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Existence and uniqueness of positive solutions for a Hadamard fractional integral boundary value problem

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Abstract

The main aim of this paper is to study a kind of boundary value problem with an integral boundary condition including Hadamard-type fractional differential equations. To do this, upper and lower solutions are used to guarantee their existence, and Schauders fixed point theorem is used to prove the uniqueness of the positive solutions to this problem. An illustrated example is presented to explain the theorems that have been proved.

Keywords. Hadamard fractional derivative, Boundary value problem, Integral boundary condition.
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1. Introduction

For some time now, fractional order differential equations due to their wide applications in sciences such as physics, engineering, economy, control systems, chemistry, biology, medicine, etc. (see [8, 11, 12]) have attracted the attention of many scientists and mathematicians. During this time, several operators as a generalization of the derivative of the natural order to the derivative of each order, such as Riemann-Liouville, Caputo, Graunwald-Letnikov, Hadamard, Atangana-Baleanu, etc. have been proposed. Many studies have been done on boundary and initial value problems, including fractional differential equations, with some operators such as Riemann-Liouville and Caputo (see [1–3, 7–13]). However, relatively few studies have been performed on boundary and initial value problems, including Hadamard-type fractional-order differential equations. See [4, 14] and the references there in.

In this article we will investigate the existence and uniqueness of positive solutions for the following integral boundary value problem on $[1, e]$

$$ {}^{\mathcal{CH}}\mathfrak{D}_1^{\varrho} u(t) = f(t, u(t)), \tag{1.1} $$

$$ u(1) = \frac{1}{\lambda}\int_1^{\eta} u(t)dt, u'(1) = 0, \tag{1.2} $$

where ${}^{\mathcal{CH}}\mathfrak{D}^{\varrho}$ the Caputo-Hadamard fractional derivative, $\varrho \in (1, 2]$, $1 < \eta < e$, $\lambda > \eta$. At first, we need some preliminary facts about the subject of the paper.

2. Preliminaries

Throughout the paper, we use $C([1, e])$ to indicate the Banach space of all real-valued continuous functions defined on the closed interval $[1, e]$, which equipped with the maximum norm. Also we define the cone

$$ \Sigma = \{x \in X : x(t) \geq 0, \forall t \in [1, e]\}. $$

Here we recall some definitions, lemmas and theorems that will be used in this paper. One can find more details in [8, 12].

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Definition 2.1. ([8]). Let $f : [1, +\infty) \to \mathbb{R}$ be a continuous function and $\varrho > 0$, the Hadamard fractional integral of order ϱ is defined as

$$^{H}\mathfrak{J}_1^{\varrho} f(t) = \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} f(s) \frac{ds}{s}.$$

Definition 2.2. ([8]). Let $f : [1, +\infty) \to \mathbb{R}$ be a continuous function and $\varrho > 0$, the Caputo-Hadamard fractional derivative of order ϱ is defined as

$$^{\mathcal{CH}}\mathfrak{D}_1^{\varrho} f(t) = \frac{1}{\Gamma(n-\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\varrho-1} \left(s\frac{d}{ds}\right)^n f(s) \frac{ds}{s}.$$

Lemma 2.3. *([8]) Let $n-1 < \varrho \leq n, n \in \mathbb{N}$ and $f \in C^n([1,T])$. Then*

$$(^{H}\mathfrak{J}_1^{\varrho}\, {}^{\mathcal{CH}}\mathfrak{D}_1^{\varrho})(f(t)) = f(t) - \sum_{j=0}^{n-1} \frac{f^j(1)}{\Gamma(j+1)} \log(t)^j.$$

3. Main Results

The main purpose of this section is to prove uniqueness of problem (1.1)-(1.2) using fixed point theorem. The following auxiliary lemma is quite useful in concluding the desired results.

Lemma 3.1. *Suppose $h : [0,1] \to [0,\infty)$ be a continuous function, then the unique solution of the fractional boundary value problem (FBVP)*

$$^{\mathcal{CH}}\mathfrak{D}_1^{\varrho} u(t) = h(t), \tag{3.1}$$

$$u(1) = \frac{1}{\lambda} \int_1^{\eta} u(t)dt, \quad u'(1) = 0. \tag{3.2}$$

is expressed by

$$u(t) = \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} h(s)\frac{ds}{s} + \frac{1}{(\lambda-\eta)\Gamma(\varrho)} \int_1^{\eta}\int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} h(s)\frac{ds}{s}dt. \tag{3.3}$$

Proof. In view of lemma 2.3 we have

$$u(t) = \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} h(s)\frac{ds}{s} + u(1) + u'(1)\log t.$$

By applying initial conditions to the above relation we get

$$u(t) = \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} h(s)\frac{ds}{s} + \frac{1}{\lambda}\int_1^{\eta} u(s)ds. \tag{3.4}$$

Integrating relation (3.4) yields

$$\int_1^{\eta} u(t)dt = \frac{1}{\Gamma(\varrho)} \int_1^{\eta}\int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} h(s)\frac{ds}{s}dt + \frac{\eta}{\lambda}\int_1^{\eta} u(s)ds. \tag{3.5}$$

So

$$(1-\frac{\eta}{\lambda})\int_1^{\eta} u(t)dt = \frac{1}{\Gamma(\varrho)} \int_1^{\eta}\int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} h(s)\frac{ds}{s}dt. \tag{3.6}$$

By substituting relation (3.6) in relation (3.4), the necessary result is obtained. □

Corollary 3.2. *The fractional boundary value problem* (1.1)-(1.2) *is equivalent to the integral equation*

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} f(s,u(s)) \frac{ds}{s} \\ &+ \frac{1}{(\lambda-\eta)\Gamma(\varrho)} \int_1^t \int_1^\eta \left(\log \frac{t}{s}\right)^{\varrho-1} f(s,u(s)) \frac{ds}{s} dt. \end{aligned} \tag{3.7}$$

Now we can transform problem (1.1)-(1.2) into a fixed point problem, for this purpose we define

$$\begin{aligned} (\Psi u)(t) &= \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho-1} f(s,u(s)) \frac{ds}{s} \\ &+ \frac{1}{(\lambda-\eta)\Gamma(\varrho)} \int_1^t \int_1^\eta \left(\log \frac{t}{s}\right)^{\varrho-1} f(s,u(s)) \frac{ds}{s} dt. \end{aligned} \tag{3.8}$$

Clearly, the fixed points of operator (3.8) are equal to the solutions of problem (1.1)-(1.2).

Definition 3.3. Let $a, b \in \mathbb{R}^+$ and

$$\overline{\Xi}(t,u) = \sup_{a \le \upsilon \le u} f(t,\upsilon), \quad \underline{\Xi}(t,u) = \inf_{u \le \upsilon \le b} f(t,\upsilon),$$

then $\overline{\Xi}(t,u)$, $\underline{\Xi}(t,u)$ are called respectively the upper and lower control functions.

Definition 3.4. Let $u^*(t), u_*(t) \in \Sigma$, and there exists $a, b \in \mathbb{R}$, such that for $a \le u_*(t) \le u^*(t) \le b$ we have

$$\begin{aligned} &{}^{\mathcal{CH}}\mathfrak{D}_1^{\varrho} u^*(t) \ge \overline{\Xi}(t, u^*(t)), \\ &u^*(1) \ge \frac{1}{\lambda} \int_1^\eta u^*(t) dt, \quad \frac{d}{dt} u^*(1) \ge 0, \end{aligned}$$

and

$$\begin{aligned} &{}^{\mathcal{CH}}\mathfrak{D}_1^{\varrho} u_*(t) \le \underline{\Xi}(t, u_*(t)), \\ &u_*(1) \le \frac{1}{\lambda} \int_1^\eta u_*(t) dt, \quad \frac{d}{dt} u_*(1) \le 0. \end{aligned}$$

Then $u^*(t)$ and $u_*(t)$ are called upper and lower solutions for the fractional boundary value problem (1.1)-(1.2).

The following hypothesists will be need for our results.

(H1) $f : [1, e] \times [0, \infty) \to [0, \infty)$ is continuous.
(H2) $u^*(t), u_*(t)$ are upper and lower solutions respectively for the problem (1.1)-(1.2).

Theorem 3.5. *Let* $(H1)$ *hold, then the operator* $\Psi : \Sigma \to \Sigma$ *is completely continuous.*

Proof. Since f is continuous function, it is concluded that $\Psi : \Sigma \to \Sigma$ is a continuous operator. Now if we define B_δ as

$$B_\delta = \{u \in \Sigma, \|u\| \le \delta\},$$

then $f : [1, e] \times B_\delta \to [0, \infty)$ will be bounded. That is there exists $\zeta > 0$ such that for all $u \in B_\delta$ we have $0 \leq f(t, u(t)) \leq \zeta$. So we obtain

$$\begin{aligned}
&|(\Psi u)(t)| \\
\leq\ & \left| \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s} \right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s} \right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} dt \right| \\
\leq\ & \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s} \right)^{\varrho - 1} |f(s, u(s))| \frac{ds}{s} + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s} \right)^{\varrho - 1} |f(s, u(s))| \frac{ds}{s} dt \\
\leq\ & \frac{\zeta}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s} \right)^{\varrho - 1} \frac{ds}{s} + \frac{\zeta}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s} \right)^{\varrho - 1} \frac{ds}{s} dt \\
\leq\ & \frac{\zeta}{\Gamma(\varrho + 1)} (\log t)^\varrho + \frac{\zeta}{(\lambda - \eta)\Gamma(\varrho + 1)} \int_1^\eta (\log t)^\varrho dt \\
\leq\ & \frac{\zeta}{\Gamma(\varrho + 1)} (\log t)^\varrho + \frac{\zeta}{(\lambda - \eta)\Gamma(\varrho + 1)} \int_1^\eta (\log t) dt \\
\leq\ & \frac{\zeta}{\Gamma(\varrho + 1)} (\log t)^\varrho + \frac{\zeta}{(\lambda - \eta)\Gamma(\varrho + 1)} (\eta \log \eta + \eta + 1).
\end{aligned}$$

That is $\Psi(B_\delta)$ is uniformly bounded.

Now it is necessary to show $\Psi(\Sigma)$ is equi-continuous. For this purpose, we assume $t_1, t_2 \in [1, e]$ with $t_1 \leq t_2$ and $u \in \Sigma$, then

$$\begin{aligned}
&|(\Psi u)(t_2) - (\Psi u)(t_1)| \\
=\ & \left| \frac{1}{\Gamma(\varrho)} \int_1^{t_2} \left(\log \frac{t_2}{s} \right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^{t_2} \left(\log \frac{t_2}{s} \right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} dt \right. \\
& \left. - \frac{1}{\Gamma(\varrho)} \int_1^{t_1} \left(\log \frac{t_1}{s} \right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^{t_1} \left(\log \frac{t_1}{s} \right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} dt \right| \\
\leq\ & \left| \frac{1}{\Gamma(\varrho)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\varrho - 1} - \left(\log \frac{t_1}{s} \right)^{\varrho - 1} \right] f(s, u(s)) \frac{ds}{s} \right. \\
& \left. + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\varrho - 1} - \left(\log \frac{t_1}{s} \right)^{\varrho - 1} \right] f(s, u(s)) \frac{ds}{s} dt \right| \\
& + \left| \frac{1}{\Gamma(\varrho)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} dt \right| \\
\leq\ & \frac{\zeta}{\Gamma(\varrho)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\varrho - 1} - \left(\log \frac{t_1}{s} \right)^{\varrho - 1} \right] \frac{ds}{s} \\
& + \frac{(\eta - 1)\zeta}{(\lambda - \eta)\Gamma(\varrho)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\varrho - 1} - \left(\log \frac{t_1}{s} \right)^{\varrho - 1} \right] \frac{ds}{s} \\
& + \frac{\zeta}{\Gamma(\varrho)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\varrho - 1} \frac{ds}{s} + \frac{(\eta - 1)\zeta}{(\lambda - \eta)\Gamma(\varrho)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\varrho - 1} \frac{ds}{s} \\
\leq\ & \frac{\zeta}{\Gamma(\varrho + 1)} |(\log t_2)^\varrho - (\log t_1)^\varrho| + \frac{(\eta - 1)\zeta}{(\lambda - \eta)\Gamma(\varrho)} |(\log t_2)^\varrho - (\log t_1)^\varrho| \\
& + \frac{\zeta}{\Gamma(\varrho + 1)} \left| \log \frac{t_2}{t_1} \right|^\varrho + \frac{(\eta - 1)\zeta}{(\lambda - \eta)\Gamma(\varrho + 1)} \left| \log \frac{t_2}{t_1} \right|^\varrho.
\end{aligned}$$

Therefore $|(\Psi u)(t_2) - (\Psi u)(t_1)| \to 0$, when $t_2 \to t_1$. Consequently, under the conditions stated above, $\Psi(B_\delta)$ is equi-continuous and using the Arzela-Ascoli theorem [5, 6] we conclude that $\Psi : \Sigma \to \Sigma$ is completely continuous. □

Theorem 3.6. *Assume $(H1) - (H2)$ hold. Then the fractional boundary value problem (1.1)-(1.2) has at least one positive solution $u(t)$ such that $u_*(t) \le u(t) \le u^*(t)$.*

Proof. We will apply the Schauder's fixed point theorem to show the existence of positive solutions for the problem (1.1)-(1.2). Firstly, we need to define

$$K = \{u(t) | u(t) \in \Sigma, u_*(t) \le u(t) \le u^*(t), t \in [1, e]\}.$$

It is clear that K is closed, convex and bounded subset of $C([1, e], \mathbb{R}^+)$. Also for any $u(t) \in K$, we have $u_*(t) \le u(t) \le u^*(t)$. Now we show that $\Psi(K) \subset K$. Let $u \in K$ then we have

$$\begin{aligned}
(\Psi u)(t) &= \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} \\
&\quad + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} dt \\
&\le \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} \Xi(s, u(s)) \frac{ds}{s} \\
&\quad + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} \Xi(s, u(s)) \frac{ds}{s} dt \\
&\le \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} \Xi(s, u^*(s)) \frac{ds}{s} \\
&\quad + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} \Xi(s, u^*(s)) \frac{ds}{s} dt \\
&\le u^*(t),
\end{aligned}$$

and

$$\begin{aligned}
(\Psi u)(t) &= \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} \\
&\quad + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} f(s, u(s)) \frac{ds}{s} dt \\
&\ge \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} \Xi(s, u(s)) \frac{ds}{s} \\
&\quad + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} \Xi(s, u(s)) \frac{ds}{s} dt \\
&\ge \frac{1}{\Gamma(\varrho)} \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} L(s, u_*(s)) \frac{ds}{s} \\
&\quad + \frac{1}{(\lambda - \eta)\Gamma(\varrho)} \int_1^\eta \int_1^t \left(\log \frac{t}{s}\right)^{\varrho - 1} \Xi(s, u_*(s)) \frac{ds}{s} dt \\
&\ge u_*(t).
\end{aligned}$$

Thus we can conclude that $\Psi(u) \subset K$. So all the conditions of the Schauder's fixed point theorem are satisfied and there exists at least one fixed point in K. Therefore the problem (1.1)-(1.2) will have at least one positive solution in K.

□

Corollary 3.7. *Assume that, there exist constants $\zeta_1, \zeta_2 > 0$, such that for all $(t, u(t)) \in J \times [0, \infty)$ we have $0 \le \zeta_1 \le f(t, u(t)) \le \zeta_2 < \infty$. Then fractional boundary value problem (1.1)-(1.2) has at least one positive solution.*

Proof. From the hypothesis and definition of functions $\underline{\Xi}(t,u), \overline{\Xi}(t,u)$, we have

$$\zeta_1 \le \underline{\Xi}(t,u) \le \overline{\Xi}(t,u) \le \zeta_2. \tag{3.9}$$

Let

$$^{\mathcal{CH}}\mathfrak{D}_1^{\varrho} u^*(t) = \zeta_2, \quad u^*(1) = \frac{1}{\lambda}\int_1^{\eta} u^*(t)dt. \tag{3.10}$$

Equation (3.10) has at least one positive solution like

$$u^*(t) = \frac{\zeta_2}{\Gamma(\varrho)}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho-1}\frac{ds}{s} + \frac{\zeta_2}{(\lambda-\eta)\Gamma(\varrho)}\int_1^{\eta}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho-1}\frac{ds}{s}dt. \tag{3.11}$$

Now from relation (3.9) we can see

$$u^*(t) \ge \frac{1}{\Gamma(\varrho)}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho-1}\overline{\Xi}(s,u^*)\frac{ds}{s} + \frac{1}{(\lambda-\eta)\Gamma(\varrho)}\int_1^{\eta}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho-1}\overline{\Xi}(s,u^*)\frac{ds}{s}dt. \tag{3.12}$$

Consequently u^* is the upper solution of fractional boundary value problem (1.1)-(1.2). By the similar way if we let

$$^{\mathcal{CH}}\mathfrak{D}_1^{\varrho} u_*(t) = \zeta_1, \quad u_*(1) = \frac{1}{\lambda}\int_1^{\eta} u_*(t)dt. \tag{3.13}$$

It is clear equation (3.13) has a positive solution

$$u_*(t) = \frac{\zeta_1}{\Gamma(\varrho)}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho-1}\frac{ds}{s} + \frac{\zeta_1}{(\lambda-\eta)\Gamma(\varrho)}\int_1^{\eta}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho-1}\frac{ds}{s}dt. \tag{3.14}$$

From relation (3.9) it is concluded that $u_*(t)$ is the lower solution of fractional boundary value problem (1.1)-(1.2), Now all conditions of Theorem hold and fractional boundary value problem (1.1)-(1.2) has at least one positive solution. □

Theorem 3.8. *Let $(H1)-(H2)$ hold and there exists constant $q > 0$ such that*

$$|f(t,u) - f(t,v)| \le q|u-v|, \quad t \in [1,e], u, v \ge 0, \tag{3.15}$$

where $\frac{q}{(\lambda-\eta)\Gamma(\varrho+1)}[\lambda + \eta\log\eta + 1] < 1$ then the problem (1.1)-(1.2) has a unique positive solution $u \in K$

Proof. Let $u, v \in K$, in view of the relation (3.8) we have

$$\begin{aligned}
&|(\Psi u)(t) - (\Psi v)(t)| \\
\le\;& \frac{1}{\Gamma(\varrho)}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho-1}|f(s,u(s)) - f(s,v(s))|\frac{ds}{s} \\
\le\;& \frac{1}{(\lambda-\eta)\Gamma(\varrho)}\int_1^{\eta}\int_1^t \left(\log\frac{t}{s}\right)^{\varrho-1}|f(s,u(s)) - f(s,v(s))|\frac{ds}{s}dt \\
\le\;& \frac{(\log t)^{\varrho}}{\Gamma(\varrho+1)}q\|u-v\| + \frac{q\|u-v\|}{(\lambda-\eta)\Gamma(\varrho+1)}\int_1^{\eta}(\log t)^{\varrho}dt \\
\le\;& \frac{q}{\Gamma(\varrho+1)}\left[(\log t)^{\varrho} + \frac{\eta\log\eta + \eta + 1}{\lambda-\eta}\right]\|u-v\| \\
\le\;& \frac{q}{(\lambda-\eta)\Gamma(\varrho+1)}[\lambda + \eta\log\eta + 1]\|u-v\|.
\end{aligned}$$

Thus

$$\|(\Psi u_1)(t) - (\Psi v)(t)\| \le \frac{q}{(\lambda-\eta)\Gamma(\varrho+1)}[\lambda + \eta\log\eta + 1]\|u-v\|.$$

So the operator Ψ is a contraction mapping and by the contraction mapping principle, the problem (1.1)-(1.2) has a unique positive solution $u \in K$. □

Example 3.9. Consider the Hadamard fractional boundary value problem

$$\begin{cases} {}^{\mathcal{CH}}\mathfrak{D}_1^{\frac{3}{2}}u(t) = \frac{1}{t+2}\left(\frac{t\sin x(t)}{e+x(t)}+2\right), \\ u(1) = \frac{1}{e}\int_1^{\frac{e}{2}} u(t)dt, u'(1) = 0, \end{cases} \tag{3.16}$$

where $\eta = \frac{e}{2}$, $\lambda = e$ and $\frac{1}{t+2}\left(\frac{t\sin x(t)}{e+x(t)}+2\right)$. It is easy to see that $\frac{2}{3} \le f(t,x(t)) \le 1$. Hence by Corollary 3.7 the fractional boundary value problem (3.16) has at least one positive solution $u(t)$, with $u_*(t) \le u(t) \le u^*(t)$, where

$$u^*(t) = \frac{1}{\Gamma(\frac{5}{2})}(\log t)^{\frac{3}{2}} + \frac{4}{e\Gamma(\frac{5}{2})}\int_1^{\frac{e}{2}}(\log t)^{\frac{3}{2}}dt,$$

$$u_*(t) = \frac{2}{3\Gamma(\frac{5}{2})}(\log t)^{\frac{3}{2}} + \frac{8}{3e\Gamma(\frac{5}{2})}\int_1^{\frac{e}{2}}(\log t)^{\frac{3}{2}}dt.$$

4. Conclusion

This work investigated a boundary value problem involving fractional differential equations with the Caputo-Hadamard and an integral boundary condition. We presented two theorems about the existence of solutions and the uniqueness of positive solutions. For this purpose, we used different methods, such as the upper-lower solution method and the fixed-point method. In the future one can extend the problem and combine it with some operators such as the p-laplacian operator and show the existence and uniqueness of positive solutions for the Caputo-Hadamard fractional boundary value problem with an integral boundary condition.

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