.1, concludes:

Corollary 2.2. *Suppose that the following condition holds for a gradient Ricci soliton*

$$\mathrm{Ric} + Hessh \geq -\lambda g,$$

and moreover, consider two condition for the potential function h as follows:

$$|h(y) - h(z)| \leq K_1 d(y,z)^{\alpha}, \ \ and \quad \sup_{x\in M, 0\leq r\leq 1} \left(r^{\beta}\|\nabla h\|^*_{q,B(x,r)}\right) \leq K_2,$$

for any $y, z \in M$ with $d(y,z) \leq 1$. Here $K_1, K_2 \geq 0$, $0 < \alpha < 1$, $0 \leq \beta < 1$, and $q \geq 1$ are constants. Then there is a constant $r_0 = r_0(n, K_1, K_2, \alpha, \beta, l_1, l_2, l_3)$, such that the solution of (1.5) satisfies

$$\sup_{B(x,\frac{r}{2})} |\nabla u|^2 \leq C(n, K_1, K_2, \alpha, \beta, l_1, l_2, l_3)\left[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|h\|^*_{2q,B(x,r)})^2\right],$$

for any $q > \dfrac{n}{2}$.

Now, we are ready to prove the Theorem 2.1.

Proof of Theorem 2.1. Set $v = |\nabla u|^2 + \|f^2\|^*_{q,B(x,r)}$. Then, the Bochner formula gives

$$\Delta v = 2|\nabla^2 u|^2 + 2 < \nabla u, \nabla \Delta u > + 2\mathrm{Ric}(\nabla u, \nabla u). \tag{2.1}$$

Since $\Delta u + au(\log u)^p + bu = f$, substituting (1.1) into (2.1), we get

$$\begin{aligned}
\Delta v &\geq 2 < \nabla u, \nabla f > -2 < \nabla u, \nabla(au(\log u)^p + bu) > -2\lambda v - (\mathcal{L}_V g)(\nabla u, \nabla u) \\
&\geq 2 < \nabla u, \nabla f > -2av(\log u)^p - 2apv(\log u)^{p-1} - 2v(b+\lambda) - (\mathcal{L}_V g)(\nabla u, \nabla u),
\end{aligned}$$

and for any $q > 0$, we get

$$\begin{aligned}
\Delta v^q &= qv^{q-1}\Delta v + q(q-1)v^{q-2}|\nabla v|^2 \\
&\geq 2qv^{q-1} < \nabla u, \nabla f > -2aqv^q(\log u)^p - 2apqv^q(\log u)^{p-1} - 2qv^q(b+\lambda) \\
&\quad -qv^{q-1}(\mathcal{L}_V g)(\nabla u, \nabla u) + \frac{q-1}{q}v^{-q}|\nabla v^q|^2.
\end{aligned} \tag{2.2}$$

Let $B = B(x,r)$, then by (2.2) for any $\eta \in C_0^{\infty}(B_x(1))$ and $q \geq 1$, in the local coordinate we compute

$$\begin{aligned}
\int_B |\nabla(\eta v^q)|^2 &= \int_B |\eta \nabla v^q + v^q \nabla \eta|^2 \\
&= \int_B v^{2q}|\nabla \eta|^2 - \eta^2 v^q \Delta v^q \\
&\leq \int_B v^{2q}|\nabla \eta|^2 - 2q\eta^2 v^{2q-1}u_i f_i + 2aq\eta^2(\log u)^p v^{2q} + 2apq\eta^2(\log u)^{p-1}v^{2q} \\
&\quad +2(b+\lambda)q\eta^2 v^{2q} + q\eta^2 v^{2q-1}(\mathcal{L}_V g)_{ij}u_i u_j.
\end{aligned} \tag{2.3}$$

We know $(\mathcal{L}_V g)_{ij} = \nabla_i V_j + \nabla_j V_i$, so using integration by parts we get

$$
\begin{aligned}
&\frac{1}{2}\int_B \eta^2 v^{2q-1}(\mathcal{L}_V g)_{ij} u_i u_j \\
&= -\int_B 2\eta v^{2q-1}\eta_j V_i u_i u_j + (2q-1)\eta^2 v^{2q-2} v_j V_i u_i u_j \\
&+\eta^2 v^{2q-1} V_i u_{ij} u_j + \eta^2 v^{2q-1} V_i u_i u_{jj} \\
\leq\ & \int_B v^{2q}|\nabla\eta|^2 + \eta^2 v^{2q-2}|V|^2|\nabla u|^4 - \frac{2q-1}{q}\eta v^{q-1} V_i u_i u_j[(\eta v^q)_j - v^q\eta_j] \\
&-\frac{1}{2}\eta^2 v^{2q-1} V_i v_i + \frac{1}{2}\eta^2 v^{2q-2} f^2|\nabla u|^2 + \frac{1}{2}\eta^2 v^{2q}|V|^2 - \eta^2 v^{2q-1} V_i u_i(au(\log u)^p + bu). \\
\leq\ & \int_B v^{2q}|\nabla\eta|^2 + \frac{3}{2}\eta^2 v^{2q}|V|^2 - \frac{2q-1}{q}\eta v^{q-1} V_i u_i u_j[(\eta v^q)_j - v^q\eta_j] \\
&-\frac{1}{2q}\eta v^q V_i[(\eta v^q)_i - v^q\eta_i] + \frac{1}{2}\eta^2 v^{2q-2} f^2|\nabla u|^2 - \eta^2 v^{2q-1} V_i u_i(au(\log u)^p + bu).
\end{aligned}
\tag{2.4}
$$

With the boundary condition stated in the theorem for λ, u, and $(\log u)^p$, (2.4) becomes

$$
\begin{aligned}
&\int_B \eta^2 v^{2q-1}(\mathcal{L}_V g)_{ij} u_i u_j \\
\leq\ & \int_B 2v^{2q}|\nabla\eta|^2 + 3\eta^2 v^{2q}|V|^2 + \frac{1}{2q}|\nabla(\eta v^q)|^2 + \frac{2(2q-1)^2}{q}\eta^2 v^{2q}|V|^2 \\
&+\frac{2q-1}{q} v^{2q}|\nabla\eta|^2 + \frac{2q-1}{q}\eta^2 v^{2q}|V|^2 + \frac{1}{2q}|\nabla(\eta v^q)|^2 + \frac{1}{2q}\eta^2 v^{2q}|V|^2 \\
&+\frac{1}{2q}\eta^2 v^{2q}|V|^2 + \frac{1}{2q} v^{2q}|\nabla\eta|^2 + \eta^2 v^{2q-1} f^2 + l_2 l_3\eta^2 v^{2q}|V|^2 + l_2 l_3\eta^2 v^{2q-2}|\nabla u|^2 \\
&+b l_2\eta^2 v^{2q}|V|^2 + b l_2\eta^2 v^{2q-2}|\nabla u|^2 \\
\leq\ & \int_B \frac{8q-1}{4q} v^{2q}|\nabla\eta|^2 + \frac{2(2q-1)^2 + 5q + 2q l_2(b + l_3)}{2q}\eta^2 v^{2q}|V|^2 \\
&+\frac{1}{2q}|\nabla(\eta v^q)|^2 + \frac{1}{2}\eta^2 v^{2q-1} f^2 + l_2(b + l_3)\eta^2 v^{2q-1}.
\end{aligned}
\tag{2.5}
$$

On the other hand, simple computation gives

$$
\begin{aligned}
& -2q\int_B \eta^2 v^{2q-1}u_i f_i \\
= \;& 2q\int_B \eta^2 v^{2q-1} f^2 - au(\log u)^p \eta^2 v^{2q-1} f + 2\eta v^{2q-1} f u_i\eta_i + (2q-1)\eta^2 v^{2q-2} f u_i v_i \\
= \;& 2q\int_B \eta^2 v^{2q-1} f^2 + 2\eta v^{2q-1} f u_i \eta_i + \frac{2q-1}{q}\eta v^{q-1} f u_i((\eta v^q)_i - v^q\eta_i) - (al_2 l_3 \\
& + + bl_2)\eta^2 v^{2q-1} f \\
= \;& \int_B \eta^2 v^{2q-1} f^2 + \frac{1}{q}\eta v^{2q-1} f u_i\eta_i + \frac{2q-1}{q}\eta v^{q-1} f u_i(\eta v^q)_i - (al_2l_3 + bl_2)\eta^2 v^{2q-1} f \\
\leq \;& 2q\int_B \eta^2 v^{2q-1} f^2 + \frac{1}{2q}\eta^2 v^{2q-2} f^2|\nabla u|^2 + \frac{1}{2q} v^{2q}|\nabla\eta|^2 + \frac{1}{8q}|\nabla(\eta v^q)|^2 \\
& + \frac{2(2q-1)^2}{q}\eta^2 v^{2q-2} f^2 |\nabla u|^2 + \frac{1}{2}\eta^2 v^{2q-2} f^2 + \frac{(al_2l_3 + bl_2)^2}{2}\eta^2 v^{2q} \\
= \;& 2q\int_B \frac{4(2q-1)^2 + 3q + 1}{2q}\eta^2 v^{2q-1} f^2 + \frac{1}{2q} v^{2q}|\nabla\eta|^2 + \frac{1}{8q}|\nabla(\eta v^q)|^2 \\
& + \frac{(al_2l_3 + bl_2)^2}{2}\eta^2 v^{2q}.
\end{aligned}
\tag{2.6}
$$

Putting (2.5) and (2.6) into (2.3), follows that

$$
\begin{aligned}
\int_B |\nabla(\eta v^q)|^2 \;\leq\; & \int_B 4v^{2q}|\nabla\eta|^2 + (16(2q-1)^2 + 12q + 4)\eta^2 v^{2q-1} f^2 + 4v^{2q}|\nabla\eta|^2 \\
& + (8q-1)v^{2q}|\nabla\eta|^2 + (4(2q-1)^2 + 10q + 2ql_2(b + l_3))\eta^2 v^{2q}|V|^2 \\
& + 2q\eta^2 v^{2q-1} f^2 + 2ql_2(b + l_3)\eta^2 v^{2q-1} + 4q(al_2l_3 + bl_3)^2\eta^2 v^{2q} \\
& + 8aql_3\eta^2 v^{2q} + 8apql_3\eta^2 v^{2q} + 8(b + l_1)q\eta^2 v^{2q} \\
\leq\; & \int_B 16qv^{2q}|\nabla\eta|^2 + 70q^2\eta^2 v^{2q-1} f^2 + [30q^2 + 2ql_2(b + l_3)]\eta^2 v^{2q}|V|^2 \\
& + [4(al_2l_3 + bl_3)^2 + 8al_3 + 8apl_3 + 8(b + l_1)]q\eta^2 v^{2q} \\
& + 2ql_2(b + l_3)\eta^2 v^{2q-1}.
\end{aligned}
\tag{2.7}
$$

Constructing a cut-off function $\psi_i(s)$ such that for $r_i = (\frac{1}{2}, \frac{1}{2^{i+2}})$, $i = 0, 1, 2, ...$, $\psi_i(t) \equiv 1$ for $t \in [0, r_{i+1}]$, $supp\psi_i \subseteq [0, r_i]$ and $-\frac{52^i}{r} \leq \psi_i^{'} \leq 0$. Then define $\eta_i(y) = \psi_i(s)$. Thus, (2.7) becomes

$$
\begin{aligned}
\int_{B(x,r_i)} |\nabla(\eta_i v^q)|^2 \;\leq\; & \int_{B(x,r_i)} 16qv^{2q}|\nabla\eta_i|^2 + 70q^2\eta_i^2 v^{2q-1} f^2 + [30q^2 + 2ql_2(b + l_3)]\eta_i^2 v^{2q}|V|^2 \\
& + [4(al_2l_3 + bl_3)^2 + 8al_3 + 8apl_3 + 8(b + l_1)]q\eta_i^2 v^{2q} \\
& + 2ql_2(b + l_3)\eta_i^2 v^{2q-1}.
\end{aligned}
\tag{2.8}
$$

Using volume comparison Theorem 1.2, for $\frac{r}{2} \leq r_i \leq \frac{3r}{4}$, and Young's inequality we can conclude

$$
\begin{aligned}
& 70q^2 \oint_{B(x,r_i)} \eta_i^2 v^{2q-1} f^2 \\
& \leq \frac{70q^2}{\|f^2\|^*_{q,B(x,r)}} \oint_{B(x,r_i)} \eta_i^2 v^{2q} f^2 \\
\leq \; & C(n,l_1,K,\alpha,\rho) q^2 \left(\oint_{B(x,r_i)} (\eta_i v^q)^{\frac{2q}{q-1}} \right)^{\frac{q-1}{q}} \\
\leq \; & C(n,l_1,K,\alpha,\rho) q^2 \left(\oint_{B(x,r_i)} (\eta_i v^q)^{a.\frac{2q}{q-1}.b} \right)^{\frac{q-1}{q^b}} \\
& \times \left(\oint_{B(x,r_i)} (\eta_i v^q)^{(1-a).\frac{2q}{q-1}.\frac{b}{b-1}} \right)^{\frac{(q-1)(b-1)}{q^b}} \\
\leq \; & \epsilon \left(\oint_{B(x,r_i)} (\eta_i v^q)^{a.\frac{2q}{q-1}.b} \right)^{\frac{q-1}{q^{ba}}} \\
& + \epsilon^{-\frac{a}{1-a}} C^{\frac{1}{1-a}} q^{\frac{2}{1-a}} \left(\oint_{B(x,r_i)} (\eta_i v^q)^{(1-a).\frac{2q}{q-1}.\frac{b}{b-1}} \right)^{\frac{(q-1)(b-1)}{q^{b(1-a)}}}.
\end{aligned}
$$

By choosing $a = \frac{n}{2q}$, and $b = \frac{2q-2}{n-2}$, it follows

$$
\begin{aligned}
& 70q^2 \oint_{B(x,r_i)} \eta_i^2 v^{2q-1} f^2 \\
\leq \; & \epsilon \left(\oint_{B(x,r_i)} (\eta_i v^q)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + \epsilon^{-\frac{n}{2q-n}} C^{\frac{2q}{2q-n}} q^{\frac{4q}{2q-n}} \oint_{B(x,r_i)} \eta_i^2 v^{2q}.
\end{aligned}
\tag{2.9}
$$

With the same argument for $q \in (\frac{n}{2}, \frac{n}{2\alpha})$, we have

$$
\begin{aligned}
& 30q^2 \oint_{B(x,r_i)} \eta_i^2 v^{2q} |V|^2 \\
\leq \; & \epsilon r_i^{-2\alpha} \left(\oint_{B(x,r_i)} (\eta_i v^q)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\
& + \epsilon^{-\frac{n}{2q-n}} q^{\frac{4q}{2q-n}} C^{\frac{2q}{2q-n}} r_i^{-2\alpha} \oint_{B(x,r_i)} \eta_i^2 v^{2q},
\end{aligned}
\tag{2.10}
$$

and

$$\begin{aligned}
&2ql_2(b+l_3)\oint_{B(x,r_i)}\eta_i^2 v^{2q}|V|^2\\
\leq\ &\epsilon r_i^{-2\alpha}\left(\oint_{B(x,r_i)}(\eta_i v^q)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&+\epsilon^{-\frac{n}{2q-n}} q^{\frac{2q}{2q-n}} C_1^{\frac{2q}{2q-n}} r_i^{-2\alpha}\oint_{B(x,r_i)}\eta_i^2 v^{2q}. \qquad (2.11)
\end{aligned}$$

Here $C_1 = C_1(n, K, \alpha, \rho, l_1, l_2, l_3)$. Now, substituting (2.9), (2.10), and (2.11) in (2.8), and using Sobolev inequality (1.4), we obtain

$$\begin{aligned}
&\left(\oint_{B(x,r_i)}(\eta_i v^q)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
\leq\ &C(n)r_i^2\oint_{B(x,r_i)}|\nabla(\eta_i v^q)|^2\\
\leq\ &C(n)r_i^2\Big[16qv^{2q}|\nabla\eta_i|^2+[4(al_2l_3+bl_3)^2+8al_3+8apl_3+8(b+l_1)]q\eta_i^2v^{2q}\\
&+2ql_2(b+l_3)\eta_i^2v^{2q-1}\Big]\\
&+C(n)\epsilon r_i^{2-2\alpha}\left(\oint_B(\eta v^q)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&+C(n)\epsilon^{-\frac{a}{1-a}}C^{\frac{2q}{2q-n}}q^{\frac{4q}{2q-n}}r_i^{2-2\alpha}\oint_{B(x,r_i)}\eta_i^2v^{2q}\\
&+C(n)r_i^2\epsilon\left(\oint_{B(x,r_i)}(\eta_i v^q)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&+C(n)r_i^2\epsilon^{-\frac{a}{1-a}}\left(q^{\frac{4q}{2q-n}}+q^{\frac{2q}{2q-n}}\right)\oint_{B(x,r_i)}\eta_i^2v^{2q}. \qquad (2.12)
\end{aligned}$$

Due to $r_i \leq r \leq 1$ and $\alpha < 1$, we choose $\epsilon = \epsilon(n)$ so small that (2.12) changes as

$$\begin{aligned}
\left(\oint_{B(x,r_i)}(\eta_i v^q)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq\ &C(n,K,\alpha,\rho,l_1,l_2,l_3)r_i^2\oint_{B(x,r_i)}qv^{2q}|\nabla\eta_i|^2\\
&+q\eta_i^2v^{2q}+q\eta_i^2v^{2q-1}.
\end{aligned}$$

With volume comparison Theorem 1.2, we get

$$\begin{aligned}
\left(\oint_{B(x,r_{i+1})}(v^q)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq\ &C(n,l_3,K,\alpha,\rho)\left(\oint_{B(x,r_i)}(\eta_i v^q)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&\leq C(n,K,\alpha,\rho,l_1,l_2,l_3)\oint_{B(x,r_i)}2^{2i}qv^{2q}+2qv^{2q}.
\end{aligned}$$

Now, we choose $q = \frac{\mu^i}{2}$ for $i = 0, 1, 2, \cdots$, where $\mu = \frac{n}{n-2}$. Therefore

$$\begin{aligned}\left(\fint_{B(x,r_{i+1})} v^{\mu^{i+1}}\right)^{\frac{n-2}{n}} &= \left(\fint_{B(x,r_{i+1})} (v^q)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq C(n,K,\alpha,\rho,l_1,l_2,l_3)(2^{2i-1}\mu^i + 2\mu^i)\fint_{B(x,r_i)} v^{\mu^i} \\ &\leq C(n,K,\alpha,\rho,l_1,l_2,l_3)2(2^{2i-2}+1)2^{2i}\fint_{B(x,r_i)} v^{\mu^i},\end{aligned}$$

here in the last step we use the fact that $\mu \leq 3$. So

$$\|v\|^*_{\mu^{i+1},B(x,r_{i+1})} \leq C^{\mu^{-i}}(2^{4i-1}+2^{2i+1})^{\mu^{-i}}\|v\|^*_{\mu^i,B(x,r_i)}.$$

Using powerful Nash-Moser iteration, we conclude

$$\sup_{B(x,\frac{1}{2}r)} v \leq C^{\Sigma\mu^{-i}}(2^{4i-1}+2^{2i+1})^{\Sigma\mu^{-i}}\|v\|^*_{1,B(x,\frac{3}{4}r)} \leq C(n,K,\alpha,\rho,l_1,l_2,l_3)\|v\|^*_{1,B(x,\frac{3}{4}r)}. \tag{2.13}$$

Since

$$\begin{aligned}\int_{B(x,r)} \eta^2|\nabla u|^2 &= \int_{B(x,r)} -\eta^2 u(f - au(\log u)^p - bu) - 2\eta u\nabla u\nabla\eta \\ &\leq \int_{B(x,r)} \frac{1}{2}u^2\eta^2 + \frac{1}{2}f^2\eta^2 + a\eta^2 l_1^2 + \eta^2 b l_1^2 + \frac{1}{2}\eta^2|\nabla u|^2 + 2u^2|\nabla\eta|^2.\end{aligned}$$

This with the definition of η, imply that

$$\begin{aligned}\fint_{B(x,r)} \eta^2|\nabla u|^2 &\leq 4\fint_{B(x,r)} u^2\eta^2 + f^2\eta^2 + a\eta^2 l_1^2 l_2 + b\eta^2 l_1^2 + u^2|\nabla\eta|^2 \\ &\leq 100r^{-2}(\|u\|^*_{2,B(x,r)})^2 + 4\|f^2\|^*_{q,B(x,r)} + al_1^2 l_2 + bl_1^2.\end{aligned}$$

Thus

$$\begin{aligned}\|v\|^*_{1,B(x,\frac{3}{4}r)} &\leq \frac{Vol(B(x,r))}{Vol(B(x,\frac{3}{4}r))}\fint_{B(x,r)} \eta^2(|\nabla u|^2 + \|f^2\|^*_{q,B(x,r)}) \\ &\leq C(n,K,\alpha,\rho,l_1,l_2,l_3)\big[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|f\|^*_{2q,B(x,r)})^2\big].\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \leq \|v\|_{\infty,B(x,\frac{1}{2}r)} \leq C(n,K,\alpha,\rho,l_1,l_2,l_3)\big[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|f\|^*_{2q,B(x,r)})^2\big].$$

This completes the proof. □

The proof of Corollary 2.2 is the same, so we omit it.

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