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Plasma particles dispersion based on Bogoyavlensky-Konopelchenko mathematical model

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Abstract

An optimal system of Lie infinitesimals has been used in an investigation to find a solution to the (2+1)-dimensional Bogoyavlensky-Konopelchenko equation (BKE). This investigation was conducted to characterize certain fantastic characteristics of plasma-particle dispersion. A careful investigation into the Lie space with an unlimited number of dimensions was carried out to locate the relevant arbitrary functions. When developing accurate solutions for the BKE, it was necessary to establish an optimum system that could be employed in single, double, and triple combination forms. There were some fantastic wave solutions developed, and these were depicted visually. The Optimal Lie system demonstrates that it can obtain many accurate solutions to evolution equations.

Keywords. Evolution equations, BKE, Lie infinitesimals, Optimal Lie system.
2010 Mathematics Subject Classification. 37L05, 17B45.

1. Introduction

Investigating nonlinear physical processes plays a crucial role in understanding various scientific and technological phenomena. A key aspect of this investigation is the analysis of precise traveling wave solutions to nonlinear partial differential equations. These solutions provide valuable insights into the behavior of nonlinear waves and are essential for advancing knowledge in many fields, particularly in industry. Nonlinear wave phenomena are prevalent across diverse disciplines, spanning fluid mechanics, plasma physics, optical fibers, biology, solid-state physics, chemical kinematics, chemical physics, and geochemistry, among others. By studying the intricate dynamics of nonlinear waves in these areas, researchers can uncover fundamental principles and develop practical applications. One widely recognized evolution equation used in various contexts is the Bogoyavlensky-Konopelchenko equation (BKE). This equation's versatility allows it to be applied in different settings, making it a valuable tool for investigating nonlinear wave phenomena. For instance, the BKE can help explain the dispersion of particles in plasma, which occurs due to the diffusion of plasma. By employing this theory, scientists can gain a better understanding of plasma dynamics and its implications in fields such as fusion energy research and astrophysics. Numerous methods have been employed to attain exact solutions of these equations, such as Backlund transform [1–4], auxiliary function method [5], extended tanh method [6–9], Lie group method [10–15], Darboux transformation [16–19], the projective Riccati equations method [20, 21], sine-cosine method [9, 22, 23] and the variation iteration method [24–26]. Moreover, sub-equation method [27, 28], nonlocal conservation laws [29], and Painlevé analysis via WTC–Kruskal algorithm [30] are also some useful methods. Similarity transformation methods, which can be used to construct either exact solutions or reduce the partial differential equation (PDE) model into an ordinary differential equation (ODE) model, are one of the most powerful techniques. These methods include Lie infinitesimals, group transformation methods, and hidden symmetries, among others, and can be used to construct either exact solutions or reduce the PDE model into an ODE model [31–42]. The motivation of this manuscript is to discover new families of solutions compatible with describing the dispersion

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of plasma particles. The technique is to investigate and create an optimal system of Lie space instead of using the typical Lie infinitesimals directly. Moreover, the hidden symmetry vectors are detected during the reduction process, which will lead to totally new families of solutions. The paper is arranged as follows. In section 2, the mathematical formulation of the problem is introduced with Lie infinitesimals vectors. Then, the commutator and adjoint tables are constructed in section 3. The optimal system is discovered in section 4, and the results are illustrated in section 5. Finally, the paper is terminated by conclusion remarks.

2. Mathematical formulation

BKE in (2+1) dimensions may describe the dispersion of plasma particles. One of its forms can be written as:

$$u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx} u_y = 0, \tag{1}$$

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t} + F_1(t)\frac{\partial}{\partial x} + \left(\frac{y}{2}\frac{d}{dt}F_1(t) + F_2(t)\right)\frac{\partial}{\partial u},\\
X_2 &= F_3(t)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \left(\frac{y}{2}\frac{d}{dt}F_3(t) + F_4(t)\right)\frac{\partial}{\partial u},\\
X_3 &= F_5(t)\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}\left(\frac{y}{2}\frac{d}{dt}F_5(t) + F_6(t) + \frac{x}{4}\right)\frac{\partial}{\partial u},\\
X_4 &= t\frac{\partial}{\partial t} + \left(F_7(t) + \frac{x}{2}\right)\frac{\partial}{\partial x} + \left(\frac{y}{2}\frac{d}{dt}F_7(t) + F_8(t) - \frac{u}{2}\right)\frac{\partial}{\partial u},\\
X_5 &= \left(F_9(t) - \frac{x}{2}\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + y\frac{\partial}{\partial y} + \left(\frac{y}{2}\frac{d}{dt}F_9(t) + F_{10}(t) + \frac{u}{2}\right)\frac{\partial}{\partial u},\\
X_6 &= \frac{t^2}{2}\frac{\partial}{\partial t} + \left(F_{11}(t) + \frac{xt}{4}\right)\frac{\partial}{\partial x} + \frac{yt}{2}\frac{\partial}{\partial y} + \left(\frac{y}{2}\frac{d}{dt}F_{11}(t) + F_{12}(t) - \frac{ut}{4} + \frac{xy}{8}\right)\frac{\partial}{\partial u}.
\end{aligned} \tag{2}$$

Where, $F_i(t),\ i = 1..12$ are arbitrary functions to be determined later.

3. Commutator and adjoint tables

To evaluate the appropriate functions, the commutator table of the vectors X_1, X_2, X_3, X_4, X_5 *and* X_6 is constructed using the relation $[X_i, X_j] = X_i(X_j) - X_j(X_i)$. For example, the following commutators are computed.

$$[X_1, X_2] = \left(\frac{d}{dt}F_3(t)\right)\frac{\partial}{\partial x} + \left(\frac{y}{2}\frac{d^2}{dt^2}F_3(t) + \frac{d}{dt}F_4(t) - \frac{1}{2}\frac{d}{dt}F_1(t)\right),$$

$$[X_1, X_3] = \left(\frac{d}{dt}F_5(t)\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \left(\frac{y}{2}\frac{d^2}{dt^2}F_5(t) + \frac{d}{dt}F_6(t) + \frac{1}{4}F_1(t) - \frac{t}{2}\frac{d}{dt}F_1(t)\right)\frac{\partial}{\partial u},$$

$$\begin{aligned}
[X_1, X_4] &= \frac{\partial}{\partial t} + \left(\frac{d}{dt}F_7(t) + \frac{1}{2}F_1(t) - t\frac{d}{d}F_1(t)\right)\frac{\partial}{\partial x}\\
&+ \left(\frac{y}{2}\frac{d^2}{dt^2}F_7(t) + \frac{d}{dt}F_8(t) - \frac{y}{4}\frac{d}{dt}F_1(t) - \frac{1}{2}F_2(t) - \frac{ty}{2}\frac{d}{dt}F_1(t) - t\frac{d}{dt}F_2(t)\right)\frac{\partial}{\partial u},
\end{aligned}$$

$$[X_1,\ X_5] = \left(\frac{d}{dt}F_9(t) - \frac{1}{2}F_1(t)\right)\frac{\partial}{\partial x} + \left(\frac{y}{2}\frac{d^2}{dt^2}F_9(t) + \frac{d}{dt}F_{10}(t) + \frac{y}{4}\frac{d}{dt}F_1(t) + \frac{1}{2}F_2(t) - \frac{y}{2}\frac{d}{dt}F_1(t)\right)\frac{\partial}{\partial u},$$

$$
\begin{aligned}
[X_1, X_6] &= t\frac{\partial}{\partial t} + \left(\frac{d}{dt}F_{11}(t) + \frac{x}{4} + \frac{t}{4}F_1(t) - \frac{t^2}{2}\frac{d}{dt}F_1(t)\right)\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} \\
&+ \left(\frac{y}{2}\frac{d^2}{dt^2}F_{11}(t) + \frac{d}{dt}F_{12}(t) - \frac{u}{4} + \frac{y}{8}F_1(t) - \frac{ty}{8}\frac{d}{dt}F_1(t) - \frac{t}{4}F_2(t) - \frac{t^2 y}{4}\frac{d^2}{dt^2}F_1(t)\right)\frac{\partial}{\partial u} \\
&\left(-\frac{t^2}{2}\frac{d}{dt}F_2(t) - \frac{yt}{4}\frac{d}{dt}F_1(t)\right)\frac{\partial}{\partial u}.
\end{aligned} \tag{3}
$$

To get a solvable system, the following condition must be satisfied. $[X_i, X_j] = \lambda X_n$ where $n = 1 \ldots 6$.

Setting $[X_1, X_2] = 0$ leads to:

$$F_3(t) = 0 \quad and \quad F_4(t) = \frac{1}{2}F_1(t). \tag{4}$$

Setting $[X_1, X_3] = X_2$ leads to:

$$F_5(t) = 0, \quad F_4(t) = \frac{1}{2}F_1(t) \quad and \quad \frac{d}{dt}F_6(t) = \frac{1}{4}F_1(t) + \frac{t}{2}\frac{d}{dt}\ F_1(t). \tag{5}$$

Moreover, setting $[X_1, X_4] = X_1$ leads to:

$$\frac{d}{dt}F_7(t) = \frac{1}{2}F_1(t) + t\frac{d}{dt}F_1(t), \tag{6}$$

$$\frac{y}{2}\frac{d^2F_7}{dt^2} + \frac{dF_8}{dt} - \frac{y}{4}\frac{dF_1}{dt} - \frac{F_2}{2} - \frac{ty}{2}\frac{d^2F_1}{dt^2} - t\frac{dF_2}{dt} = \frac{y}{2}\frac{dF_1}{dt} + F_2. \tag{7}$$

Also, for $[X_1, X_5] = 0$, one can get

$$\frac{dF_9}{dt} - \frac{F_1}{2} = 0, \quad \frac{y}{2}\frac{d^2F_9}{dt^2} + \frac{dF_{10}}{dt} - \frac{y}{4}\frac{dF_1}{dt} + \frac{F_2}{2} = 0. \tag{8}$$

In the same way, the remaining commutators are simplified, and the resultant ordinary differential equations are solved simultaneously to get:

$$
\begin{aligned}
&F_1 = 2,\ F_2 = 2,\ F_3 = 0,\ F_4 = 1,\ F_5 = 0,\ F_6 = \frac{t}{2},\ F_7 = t,\ F_8 = 3t,\ F_9 = t, F_{10} = -t, \\
&F_{11} = \frac{t^2}{2}, \quad F_{12} = \frac{3}{2}t^2.
\end{aligned} \tag{9}
$$

Thus, the space of Lie vectors will be:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}, \\
X_2 &= \frac{\partial}{\partial y} + \frac{\partial}{\partial u}, \\
X_3 &= t\frac{\partial}{\partial y} + \left(\frac{t}{2} + \frac{x}{4}\right)\frac{\partial}{\partial u}, \\
X_4 &= t\frac{\partial}{\partial t} + \left(t + \frac{x}{2}\right)\frac{\partial}{\partial x} + \left(\frac{y}{2} + 3t - \frac{u}{2}\right)\frac{\partial}{\partial u}, \\
X_5 &= \left(t - \frac{x}{2}\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \left(\frac{y}{2} - t + \frac{u}{2}\right)\frac{\partial}{\partial u},
\end{aligned}
$$

$$X_6 = \frac{t^2}{2}\frac{\partial}{\partial t} + \left(\frac{t^2}{2} + \frac{xt}{4}\right)\frac{\partial}{\partial x} + \frac{yt}{2}\frac{\partial}{\partial y} + \left(\frac{yt}{2} + \frac{3}{2}t^2 - \frac{ut}{4} + \frac{xy}{8}\right)\frac{\partial}{\partial u}. \tag{10}$$

Finally, the commutator table is constructed hereafter:

Table 1. Commutator Table.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
$\mathbf{X_1}$	0	0	X_2	X_1	0	$X_4 + \frac{1}{2}X_5$
$\mathbf{X_2}$	0	0	0	0	X_2	$\frac{1}{2}X_3$
$\mathbf{X_3}$	$-X_2$	0	0	$-X_3$	X_3	0
$\mathbf{X_4}$	$-X_1$	0	X_3	0	0	X_6
$\mathbf{X_5}$	0	$-X_2$	$-X_3$	0	0	0
$\mathbf{X_6}$	$-X_4 - \frac{1}{2}X_5$	$\frac{-1}{2}X_3$	0	$-X_6$	0	0

The adjoint table is to be constructed using the relation:

$$Ad\left(e^{\varepsilon V}\right)W_0 = W_0 - \varepsilon\left[V, W_0\right] + \frac{\varepsilon^2}{2}\left[V, \left[V, W_0\right]\right] - \dots \tag{11}$$

Using this relation, the adjoint table is constructed in the form:

Table 2. Adjoint Table.

Ad	X_1	X_2	X_3	X_4	X_5	X_6
$\mathbf{X_1}$	X_1	X_2	$X_3 - \varepsilon X_2$	$X_4 - \varepsilon X_1$	X_5	$X_6 - \varepsilon\left(X_4 + \frac{1}{2}X_5\right) + \frac{\varepsilon^2}{2}X_1$
$\mathbf{X_2}$	X_1	X_2	X_3	X_4	$X_5 - \varepsilon X_2$	$X_6 - \frac{\varepsilon}{2}X_3$
$\mathbf{X_3}$	$X_1 + \varepsilon X_2$	X_2	X_3	$X_4 + \varepsilon X_3$	$X_5 - \varepsilon X_3$	X_6
$\mathbf{X_4}$	$e^{-\varepsilon}X_1$	X_2	$e^{-\varepsilon}X_3$	x_4	X_5	$e^{-\varepsilon}X_6$
$\mathbf{X_5}$	X_1	$e^{\varepsilon}X_2$	$e^{\varepsilon}X_3$	X_4	X_5	X_6
$\mathbf{X_6}$	$X_1 + \varepsilon\left(X_4 + \frac{1}{2}X_5\right) + \frac{\varepsilon^2}{2}X_6$	$X_2 + \varepsilon X_3$	X_3	$X_4 + \varepsilon X_6$	X_5	X_6

4. Optimal Lie system

The following adjoint matrices are constructed from Table 2.

$$\mathbf{Ad}(\mathbf{e}^{\varepsilon_1 \mathbf{X_1}}) = \begin{bmatrix} 1 & 0 & 0 & -\varepsilon_1 & 0 & \frac{\varepsilon^2}{2} \\ 0 & 1 & -\varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\varepsilon_1 \\ 0 & 0 & 0 & 0 & 1 & \frac{-\varepsilon_1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{Ad}(\mathbf{e}^{\varepsilon_2 \mathbf{X_2}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & --\varepsilon_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{-\varepsilon_2}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{Ad}(\mathbf{e}^{\varepsilon_3 \mathbf{X}_3}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon_3 & -\varepsilon_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{Ad}(\mathbf{e}^{\varepsilon_4 \mathbf{X}_4}) = \begin{bmatrix} e^{-\varepsilon_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\varepsilon_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\varepsilon_4} \end{bmatrix},$$

$$\mathbf{Ad}(\mathbf{e}^{\varepsilon_5 \mathbf{X}_5}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\varepsilon_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\varepsilon_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Ad(e^{\varepsilon_6 X_6}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_6 & 1 & 0 & 0 & 0 \\ \varepsilon_6 & 0 & 0 & 1 & 0 & 0 \\ \frac{\varepsilon_6}{2} & 0 & 0 & 0 & 1 & 0 \\ \frac{\varepsilon_6}{2} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

Now $Ad_g = Ad\left(e^{\varepsilon_1 X_1}\right) * Ad\left(e^{\varepsilon_2 X_2}\right) * \ldots * Ad(e^{\varepsilon_6 X_6})$.
Let:

$$\frac{1}{a} Ad_g \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix}. \tag{13}$$

To obtain the optimal system, the following cases are studied.
Case 1:
For $\alpha_6 \neq 0$, there are two subcases:
If $\varepsilon_1 = 0$, the vector X_3 is an optimal vector,
If $\varepsilon_2 = 0$, the vector $X_1 + X_4 + X_5$ is an optimal vector.
Case 2:
For $\alpha_5 \neq 0$, there are three subcases:

If $\varepsilon_3 \neq 0$, the vector $X_2 + X_3 + X_5$ is an optimal vector,
If $\varepsilon_3 = 0$, the vector $X_2 + X_5$ is an optimal vector,
If $\varepsilon_2 = \varepsilon_3 = 0$, the vector X_5 is an optimal vector.
Case 3:
For $\alpha_4 \neq 0$, the vector X_4 is an optimal vector.
Case 4:
For $\alpha_3 \neq 0$, the vector X_3 is an optimal vector.
Case 5:
For $\alpha_2 \neq 0$, the vector X_2 is an optimal vector.
Case 6:
For $\alpha_1 \neq 0$, the vector X_1 is an optimal vector.
Case 7:
For $\alpha_4 \neq 0, \alpha_5 \neq 0,$, the vector $X_1 + X_4$ is an optimal vector.
Following the same procedures, all cases were studied and the optimal system of Lie vectors are:
Single Vectors: $X_1,\ X_2,\ X_3,\ X_4,\ X_5$.
Double linear combinations: $X_1 + X_4,\ X_2 + X_4,\ X_1 + X_3,\ X_4 + X_5,\ X_1 + X_5$.
Triple linear combinations: $X_1 + X_4 + X_5,\ X_2 + X_3 + X_5$.
These vectors are used to reduce the Bogoyavlensky-Konopelchenko equation to an invariant ordinary differential equation to get the exact solutions.

5. Results and discussion

5.1. **Using $\mathbf{X_1}$.** The Equation (1) is reduced to:

$$w_{rrrs} + (2w_s - 2) + 4w_r w_{rs} = 0, \tag{14}$$

where, $r = -2t + x,\ s = y,\ w(r,\ s) = u(t,\ x,\ y) - 2t.$
Equation (14) has the following exact solution:

$$w(r,\ s) = 2C_2 \tanh\left(C_2\ r + \frac{s}{2\ C_2} + C_1\right) + C_3. \tag{15}$$

Then, by back substitution, the solution of Equation (1) is:

$$u(x,y,t) = 2t + 2C_2 \tanh\left(C_2(x-2t) + \frac{y}{2C_2} + C_1\right) + C_3. \tag{16}$$

Moreover, when studying the symmetry vectors of (14), these vectors can be used to reduce the equation to an ODE. Using the vector $\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + \frac{\partial}{\partial w}$, Equation (14) is reduced to:

$$\frac{d^4V(\alpha)}{d\alpha^4} - \left(6\frac{dV(\alpha)}{d\alpha} - 6\right)\frac{d^2V(\alpha)}{d\alpha^2} = 0, \tag{17}$$

where, $\alpha = -r + s,\ V(\alpha) = w(r,s) - r.$
Now, one of the symmetry vectors of (17) is:
$\frac{\partial}{\partial V}$ which reduces its order to:

$$\psi^{'''} = (6\psi - 6)\psi^{'}, where, \varepsilon = \alpha,\ \psi(\alpha) = \frac{dV(\alpha)}{d\alpha}. \tag{18}$$

Moreover, this Equation, (18), has one of its vectors as $\frac{\partial}{\partial \varepsilon}$ which reduces the order of (18) to:

$$TT^{''} = -T^{''2} + 6\delta - 6, \tag{19}$$

where

$$\delta = \psi(\varepsilon),\ T(\delta) = \frac{d\psi(\varepsilon)}{d\varepsilon},$$

which has a solution in the form:

$$T(\delta) = \pm\sqrt{2\delta^3 - 6\delta^2 - C_1\delta + C_2}. \tag{20}$$

Then, by back substitution, the solution of Equation (1) is written as:

$$u(x,\ y,t) = \ x\ +\ \sqrt{6}\ tan\left(\frac{\sqrt{6}}{2}\ (y\ +\ 2t\ -\ x)\ \ +\ C\right). \tag{21}$$

5.2. **Using $\mathbf{X_2}$.** By the same procedures, X_2 is used to get the following solution:

$$u(x,\ y,t)\ =\ y\ +\ F_1(t)\ +\ F_2(-2t\ +\ x), \tag{22}$$

where, $F_1 and\ F_2$ are arbitrary functions in their arguments.

5.3. **Using $\mathbf{X_3}$.** By the same procedures, X_3 is used to attain the following solution:

$$u(x,\ y,t) = \frac{(2t\ +\ x)y}{4t}\ +\ F_2(t)\ +\left(\frac{1}{\sqrt{t}}\right)F_1\left(\frac{x-\ 2t}{\sqrt{t}}\right), \tag{23}$$

where, $F_1 and\ F_2$ are arbitrary functions in their arguments.
Using the other optimal vectors, one can get the following solutions:

$$u(x,\ y,t) = \ t\left(2\ +\frac{x}{4}\right)\ +\ C_2\left(-\frac{t^2}{2}\ +\ y\right) - \frac{(-2t\ +\ x)^2}{8\,(2\ C_2\ -\ 2)}\ + C_3(-2t\ +\ x)\ +\ C_4, \tag{24}$$

$$u(x,\ y,t) = \frac{(2t\ +\ x+4)y}{4t+4}\ +\ F_2(t)\ +\left(\frac{1}{\sqrt{t+1}}\right)F_1\left(\frac{x-\ 2t}{\sqrt{t+1}}\right), \tag{25}$$

$$u(x,y,t) = 3t + 2\ C_2 \tanh\left(C_2(y-2t) +\ 2\ C_2\frac{y-t}{4\ C_2^2\ -\ 1}\ +\ C_1\right)\ +\ C_3, \tag{26}$$

Some of the previous results are depicted in Figures 1-4. These figures illustrate some interpretation of plasma particles dispersion in different cases of boundary conditions. These behaviors vary between kink solutions, trigonometric and arbitrary functions.

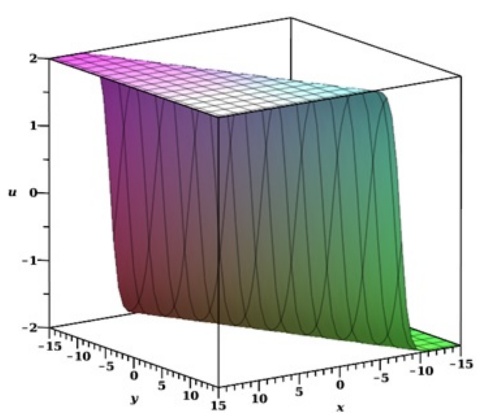

Figure 1. Solution behavior of Equation (16).

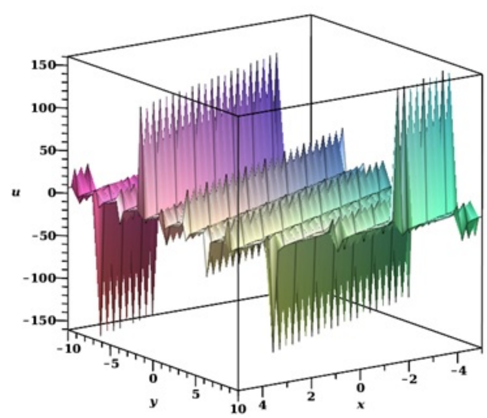

Figure 2. Solution behavior of Equation (21).

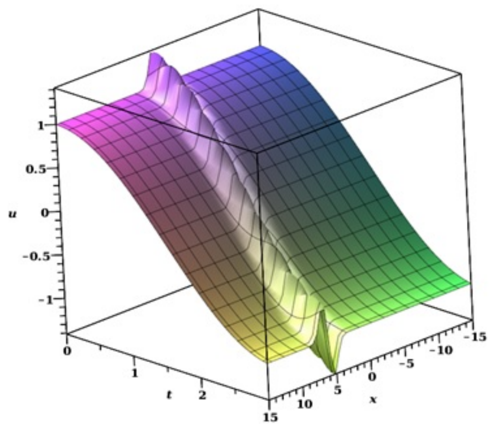

FIGURE 3. Solution behavior of Equation (22) with $F_1(t) = cos(t), F_2(x-2t) = (x-2t)e^{-(x\ -\ 2t)^2}$.

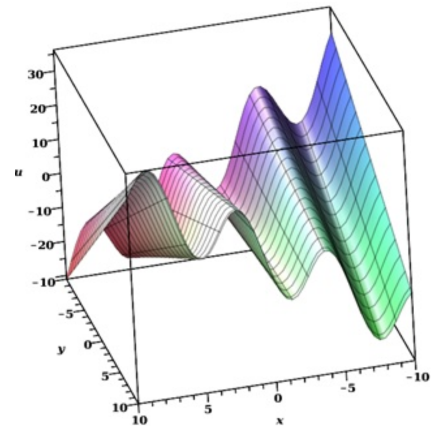

FIGURE 4. Solution behavior of Equation (23) with $F_2(t) = e^{-t}$, $F1\left(\frac{x-2t}{\sqrt{(t)}}\right) = 10*cos\left(\frac{x-\ 2t}{\sqrt{(t)}}\right)$.

The illustrations shown in Figures 1-4 show the behavior of plasma particles in different cases. For example, in Figure 1, the graph shows the case of limited dispersion of the particles. In Figure 2, the periodic behavior of the plasma particles is illustrated. Some chaotic arbitrary dispersion of the particles are shown in Figure 3 and 4 due to the arbitrary functions in the solution.

6. Conclusion

The solution of the (2+1)-dimensional BKE has been investigated using an optimal system of Lie infinitesimals to describe some fabulous behaviors of plasma-particle dispersion. The infinite-dimensional Lie space was precisely analyzed to detect the appropriate arbitrary functions. An optimal system was obtained as single, double, and triple combinations to be used in constructing exact solutions of the BKE. Some fabulous wave solutions were constructed and illustrated graphically. The Optimal Lie system proves its ability to get numerous exact solutions to evolution equations.

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