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An innovative computational approach for fuzzy space-time fractional Telegraph equation via the new iterative transform method

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Abstract

In this paper, the Fuzzy Sumudu Transform Iterative method (FSTIM) was applied to find the exact fuzzy solution of the fuzzy space-time fractional telegraph equations using the Fuzzy Caputo Fractional Derivative operator. The Telegraph partial differential equation is a hyperbolic equation representing the reaction-diffusion process in various fields. It has applications in engineering, biology, and physics. The FSTIM provides a reliable and efficient approach for obtaining approximate solutions to these complex equations improving accuracy and allowing for fine-tuning and optimization for better approximation results. The work introduces a fuzzy logic-based approach to Sumudu transform iterative methods, offering flexibility and adaptability in handling complex equations. This innovative methodology considers uncertainty and imprecision, providing comprehensive and accurate solutions, and advancing numerical methods. Solving the fuzzy space-time fractional telegraph equation used a fusion of the Fuzzy Sumudu transform and iterative approach. Solution of fuzzy fractional telegraph equation finding analytically and interpreting its results graphically. Throughout the article, whenever we draw graphs, we use Mathematica Software. We successfully employed FSTIM, which is elegant and fast to convergence.

Keywords. Fuzzy Sumudu Transform, Fuzzy Fractional differential equations, Fuzzy Telegraph equations, Fuzzy Sumudu Iterative transform method.

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1. Introduction

Fuzzy set theory plays a pivotal role in the simulation of uncertain problems, making it a versatile tool for representing a wide array of natural phenomena. One prominent application of this theory is the fractional fuzzy differential equation, which finds extensive utility across diverse scientific domains such as civil engineering, weapon systems assessment, population modeling, and electro-hydraulics[17]. In fuzzy calculus, the concept of fractional derivatives assumes paramount importance. Consequently, the study of fuzzy fractional differential equations has garnered significant attention within the scientific and mathematical communities [5–7, 10, 11, 14]. Notably, Agarwal et al.[46] have made notable contributions to understanding fuzzy fractional differential equations. They developed the Riemann-Liouville's concept within the framework of the Hukuhara notion, enables the analysis of fractional fuzzy differential equations. In our ever-ambiguous world, where uncertainty abounds, individuals often question the validity and clarity of information [7, 22, 50]. This prevalence of misinformation underscores the necessity for a mindset that embraces ambiguity, particularly among scientists [14, 42].

There has been a growing focus on employing fractional calculus as a potent method to achieve accurate solutions for a wide range of scientific and mathematical challenges, encompassing fields like communication systems, viscoelasticity, aerodynamics, process control, and bio-mathematics [1, 5, 7, 17, 24, 58, 60]. Talib et al. introduced a numerical method using the decomposition algorithm coupled with operational matrices to solve linear and nonlinear fractional initial value problems of various orders [55]. In mathematical physics and engineering, Alam introduced

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the innovative $\frac{G'}{G}$ expansion method for finding exact solutions to nonlinear evolution equations (NLEEs). His work also delved into the (3, 3, 3) time-fractional Zakharov-Kuznetsov (TFZK) equation and its wide-ranging applications in fields such as traffic flows, material science, sound waves, signal processing, and financial mathematics [9, 12, 59]. Alam examined soliton solutions of electric signals in telegraph lines [8], utilizing the modified expansion method to generate rational, hyperbolic, and trigonometric solutions. Furthermore, the study included the implementation of the expansion scheme to derive precise solutions for the fractional Clannish Random Walker's parabolic (FCRWP) equation and the nonlinear fractional Cahn-Allen (NFCA) equation [6]. Additionally, Alam investigated new results involving three nonlinear conformable models using the generalized Kudryashov method [5], defined with exponential and rational functions. Hoa et al. demonstrated the solvability of fuzzy fractional differential equations [32], while Salahshour et al. utilized Laplace fuzzy transforms [15] to explore related topics, among their various contributions.

In this article, authors consider the fuzzy space-time fractional telegraph equations in the following form:

$$\mathfrak{D}_{\hbar}^{\beta}\tilde{\Lambda}(\hbar,\wp) = \mathfrak{D}_{\wp}^{k\theta}\tilde{\Lambda}(\hbar,\wp) + \gamma_1\mathfrak{D}_{\wp}^{n\theta}\tilde{\Lambda}(\hbar,\wp) + \gamma_2\tilde{\Lambda}(\hbar,\wp) + \tilde{g}(\hbar,\wp), \quad 0 < \hbar \leq 1, \wp > 0, \tag{1.1}$$

and fuzzy initial conditions

$$\tilde{\Lambda}^{u}(0,\wp) = \tilde{\kappa}(r)\tilde{h}_u(\wp), u = 0, 1, 2, \ldots, u-1, \tag{1.2}$$

where, $\theta = \dfrac{1}{m}, k, m, n \in \mathbb{N},\ 1 < \beta \leq 2,\ 1 < k\theta \leq 2,\ 0 < n\theta \leq 1,$
$\mathfrak{D}_{\wp}^{k\theta} \equiv \mathfrak{D}_{\wp}^{\theta}\mathfrak{D}_{\wp}^{\theta}...\mathfrak{D}_{\wp}^{\theta}$ (k-times),
$\mathfrak{D}_{\wp}^{n\theta} \equiv \mathfrak{D}_{\wp}^{\theta}\mathfrak{D}_{\wp}^{\theta}...\mathfrak{D}_{\wp}^{\theta}$ (n-times),
$\mathfrak{D}_{\hbar}^{\beta}, \mathfrak{D}_{\wp}^{\theta}$- are Fuzzy Caputo fractional derivatives, $\mathfrak{E}_{\theta}$ is the Mittag-Leffler function, γ_1, γ_2 are the constants, $\tilde{g}(\hbar,\wp)$ is given fuzzy function and $\tilde{\kappa}(r) = \left[\underline{\kappa}(r), \overline{\kappa}(r)\right] = [r-1, 1-r],\ 0 \leq r \leq 1.$
In this case of $\beta = 2, m = 1, k = 2, n = 1$, and $\tilde{g} = 0$ fuzzy space-time Fractional Telegraph Equation (1.1) reduces to Fuzzy Telegraph Equation.

The Fuzzy Fractional Telegraph Equation (FFTE) is a complex mathematical model that plays a critical role in various real-world applications, including signal processing, bioinformatics, finance, and physics [8, 30, 56]. Solving FFTE is a challenge due to its fractional order and inherent fuzziness, requiring innovative computational methods for precision and efficiency. This study introduces a novel computational approach, the New Iterative Transform Method (NITM), to tackle FFTE. NITM transforms FFTE into a manageable form, reducing it to a system of ordinary differential equations by integrating fractional calculus and fuzzy logic principles. This enhances the representation of real-world uncertainties and complexities. The fractional order, a crucial aspect of FFTE, is treated using Caputo fractional derivatives, allowing us to accurately model non-local and memory-dependent behaviours. Furthermore, fuzzy logic is incorporated to account for the imprecisions and uncertainties inherent in real-world systems, providing a more realistic representation of the problem. The fuzzy logic-based approach enables the expression of vague parameters as fuzzy numbers, resulting in a more flexible and comprehensive computational framework for FFTE. The proposed method ensures improved accuracy and convergence rates compared with traditional techniques, providing a practical approach to solving FFTE.

The fuzzy Sumudu transform (FST) is first proposed in [2]. Abdul Rahman N. et al., Ahmed M. et al, and others solve linear FFDEs using FST [31]. The fractional Sumudu Transform examined semi-analytical solutions to fuzzy fractional-order linear and nonlinear dynamical problems [47, 51]. Agarwal proposed fuzzy fractional differential equations [1, 46]. In physics and engineering, Telegraph equations optimize directed communication systems, propagate electrical signals, and calculate quaternionic momentum eigenvalues [27, 52]. Ibrahim et al., Liu et al., Odibat et al., Yildirim et al., and Khastan et al. offer an iterative strategy for solving fractional differential equations [33, 35, 41, 44, 62]. In 2006, Jafari and Daftardar-Gejji devised the iterative strategy to solve linear and nonlinear fractional differential equations and achieved the series solution [3, 18, 21, 23, 29, 42]. Momani uses ADM to solve the space-and-time-fractional telegraph problem[43]. Ahmad and Ibrahim addressed the same type of problem [13, 26]. Zhao et al., Tapaswini, and Alawad employed a different approach and finite element method to approximate the space-time fractional telegraph equation [28, 56, 63]. Khan et al. created a new perturbative Laplace approach, while Garg et al.

utilized the transform method to solve the space-time-fractional telegraph equation [27, 39, 40]. Yldrm discussed the HPM [62], and Sevimlican used the VIM [52] to solve the governing equation.

The article's authors obtained numerical and analytical solutions to the fuzzy space-time Fractional Telegraph Equations using the Fuzzy Sumudu Transform Iterative method (FSTIM). The benefit of this new method is that it makes the calculations easy and gives the most accurate estimate of the exact solution [16, 21]. There are many problems in fractional derivatives [4, 34], hydrodynamics[61], chemical diffusion [22], and option pricing [3]. Nonlinear FPDEs and processes for finding numerical solutions to nonlinear problems have gotten much attention recently [45]. The main theme of this research is focused on the solution and analysis of fuzzy space-time fractional Telegraph equations with specific boundary conditions. The analytical method focuses on finding an exact solution to Fuzzy partial differential equations [38], but solving the fuzzy space-time fractional Telegraph equations is challenging due to the fractional derivative. The iterative approach discretizes the equation in space and time using techniques like the Fuzzy Sumudu Transform Iterative method (FSTIM). It updates it iteratively until it converges to the desired accuracy. The analytical method approximates the fractional derivative term, while the iterative approach transforms the equation into linear or nonlinear algebraic equations that can be solved iteratively. The proposed method's robustness allows it to handle uncertainty and imprecision without compromising solution accuracy, making it reliable and effective in unpredictable scenarios. Its adaptability makes it applicable across various fields, making it a powerful tool for complex equations.

Mathematica software provides powerful tools for creating visualizations and graphical representations of fuzzy space-time fractional Telegraph equations. It supports 2D and 3D plotting, graphing, and interactive visualizations, which aid in a better understanding of solutions to fuzzy space-time fractional Telegraph equations.

2. Basics

Definition 2.1. [25] Consider a continuous fuzzy function $\tilde{\Lambda}$ in $[0,\beta] \in \mathbb{R}$. The Fuzzy Fractional Riemann- Lioville integral(FFRLI) is defined as

$$\mathfrak{I}^{\beta}\tilde{\Lambda}(\hbar) = \frac{1}{\Gamma(\beta)}\int_0^{\hbar}(\hbar-\eta)^{\beta-1}\tilde{\Lambda}(\eta)d\eta, \qquad \beta,\eta \in (0,\infty), \tag{2.1}$$

moreover, if $\tilde{\Lambda} \in \mathbb{C}^F[0,\beta] \cap \mathbb{L}^F[0,\beta]$.
It is possible to derive the fuzzy fractional integral of a function as follows:

$$\left[\mathfrak{I}^{\beta}\tilde{\Lambda}(\hbar)\right]_r = \left[\mathfrak{I}^{\beta}\underline{\Lambda}(\hbar;r), \mathfrak{I}^{\beta}\overline{\Lambda}(\hbar;r)\right], \qquad 0 \le r \le 1, \tag{2.2}$$

such that,

$$\mathfrak{I}^{\beta}\underline{\Lambda}(\hbar;r) = \frac{1}{\Gamma(\beta)}\int_0^{\hbar}(\hbar-\eta)^{\beta-1}\underline{\Lambda}(\eta;r)d\eta, \qquad \beta,\eta \in (0,\infty), \tag{2.3}$$

and

$$\mathfrak{I}^{\beta}\overline{\Lambda}(\hbar;r) = \frac{1}{\Gamma(\beta)}\int_0^{\hbar}(\hbar-\eta)^{\beta-1}\overline{\Lambda}(\eta;r)d\eta, \qquad \beta,\eta \in (0,\infty), \tag{2.4}$$

Definition 2.2. [53] If the function $\tilde{\Lambda} \in \mathbb{C}^F[0,\beta] \cap \mathbb{L}^F[0,\beta]$ such that $\tilde{\Lambda} = \left[\underline{\Lambda}(\hbar;r), \overline{\Lambda}(\hbar;r)\right]$, $r \in [0,1]$ and $\hbar_0 \in (0,\beta)$ then, the Fuzzy Caputo fractional derivative (FCFD) is given as

$$\left[{}^{c}\mathfrak{D}^{\beta}\tilde{\Lambda}(\hbar_0)\right]_r = \left[{}^{c}\mathfrak{D}^{\beta}\underline{\Lambda}(\hbar_0;r), {}^{c}\mathfrak{D}^{\beta}\overline{\Lambda}(\hbar_0;r)\right], \qquad 0 < \beta \le 1, \tag{2.5}$$

where,

$$ {}^{c}\mathfrak{D}^{\beta}\underline{\Lambda}(\hbar_0;r)=\left[\frac{1}{\Gamma(\beta)}\int_0^{\hbar}(\hbar-\eta)^{m-\beta-1}\left(\frac{d^m}{d\eta^m}\right)\underline{\Lambda}(\eta;r)d\eta\right]_{\hbar=\hbar_0}, \tag{2.6}$$

$$ {}^{c}\mathfrak{D}^{\beta}\overline{\Lambda}(\hbar_0;r)=\left[\frac{1}{\Gamma(\beta)}\int_0^{\hbar}(\hbar-\eta)^{m-\beta-1}\left(\frac{d^m}{d\eta^m}\right)\overline{\Lambda}(\eta;r)d\eta\right]_{\hbar=\hbar_0}, \tag{2.7}$$

right side integral converges and $m=\lceil\beta\rceil$. Since $\beta\in(0,1]$, $m=1$.

Definition 2.3. [47] The fuzzy Sumudu transform (FST) for $\tilde{\Lambda}(\hbar)$, is defined as

$$\mathfrak{G}(p)=\mathfrak{S}\big[\tilde{\Lambda}(\hbar)\big](p)=\int_0^{\infty}e^{-\hbar}\odot\tilde{\Lambda}(p\hbar)d\hbar,\quad \hbar>0, p\in[-\tau_1,\tau_2], \tag{2.8}$$

where $\tilde{\Lambda}$ is value fuzzy term and $\tau_1,\tau_2>0$.

Definition 2.4. The "Mittag-Leffler" function[37] $\mathfrak{E}_{\beta}(\hbar)$ is defined as,

$$\mathfrak{E}_{\beta}(\hbar)=\sum_{n=0}^{\infty}\frac{\hbar^n}{\Gamma(1+n\beta)},\qquad \text{in one parameter,} \tag{2.9}$$

and

$$\mathfrak{E}_{\alpha,\beta}(\hbar)=\sum_{n=0}^{\infty}\frac{\hbar^n}{\Gamma(\alpha+n\beta)},\qquad \text{in two parameter,} \tag{2.10}$$

where $\alpha,\beta>0$.

Definition 2.5. [49] If $\mathfrak{G}(p)$ is the FST of the fuzzy function $\tilde{\Lambda}(\hbar)$, then $\mathfrak{S}\Big[\mathfrak{I}^{\beta}\tilde{\Lambda}(\hbar)\Big]$ is the FST of the FFRLI of $\tilde{\Lambda}(\hbar)$ of the order β

$$\mathfrak{S}\Big[\mathfrak{D}^{-\beta}\tilde{\Lambda}(\hbar)\Big](p)=\mathfrak{S}\Big[\mathfrak{I}^{\beta}\tilde{\Lambda}(\hbar)\Big](p)=p^{\beta}\mathfrak{G}(p),\quad \beta>0. \tag{2.11}$$

Theorem 2.6. *[19, 48] Let $\mathfrak{F}(p)=\mathfrak{S}\big[\tilde{f}(\hbar)\big]$ and $\mathfrak{G}(p)=\mathfrak{S}\big[\tilde{g}(\hbar)\big]$ be the FST. If*

$$\tilde{H}(\hbar)=\big(\tilde{f}*\tilde{g}\big)(\hbar)=\int_0^{\hbar}\tilde{f}(t)*\tilde{g}(\hbar-t)dt, \tag{2.12}$$

then the FST of $\tilde{H}(\hbar)$ is

$$\mathfrak{S}\Big[\tilde{H}(\hbar)\Big]=p\mathfrak{F}(p)\mathfrak{G}(p). \tag{2.13}$$

where '$*$' represents the fuzzy convolution that occurs between $\tilde{f}(\hbar)$ and $\tilde{g}(\hbar)$.

Theorem 2.7. *[36, 49] If $\mathfrak{G}(p)=\mathfrak{S}\big[\tilde{f}(\hbar)\big]$ be the FST of the fuzzy function. Then FST of m^{th} derivative of $\tilde{f}(\hbar)$ denoted by*

$$\mathfrak{S}\big[\tilde{f}^{\beta}(\hbar)\big](p)=\mathfrak{G}_{\beta}(p)=\frac{\mathfrak{G}(p)}{p^{\beta}}-\sum_{z=0}^{m-1}\frac{\tilde{f}^{(z)}(0)}{p^{(\beta-z)}},\qquad m\geq 1. \tag{2.14}$$

Theorem 2.8. *[36, 49] If $\beta > 0, m \in \mathbb{N}$, $m-1 < \beta \leq m$ and $\mathfrak{G}(p) = \mathfrak{S}\big[\tilde{f}(\hbar)\big]$ be the FST of the function then the FST of the FCFD of $f(\hbar)$ of order β is given by*

$$\mathfrak{S}\big[{}^{c}\mathfrak{D}_{\hbar}^{\beta}\tilde{f}(\hbar)\big](p) = \mathfrak{G}_{\beta}^{c}(p) = p^{-\beta}\Big[\mathfrak{G}(p) - \sum_{z=0}^{m-1} p^{z}\big(\tilde{f}^{z}(0)\big)\Big], \quad -1 < m-1 < \beta \leq m. \tag{2.15}$$

Definition 2.9. [53] A count of fuzzy $\kappa : \mathbb{R} \to [0,1]$ with the necessary characteristics

(1) κ is an upper semi-continuous number
(2) $\kappa\{\Theta(r_1) + \Theta(r_2)\} \geq \min\{\kappa(r_1), \kappa(r_2)\}$
(3) $\exists r_0 \in \mathbb{R}$ such that $\kappa(r_0) = 1$, that is, κ is normal
(4) $cl\{r \in \mathbb{R}, \kappa(r) > 0\}$ is compact.

E denotes the collection of all fuzzy counts.

Definition 2.10. [25, 54] The number in question can be stated mathematically as $\big[\underline{\kappa}(r), \overline{\kappa}(r)\big]$ in conjunction with the values

(1) $\overline{\kappa}(r)$ is a continuous, bounded, rising function from left to right over $[0,1]$
(2) $\underline{\kappa}(r)$ is a continuous, bounded, rising function from right over $[0,1]$
(3) $\overline{\kappa}(r) \leq \underline{\kappa}(r)$, $r \in [0,1]$.

3. Implementation of FSITM

Consider a nonlinear and non-homogeneous fuzzy Caputo fractional differential equation[23, 33, 57]

$${}^{c}\mathfrak{D}^{\beta}\tilde{\Lambda}(\hbar,\wp) + \mathfrak{L}\big[\tilde{\Lambda}(\hbar,\wp)\big] + \mathfrak{N}\big[\tilde{\Lambda}(\hbar,\wp)\big] = \tilde{g}(\hbar,\wp), \quad m-1 < \beta \leq m, m \in \mathbb{N}, \tag{3.1}$$

with initial conditions

$$\tilde{\Lambda}^{j}(\hbar,0) = \tilde{H}_{j}(\hbar), j = 0,1,2,\dots m-1, \tag{3.2}$$

where ${}^{c}\mathfrak{D}^{\beta}$ stands for the FCFD, $\mathfrak{N}$ is non-linear operator, L stands for the Linear, and $\tilde{g}(\hbar,\wp)$ is the fuzzy function.
Applying the FST on Equation (3.1) as

$$\mathfrak{S}\big[{}^{c}\mathfrak{D}^{\beta}\tilde{\Lambda}(\hbar,\wp)\big] + \mathfrak{S}\Big[\mathfrak{L}\big(\tilde{\Lambda}(\hbar,\wp)\big) + \mathfrak{N}\big(\tilde{\Lambda}(\hbar,\wp)\big)\Big] = \mathfrak{S}\Big[\tilde{g}(\hbar,\wp)\Big], \tag{3.3}$$

from Equation (2.15) with the initial condition, we get

$$\mathfrak{S}\Big[\tilde{\Lambda}(\hbar,\wp)\Big] = \frac{1}{p^{-\beta}}\sum_{z=0}^{m-1} p^{z-\beta}\big[\tilde{\Lambda}^{z}(\hbar,0)\big] - \frac{1}{p^{-\beta}}\mathfrak{S}\Big[\mathfrak{L}\big(\tilde{\Lambda}(\hbar,\wp)\big) + \mathfrak{N}\big(\tilde{\Lambda}(\hbar,\wp)\big)\Big] + \frac{1}{p^{-\beta}}\mathfrak{S}\Big[\tilde{g}(\hbar,\wp)\Big]. \tag{3.4}$$

Suppose that the solution is $\tilde{\Lambda}(\hbar,\wp) = \sum_{n=0}^{\infty}\tilde{\Lambda}_{n}(\hbar,\wp)$ then Equation (3.4) expressed

$$\mathfrak{S}\Big[\sum_{n=0}^{\infty}\tilde{\Lambda}_{n}(\hbar,\wp)\Big] = \begin{bmatrix} \frac{1}{p^{-\beta}}\sum_{z=0}^{m-1} p^{z-\beta}\big[\tilde{\Lambda}^{z}(\hbar,0)\big] - \frac{1}{p^{-\beta}}\mathfrak{S}\Big[\mathfrak{L}\Big(\sum_{n=0}^{\infty}\tilde{\Lambda}_{n}(\hbar,\wp)\Big) \\ +\mathfrak{N}\Big(\sum_{n=0}^{\infty}\tilde{\Lambda}_{n}(\hbar,\wp)\Big)\Big] + \frac{1}{p^{-\beta}}\mathfrak{S}\Big[\tilde{g}(\hbar,\wp)\Big] \end{bmatrix}. \tag{3.5}$$

When we look at things from both perspectives, we get the following:

$$\begin{aligned}
\mathfrak{S}\Big[\tilde{\Lambda}_0(\hbar,\wp)\Big] &= \frac{1}{p^{-\beta}}\sum_{z=0}^{m-1} p^{z-\beta}\big[\tilde{\Lambda}^z(\hbar,0)\big] + \frac{1}{p^{-\beta}}\mathfrak{S}\Big[\tilde{g}(\hbar,\wp)\Big],\\
\mathfrak{S}\Big[\tilde{\Lambda}_1(\hbar,\wp)\Big] &= -\frac{1}{p^{-\beta}}\mathfrak{S}\Big[\mathfrak{L}\Big(\tilde{\Lambda}_0(\hbar,\wp)\Big) + \mathfrak{N}\Big(\tilde{\Lambda}_0(\hbar,\wp)\Big)\Big],\\
\mathfrak{S}\Big[\tilde{\Lambda}_2(\hbar,\wp)\Big] &= -\frac{1}{p^{-\beta}}\mathfrak{S}\Big[\mathfrak{L}\Big(\tilde{\Lambda}_1(\hbar,\wp)\Big) + \mathfrak{N}\Big(\tilde{\Lambda}_1(\hbar,\wp)\Big)\Big],\\
&\vdots\\
\mathfrak{S}\Big[\tilde{\Lambda}_{n+1}(\hbar,\wp)\Big] &= -\frac{1}{p^{-\beta}}\mathfrak{S}\Big[\mathfrak{L}\Big(\tilde{\Lambda}_n(\hbar,\wp)\Big) + \mathfrak{N}\Big(\tilde{\Lambda}_n(\hbar,\wp)\Big)\Big], n \geq 0.
\end{aligned} \tag{3.6}$$

Applying the Inverse Fuzzy Sumudu transform, we have

$$\begin{aligned}
\tilde{\Lambda}_0(\hbar,\wp) &= \mathfrak{S}^{-1}\Big[\frac{1}{p^{-\beta}}\sum_{z=0}^{m-1} p^{z-\beta}\big[\tilde{\Lambda}^z(\hbar,0)\big] + \frac{1}{p^{-\beta}}\mathfrak{S}\Big[\tilde{g}(\hbar,\wp)\Big]\Big],\\
\tilde{\Lambda}_1(\hbar,\wp) &= -\mathfrak{S}^{-1}\Big[\frac{1}{p^{-\beta}}\mathfrak{S}\Big[\mathfrak{L}\Big(\tilde{\Lambda}_0(\hbar,\wp)\Big) + \mathfrak{N}\Big(\tilde{\Lambda}_0(\hbar,\wp)\Big)\Big]\Big],\\
\tilde{\Lambda}_2(\hbar,\wp) &= -\mathfrak{S}^{-1}\Big[\frac{1}{p^{-\beta}}\mathfrak{S}\Big[\mathfrak{L}\Big(\tilde{\Lambda}_1(\hbar,\wp)\Big) + \mathfrak{N}\Big(\tilde{\Lambda}_1(\hbar,\wp)\Big)\Big]\Big],\\
&\vdots\\
\tilde{\Lambda}_{n+1}(\hbar,\wp) &= -\mathfrak{S}^{-1}\Big[\frac{1}{p^{-\beta}}\mathfrak{S}\Big[\mathfrak{L}\Big(\tilde{\Lambda}_n(\hbar,\wp)\Big) + \mathfrak{N}\Big(\tilde{\Lambda}_n(\hbar,\wp)\Big)\Big]\Big], n \geq 0.
\end{aligned} \tag{3.7}$$

Therefore, the necessary solution to the series may be accomplished by

$$\tilde{\Lambda}(\hbar,\wp) = \tilde{\Lambda}_0(\hbar,\wp) + \tilde{\Lambda}_1(\hbar,\wp) + \tilde{\Lambda}_2(\hbar,\wp) + \dots. \tag{3.8}$$

4. Convergence and Error Analysis

Theorem 4.1. *[20] Let $\tilde{\Lambda}_p(\hbar,\wp)$ and $\tilde{\Lambda}_n(\hbar,\wp)$ be the members of fuzzy Banach space H, and the exact solution of (1.1) be $\tilde{\Lambda}(\hbar,\wp)$. The Series solution $\sum_{p=0}^{\infty}\tilde{\Lambda}_p(\hbar,\wp)$ converges to $\tilde{\Lambda}(\hbar,\wp)$, if $||\tilde{\Lambda}_p(\hbar,\wp)|| \leq \lambda||\tilde{\Lambda}_{p-1}(\hbar,\wp)||$ for $\lambda \in (0,1)$, that is for any $\Lambda > 0, \exists I$ such that $||\tilde{\Lambda}_{p+n}(\hbar,\wp)|| \leq \Lambda, \forall p, n > I$.*

Proof. Let $u_p(\hbar,\wp) = \tilde{\Lambda}_0(\hbar,\wp) + \tilde{\Lambda}_1(\hbar,\wp) + \tilde{\Lambda}_2(\hbar,\wp) + \ldots + \tilde{\Lambda}_p(\hbar,\wp)$ be the sequence of p^{th} partial sum of series $\sum_{p=0}^{\infty} \tilde{\Lambda}_p(\hbar,\wp)$. Now consider

$$\begin{aligned} ||u_{p+1}(\hbar,\wp) - u_p(\hbar,\wp)|| &= ||\tilde{\Lambda}_{p+1}(\hbar,\wp)|| \\ &\leq \lambda||\tilde{\Lambda}_p(\hbar,\wp)|| \\ &\leq \lambda^2||\tilde{\Lambda}_{p-1}(\hbar,\wp)|| \\ &\leq \lambda^3||\tilde{\Lambda}_{p-2}(\hbar,\wp)|| \\ &\vdots \\ &\leq \lambda^{p+1}||\tilde{\Lambda}_0(\hbar,\wp)||. \end{aligned} \tag{4.1}$$

for $\forall n, p \in E$
Consider,

$$\begin{aligned} ||u_p(\hbar,\wp) - u_n(\hbar,\wp)|| &= ||\tilde{\Lambda}_{p+n}(\hbar,\wp)|| \\ &= ||\big(u_p(\hbar,\wp) - u_{p-1}(\hbar,\wp)\big) \\ &+ \big(u_{p-1}(\hbar,\wp) - u_{p-2}(\hbar,\wp)\big) \\ &+ \big(u_{p-2}(\hbar,\wp) - u_{p-3}(\hbar,\wp)\big) \\ &+ \ldots + \big(u_{n+1}(\hbar,\wp) - u_n(\hbar,\wp)\big)|| \\ &\leq ||\big(u_p(\hbar,\wp) - u_{p-1}(\hbar,\wp)\big)|| \\ &+ ||\big(u_{p-1}(\hbar,\wp) - u_{p-2}(\hbar,\wp)\big)|| \\ &+ ||\big(u_{p-2}(\hbar,\wp) - u_{p-3}(\hbar,\wp)\big)|| \\ &+ \ldots + ||\big(u_{n+1}(\hbar,\wp) - u_n(\hbar,\wp)\big)|| \\ &\leq \lambda^p||\tilde{\Lambda}_0(\hbar,\wp)|| \\ &+ \lambda^{p-1}||\tilde{\Lambda}_0(\hbar,\wp)|| \\ &+ \lambda^{p-2}||\tilde{\Lambda}_0(\hbar,\wp)|| \\ &+ \ldots + \lambda^{p-1}||\tilde{\Lambda}_0(\hbar,\wp)|| \\ &= ||\tilde{\Lambda}_0(\hbar,\wp)||\big(\lambda^p + \lambda^{p-1} + \ldots + \lambda^{p+1}\big) \\ &= ||\tilde{\Lambda}_0(\hbar,\wp)||\big(\frac{1-\lambda^{p-n}}{1-\lambda}\big)\lambda^{n+1}. \end{aligned} \tag{4.2}$$

$\tilde{\Lambda}_0(\hbar,\wp)$ is bounded, since $0 < \lambda < 1$. Also $\sum_{p=0}^{\infty} \tilde{\Lambda}_p(\hbar,\wp)$ is a Cauchy sequence in H, which implies that there exists $\tilde{\Lambda}_0(\hbar,\wp) \in H$ such that $\lim_{p\to\infty} \tilde{\Lambda}_p(\hbar,\wp) = \tilde{\Lambda}(\hbar,\wp)$. By assuming,

$$\Lambda = ||\tilde{\Lambda}_0(\hbar,\wp)||\big(\frac{1-\lambda^{p-n}}{1-\lambda}\big)\lambda^{n+1},$$

we get the desired result. Hence complete the proof of theorem. □

Theorem 4.2. *[20] If the finite and approximate solution of Equation (1.1) is* $\sum_{p=0}^{q} \tilde{\Lambda}_p(\hbar, \wp)$ *and* $||\tilde{\Lambda}_{p+1}(\hbar, \wp)|| \leq \lambda ||\tilde{\Lambda}_0(\hbar, \wp)||$ *for* $\lambda \in (0,1)$, *then the maximum absolute error is*

$$||\tilde{\Lambda}(\hbar, \wp) - \sum_{p=0}^{q} \tilde{\Lambda}_p(\hbar, \wp)|| \leq \frac{\lambda^{q+1}}{1-\lambda} ||\tilde{\Lambda}_0(\hbar, \wp)||.$$

Proof. Consider

$$\begin{aligned} ||\tilde{\Lambda}(\hbar, \wp) - \sum_{p=0}^{q} \tilde{\Lambda}_p(\hbar, \wp)|| &= ||\sum_{p=0}^{\infty} \tilde{\Lambda}_p(\hbar, \wp)|| \\ &\leq \sum_{p=q+1}^{\infty} ||\tilde{\Lambda}_p(\hbar, \wp)|| \\ &\leq \sum_{p=q+1}^{\infty} \lambda^q ||\tilde{\Lambda}_0(\hbar, \wp)|| \\ &= \lambda^{q+1}\big(1 + \lambda + \lambda^2 + \dots\big) ||\tilde{\Lambda}_0(\hbar, \wp)|| \\ &= \frac{\lambda^{q+1}}{1-\lambda} ||\tilde{\Lambda}_0(\hbar, \wp)|| \end{aligned} \tag{4.3}$$

i.e. $||\tilde{\Lambda}(\hbar, \wp) - \sum_{p=0}^{q} \tilde{\Lambda}_p(\hbar, \wp)|| \leq \frac{\lambda^{q+1}}{1-\lambda} ||\tilde{\Lambda}_0(\hbar, \wp)||$,
hence complete the proof of the theorem. □

5. Numerical Result

Example 5.1. Consider linear homogeneous fuzzy space-time fractional Telegraph equation

$$\mathfrak{D}_{\hbar}^{\beta} \tilde{\Lambda}(\hbar, \wp) = \mathfrak{D}_{\wp}^{k\theta} \tilde{\Lambda}(\hbar, \wp) + \mathfrak{D}_{\wp}^{n\theta} \tilde{\Lambda}(\hbar, \wp) + \tilde{\Lambda}(\hbar, \wp), \quad 0 < \hbar \leq 1, \wp > 0, \tag{5.1}$$

and fuzzy initial conditions

$$\tilde{\Lambda}(0, \wp) = \tilde{\kappa}(r) \mathfrak{E}_{\theta}(-\wp^{\theta}) \quad \text{and } \tilde{\Lambda}_{\hbar}(0, \wp) = \tilde{\kappa}(r) \mathfrak{E}_{\theta}(-\wp^{\theta}), \tag{5.2}$$

where, $\theta = \dfrac{1}{m}, k, m, n \in \mathbb{N},\ 1 < \beta \leq 2,\ 1 < k\theta \leq 2,\ 0 < n\theta \leq 1$,
$\mathfrak{D}_{\wp}^{k\theta} \equiv \mathfrak{D}_{\wp}^{\theta} \mathfrak{D}_{\wp}^{\theta} ... \mathfrak{D}_{\wp}^{\theta}$ (k-times),
$\mathfrak{D}_{\wp}^{n\theta} \equiv \mathfrak{D}_{\wp}^{\theta} \mathfrak{D}_{\wp}^{\theta} ... \mathfrak{D}_{\wp}^{\theta}$ (n-times),
$\mathfrak{D}_{\hbar}^{\beta}, \mathfrak{D}_{\wp}^{\theta}$- are Fuzzy Caputo fractional derivaties, $\mathfrak{E}_{\theta}$ is the Mittag-Leffler function, $k+n$ is odd, and $\tilde{\kappa}(r) = \big[\underline{\kappa}(r), \overline{\kappa}(r)\big] = [r-1, 1-r],\ 0 \leq r \leq 1$.

Applying Equation (3.7), the results are as follows.

$$\underline{\Lambda}_0(\hbar, \wp) = \underline{\kappa}(r)\big(1+\hbar\big) \mathfrak{E}_{\theta}(-\wp^{\theta}), \quad \overline{\Lambda}_0(\hbar, \wp) = \overline{\kappa}(r)\big(1+\hbar\big) \mathfrak{E}_{\theta}(-\wp^{\theta}),$$

$$\underline{\Lambda}_1(\hbar, \wp) = \underline{\kappa}(r)\left(\frac{\hbar^{\beta}}{\Gamma(1+\beta)} + \frac{\hbar^{\beta+1}}{\Gamma(2+\beta)}\right) \mathfrak{E}_{\theta}(-\wp^{\theta}), \quad \overline{\Lambda}_1(\hbar, \wp) = \overline{\kappa}(r)\left(\frac{\hbar^{\beta}}{\Gamma(1+\beta)} + \frac{\hbar^{\beta}}{\Gamma(2+\beta)}\right) \mathfrak{E}_{\theta}(-\wp^{\theta}),$$

$$\underline{\Lambda}_2(\hbar, \wp) = \underline{\kappa}(r)\left(\frac{\hbar^{2\beta}}{\Gamma(1+2\beta)} + \frac{\hbar^{2\beta+1}}{\Gamma(2+2\beta)}\right) \mathfrak{E}_{\theta}(-\wp^{\theta}), \quad \overline{\Lambda}_2(\hbar, \wp) = \overline{\kappa}(r)\left(\frac{\hbar^{2\beta}}{\Gamma(1+2\beta)} + \frac{\hbar^{2\beta+1}}{\Gamma(2+2\beta)}\right) \mathfrak{E}_{\theta}(-\wp^{\theta}),$$

and so on. The required series solution can be stated as an infinite series.

$$\tilde{\Lambda}(\hbar, \wp) = \tilde{\Lambda}_0(\hbar, \wp) + \tilde{\Lambda}_1(\hbar, \wp) + \dots, \tag{5.3}$$

such that

$$\begin{aligned}\underline{\Lambda}(\hbar,\wp) &= \underline{\Lambda}_0(\hbar,\wp)+\underline{\Lambda}_1(\hbar,\wp)+\underline{\Lambda}_2(\hbar,\wp)+\dots,\\ \overline{\Lambda}(\hbar,\wp) &= \overline{\Lambda}_0(\hbar,\wp)+\overline{\Lambda}_1(\hbar,\wp)+\overline{\Lambda}_2(\hbar,\wp)+\dots.\end{aligned}\tag{5.4}$$

In general, we can write the series solution as

$$\begin{aligned}\underline{\Lambda}(\hbar,\wp) &= \begin{bmatrix} \underline{\kappa}(r)\big(1+\hbar\big)\mathfrak{E}_\theta(-\wp^\theta) \\ +\underline{\kappa}(r)\left(\dfrac{\hbar^\beta}{\Gamma(1+\beta)}+\dfrac{\hbar^{\beta+1}}{\Gamma(2+\beta)}\right)\mathfrak{E}_\theta(-\wp^\theta) \\ +\underline{\kappa}(r)\left(\dfrac{\hbar^{2\beta}}{\Gamma(1+2\beta)}+\dfrac{\hbar^{2\beta+1}}{\Gamma(2+2\beta)}\right)\mathfrak{E}_\theta(-\wp^\theta)\dots \end{bmatrix},\\ \overline{\Lambda}(\hbar,\wp) &= \begin{bmatrix} \overline{\kappa}(r)\big(1+\hbar\big)\mathfrak{E}_\theta(-\wp^\theta) \\ +\overline{\kappa}(r)\left(\dfrac{\hbar^\beta}{\Gamma(1+\beta)}+\dfrac{\hbar^{\beta+1}}{\Gamma(2+\beta)}\right)\mathfrak{E}_\theta(-\wp^\theta) \\ +\overline{\kappa}(r)\left(\dfrac{\hbar^{2\beta}}{\Gamma(1+2\beta)}+\dfrac{\hbar^{2\beta+1}}{\Gamma(2+2\beta)}\right)\mathfrak{E}_\theta(-\wp^\theta)\dots \end{bmatrix}.\end{aligned}\tag{5.5}$$

The Exact solution is

$$\tilde{\Lambda}(\hbar,\wp)=\tilde{\kappa}(r)\Big[\mathfrak{E}_\beta(\hbar^\beta)+\hbar\mathfrak{E}_{\beta,2}(\hbar^\beta)\Big]\mathfrak{E}_\theta(-\wp^\theta).\tag{5.6}$$

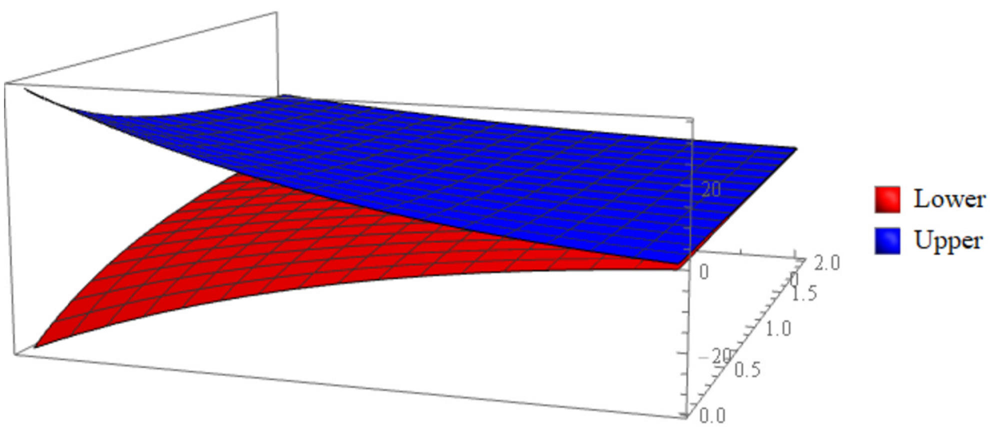

FIGURE 1. Exact solution at $\beta = 2$.

Remark 5.2. The fuzzy space-time fractional Telegraph Equation (5.1) may be simplified by putting $m = n = 1, k = 2,$ to fuzzy space fractional Telegraph equation

$$\mathfrak{D}_\hbar^\beta\tilde{\Lambda}(\hbar,\wp)=\mathfrak{D}_\wp^2\tilde{\Lambda}(\hbar,\wp)+\mathfrak{D}_\wp\tilde{\Lambda}(\hbar,\wp)+\tilde{\Lambda}(\hbar,\wp),\quad 0<\hbar\le 1,\wp>0,\tag{5.7}$$

with solution

$$\tilde{\Lambda}(\hbar,\wp)=\tilde{\kappa}(r)\mathfrak{E}_\beta\big(\hbar^\beta\big)e^{-\wp}.\tag{5.8}$$

Remark 5.3. The fuzzy space-time fractional Telegraph Equation (5.1) may be simplified by putting $\beta = 2$ to fuzzy time fractional Telegraph equation

$$\mathfrak{D}_{\hbar}^{2}\tilde{\Lambda}(\hbar,\wp) = \mathfrak{D}_{\wp}^{k\theta}\tilde{\Lambda}(\hbar,\wp) + \mathfrak{D}_{\wp}^{n\theta}\tilde{\Lambda}(\hbar,\wp) + \tilde{\Lambda}(\hbar,\wp), \quad 0 < \hbar \leq 1, \wp > 0, \tag{5.9}$$

with solution

$$\tilde{\Lambda}(\hbar,\wp) = \tilde{\kappa}(r)e^{\hbar}\mathfrak{E}_{\theta}\big(-\wp^{\theta}\big). \tag{5.10}$$

Remark 5.4. The fuzzy space-time fractional Telegraph Equation (5.1) may be simplified by putting $m = n = 1, \beta = 2, k = 2$, to fuzzy fractional Telegraph equation

$$\mathfrak{D}_{\hbar}^{2}\tilde{\Lambda}(\hbar,\wp) = \mathfrak{D}_{\wp}^{2}\tilde{\Lambda}(\hbar,\wp) + \mathfrak{D}_{\wp}\tilde{\Lambda}(\hbar,\wp) + \tilde{\Lambda}(\hbar,\wp), \quad 0 < \hbar \leq 1, \wp > 0, \tag{5.11}$$

with solution

$$\tilde{\Lambda}(\hbar,\wp) = \tilde{\kappa}(r)e^{\hbar-\wp}. \tag{5.12}$$

The physical behavior of the obtained solution at $\wp \in [0,2]$ is depicted in 5. The solution graphs of Example 5.1 can be seen to be friendly with each other in this illustration. Figure 5 shows how, with more iterations, the FSTIM solution approaches the precise answer more quickly. Figure 5 shows the 2D graphs of $\tilde{\Lambda}(\hbar,\wp)$ at several fractional orders of $\beta = 1.2, 1.6, 1.8$, and 2. Figure 5 depicts evaluative solution estimations. The charts show that as the fractional order β approaches its integer value, the curve will end at classical order 2.

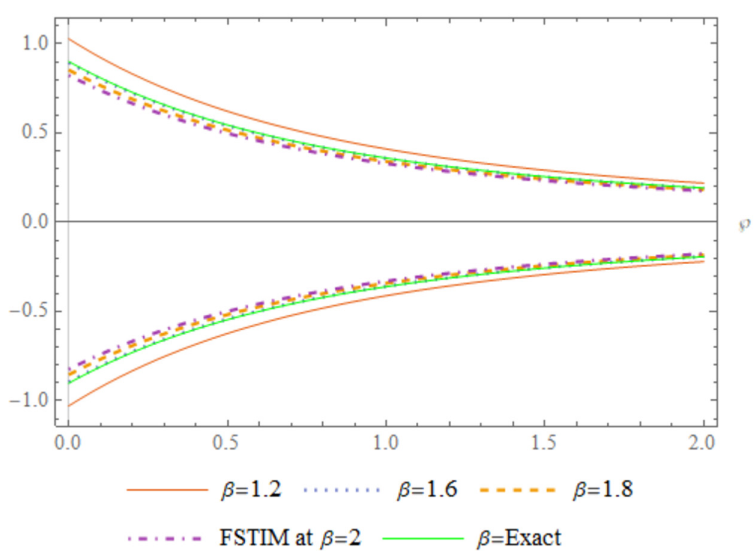

Figure 2. Example 1's upper and lower fuzzy figures for different fractional orders β.

The FSTIM is an effective method for discovering precise and approximate solutions to various linear and nonlinear partial differential equations involving multiple fractional order 'β '. The convergence analysis is sufficiently reliable to estimate the maximum absolute error of the Example 5.1 at $\beta = 2, \hbar = 0.2, 0.4, 0.6, 0.8$ and $\wp = 0.3, 0.5, 0.7, 0.9$

TABLE 1. Comparison and error analysis between the value $\tilde{\Lambda}(\hbar,\wp)$ for the Exact and Approximate solution of the Equation (5.1).

$\hbar$	$\wp$	$\underline{\Lambda}_{FSTIM}$	$\underline{\Lambda}_{Exact}$	$\underline{\Lambda}_{Error}$	$\overline{\Lambda}_{FSTIM}$	$\overline{\Lambda}_{Exact}$	$\overline{\Lambda}_{Error}$
0.2	0.3	-0.672811	-0.77362	0.100809	0.672811	0.77362	0.100809
0.4	0.5	-0.70421	-0.77362	0.0694097	0.70421	0.77362	0.0694097
0.6	0.7	-0.727883	-0.77362	0.0457371	0.727883	0.77362	0.0457371
0.8	0.9	-0.747289	-0.77362	0.026331	0.747289	0.77362	0.026331

Example 5.5. Consider non-homogeneous fuzzy space-time fractional Telegraph equation

$$\mathfrak{D}_{\hbar}^{\beta}\tilde{\Lambda}(\hbar,\wp)=\mathfrak{D}_{\wp}^{k\theta}\tilde{\Lambda}(\hbar,\wp)+\mathfrak{D}_{\wp}^{n\theta}\tilde{\Lambda}(\hbar,\wp)+\tilde{\Lambda}(\hbar,\wp)-2\mathfrak{E}_{\beta}(\hbar^{\beta})\mathfrak{E}_{\theta}(-\wp^{\theta}),\quad 0<\hbar\leq 1,\wp>0, \tag{5.13}$$

and fuzzy initial conditions

$$\tilde{\Lambda}(0,\wp)=\tilde{\kappa}(r)\mathfrak{E}_{\theta}(-\wp^{\theta})\quad\text{and }\tilde{\Lambda}_{\hbar}(0,\wp)=0, \tag{5.14}$$

where, $\theta=\dfrac{1}{m}, k,m,n\in\mathbb{N},\ 1<\beta\leq 2,\ 1<k\theta\leq 2,\ 0<n\theta\leq 1,$
$\mathfrak{D}_{\wp}^{k\theta}\equiv\mathfrak{D}_{\wp}^{\theta}\mathfrak{D}_{\wp}^{\theta}...\mathfrak{D}_{\wp}^{\theta}$ (k-times),
$\mathfrak{D}_{\wp}^{n\theta}\equiv\mathfrak{D}_{\wp}^{\theta}\mathfrak{D}_{\wp}^{\theta}...\mathfrak{D}_{\wp}^{\theta}$ (n-times),
$\mathfrak{D}_{\hbar}^{\beta},\mathfrak{D}_{\wp}^{\theta}$- are Fuzzy Caputo fractional derivatives, $\mathfrak{E}_{\theta}$ is the Mittag-Leffler function, k and n is odd and $\tilde{\kappa}(r)=\left[\underline{\kappa}(r),\overline{\kappa}(r)\right]=[r-1,1-r],0\leq r\leq 1.$

Applying Equation (3.7), results are as follow

$$\underline{\Lambda}_0(\hbar,\wp)=\underline{\kappa}(r)\begin{pmatrix}\mathfrak{E}_{\beta}\big(\hbar^{\beta}\big)\mathfrak{E}_{\theta}\big(-\wp^{\theta}\big)\\ -3\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+1)}}{\Gamma\big(\beta(u+1)+1\big)}\end{pmatrix},\quad \overline{\Lambda}_0(\hbar,\wp)=\overline{\kappa}(r)\begin{pmatrix}\mathfrak{E}_{\beta}\big(\hbar^{\beta}\big)\mathfrak{E}_{\theta}\big(-\wp^{\theta}\big)\\ -3\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+1)}}{\Gamma\big(\beta(u+1)+1\big)}\end{pmatrix},$$

$$\underline{\Lambda}_1(\hbar,\wp)=3\underline{\kappa}(r)\begin{pmatrix}\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+1)}}{\Gamma\big(\beta(u+1)+1\big)}\\ -3\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+2)}}{\Gamma\big(\beta(u+2)+1\big)}\end{pmatrix},\quad \overline{\Lambda}_1(\hbar,\wp)=3\overline{\kappa}(r)\begin{pmatrix}\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+1)}}{\Gamma\big(\beta(u+1)+1\big)}\\ -3\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+2)}}{\Gamma\big(\beta(u+2)+1\big)}\end{pmatrix},$$

$$\underline{\Lambda}_2(\hbar,\wp)=3^2\underline{\kappa}(r)\begin{pmatrix}\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+2)}}{\Gamma\big(\beta(u+2)+1\big)}\\ -3\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+3)}}{\Gamma\big(\beta(u+3)+1\big)}\end{pmatrix},\quad \overline{\Lambda}_2(\hbar,\wp)=3^2\overline{\kappa}(r)\begin{pmatrix}\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+2)}}{\Gamma\big(\beta(u+2)+1\big)}\\ -3\mathfrak{E}_{\theta}(-\wp^{\theta})\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+3)}}{\Gamma\big(\beta(u+3)+1\big)}\end{pmatrix},$$

and so on. The required series solution can be stated as an infinite series.

$$\tilde{\Lambda}(\hbar,\wp)=\tilde{\Lambda}_0(\hbar,\wp)+\tilde{\Lambda}_1(\hbar,\wp)+\ldots, \tag{5.15}$$

such that

$$\begin{aligned}\underline{\Lambda}(\hbar,\wp)&=\underline{\Lambda}_0(\hbar,\wp)+\underline{\Lambda}_1(\hbar,\wp)+\underline{\Lambda}_2(\hbar,\wp)+\ldots,\\ \overline{\Lambda}(\hbar,\wp)&=\overline{\Lambda}_0(\hbar,\wp)+\overline{\Lambda}_1(\hbar,\wp)+\overline{\Lambda}_2(\hbar,\wp)+\ldots.\end{aligned} \tag{5.16}$$

In general, we can write the series solution as

$$\underline{\Lambda}(\hbar,\wp)=\underline{\kappa}(r)\left[\begin{array}{c}\left(\begin{array}{c}\mathfrak{E}_\beta\big(\hbar^\beta\big)\mathfrak{E}_\theta\big(-\wp^\theta\big)\\ -3\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+1)}}{\Gamma\big(\beta(u+1)+1\big)}\end{array}\right)+3\left(\begin{array}{c}\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+1)}}{\Gamma\big(\beta(u+1)+1\big)}\\ -3\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+2)}}{\Gamma\big(\beta(u+2)+1\big)}\end{array}\right)\\ +3^2\left(\begin{array}{c}\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+2)}}{\Gamma\big(\beta(u+2)+1\big)}\\ -3\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+3)}}{\Gamma\big(\beta(u+3)+1\big)}\end{array}\right)+\dots\end{array}\right],$$

$$\overline{\Lambda}(\hbar,\wp)=\overline{\kappa}(r)\left[\begin{array}{c}\left(\begin{array}{c}\mathfrak{E}_\beta\big(\hbar^\beta\big)\mathfrak{E}_\theta\big(-\wp^\theta\big)\\ -3\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+1)}}{\Gamma\big(\beta(u+1)+1\big)}\end{array}\right)+3\left(\begin{array}{c}\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+1)}}{\Gamma\big(\beta(u+1)+1\big)}\\ -3\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+2)}}{\Gamma\big(\beta(u+2)+1\big)}\end{array}\right)\\ +3^2\left(\begin{array}{c}\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+2)}}{\Gamma\big(\beta(u+2)+1\big)}\\ -3\mathfrak{E}_\theta(-\wp^\theta)\sum\limits_{u=0}^{\infty}\dfrac{\hbar^{\beta(u+3)}}{\Gamma\big(\beta(u+3)+1\big)}\end{array}\right)+\dots\end{array}\right]. \tag{5.17}$$

The Exact solution is

$$\tilde{\Lambda}(\hbar,\wp)=\tilde{\kappa}(r)\Big[\mathfrak{E}_\beta\big(\hbar^\beta\big)\mathfrak{E}_\theta\big(-\wp^\theta\big)\Big]. \tag{5.18}$$

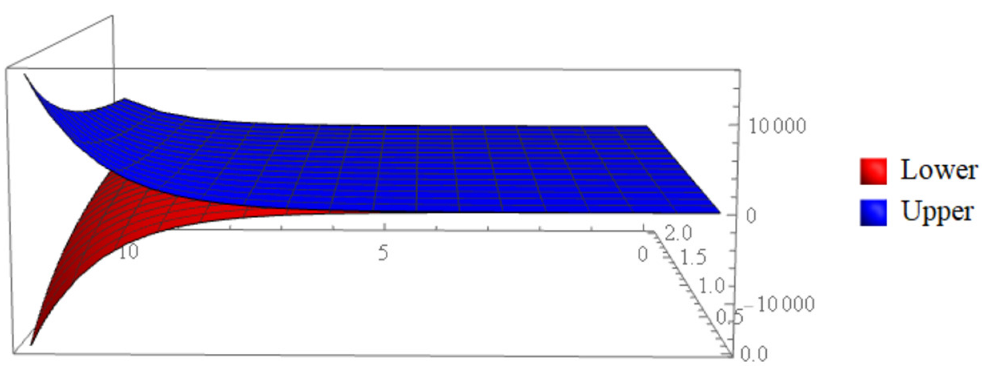

FIGURE 3. Exact solution at $\beta = 2$.

Remark 5.6. The non-homogeneous fuzzy space-time fractional Telegraph Equation (5.11) may be simplified by putting $m = n = 2, k = 4,$ to non-homogeneous fuzzy space fractional Telegraph equation

$$\mathfrak{D}_\hbar^\beta\tilde{\Lambda}(\hbar,\wp)=\mathfrak{D}_\wp^2\tilde{\Lambda}(\hbar,\wp)+\mathfrak{D}_\wp\tilde{\Lambda}(\hbar,\wp)+\tilde{\Lambda}(\hbar,\wp)-2\mathfrak{E}_\beta\big(\hbar^\beta\big)e^{-\wp},\quad 0<\hbar\le 1,\wp>0, \tag{5.19}$$

with solution

$$\tilde{\Lambda}(\hbar,\wp)=\mathfrak{E}_\beta\big(\hbar^\beta\big)e^{-\wp}. \tag{5.20}$$

Remark 5.7. The non-homogeneous fuzzy space-time fractional Telegraph equation (5.11) may be simplified by puttings $\beta = 2$ to non-homogeneous fuzzy time fractional Telegraph equation

$$\mathfrak{D}_\hbar^2\tilde{\Lambda}(\hbar,\wp)=\mathfrak{D}_\wp^{k\theta}\tilde{\Lambda}(\hbar,\wp)+\mathfrak{D}_\wp^{n\theta}\tilde{\Lambda}(\hbar,\wp)+\tilde{\Lambda}(\hbar,\wp)-2e^\hbar\mathfrak{E}_\theta(-\wp^\theta),\quad 0<\hbar\le 1,\wp>0, \tag{5.21}$$

$$\tilde{\Lambda}(\hbar,\wp) = \tilde{\kappa}(r)e^{\hbar}\mathfrak{E}_{\theta}\big(-\wp^{\theta}\big). \tag{5.22}$$

Remark 5.8. The non-homogeneous fuzzy space-time fractional Telegraph equation (5.11) may be simplified by putting $\beta = 2, m = n = 2, k = 4,$ to non-homogeneous fuzzy fractional Telegraph equation

$$\mathfrak{D}^2_{\hbar}\tilde{\Lambda}(\hbar,\wp) = \mathfrak{D}^2_{\wp}\tilde{\Lambda}(\hbar,\wp) + \mathfrak{D}_{\wp}\tilde{\Lambda}(\hbar,\wp) + \tilde{\Lambda}(\hbar,\wp) - 2e^{\hbar}\mathfrak{E}_{1/2}\big(-\wp^{1/2}\big), \quad 0 < \hbar \leq 1, \wp > 0, \tag{5.23}$$

with solution

$$\tilde{\Lambda}(\hbar,\wp) = \tilde{\kappa}(r)\ e^{\hbar}\mathfrak{E}_{1/2}\big(-\wp^{1/2}\big). \tag{5.24}$$

The physical behavior of the obtained solution at $\wp \in [0, 2]$ is depicted in 4. The solution graphs of Example 5.5 can be seen to be friendly with each other in this illustration. Figure 4 shows how, with more iterations, the FSTIM solution approaches the precise answer more quickly. Figure 4 shows the 2D graphs of $\tilde{\Lambda}(\hbar,\wp)$ at several fractional orders of $\beta = 1.2, 1.6, 1.8,$ and 2. Figure 4 depicts evaluative solution estimations. The charts show that as the fractional order β approaches its integer value, the curve will end at classical order 2.

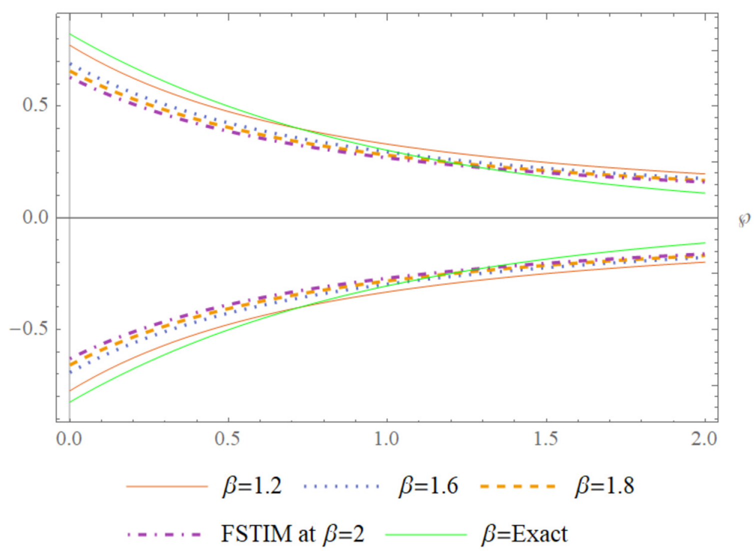

Figure 4. Example 2's upper and lower fuzzy figures for different fractional orders β.

The convergence analysis is sufficiently reliable to estimate the maximum absolute error of the Example 5.5 at $\beta = 2, \hbar = 0.2, 0.4, 0.6, 0.8$ and $\wp = 0.3, 0.5, 0.7, 0.9$.

Table 2. Comparison and error analysis between the value $\tilde{\Lambda}(\hbar,\wp)$ for the Exact and Approximate solution of the Equation (5.13).

$\hbar$	$\wp$	$\underline{\Lambda}_{FSTIM}$	$\underline{\Lambda}_{Exact}$	$\underline{\Lambda}_{Error}$	$\overline{\Lambda}_{FSTIM}$	$\overline{\Lambda}_{Exact}$	$\overline{\Lambda}_{Error}$
0.2	0.3	-0.408072	-0.452419	0.0443472	0.408072	0.452419	0.0443472
0.4	0.5	-0.365967	-0.452419	0.0864514	0.365967	0.452419	0.0864514
0.6	0.7	-0.32658	-0.452419	0.125839	0.32658	0.452419	0.125839
0.8	0.9	-0.290166	-0.452419	0.162252	0.290166	0.452419	0.162252

6. Conclusion

In this study, FSTIM was employed to obtain approximate solutions for linear and nonlinear space-time fuzzy fractional Telegraph equations with the Caputo operator. The use of fuzzy operators proved more suitable for capturing physical phenomena in these contexts. The authors demonstrated the application of the space-time fuzzy fractional Telegraph equation through a fuzzy technique, addressing uncertainty in the initial condition. The research extended the fuzzy fractionalization of space-time Telegraph equations, utilizing FSTIM to derive the approximate parametric formulation of the proposed problem. Various illustrations supported the methodology, yielding a parametric solution for each scenario. Analytical solutions for many types of fuzzy fractional partial differential equations are challenging to find; however, FSTIM's rapid convergence to a power series approximation makes it a valuable tool for addressing nonlinear challenges in science and engineering without reliance on linearization, discretization, or perturbation techniques. This approach uniquely tackles both linear and nonlinear space-time fuzzy fractional Telegraph equations simultaneously. Graphical analysis and computations were conducted using Mathematica package 11.3.0.0 in this study, and readers are encouraged to explore this approach, considering alternatives such as the Katugampola fractional derivative alongside the Caputo-Fabrizio operator.

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